A novel concept-cognitive learning model oriented to three-way concept for knowledge acquisition

Weihua Xu, Di Jiang

***Abstract*—Concept-cognitive learning(CCL) is the process of enabling machines to simulate the concept learning of the human brain. Existing CCL models focus on formal context while neglecting the importance of skill context. Furthermore, CCL models, which solely focus on positive information, restrict the learning capacity by neglecting negative information, and greatly impeding the acquisition of knowledge. To overcome these issues, we proposes a novel concept-cognitive learning model oriented to three-way concept for knowledge acquisition. Firstly, this paper explains and investigates the relationship between skills and knowledge based on the three-way concept and its properties. Then, in order to simultaneously consider positive and negative information, describe more detailed information, learn more skills, and acquire accurate knowledge, a three-way information granule is described from the perspective of cognitive learning. Then, a transformation method is proposed to transform between different three-way information granules, allowing for the transformation of arbitrary three-way information granule into necessary, sufficient, sufficient and necessary three-way information granules. Finally, algorithm corresponding to the transformation method is designed, and subsequently tested across diverse UCI datasets. The experimental outcomes affirm the effectiveness and excellence of the suggested model and algorithm.**

***Index Terms*—Three-way concept; Concept-cognitive learning; Granular computing; Formal concept analysis.**

I. Introduction

In concept learning, concept serve as the foundation of human cognition, the speed of learning processing in human brain is remarkably fast, and the effectiveness is significant, humans are capable of quickly comprehending and recognizing a wide range of concepts from limited information. In contrast, machine learning lags behind the human brain in terms of both speed and effectiveness. Concept-cognitive learning(CCL), as a part of artificial intelligence, aims to simulate the learning processes of the human brain through computational systems. It involves concept learning using specific cognitive models based on given uncertain and inaccurate clue [1]. As Wille [2] proposed formal concept analysis (FCA) based on formal context, it provided mathematical tool for obtaining concepts. Through continuous development, CCL has become an efficient method and has gained considerable attention from researchers, resulting in numerous effective outcomes such as Wille formal concept [2], three-way concept [3], [4], fuzzy concept [5], [6], and weighted concept [7]. Furthermore, with the rapid advancement of the era of artificial intelligence, machine learning [8], [9] and big data [10], [11] has attracted numerous researchers to delve into its study.

Granular computing (GrC) serves as a vital instrument for addressing intricate problems [12]. Granular computing simulates the way the human brain processes problems, by using information granules to break down complex problems into simpler ones, thereby effectively reducing the complexity of the problem. Granular computing aims to assist humans in better cognition, reasoning, and decision-making [13], [14]. With ongoing research by scholars, the concept of granular computing has found application in diverse domains. Zhang et al. [15] discussed the double-quantitative rough set models using GrC. Wu et al. [16] discussed granular computing and its applications in formal contexts. Xu et al. [17] introduced a novel granular computing model and developed a two-way learning system using information granules. Cabrerizo et al. [18] utilized information granules to replace numerical values, and estimate missing values in incomplete preference relations, thereby improving the consistency of relationships.

Concept-cognitive learning(CCL) is the science of cognition and learning things via concepts [19], different learning methods enable the recognition of different concepts, thereby giving them different semantic interpretations. Guo et al. [20] combined the memory mechanism with CCL for knowledge fusion. Shi et al. [21] proposed new CCL models based on concept space building upon the classical CCL method, to integrate newly added objects with acquired knowledge, thereby enhancing concept learning ability, and incremental learning has been achieved. Mi et al. [22] introduced a learning method for fuzzy concepts to focus on object information and handle continuous data. Yuan et al. [23] focused on incremental learning, combining positive and negative operators to form fuzzy three-way concept, this approach enables concepts to capture more detailed information, and facilitates dynamic object classification by accommodating the addition of new data. Wang et al. [24] conducted fuzzy concept-cognitive learning from multiple views. Zhang et al. [7] proposed a CCL approach that assigns different weights to each attribute in fuzzy formal context, this approach introduced the weighted fuzzy concept, which allows for the acquisition of interesting knowledge and the removal of redundant information. Guo et al. [4] proposed Fuzzy-granular concept-cognitive learning via three-way decision. Xu et al. [25] proposed a CCL

This paper was supported by the National Natural Science Foundation of China under Grant 62376229. (*Corresponding author: Weihua Xu.*)

Weihua Xu and Di Jiang are with the College of Artificial Intelligence, Southwest University, Chongqing 400715, China (e-main:chxuwh@gmail.com, 1754105726@qq.com).

model that takes into account the influence of attention, this model generates pseudo-concept with higher expectation and improves the efficiency of classification task. Moreover, over time, researchers have proposed many practical and efficient methods, models, and frameworks for CCL [26]–[28]. Xu et al. [29] discussed the connection between objects and attributes within formal context, they proposed a transformation model to transform a general information granule into necessary, sufficient, and sufficient and necessary information granules, this paper draws inspiration from its ideas. Ref [30] and [31] conducted research on relationships. Hu et al. [32] introduced a transformation model in interval-valued formal context to transform between different interval-valued information granules. Based on the framework of two-way learning, Xu et al. [33], [34] proposed a new CCL method that addresses the challenges of concept generation and concept evolution by considering the concept movement perspective. Furthermore, they introduced a progressive learning mechanism in dynamic environment for fuzzy context, which reduces the time required for granular concept learning. Two-way concept-cognitive learning has attracted the attention of scholars [35]–[37]. But these only focuses on the formal context, and does not consider the skill context, different from the aforementioned learning methods and models, Xie et al. [38] proposed a CCL model by combining FCA and knowledge space theory(KST), emphasizing the aspect of competence, this model focuses on problem-solving and skill learning. However, in the skill context, it only considers positive information, while this paper simultaneously considers both positive and negative information, enabling it to take into account more detailed information. and the knowledge obtained is more accurate. Table I presents a comparative analysis of different CCL models. Ref [39], [40] conducted a detailed survey about concept-cognitive learning.

TABLE I
COMPARATIVE ANALYSIS AMONG DIFFERENT CCL MODELS

Model	F	S	P and N
this paper	×	✓	✓
[29]	✓	×	×
[33]	✓	×	×
[37]	✓	×	×
[38]	×	✓	×
[36]	×	✓	×

Note: F represent Formal context, S represent Skill context, P and N represent consider Positive and Negative information.

Knowledge space theory (KST), as formulated by Doignon [41], represents a mathematical theory, it has been widely applied in knowledge assessment, adaptive testing [42], [43]. Rusch et al. [44] proved the mutual transformation between formal context and knowledge space, thus linking FCA with KST. Li et al. [45] introduced the method and theory of knowledge base in knowledge space and formal context, and further established a connection between the two through knowledge base. Doble et al. [46] applied the theory of knowledge space to the simulation study of adaptive assessment. In recent decades, an increasing number of scholars have been paying attention to the relationship between skill and problem. Duntsch et al. [47] introduced the skill function and problem function. Heller et al. [48] described two scenarios for skill mapping: conjunction model and disjunction model. Zhou et al. [49], [50] established a skill context and discussed skill reduction and skill evaluation. Sun et al. [51] constructed a knowledge space through fuzzy skill map. Skills are crucial for solving problems, mastering simple skills allows one to solve simple problems, and as more complex skills are learned, one can tackle more complex problems, therefore, as skills are acquired, more problems can be solved, and more knowledge can be obtained, additionally, there are relationships between skills.

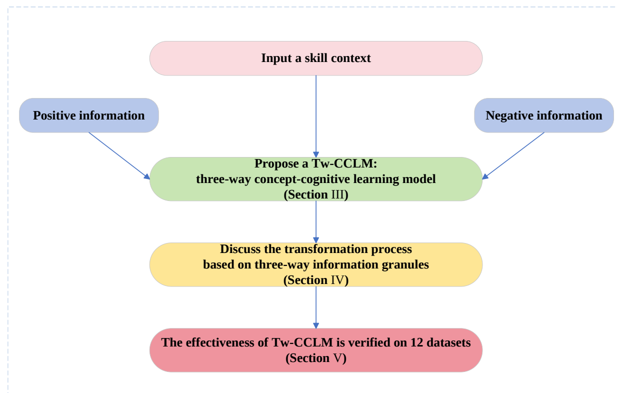


Fig. 1. Block diagram of the proposed approach.

The existing concept-cognitive learning models focus on formal context while neglecting the importance of skill context. In addition, models that only considers positive information, thereby overlooking many important details, significantly limiting the learning capability and knowledge acquisition. To tackle these limitations, this paper proposes a novel concept-cognitive learning model oriented to three-way concept for knowledge acquisition, in the skill context, considering both positive and negative information allows for a more detailed description of information, leading to more accurate knowledge acquisition. Fig. 1 depicts the block diagram of the proposed approach. The primary contributions of this model are delineated as follows:

1) The three-way concept-cognitive learning model(Tw-CCLM) is proposed. The study utilizes the skill context, and explores the relationship between skills and knowledge.

2) Based on FCA and GrC, the information granule is described in three-way form. The three-way concept is represented as sufficient and necessary three-way information granule, allowing the cognitive process to consider more detailed information and learn more skills, acquiring more accurate knowledge, thus aligning more closely with human cognition.

3) A transformation method is proposed for Tw-CCLM, from arbitrary three-way information granule into sufficient

and necessary three-way information granules (i.e., three-way concepts). Furthermore, an algorithm for the method is subsequently developed, followed by empirical validations to demonstrate its efficacy.

The following sections of this paper are organized as outlined below: Section II provides an introduction to formal concept analysis and knowledge space theory. Section III presents a novel concept-cognitive learning model called Tw-CCLM. Section IV introduces a transformation method for three-way information granule and presents the respective algorithm. Section V is dedicated to validating the efficiency of the proposed algorithm through experiments. Section VI serves as a conclusion to the research.

II. Related Word

This section will cover essential concepts, including FCA and KST. This will help us gain a better understanding of the learning process that follows.

A. Formal concept analysis

Let (U, A, R) as a formal context, where $U = \{y_1, y_2, \ldots, y_n\}$ denotes the object set and $A = \{m_1, m_2, \ldots, m_n\}$ denotes the attribute set, and $R \subseteq U \times A$ represents the binary relation between U and A. If $(y, m) \in R$ denotes that attribute m is possessed by object y, meaning $R(y, m) = 1$, while $(y, m) \notin R$ indicates the absence of attribute m in object y, meaning $R(y, m) = 0$. The attribute set $R(y) = \{m \in A | (y, m) \in R\}$ represents the attribute that object y possesses, and the object set $R(m) = \{y \in U | (y, m) \in R\}$ represents the object possessing attribute m. Further, the complement of R is marked as $R^C = U \times A - R$.

In a formal context (U, A, R), for $Y \subseteq U$ and $M \subseteq A$, a pair of operators is introduced for further study:

$Y^* = \{m \in A | \forall y \in Y, (y, m) \in R\}$,

$M^* = \{y \in U | \forall m \in M, (y, m) \in R\}$.

where Y^* signifies a set of attributes representing each attribute possessed by all objects in set Y, and M^* represents a set of objects, each of which possesses all the attributes within M. Therefore, this pair of operators $(*, *)$ is referred to as the positive operators, capable of describing positive information.

In a formal context, when $Y^* = M$ and $M^* = Y$ are satisfied, the pair (Y, M) is termed a formal concept, where Y represents the extent and M represents the intent. In practical problems, for a set of objects, there are only a few common attributes shared among them, while the attributes that they individually lack are significantly different. When dealing with data, analyzing only the shared attributes is not sufficient and may overlook valuable information, the introduction of three-way concept addresses this limitation effectively.

To construct three-way concept, an additional pair of negative operators $(*-, *-)$ is introduced to capture negative information. In a formal context (U, A, R), a pair of negative operators as follows:

$Y^{*-} = \{m \in A | \forall y \in Y, (y, m) \notin R\}$,

$M^{*-} = \{y \in U | \forall m \in M, (y, m) \notin R\}$.

The negative operators describe information from the perspective of objects and attributes that are not jointly possessed.

In a formal context (U, A, R), for $Y \subseteq U$ and $M, T \subseteq A$, considering the aforementioned positive and negative operators, then the three-way operator and its inverse operator as follows:$Y \uparrow = (Y^*, Y^{*-})$ and $(M, T) \downarrow = M^* \cap T^{*-}$.The pair $(Y, (M, T))$ is denoted as a three-way concept when (see [52])

$Y \uparrow = (M, T)$, $(M, T) \downarrow = Y$.

The Y and (M, T) are commonly known as the extent and intent, respectively. As the three-way concept takes into account both positive and negative information, it is capable of providing a more detailed description.

In a formal context (U, A, R), for $Y \subseteq U$ and $M \subseteq A$, four operators is defined as follows: (see [53], [54])

$Y^\Diamond = \{m \in A | Y \cap R(m) \neq \emptyset\}$,

$M^\Box = \{y \in U | R(y) \subseteq M\}$,

$Y^{\overline{\Diamond}} = \{m \in A | Y \cap R^c(m) \neq \emptyset\}$,

$M^{\overline{\Box}} = \{y \in U | R^c(y) \subseteq M\}$.

the semantics expressed by $(*, *)$ and $(*-, *-)$ are different. Where $Y^\Diamond$ is a set of attributes representing the attribute possessed by at least one object in Y, $M^\Box$ is a set of objects representing the object in which all attributes are included in M, $Y^{\overline{\Diamond}}$ is a set of attributes representing the attribute not possessed by at least one object in Y, and $M^{\overline{\Box}}$ is a set of objects representing object where all attributes not possessed by the object are included in M.$(\Diamond, \Box)$ and $(\overline{\Diamond}, \overline{\Box})$ are termed positive and negative operators, capable of describing positive and negative information, respectively, and the subsequent works rely on them precisely.

In a formal context, when $Y^\Diamond = M$ and $M^\Box = Y$ are satisfied, the pair (Y, M) is termed a formal concept, where Y represents the extent and M represents the intent.

Definition 1: (see [55]) In a formal context (U, A, R), for $Y \subseteq U$ and $M, T \subseteq A$, considering the positive and negative operators, then the three-way operator and its inverse operators are defined as follows:

$Y^\blacktriangledown = (Y^\Diamond, Y^{\overline{\Diamond}})$ and $(M, T)^\blacktriangle = M^\Box \cap T^{\overline{\Box}}$.

Lemma 1: (see [54], [56]) In a formal context (U, A, R), for $Y, Y_1, Y_2 \subseteq U$; $M, T \subseteq A$. For positive operators $(\Diamond, \Box)$, we have:

1) $Y_1 \subseteq Y_2 \Rightarrow Y_1^\Diamond \subseteq Y_2^\Diamond$, $M \subseteq T \Rightarrow M^\Box \subseteq T^\Box$.
2) $Y \subseteq Y^{\Diamond\Box}$, $M^{\Box\Diamond} \subseteq M$.
3) $Y^\Diamond = Y^{\Diamond\Box\Diamond}$, $M^\Box = M^{\Box\Diamond\Box}$.
4) $(Y_1 \cup Y_2)^\Diamond = Y_1^\Diamond \cup Y_2^\Diamond$, $(M \cap T)^\Box = M^\Box \cap T^\Box$.
5) $(Y_1 \cap Y_2)^\Diamond \subseteq Y_1^\Diamond \cap Y_2^\Diamond$, $(M \cup T)^\Box \supseteq M^\Box \cup T^\Box$.
6) $Y \subseteq M^\Box \Leftrightarrow Y^\Diamond \subseteq M$.

Lemma 2: (see [57]) In a formal context (U, A, R), for $Y, Y_1, Y_2 \subseteq U$; $M, T \subseteq A$. For negative operators $(\overline{\Diamond}, \overline{\Box})$, we have:

1) $Y_1 \subseteq Y_2 \Rightarrow Y_1^{\overline{\Diamond}} \subseteq Y_2^{\overline{\Diamond}}$, $M \subseteq T \Rightarrow M^{\overline{\Box}} \subseteq T^{\overline{\Box}}$.
2) $Y \subseteq Y^{\overline{\Diamond}\overline{\Box}}$, $M^{\overline{\Box}\overline{\Diamond}} \subseteq M$.
3) $Y^{\overline{\Diamond}} = Y^{\overline{\Diamond}\overline{\Box}\overline{\Diamond}}$, $M^{\overline{\Box}} = M^{\overline{\Box}\overline{\Diamond}\overline{\Box}}$.
4) $(Y_1 \cup Y_2)^{\overline{\Diamond}} = Y_1^{\overline{\Diamond}} \cup Y_2^{\overline{\Diamond}}$, $(M \cap T)^{\overline{\Box}} = M^{\overline{\Box}} \cap T^{\overline{\Box}}$.
5) $(Y_1 \cap Y_2)^{\overline{\Diamond}} \subseteq Y_1^{\overline{\Diamond}} \cap Y_2^{\overline{\Diamond}}$, $(M \cup T)^{\overline{\Box}} \supseteq M^{\overline{\Box}} \cup T^{\overline{\Box}}$.
6) $Y \subseteq M^{\overline{\Box}} \Leftrightarrow Y^{\overline{\Diamond}} \subseteq M$.

Proposition 1: (see [57]) In a formal context (U, A, R), for $Y, Y_1, Y_2 \subseteq U$; $M, M_1, M_2, T, T_1, T_2 \subseteq A$. According to

Lemma 1 and 2, for the three-way operator and its inverse operators $(\blacktriangledown, \blacktriangle)$, we have:

1) $Y_1 \subseteq Y_2 \Rightarrow Y_1^{\blacktriangledown} \subseteq Y_2^{\blacktriangledown}$,
$(M_1, T_1) \subseteq (M_2, T_2) \Rightarrow (M_1, T_1)^{\blacktriangle} \subseteq (M_2, T_2)^{\blacktriangle}$.
2) $Y \subseteq Y^{\blacktriangledown\blacktriangle}$, $(M, T)^{\blacktriangle\blacktriangledown} \subseteq (M, T)$.
3) $Y^{\blacktriangledown} = Y^{\blacktriangledown\blacktriangle\blacktriangledown}$, $(M, T)^{\blacktriangle} = (M, T)^{\blacktriangle\blacktriangledown\blacktriangle}$.
4) $(Y_1 \cup Y_2)^{\blacktriangledown} = Y_1^{\blacktriangledown} \cup Y_2^{\blacktriangledown}$,
$((M_1, T_1) \cap (M_2, T_2))^{\blacktriangle} = (M_1, T_1)^{\blacktriangle} \cap (M_2, T_2)^{\blacktriangle}$.
5) $(Y_1 \cap Y_2)^{\blacktriangledown} \subseteq Y_1^{\blacktriangledown} \cap Y_2^{\blacktriangledown}$,
$((M_1, T_1) \cup (M_2, T_2))^{\blacktriangle} \supseteq (M_1, T_1)^{\blacktriangle} \cup (M_2, T_2)^{\blacktriangle}$.
6) $Y \subseteq (M, T)^{\blacktriangle} \Leftrightarrow Y^{\blacktriangledown} \subseteq (M, T)$.

About Proposition 1, a detailed proof has already been provided in Ref [57].

Definition 2: (see [55]) In a formal context (U, A, R), for $Y \subseteq U$ and $M, T \subseteq A$. The pair $(Y, (M, T))$ is denoted as a three-way concept when

$Y^{\blacktriangledown} = (M, T)$, $(M, T)^{\blacktriangle} = Y$.

The Y and (M, T) are commonly known as the extent and intent, respectively, the three-way concept can simultaneously describe positive and negative information about objects and attributes simultaneously, enabling a more accurate representation of knowledge. $(Y, (M, T))$ is the three-way concept that will be used in this paper. It is evident that $(Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangledown})$ and $((M, T)^{\blacktriangle}, (M, T)^{\blacktriangle\blacktriangledown})$ are also three-way concepts.

In a formal context (U, A, R), all three-way concepts constitute the three-way concept lattice $L_T(U, A, R)$. In the three-way concept lattice, for any two three-way concepts $(Y_1, (M_1, T_1))$ and $(Y_2, (M_2, T_2))$, there exist an ordering $(Y_1, (M_1, T_1)) \leqslant (Y_2, (M_2, T_2)) \Leftrightarrow Y_1 \subseteq Y_2 \Leftrightarrow (M_1, T_1) \subseteq (M_2, T_2)$. $(Y_1, (M_1, T_1))$ is referred to as a sub-concept of $(Y_2, (M_2, T_2))$, and correspondingly, $(Y_2, (M_2, T_2))$ is identified as the super-concept of $(Y_1, (M_1, T_1))$.

If $(Y_1, (M_1, T_1))$ and $(Y_2, (M_2, T_2))$ represent two three-way concepts, then $((Y_1 \cap Y_2), ((M_1, T_1) \cap (M_2, T_2))^{\blacktriangle\blacktriangledown})$ and $((Y_1 \cup Y_2)^{\blacktriangledown\blacktriangle}, ((M_1, T_1) \cup (M_2, T_2)))$are also three-way concepts. Therefore, if the three-way concepts in the three-way concept lattice satisfy:

$(Y_1, (M_1, T_1)) \wedge (Y_2, (M_2, T_2)) = ((Y_1 \cap Y_2), ((M_1, T_1) \cap (M_2, T_2))^{\blacktriangle\blacktriangledown})$;

$(Y_1, (M_1, T_1)) \vee (Y_2, (M_2, T_2)) = ((Y_1 \cup Y_2)^{\blacktriangledown\blacktriangle}, ((M_1, T_1) \cup (M_2, T_2)))$.

then the three-way concept lattice $L_T(U, A, R)$ is called a three-way complete lattice.

TABLE II
A SKILL CONTEXT(U, A, R)

U\A	o	p	q	r	s
x_1	0	1	1	0	1
x_2	1	1	0	0	0
x_3	1	0	0	0	0
x_4	0	0	0	0	1
x_5	0	0	0	1	1

B. Knowledge space analysis

In the following part, use the non-empty set U to represent the set of items and A to represent the set of skills. For the set of items, the skills in A are relevant to the items within it, the connection between the two can be expressed through skill map.

A skill map is represented as a triplet (U, A, τ), where U denotes a non-empty set of items, A denotes a non-empty set of skills, and τ represents a mapping from U to $2^A \setminus \{\emptyset\}$. This function τ as referred to as the skill map, where for $y \in U$, $\tau(y) \subseteq A$ represents the subset of skills from A assigned to item y through the skill map. (see [58])

Consider a skill map (U, A, τ), and let $M \subseteq A$. We can assert that the knowledge state $P(M) \subseteq U$ delineated by M via the disjunctive model if it meets $P(M) = \{y \in U | \tau(y) \cap M \neq \emptyset\}$. Note that an empty skill subset delineates an empty knowledge state, while A delineates U. Another knowledge state $P(M) \subseteq U$ delineated by M via the conjunctive model is defined as $P(M) = \{y \in U | \tau(y) \subseteq M\}$. The entire set of knowledge states is represented by the knowledge structure delineated by the skill map (U, A, τ). Because the knowledge structure induced by the conjunction model forms a simple closure space, therefore, this paper adopts the conjunction model.

In the conjunction model (U, A, τ), for $y \in U$, and $B \subseteq A$, when $m \in B = \tau(y)$ if and only if $(y, m) \in R$, this conjunction model can be transformed into a formal context (U, A, R), referred to as a skill context. In the skill context, for any $y \in U$, and $\tau(y) = B$, there have $y^{\Diamond} = B$. In the skill context, (Y, M) is a concept formed by the operators $\Diamond$ and $\Box$ as defined, and the concept lattice, consisting of all such concepts, is represented as $L(U, A, R)$. Similarly, $(Y, (M, T))$ is a three-way concept formed by the operators $\blacktriangledown$ and $\blacktriangle$ as defined, the three-way concept lattice formed by all three-way concepts is denoted as $L_T(U, A, R)$.

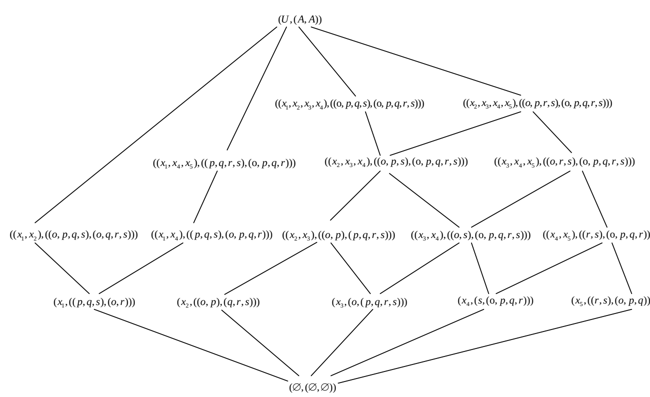


Fig. 2. The three-way concept lattice of (U, A, R).

Example 1: For the conjunction model (U, A, τ), the skill context (U, A, R) corresponds to (U, A, τ) in Table II. By using the operators $(\blacktriangledown, \blacktriangle)$ in Definition 1, we can compute the three-way concepts in the skill context, and construct a three-way concept lattice by identifying all the three-way concepts. For example, we take $Y = \{x_2, x_3, x_4\}$, $(M, T) = ((o, p, s), (o, p, q, r, s))$, then $Y^{\blacktriangledown} = (Y^{\Diamond}, Y^{\bar{\Diamond}}) =$

$(\{x_2,x_3,x_4\}^{\Diamond},\{x_2,x_3,x_4\}^{\overline{\Diamond}}) = ((o,p,s),(o,p,q,r,s))$, and $(M,T)^{\blacktriangle} = M^{\Box} \cap T^{\overline{\Box}} = (o,p,s)^{\Box} \cap (o,p,q,r,s)^{\overline{\Box}} = \{x_2,x_3,x_4\} \cap \{x_1,x_2,x_3,x_4,x_5\} = \{x_2,x_3,x_4\}$, then we have $Y^{\blacktriangledown} = (M,T)$, $(M,T)^{\blacktriangle} = Y$. Therefore, $((x_2,x_3,x_4),((o,p,s),(o,p,q,r,s)))$ is a three-way concept satisfying Definition 2, further computation yields the three-way concept lattice formed by all the three-way concepts, as shown in Figure 2.

III. Three-way concept-cognitive learning model

Nowadays, skills are indispensable to an individual, and knowledge is paramount, skills are utilized to address problems encountered, thereby acquiring corresponding knowledge. As problems are solved, one can learn more skills; with mastery of more skills, one can tackle a wider range of problems, leading to changes in knowledge status, the skills learned in the previous stage are related to the new skills, and one can only learn new skills after mastering the old ones. Therefore, the three-way concept-cognitive learning process for one is a gradual progression, constantly learning new skills and acquiring new knowledge. In a skill context, when items and skills are consistent, it is referred to as learning a complete three-way concept, at this point, the three-way concept simultaneously describes both positive and negative information, allowing one to grasp more detailed information and acquire knowledge more accurately. Therefore, for a three-way concept, items and skills are corresponding, at which point skills are both sufficient and necessary for knowledge, when items and skills cannot form a three-way concept, it indicates that the skills might be excessive, inadequate, or irrelevant to the items, at this point, skills can be considered as sufficient, necessary, or potentially neither sufficient nor necessary for knowledge. Therefore, the process of learning three-way concepts is the process of acquiring knowledge.

However, one is unable to grasp or learn three-way concept at the beginning. At the beginning, skills often have little to no impact on knowledge acquisition. As the process progresses, one gradually gains some skills and solves certain problems, at this stage, skills become beneficial for acquiring knowledge, potentially being either sufficient or necessary. Upon completing the entire process, skills become mutually correspondent with knowledge, being both sufficient and necessary, signifying that one has learned the three-way concept. Therefore, the entire process involves the transformation of skills and items. Consequently, cognitive operators among skills and knowledge have been introduced to formulate a novel three-way concept-cognitive learning model, let L be a lattice, where 0_L represents the zero element, and 1_L represents the unit element.

Definition 3: Let L_1 and L_2 represent two complete lattices of items and skills, respectively. For items $y_1,y_2 \in L_1$ and skills $m_1,m_2,t_1,t_2 \in L_2$, for $\mathcal{F}: L_1 \to L_2$ and $\mathcal{H}: L_2 \to L_1$ are two cognitive operators, if $\mathcal{F}$ and $\mathcal{H}$ satisfy the following conditions:

1) $\mathcal{F}(0_{L_1}) = 0_{L_2}, \mathcal{F}(1_{L_1}) = 1_{L_2}$.
2) $\mathcal{F}(y_1 \vee y_2) = \mathcal{F}(y_1) \vee \mathcal{F}(y_2)$.
3) $\mathcal{H}(0_{L_2}) = 0_{L_1}, \mathcal{H}(1_{L_2}) = 1_{L_1}$.
4) $\mathcal{H}((m_1,t_1) \wedge (m_2,t_2)) = \mathcal{H}(m_1,t_1) \wedge \mathcal{H}(m_2,t_2)$.

Definition 4: The quadruple $(L_1,L_2,\mathcal{F},\mathcal{H})$ constitutes a three-way concept-cognitive learning model(Tw-CCLM), if $\mathcal{F}$ and $\mathcal{H}$ denote two cognitive operators, and they meet the following conditions:

1)$\mathcal{H} \circ \mathcal{F}(y) \geqslant y$.
2)$\mathcal{F} \circ \mathcal{H}(m,t) \leqslant (m,t)$.

where $\mathcal{H} \circ \mathcal{F}(y)$ represents $\mathcal{H}(\mathcal{F}(y))$ and $\mathcal{F} \circ \mathcal{H}(m,t)$ represents $\mathcal{F}(\mathcal{H}(m,t))$.

The two cognitive operators defined above $\mathcal{F}$ and $\mathcal{H}$ describe the changing characteristics of items and skills in the cognitive process.

Proposition 2: Let $(L_1,L_2,\mathcal{F},\mathcal{H})$ constitutes a Tw-CCLM, for any items $y,y_1,y_2 \in L_1$ and skills $m,t,m_1,t_1 \in L_2$, the model possesses the following properties:

1) If $y_1 \leqslant y_2$, then $\mathcal{F}(y_1) \leqslant \mathcal{F}(y_2)$.
2) If $(m,t) \leqslant (m_1,t_1)$, then $\mathcal{H}(m,t) \leqslant \mathcal{H}(m_1,t_1)$.
3) $\mathcal{F}(y_1 \wedge y_2) \leqslant \mathcal{F}(y_1) \wedge \mathcal{F}(y_2)$.
4) $\mathcal{H}((m,t) \vee (m_1,t_1)) \geqslant \mathcal{H}(m,t) \vee \mathcal{H}(m_1,t_1)$.
5) $\mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) = \mathcal{F}(y)$.
6) $\mathcal{H} \circ \mathcal{F} \circ \mathcal{H}(m,t) = \mathcal{H}(m,t)$.
7) $\mathcal{F}(y) \leqslant (m,t) \Leftrightarrow y \leqslant \mathcal{H}(m,t)$.

Proof: 1)-4) can be proven based on the Definition 3 provided above.

5) Given 1) and $\mathcal{H} \circ \mathcal{F}(y) \geqslant y$, it can be deduced that $\mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) \geqslant \mathcal{F}(y)$, on the other hand, from $\mathcal{F} \circ \mathcal{H}(m,t) \leqslant (m,t)$, let's take $(m,t) = \mathcal{F}(y)$, then it can be deduced that $\mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) \leqslant \mathcal{F}(y)$. Thus, $\mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) = \mathcal{F}(y)$.

6) It can be proven using the same approach as 5).

7) If $\mathcal{F}(y) \leqslant (m,t)$, then $\mathcal{H} \circ \mathcal{F}(y) \leqslant \mathcal{H}(m,t)$, by Definition 4, we have $\mathcal{H} \circ \mathcal{F}(y) \geqslant y$, then $y \leqslant \mathcal{H}(m,t)$. Similarly, if $y \leqslant \mathcal{H}(m,t)$, then $\mathcal{F}(y) \leqslant \mathcal{F} \circ \mathcal{H}(m,t)$, by Definition 4, we have $\mathcal{F} \circ \mathcal{H}(m,t) \leqslant (m,t)$, then $\mathcal{F}(y) \leqslant (m,t)$. Thus, $\mathcal{F}(y) \leqslant (m,t) \Leftrightarrow y \leqslant \mathcal{H}(m,t)$.

Proposition 3: Let (U,A,R) be a skill context, if $L_1 = 2^U$, $L_2 = 2^A$, the two operators ▼ and ▲ , as defined in Definition 1, serve as two cognitive operators of (U,A,R).

Proof: This can be readily demonstrated by employing Proposition 1 and Definition 1.

Based on the preceding discussion, using a skill context to depict the connection between items and skills in the cognitive process. In order to obtain three-way concepts during the process, we need use the ▼ and ▲ cognitive operators that satisfy Definition 1.

IV. Transformation between three-way information granules

After the previous analysis, it is evident that two operators can establish a relationship between skills and knowledge, meaning that there exists a sufficient and necessary connection between them, when they are in this relationship, the laws of cognition can be understood. If skills are not a sufficient and necessary connection for knowledge, in that case, one has not yet acquired comprehensive knowledge during the process and needs to continue learning. So in the three-way concept-cognitive learning model, we construct a three-way information granule $(y,(m,t))$ to represent the relationship between skills and knowledge, in this representation, y signifies a set of items, and (m, t) represents a set of skills that consider more comprehensive information. The simultaneous use of necessity and sufficiency can clearly describe the relationship between items and skills in $(y, (m, t))$, enabling the determination of the type of $(y, (m, t))$, and reflecting the different stages of learning.

Definition 5: Let (U, A, R) be a skill context, L_1 and L_2 be two complete lattices, $\mathcal{F}$ and $\mathcal{H}$ are two cognitive operators. In other words, $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms a Tw-CCLM, for any $y \in L_1$ and any $m, t \in L_2$, the following two sets can be defined:

$\mathcal{N} = \{(y, (m, t)) | \mathcal{F}(y) \leqslant (m, t), y \leqslant \mathcal{H}(m, t)\}$,

$\mathcal{S} = \{(y, (m, t)) | \mathcal{F}(y) \geqslant (m, t), y \geqslant \mathcal{H}(m, t)\}$.

1) If $(y, (m, t)) \in \mathcal{N}$, then $(y, (m, t))$ is a necessary three-way information granule of the Tw-CCLM, (m, t) is a necessary skills of y. So $\mathcal{N}$ represents the necessary three-way information granules set.

2) If $(y, (m, t)) \in \mathcal{S}$, then $(y, (m, t))$ is a sufficient three-way information granule of the Tw-CCLM, (m, t) is a sufficient skills of y. So $\mathcal{S}$ represents the sufficient three-way information granules set.

3) If $(y, (m, t)) \in \mathcal{N} \cap \mathcal{S}$, which means $\mathcal{F}(y) = (m, t)$ and $y = \mathcal{H}(m, t)$, then $(y, (m, t))$ is a sufficient and necessary three-way information granule of the Tw-CCLM. So $\mathcal{N} \cap \mathcal{S}$ represents the sufficient and necessary three-way information granules set. In this case, $(y, (m, t))$ represents the three-way concept that we need to learn.

4) If $(y, (m, t)) \in \mathcal{N} \cup \mathcal{S}$, then $(y, (m, t))$ is a consistent information granule. So $\mathcal{N} \cup \mathcal{S}$ represents the three-way information granules set.

5) If $(y, (m, t)) \notin \mathcal{N} \cup \mathcal{S}$, then $(y, (m, t))$ is an inconsistent three-way information granule.

Analyzing the earlier discussion, it's clear that the sufficient and necessary three-way information granule in the Tw-CCLM is the three-way concept within the skill context. During the ongoing process of human cognition, one start learning from a state of not knowing, which corresponds to inconsistent information granule, then, based on this foundation, through gradual cognitive development, learning skills, solving problems, and acquiring knowledge, resulting in consistent three-way information granule that can be sufficient or necessary, upon further learning, the process of cognition is completed, and at this stage, sufficient and necessary three-way information granules is acquired, namely the three-way concepts.

Proposition 4: Let (U, A, R) be a skill context. $\blacktriangledown$ and $\blacktriangle$ are two cognitive operators. If $L_1 = P(U), L_2 = P(A)$, for any $Y \subseteq L_1$, $M, T \subseteq L_2$ the following properties hold:

1) If $Y^{\blacktriangledown} \subseteq (M, T)$ and $Y \subseteq (M, T)^{\blacktriangle}$, in that case, (M, T) is a necessary skills of Y.

2) If $Y^{\blacktriangledown} \supseteq (M, T)$ and $Y \supseteq (M, T)^{\blacktriangle}$, in that case, (M, T) is a sufficient skills of Y.

3) If $Y^{\blacktriangledown} = (M, T)$ and $Y = (M, T)^{\blacktriangle}$, in that case, (M, T) is a sufficient and necessary skills of Y.

Proof : from the above Definition 5 and Proposition 3, it can be proved that.

From the above discussion, it can be observed that in cognitive process, before the formation of three-way concept, one attempts to acquire sufficient or necessary three-way information granules.

Proposition 5: Let$(L_1, L_2, \mathcal{F}, \mathcal{H})$ is a Tw-CCLM, and $\mathcal{N}$ is a set of necessary information granules for the model, where the operators $\wedge$ and $\vee$ are defined within $\mathcal{N}$, and

$(y_1, (m_1, t_1)) \wedge (y_2, (m_2, t_2)) = (y_1 \wedge y_2, \mathcal{F} \circ \mathcal{H}((m_1, t_1) \wedge (m_2, t_2)))$;

$(y_1, (m_1, t_1)) \vee (y_2, (m_2, t_2)) = (\mathcal{H} \circ \mathcal{F}(y_1 \vee y_2), (m_1, t_1) \vee (m_2, t_2)))$.

$(\mathcal{N}, \leqslant)$ is closed with respect to the operations $\wedge$ and $\vee$.

Proof : If $(y_1, (m_1, t_1))$, $(y_2, (m_2, t_2)) \in \mathcal{N}$, then

$\mathcal{F}(y_1) \leqslant (m_1, t_1)$, $y_1 \leqslant \mathcal{H}(m_1, t_1)$;

$\mathcal{F}(y_2) \leqslant (m_2, t_2)$, $y_2 \leqslant \mathcal{H}(m_2, t_2)$,

and

$y_1 \wedge y_2 \leqslant \mathcal{H}(m_1, t_1) \wedge \mathcal{H}(m_2, t_2) = \mathcal{H}((m_1, t_1) \wedge (m_2, t_2)) = \mathcal{H} \circ \mathcal{F} \circ \mathcal{H}((m_1, t_1) \wedge (m_2, t_2))$,

furthermore, by further analyzing the Proposition 2, it can be concluded that

$\mathcal{F}(\mathcal{H}((m_1, t_1) \wedge (m_2, t_2))) = \mathcal{F}(\mathcal{H}(m_1, t_1) \wedge \mathcal{H}(m_2, t_2)) \geqslant \mathcal{F}(y_1 \wedge y_2)$.

Thus $(y_1, (m_1, t_1)) \wedge (y_2, (m_2, t_2))$ is a necessary three-way information granule, in other words, $(y_1, (m_1, t_1)) \wedge (y_2, (m_2, t_2)) \in \mathcal{N}$.

Similarly, $(y_1, (m_1, t_1)) \vee (y_2, (m_2, t_2))$ can be proven using the same method.

The proposition demonstrate that we can identify all the necessary three-way information granules by computing the set $\mathcal{N}$.

Proposition 6: Let$(L_1, L_2, \mathcal{F}, \mathcal{H})$ is a Tw-CCLM, and $\mathcal{S}$ a set of sufficient information granules for the model, where the operators $\wedge$ and $\vee$ are defined within $\mathcal{S}$, and

$(y_1, (m_1, t_1)) \wedge (y_2, (m_2, t_2)) = (y_1 \wedge y_2, \mathcal{F} \circ \mathcal{H}((m_1, t_1) \wedge (m_2, t_2)))$;

$(y_1, (m_1, t_1)) \vee (y_2, (m_2, t_2)) = (\mathcal{H} \circ \mathcal{F}(y_1 \vee y_2), (m_1, t_1) \vee (b_2, t_2)))$.

$(\mathcal{S}, \leqslant)$ is closed with respect to the operations $\wedge$ and $\vee$.

Proof : If $(y_1, (m_1, t_1))$, $(y_2, (m_2, t_2)) \in \mathcal{S}$, then

$\mathcal{F}(y_1) \geqslant (m_1, t_1)$, $y_1 \geqslant \mathcal{H}(m_1, t_1)$;

$\mathcal{F}(y_2) \geqslant (m_2, t_2)$, $y_2 \geqslant \mathcal{H}(m_2, t_2)$,

and

$y_1 \wedge y_2 \geqslant \mathcal{H}(m_1, t_1) \wedge \mathcal{H}(m_2, t_2) = \mathcal{H}((m_1, t_1) \wedge (m_2, t_2)) = \mathcal{H} \circ \mathcal{F} \circ \mathcal{H}((m_1, t_1) \wedge (m_2, t_2))$,

furthermore, by further analyzing the Proposition 2, it can be concluded that

$\mathcal{F}(\mathcal{H}((m_1, t_1) \wedge (m_2, t_2))) = \mathcal{F}(\mathcal{H}(m_1, t_1) \wedge \mathcal{H}(m_2, t_2)) \leqslant \mathcal{F}(y_1 \wedge y_2)$.

Thus $(y_1, (m_1, t_1)) \wedge (y_2, (m_2, t_2))$ is a sufficient three-way information granule, in other words, $(y_1, (m_1, t_1)) \wedge (y_2, (m_2, t_2)) \in \mathcal{S}$.

Similarly, $(y_1, (m_1, t_1)) \vee (y_2, (m_2, t_2))$ can be proven using the same method.

The proposition demonstrate that we can identify all the sufficient three-way information granules by computing the set $\mathcal{S}$.

Based on the analysis above, $\leqslant$ represents a quasi-order relation within $(\mathcal{N}, \leqslant)$ and $(\mathcal{S}, \leqslant)$ concerning the operators $\wedge$ and $\vee$. Therefore, these relations form a quasi-lattice, rather than a lattice.

This article has been accepted for publication in IEEE Transactions on Big Data. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TBDATA.2025.3556637

In general, in the initial stage of the three-way concept-cognitive learning model, there is no consistent three-way information granules. Therefore, one should gradually strive to acquire sufficient or necessary three-way information granules, ultimately obtaining the necessary and sufficient three-way information granules, in other words, it represents the three-way concept that need to be learned. The process of three-way concept-cognitive learning is essentially a process of three-way information granules transformation, in the early phases of this process, no sufficient and necessary three-way information granules are yet established, instead, one start by learning from arbitrary three-way information granule and gradually progress towards necessary, sufficient, and then sufficient and necessary three-way information granules through further learning.

Proposition 7: From arbitrary three-way information granules to necessary three-way information granules, $(L_1, L_2, \mathcal{F}, \mathcal{H})$ constitutes a Tw-CCLM, where $\mathcal{N}$ represents a set of necessary three-way information granules of the model. If any $y \in L_1$, any $m, t \in L_2$, then:

1) $(y \wedge \mathcal{H}(m,t), \mathcal{F}(y) \wedge (m,t)) \in \mathcal{N}$.
2) $(y \vee \mathcal{H}(m,t), \mathcal{F}(y) \vee (m,t)) \in \mathcal{N}$.
3) $(y \wedge \mathcal{H}(m,t), \mathcal{F}(y)) \in \mathcal{N}$.
4) $(\mathcal{H}(m,t), \mathcal{F}(y) \vee (m,t)) \in \mathcal{N}$.
5) $(\mathcal{H} \circ \mathcal{F}(y), \mathcal{F}(y) \vee (m,t)) \in \mathcal{N}$.
6) $(y \wedge \mathcal{H}(m,t), \mathcal{F} \circ \mathcal{H}(m,t)) \in \mathcal{N}$.

Proof : 1) As $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms a Tw-CCLM, deducing from Definition 4 and Proposition 2, it can be concluded that $\mathcal{F}(y \wedge \mathcal{H}(m,t)) \leqslant \mathcal{F}(y) \wedge \mathcal{F}(\mathcal{H}(m,t)) \leqslant \mathcal{F}(y) \wedge (m,t)$, and $\mathcal{H}(\mathcal{F}(y) \wedge (m,t)) = \mathcal{H}(\mathcal{F}(y)) \wedge \mathcal{H}(m,t) \geqslant y \wedge \mathcal{H}(m,t)$. Thus, $(y \wedge \mathcal{H}(m,t), \mathcal{F}(y) \wedge (m,t)) \in \mathcal{N}$.

2) The same method as described in 1) can be employed to prove.

3) As $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms Tw-CCLM, deducing from Definition 4 and Proposition 2, it can be concluded that $\mathcal{F}(y \wedge \mathcal{H}(m,t)) \leqslant \mathcal{F}(y) \wedge \mathcal{F}(\mathcal{H}(m,t)) \leqslant \mathcal{F}(y)$, and $\mathcal{H}(\mathcal{F}(y)) \geqslant y \geqslant y \wedge \mathcal{H}(m,t)$. Thus, $(y \wedge \mathcal{H}(m,t), \mathcal{F}(y)) \in \mathcal{N}$.

4) The same method as described in 3) can be employed to prove.

5) As $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms a Tw-CCLM, deducing from Definition 4 and Proposition 2, it can be concluded that $\mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) = \mathcal{F}(y) \leqslant \mathcal{F}(y) \vee (m,t)$, and $\mathcal{H}(\mathcal{F}(y) \vee (m,t)) \geqslant \mathcal{H}(\mathcal{F}(y)) \vee \mathcal{H}(m,t) \geqslant \mathcal{H} \circ \mathcal{F}(y)$. Thus, $(\mathcal{H} \circ \mathcal{F}(y), \mathcal{F}(y) \vee (m,t)) \in \mathcal{N}$.

6) The same method as described in 5) can be employed to prove.

From the above, it is evident that there are six possible ways to obtain necessary three-way information granules from arbitrary three-way information granules.

Proposition 8: From arbitrary three-way information granules to sufficient three-way information granules, $(L_1, L_2, \mathcal{F}, \mathcal{H})$ constitutes a Tw-CCLM, where $\mathcal{S}$ represents a set of sufficient three-way information granules of the model. If any $y \in L_1$, any $m, t \in L_2$, then:

1) $(\mathcal{H} \circ \mathcal{F}(y), \mathcal{F}(y) \wedge (m,t)) \in \mathcal{S}$.
2) $(y \vee \mathcal{H}(m,t), \mathcal{F} \circ \mathcal{H}(m,t)) \in \mathcal{S}$.

Proof : 1) As $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms a Tw-CCLM, deducing from Definition 4 and Proposition 2, it can be concluded that $\mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) = \mathcal{F}(y) \geqslant \mathcal{F}(y) \wedge (m,t)$, and $\mathcal{H}(\mathcal{F}(y) \wedge (m,t)) = \mathcal{H}(\mathcal{F}(y)) \wedge \mathcal{H}(m,t) \leqslant \mathcal{H} \circ \mathcal{F}(y)$. Thus, $(\mathcal{H} \circ \mathcal{F}(y), \mathcal{F}(y) \wedge (m,t)) \in \mathcal{S}$.

2) The same method as described in 1) can be employed to prove.

From the above, it is evident that there are two possible ways to obtain sufficient three-way information granules from arbitrary three-way information granules.

From the two properties mentioned above, it is evident that Tw-CCLM has the ability to transform from seemingly useless information granules to useful ones. However, to fully utilize the information within the model, further acquisition of sufficient and necessary three-way information granules is required in order to learn the three-way concepts. The following is a discussion on how to acquire sufficient and necessary three-way information granules from either sufficient or necessary three-way information granules.

Proposition 9: From necessary three-way information granules to sufficient and necessary three-way information granules,$(L_1, L_2, \mathcal{F}, \mathcal{H})$ constitutes a Tw-CCLM, and $\mathcal{N}$ and $\mathcal{S}$ are, respectively, the necessary, sufficient three-way information granules sets in the model. If any $y \in L_1$, any $m, t \in L_2$ and $(y, (m,t)) \in \mathcal{N}$, then:

1) $(\mathcal{H}(\mathcal{F}(y) \wedge (m,t)), \mathcal{F}(y) \wedge (m,t)) \in \mathcal{N} \cap \mathcal{S}$.
2) $(y \vee \mathcal{H}(m,t), \mathcal{F}(y \vee \mathcal{H}(m,t))) \in \mathcal{N} \cap \mathcal{S}$.

Proof : 1) Cause $(y, (m,t)) \in \mathcal{N}$, we can deduce that $\mathcal{F}(y) \leqslant (m,t)$, and $y \leqslant \mathcal{H}(m,t)$, then, $\mathcal{F}(y) \wedge (m,t) = \mathcal{F}(y)$, $\mathcal{H}(\mathcal{F}(y) \wedge (m,t)) = \mathcal{H}(\mathcal{F}(y))$. As $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms a Tw-CCLM, deducing from Definition 4 and Proposition 2, it can be concluded that $\mathcal{F}(\mathcal{H}(\mathcal{F}(y) \wedge (m,t))) = \mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) = \mathcal{F}(y) = \mathcal{F}(y) \wedge (m,t)$, and $\mathcal{H}(\mathcal{F}(y) \wedge (m,t)) = \mathcal{H} \circ \mathcal{F}(y) = \mathcal{H}(\mathcal{F}(y) \wedge (m,t))$. Thus, $(\mathcal{H}(\mathcal{F}(y) \wedge (m,t)), \mathcal{F}(y) \wedge (m,t)) \in \mathcal{N} \cap \mathcal{S}$.

2) The same method as described in 1) can be employed to prove.

It can be easily observed that there are two possible ways to obtain the sufficient and necessary three-way information granules from the necessary three-way information granules.

Proposition 10: From sufficient three-way information granules to sufficient and necessary three-way information granules, $(L_1, L_2, \mathcal{F}, \mathcal{H})$ constitutes a Tw-CCLM, and $\mathcal{N}$ and $\mathcal{S}$ are, respectively, the necessary, sufficient three-way information granules sets in the model. If any $y \in L_1$, any $m, t \in L_2$ and $(y, (m,t)) \in \mathcal{S}$, then:

1) $(\mathcal{H}(\mathcal{F}(y) \vee (m,t)), \mathcal{F}(y) \vee (m,t)) \in \mathcal{N} \cap \mathcal{S}$.
2) $(y \wedge \mathcal{H}(m,t), \mathcal{F}(y \wedge \mathcal{H}(m,t))) \in \mathcal{N} \cap \mathcal{S}$.

Proof : 1) Cause $(y, (m,t)) \in \mathcal{S}$, we can deduce that $\mathcal{F}(y) \geqslant (m,t)$, and $y \geqslant \mathcal{H}(m,t)$, then, $\mathcal{F}(y) \vee (m,t) = \mathcal{F}(y)$, $\mathcal{H}(\mathcal{F}(y) \vee (m,t)) = \mathcal{H}(\mathcal{F}(y))$. As $(L_1, L_2, \mathcal{F}, \mathcal{H})$ forms a Tw-CCLM, deducing from Definition 4 and Proposition 2, it can be concluded that $\mathcal{F}(\mathcal{H}(\mathcal{F}(y) \vee (m,t))) = \mathcal{F} \circ \mathcal{H} \circ \mathcal{F}(y) = \mathcal{F}(y) = \mathcal{F}(y) \vee (m,t)$, and $\mathcal{H}(\mathcal{F}(y) \vee (m,t)) = \mathcal{H} \circ \mathcal{F}(y) = \mathcal{H}(\mathcal{F}(y) \vee (m,t))$. Thus, $(\mathcal{H}(\mathcal{F}(y) \vee (m,t)), \mathcal{F}(y) \vee (m,t)) \in \mathcal{N} \cap \mathcal{S}$.

2) The same method as described in 1) can be employed to prove.

It can be easily observed that there are two possible ways to obtain the sufficient and necessary three-way information granules from the sufficient three-way information granules.

The analysis above indicates that there exist two paths to transform arbitrary three-way information granules into sufficient and necessary three-way information granules. One path involves initially acquiring the necessary three-way information granules and subsequently transform them to sufficient and necessary three-way information granules. The second path is to initially obtain the sufficient three-way information granules and subsequently transform them to sufficient and necessary three-way information granules. Importantly, for arbitrary three-way information granules, through Tw-CCLM, the number of sufficient and necessary information granules that can be obtained from the first path does not exceed 12 by Property 7 and 9, and the number of sufficient and necessary information granules that can be obtained from the second path does not exceed 4 by Property 8 and 10. Therefore, the total number of sufficient and necessary information granules (three-way concepts) that the model can ultimately learn does not exceed 16. The specific flow of the model is presented in Figure 3.

As the theoretical analysis above, we can achieve the transformation from arbitrary three-way information granules into sufficient and necessary three-way information granules. In the skill context (U, A, R), it's evident that the sufficient and necessary three-way information granules correspond to the three-way concepts, a more detailed transformation process is presented in Algorithm 1, $(y_i, (m_i, t_i))$ represents the i-th three-way information granule. Next, an analysis of the

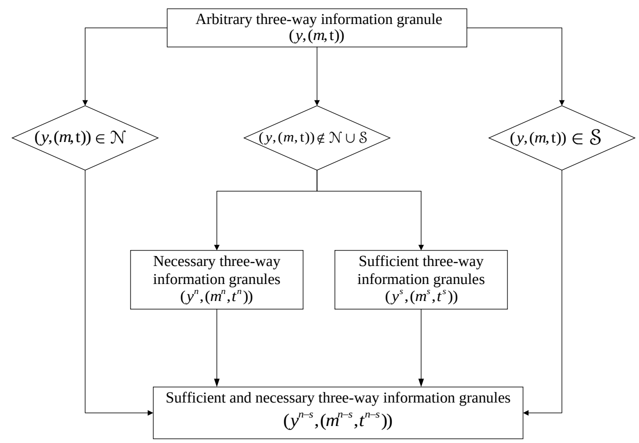


Fig. 3. The transformation flow diagram of three-way information granules.

algorithm's time complexity is conducted, where $|U|$ represents the number of items, and $|A|$ represents the number of skills. The time complexity of steps 1-6 within the while-loop is $O(|U||A|)$, and steps 7-9, 10-12, have the same time complexity. Consequently, the algorithm's overall time complexity is $O(|U||A|)$.

Example 2: In example 1, let's take a three-way information granule $(Y, (M, T)) = ((x_1, x_4), ((o, p, r), (p, q, r, s)))$. The calculation shows that $(x_1, x_4)^{\blacktriangledown} = ((p, q, s), (o, p, q, r)) \subsetneq (M, T)$, and $Y^{\blacktriangledown} \supsetneq (M, T)$, similarly, $(M, T)^{\blacktriangle} = (x_2, x_3) \subsetneq Y$, and $(M, T)^{\blacktriangle} \supsetneq Y$, therefore,

Algorithm 1: Transformation between three-way information granules.

Input: A skill context (U, A, R) and an arbitrary three-way information granule $(y, (m, t))$.
Output: Sufficient and necessary three-way information granules$(y^{n-s}(m^{n-s}, t^{n-s}))$.

1 **while** $(y, (m, t)) \notin \mathcal{N} \cup \mathcal{S}$ **do**
2 Necessary three-way information granules$(y^n, (m^n, t^n))$from $(y, (m, t))$,
3 $(y^n_1, (m^n_1, t^n_1))$,. . .,$(y^n_e, (m^n_e, t^n_e))$, e $\leqslant$6 by Proposition 7.
4 Sufficient and necessary three-way information granules $(y^{ns}, (m^{ns}, t^{ns}))$ from $(y^n, (m^n, t^n))$, $(y^{ns1}_1, (m^{ns1}_1, t^{ns1}_1))$,. . .,$(y^{ns1}_e, (m^{ns1}_e, t^{ns1}_e))$, e $\leqslant$ 12 by Proposition 9.
5 Sufficient three-way information granules $(y^s, (m^s, t^s))$from $(y, (m, t))$,
6 $(y^s_1, (m^s_1, t^s_1))$,. . .,$(y^s_e, (m^s_e, t^s_e))$, e $\leqslant$ 2 by Proposition 8.
7 Sufficient and necessary three-way information granules $(y^{sn}, (m^{sn}, t^{sn}))$ from $(y^s, (m^s, t^s))$, $(y^{sn1}_1, (m^{sn1}_1, t^{sn1}_1))$,. . .,$(y^{sn1}_e, (m^{sn1}_e, t^{sn1}_e))$, e $\leqslant$ 4 by Proposition 10.
8 **end**
9 **while** $(y, (m, t)) \in \mathcal{N}$ **do**
10 Sufficient and necessary three-way information granules $(y^{ns}, (m^{ns}, t^{ns}))$ from $(y, (m, t))$, $(y^{ns2}_1, (m^{ns2}_1, t^{ns2}_1))$,. . .,$(y^{ns2}_e, (m^{ns2}_e, t^{ns2}_e))$, e $\leqslant$ by Proposition 9.
11 **end**
12 **while** $(y, (m, t)) \in \mathcal{S}$ **do**
13 Sufficient and necessary three-way information granules $(y^{sn}, (m^{sn}, t^{sn}))$ from $(y, (m, t))$, $(y^{sn2}_1, (m^{sn2}_1, t^{sn2}_1))$,. . .,$(y^{sn2}_e, (m^{sn2}_e, t^{sn2}_e))$, e $\leqslant$ 2 by Proposition 10.
14 **end**
15 $(y^{n-s}_1, (m^{n-s}_1, t^{n-s}_1))$, . . . ,$(y^{n-s}_d, (m^{n-s}_d, t^{n-s}_d)) \leftarrow (y^{ns1}_1, (m^{ns1}_1, t^{ns1}_1))$,. . .,$(y^{ns1}_e, (m^{ns1}_e, t^{ns1}_e))$, $(y^{sn1}_1, (m^{sn1}_1, t^{sn1}_1))$,. . .,$(y^{sn1}_e, (m^{sn1}_e, t^{sn1}_e))$, $(y^{ns2}_1, (m^{ns2}_1, t^{ns2}_1))$,. . .,$(y^{ns2}_e, (m^{ns2}_e, t^{ns2}_e))$, $(y^{ns2}_1, (m^{ns2}_1, t^{ns2}_1))$,. . .,$(y^{ns2}_e, (m^{ns2}_e, t^{ns2}_e))$, d $\leqslant$ 16.
16 **return** $(y^{n-s}_1, (m^{n-s}_1, t^{n-s}_1))$, . . . ,$(y^{n-s}_d, (m^{n-s}_d, t^{n-s}_d))$.

$((x_1,x_4),((o,p,r),(p,q,r,s)))$ is an inconsistent three-way information granule. First path, by Proposition 7, necessary three-way information granules can be obtained, by Proposition 7-2), $(y \vee \mathcal{H}(m,t), \mathcal{F}(y) \vee (m,t)) = ((x_1,x_2,x_3,x_4),((o,p,q,r,s),(o,p,q,r,s)))$, the rest are $((x_1,x_4),((o,p,q,r,s),(o,p,q,r,s)))$,$(\emptyset,((o,p),(p,q,r,s)))$, $((x_2,x_3),((o,p,q,r,s),(o,p,q,r,s)))$, $(\emptyset,(p,(p,q,r)))$, $(\emptyset, ((p,q,s),(o,p,q,r)))$. Further, by Proposition 9, sufficient and necessary three-way information granules can be transformed from them, for $((x_1,x_2,x_3,x_4),((o,p,q,r,s),(o,p,q,r,s)))$, $(\mathcal{H}(\mathcal{F}(y) \wedge (m,t)), \mathcal{F}(y) \wedge (m,t)) = ((x_1,x_2,x_3,x_4),((o,p,q,s),(o,p,q,r,s)))$, $(y \vee \mathcal{H}(m,t), \mathcal{F}(y \vee \mathcal{H}(m,t))) = (U,(A,A))$, compared to the initial skills $((o,p,r),(p,q,r,s))$, when further learning and mastering the skills $((o,p,q,s),(o,p,q,r,s))$, one can solve the items (x_1,x_2,x_3,x_4); when all skills (A,A) are learned and mastered, all items U can be solved, and the rest are $(\emptyset,(\emptyset,\emptyset))$,$((x_1,x_4),((p,q,s),(o,p,q,r)))$, $((x_2,x_3),((o,p),(p,q,r,s)))$. Another path, by Proposition 8, sufficient three-way information granules can be obtained $((x_1,x_2,x_3,x_4),((o,p),(p,q,r,s)))$, $((x_1,x_4),(p,(p,q,r)))$. Further, by Proposition 10, sufficient and necessary three-way information granules can be transformed from them $(\emptyset,(\emptyset,\emptyset))$, $((x_2,x_3),((o,p),(p,q,r,s)))$, $((x_1,x_2,x_3,x_4),((o,p,q,s),(o,p,q,r,s)))$, $((x_1,x_4),((p,q,s),(o,p,q,r)))$. So, through different paths, necessary, sufficient, or sufficient and necessary three-way information granules (i.e., three-way concepts) can be transformed from an inconsistent three-way information granule. If only positive information is considered, that is, information granule $(Y,M) = ((x_1,x_4),(o,p,r))$, the learned concept are $(\emptyset,\emptyset)$, $((x_2,x_3),(o,p))$, $((x_1,x_4),(p,q,s))$, $((x_1,x_2,x_3,x_4),(o,p,q,s)$, (U,A). If only negative information is considered, that is, information granule $(Y,T) = ((x_1,x_4),(p,q,r,s))$, the learned concept are $(\emptyset,\emptyset)$, $((x_2,x_3),(p,q,r,s))$, $((x_1,x_4,x_5),(o,p,q,r))$, (U,A). Therefore, compared to considering only positive or negative information, simultaneously considering positive and negative information, i.e., input three-way information granule, the learned three-way concepts simultaneously describe both positive and negative information, allowing for a more accurate representation of knowledge.

TABLE III
DATASET DESCRIPTION

No.	Datasets	$\|U\|$	$\|A\|$
1	Zoo	101	17
2	Audit-data	776	18
3	German	1000	21
4	Waveform	5000	22
5	Sensor-readings	5456	24
6	Pen-digits	10992	17
7	Letter-recognition	20000	16
8	Credit-card-clients	30000	24
9	Statlog	58000	10
10	Sgemm	241600	18
11	Covertype	581012	54
12	Gas-sensors	919438	11

V. EXPERIMENTAL ANALYSIS

In this section, the effectiveness of the Tw-CCLM within the skill context will be verified through the use of experimental analyses. The experimental computations were carried out entirely on a personal computer using Python, the computer specifications include an Intel(R) Core(TM) i7-8750H CPU @ 2.20GHz and 16GB of memory. To ensure the proposed method's(Tw-CCLM) general applicability, we randomly downloaded 12 diverse datasets from the UCI(see https://www.uci.edu/) repository for the experiments, further details about the datasets are available in Table III.

To obtain the skill context, a data preprocessing step will be performed first, each data in the dataset will be processed using the average value of its corresponding column, if the data is greater than the average, it is set to 1; otherwise, it is set to 0, this ensures that the data in the skill context falls within {0, 1}. Additionally, to better validate the effectiveness of our method, comparisons will be made with methods that consider only positive or negative information, using P and N as abbreviations in the experiment, it is worth noting that considering only positive information(P) corresponds to the method(Cb-CCLM) used in Ref [38]. In order to ensure that the experimental results are not limited by the input information granules, we randomly select items and skills in different ratios to form various arbitrary three-way information granules$(Y,(M,T))$ as input values, select 2%, 4%, 6%, 8%, and 10% of the items from each dataset as initial Y, and 50%, 60%, 70%, 80%, and 90% of the skills as initial (M,T). For the experimental results, we will evaluate the performance of the Tw-CCLM using the number of concepts (sufficient and necessary three-way information granules) learned by the model and the running time. In order to mitigate the impact of variability in the experimental results, the experiment was run ten times, and the average values were taken as the final results.

Table IV-VI and Figure 4 show the number of concepts obtained by different methods under various ratios, and Figure 5 presents the total number of concepts obtained by different methods on various datasets(different colors represent different methods). From Table IV-VI, Figure 4 and 5, we can observe that in each dataset, for different item proportions and skill proportions of three-way information granules, the model is capable of obtain multiple sufficient and necessary three-way information granules, it indicates that the number of acquired three-way concepts is influenced by the initial three-way information granule. Specifically, when the initial item proportion is fixed, we can observe that as the skill proportion increases, the number of acquired three-way concepts also markedly increases, this suggests that the acquisition of knowledge is significantly influenced by skills. Furthermore, the number of three-way concepts is less than the theoretical maximum of 16, as different methods may lead to different three-way information granules producing the same three-way concepts. Importantly, compared to P (Cb-CCLM) and N, it can be observed that when both positive and negative information are considered, the model is able to learn multiple concepts, demonstrating the effectiveness of the proposed method.

TABLE IV
NUMBER OF THE ACQUIRED CONCEPTS(AVERAGE±SD) IN DATASETS 1-4

No.	Ratio	0.02			0.04			0.06			0.08			0.1		
		Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N
1	0.5	2.8±0.4	3.3±0.6	2.7±0.5	3.0±0.0	4.1±1.4	3.1±0.3	3.0±0.0	4.4±1.0	3.5±0.8	3.4±0.8	5.3±0.6	4.6±0.8	4.9±1.0	4.9±1.0	4.5±1.3
	0.6	2.7±0.5	3.1±0.7	2.5±0.5	2.8±0.4	3.2±0.4	2.7±0.5	3.1±0.3	4.3±1.3	2.8±0.9	3.4±0.9	5.1±0.5	4.0±1.2	4.8±0.7	4.6±1.0	3.9±1.2
	0.7	2.7±0.5	3.1±0.9	2.3±0.5	2.8±0.4	3.3±1.1	2.2±0.4	2.9±0.3	4.0±0.9	2.8±0.9	3.3±0.6	5.2±0.6	3.3±0.9	4.5±0.8	4.8±0.6	3.5±1.1
	0.8	2.9±0.3	2.9±0.5	2.3±0.5	2.5±0.5	2.6±0.7	2.2±0.4	2.7±0.5	4.1±1.3	2.2±0.4	3.2±1.0	3.7±1.0	3.1±1.0	4.3±1.1	4.2±1.1	3.0±1.0
	0.9	2.3±0.5	2.5±0.5	2.0±0.0	2.5±0.5	2.6±0.7	2.1±0.3	2.3±0.5	2.6±0.5	2.1±0.3	2.5±0.5	3.2±1.1	2.9±0.3	3.8±1.1	3.0±1.0	3.5±0.8
2	0.5	2.6±0.5	3.4±0.8	2.0±0.0	2.1±0.3	2.5±0.7	2.2±0.4	2.0±0.0	2.5±0.5	2.1±0.3	2.1±0.3	2.1±0.3	2.2±0.4	2.1±0.3	2.6±0.7	2.2±0.4
	0.6	2.6±0.5	3.8±1.2	2.5±0.5	2.3±0.5	2.8±0.7	2.6±0.5	2.3±0.5	2.7±0.5	2.3±0.5	2.1±0.3	2.6±0.5	2.2±0.4	2.0±0.0	2.5±0.5	2.4±0.5
	0.7	2.9±0.3	4.3±1.0	2.6±0.5	2.4±0.5	3.3±0.9	2.7±0.5	2.1±0.3	2.8±0.9	2.6±0.5	2.2±0.4	3.2±0.7	2.4±0.5	2.1±0.3	2.7±0.6	2.8±0.4
	0.8	3.2±0.9	4.6±1.0	2.8±0.4	2.7±0.5	4.0±1.0	2.9±0.3	2.5±0.5	3.2±1.2	2.8±0.4	2.6±0.5	2.7±0.5	3.0±0.0	2.3±0.5	2.7±0.5	2.8±0.4
	0.9	4.2±1.2	4.3±1.0	2.6±0.5	4.3±0.9	3.7±1.5	2.6±0.5	3.2±1.0	2.8±0.7	2.3±0.5	2.9±1.1	2.9±1.1	2.5±0.5	2.8±0.9	2.5±0.9	2.5±0.5
3	0.5	2.3±0.5	3.4±0.9	2.1±0.3	2.1±0.3	3.0±0.4	2.0±0.0	2.0±0.0	2.7±0.5	2.2±0.4	2.1±0.3	3.0±0.4	2.1±0.3	2.0±0.0	2.8±0.4	2.0±0.0
	0.6	2.2±0.4	3.2±0.4	2.2±0.4	2.2±0.4	3.3±0.6	2.2±0.4	2.2±0.4	3.2±0.4	2.3±0.5	2.0±0.0	3.0±0.0	2.1±0.3	2.0±0.0	3.0±0.0	2.3±0.5
	0.7	2.4±0.5	3.7±0.9	2.9±0.3	2.2±0.4	3.4±0.8	2.7±0.5	2.0±0.0	3.0±0.0	2.9±0.3	2.0±0.0	3.0±0.0	2.7±0.5	2.0±0.0	2.9±0.3	2.7±0.5
	0.8	2.6±0.5	4.0±1.0	2.9±0.3	2.6±0.9	3.3±0.9	2.7±0.5	2.2±0.4	3.0±0.0	3.0±0.0	2.3±0.5	2.7±0.5	2.8±0.4	2.2±0.4	2.9±0.3	2.9±0.3
	0.9	3.7±0.9	4.1±1.1	2.8±0.4	3.3±0.8	3.4±1.0	2.6±0.5	2.7±0.6	2.7±0.6	2.6±0.5	2.7±0.5	2.6±0.5	2.3±0.5	2.9±0.3	2.5±0.5	2.4±0.5
4	0.5	2.0±0.0	2.2±0.4	2.3±0.5	2.0±0.0	2.5±0.5	2.4±0.5	2.0±0.0	2.2±0.4	2.2±0.4	2.0±0.0	2.3±0.5	2.3±0.5	2.0±0.0	2.7±0.5	2.3±0.5
	0.6	2.0±0.0	2.6±0.5	2.7±0.5	2.0±0.0	2.8±0.4	2.9±0.3	2.0±0.0	3.0±0.0	2.8±0.4	2.0±0.0	2.9±0.3	2.4±0.5	2.0±0.0	2.7±0.5	2.5±0.5
	0.7	2.0±0.0	3.0±0.0	3.0±0.0	2.0±0.0	2.8±0.4	2.8±0.4	2.1±0.3	3.0±0.0	3.0±0.0	2.0±0.0	2.9±0.3	2.9±0.3	2.0±0.0	2.9±0.3	2.5±0.5
	0.8	2.5±0.5	2.8±0.4	3.0±0.0	2.3±0.5	2.4±0.5	2.3±0.5	2.3±0.5	2.6±0.5	2.5±0.5	2.3±0.5	2.0±0.0	2.4±0.5	2.3±0.5	2.0±0.0	2.1±0.3
	0.9	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.3±0.5	2.0±0.0	2.1±0.3	2.2±0.4	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0

TABLE V
NUMBER OF THE ACQUIRED CONCEPTS(AVERAGE±SD) IN DATASETS 5-8

No.	Ratio	0.02			0.04			0.06			0.08			0.1		
		Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N
5	0.5	2.0±0.0	3.0±0.0	2.0±0.0	2.0±0.0	3.0±0.0	2.0±0.0	2.0±0.0	3.0±0.0	2.0±0.0	2.0±0.0	2.8±0.4	2.1±0.3	2.0±0.0	2.9±0.3	2.0±0.0
	0.6	2.0±0.0	3.0±0.0	2.0±0.0	2.0±0.0	3.0±0.0	2.0±0.0	2.0±0.0	2.9±0.3	2.0±0.0	2.0±0.0	2.9±0.3	2.0±0.0	2.0±0.0	2.8±0.4	2.0±0.0
	0.7	2.0±0.0	2.9±0.3	2.2±0.4	2.0±0.0	3.0±0.0	2.3±0.5	2.0±0.0	2.7±0.5	2.2±0.4	2.0±0.0	2.6±0.5	2.2±0.4	2.0±0.0	2.8±0.4	2.2±0.4
	0.8	2.6±0.5	2.6±0.5	2.9±0.3	2.1±0.3	2.4±0.5	2.8±0.4	2.3±0.5	2.2±0.4	2.8±0.4	2.5±0.5	2.0±0.0	2.6±0.5	2.3±0.5	2.0±0.0	2.7±0.5
	0.9	2.8±0.4	2.0±0.0	2.1±0.3	2.8±0.4	2.0±0.0	2.2±0.4	2.7±0.5	2.0±0.0	2.2±0.4	2.0±0.0	2.0±0.0	2.0±0.0	2.6±0.5	2.0±0.0	2.2±0.4
6	0.5	2.0±0.0	2.6±0.5	2.8±0.4	2.0±0.0	2.6±0.5	2.8±0.4	2.0±0.0	2.3±0.5	2.6±0.5	2.0±0.0	2.5±0.5	2.5±0.5	2.0±0.0	2.5±0.5	2.8±0.4
	0.6	2.0±0.0	2.7±0.5	2.9±0.3	2.0±0.0	2.8±0.4	2.8±0.4	2.0±0.0	2.7±0.5	2.8±0.4	2.0±0.0	2.6±0.5	2.4±0.5	2.0±0.0	2.9±0.3	2.6±0.5
	0.7	2.1±0.3	2.9±0.3	3.0±0.0	2.1±0.3	2.5±0.5	2.9±0.3	2.0±0.0	2.9±0.3	2.5±0.5	2.1±0.3	2.7±0.5	2.2±0.4	2.0±0.0	2.8±0.4	2.5±0.5
	0.8	2.6±0.5	2.4±0.5	2.3±0.5	2.1±0.3	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.3±0.5	2.1±0.3	2.0±0.0	2.3±0.5	2.2±0.4	2.2±0.4
	0.9	2.0±0.0	2.0±0.0	2.0±0.0	2.1±0.3	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0
7	0.5	2.0±0.0	2.9±0.3	3.0±0.0	2.0±0.0	2.8±0.4	3.0±0.0	2.0±0.0	2.8±0.4	2.5±0.5	2.0±0.0	2.6±0.5	2.9±0.3	2.0±0.0	2.9±0.3	2.6±0.5
	0.6	2.1±0.3	2.8±0.4	2.7±0.5	2.0±0.0	2.7±0.5	2.7±0.5	2.0±0.0	2.5±0.5	2.7±0.5	2.0±0.0	2.4±0.5	2.3±0.5	2.1±0.3	2.4±0.5	2.2±0.4
	0.7	2.1±0.3	2.7±0.5	2.7±0.5	2.0±0.0	2.4±0.5	2.4±0.5	2.0±0.0	2.4±0.5	2.1±0.3	2.2±0.4	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.1±0.3
	0.8	2.2±0.4	2.0±0.0	2.2±0.4	2.3±0.5	2.0±0.0	2.0±0.0	2.3±0.5	2.0±0.0	2.0±0.0	2.3±0.5	2.0±0.0	2.0±0.0	2.3±0.5	2.0±0.0	2.0±0.0
	0.9	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0
8	0.5	2.0±0.0	3.0±0.0	3.0±0.0	2.0±0.0	3.0±0.0	3.0±0.0	2.0±0.0	2.7±0.5	3.0±0.0	2.0±0.0	2.8±0.4	3.0±0.0	2.0±0.0	2.6±0.5	2.8±0.4
	0.6	2.0±0.0	3.0±0.0	3.0±0.0	2.0±0.0	2.9±0.3	3.0±0.0	2.0±0.0	2.9±0.3	3.0±0.0	2.0±0.0	2.8±0.4	3.0±0.0	2.0±0.0	2.6±0.5	2.8±0.4
	0.7	2.0±0.0	3.0±0.0	3.0±0.0	2.0±0.0	2.7±0.5	3.0±0.0	2.0±0.0	2.8±0.4	3.0±0.0	2.0±0.0	2.7±0.5	2.8±0.4	2.0±0.0	2.7±0.5	2.9±0.3
	0.8	2.2±0.4	2.5±0.5	2.8±0.4	2.4±0.5	2.4±0.5	2.2±0.4	2.2±0.4	2.5±0.5	2.3±0.5	2.4±0.5	2.1±0.3	2.2±0.4	2.6±0.5	2.0±0.0	2.2±0.4
	0.9	2.5±0.5	2.5±0.5	2.1±0.3	2.3±0.5	2.0±0.0	2.0±0.0	2.3±0.5	2.0±0.0	2.3±0.5	2.0±0.0	2.0±0.0	2.0±0.0	2.4±0.5	2.0±0.0	2.0±0.0

Figure 6 illustrates the impact of different initial proportions of items and skills, forming the initial information granules, on the running time. From this, it can be observed that as the size of initial three-way information granule increases, the running time required to obtain three-way concepts also increases, furthermore, changes in the skill proportion have a more pronounced impact on running time compared to changes in the item proportion, which is consistent with the cognitive pattern, more importantly, in high-dimensional datasets, the model continues to demonstrate efficient learning speed. Therefore, based on the experimental results, this model appears to be highly excellent and effective, it aligns with the rapid learning of a significant amount of knowledge and skills acquisition observed in the human brain.

VI. CONCLUSIONS

This paper primarily presents a novel model for concept-cognitive learning called Tw-CCLM. Simultaneously considering positive and negative information in the skill context allows for a more detailed description of information, resulting in the acquisition of more accurate knowledge, and provides a detailed description of the relationship between skills and knowledge. By proposing a transformation method between different three-way information granules, arbitrary three-way information granules can be transformed into necessary, sufficient, sufficient and necessary three-way information granules, thereby facilitating the learning of three-way concepts. Furthermore, the outcomes of the algorithm on various datasets during the experiments validate the effectiveness of the proposed method, which also provide evidence for the feasibility

TABLE VI
NUMBER OF THE ACQUIRED CONCEPTS(AVERAGE±SD) IN DATASETS 9-12

No.	Ratio	0.02			0.04			0.06			0.08			0.1		
		Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N	Tw-CCLM	P	N
9	0.5	2.0±0.0	2.7±0.5	2.4±0.5	2.0±0.0	2.6±0.5	2.5±0.5	2.0±0.0	2.3±0.5	2.2±0.4	2.0±0.0	2.6±0.5	2.6±0.5	2.0±0.0	2.5±0.5	2.2±0.4
	0.6	2.0±0.0	2.7±0.5	2.4±0.5	2.0±0.0	2.2±0.4	2.2±0.4	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.2±0.4	2.2±0.4	2.0±0.0	2.2±0.4	2.1±0.3
	0.7	2.1±0.3	2.4±0.5	2.4±0.5	2.0±0.0	2.3±0.5	2.3±0.5	2.1±0.3	2.1±0.3	2.3±0.5	2.2±0.4	2.0±0.0	2.1±0.3	2.0±0.0	2.2±0.4	2.0±0.0
	0.8	2.1±0.3	2.1±0.3	2.0±0.0	2.3±0.5	2.0±0.0	2.3±0.5	2.1±0.3	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.2±0.4	2.0±0.0	2.0±0.0
	0.9	2.1±0.3	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.2±0.4	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.1±0.3
10	0.5	2.0±0.0	2.7±0.5	2.9±0.3	2.0±0.0	2.7±0.5	2.5±0.5	2.0±0.0	2.6±0.5	2.4±0.5	2.0±0.0	2.5±0.5	2.2±0.4	2.0±0.0	2.5±0.5	2.0±0.0
	0.6	2.0±0.0	2.6±0.5	2.5±0.5	2.0±0.0	2.9±0.3	2.7±0.5	2.0±0.0	2.6±0.5	2.5±0.5	2.0±0.0	2.4±0.5	2.3±0.5	2.0±0.0	2.4±0.5	2.4±0.5
	0.7	2.1±0.3	2.9±0.3	2.7±0.5	2.0±0.0	2.8±0.4	3.0±0.0	2.2±0.4	2.6±0.5	2.6±0.5	2.0±0.0	2.4±0.5	2.5±0.5	2.0±0.0	2.5±0.5	2.6±0.5
	0.8	2.6±0.5	2.2±0.4	2.6±0.5	2.4±0.5	2.2±0.4	2.2±0.4	2.3±0.5	2.2±0.4	2.1±0.3	2.1±0.3	2.0±0.0	2.2±0.4	2.5±0.5	2.2±0.4	2.3±0.5
	0.9	2.2±0.4	2.2±0.4	2.1±0.3	2.1±0.3	2.1±0.3	2.0±0.0	2.1±0.3	2.0±0.0	2.1±0.3	2.1±0.3	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0
11	0.5	2.2±0.4	3.0±0.6	2.4±0.5	2.0±0.0	2.8±0.4	2.0±0.0	2.0±0.0	2.6±0.5	2.0±0.0	2.0±0.0	2.7±0.5	2.0±0.0	2.0±0.0	3.0±0.6	2.1±0.3
	0.6	2.1±0.3	3.3±0.8	2.2±0.4	2.0±0.0	2.9±0.3	2.0±0.0	2.0±0.0	2.7±0.5	2.0±0.0	2.0±0.0	2.7±0.5	2.0±0.0	2.0±0.0	3.3±0.8	2.0±0.0
	0.7	2.2±0.4	3.5±1.1	2.3±0.5	2.0±0.0	2.7±0.5	2.1±0.3	2.0±0.0	2.6±0.5	2.0±0.0	2.0±0.0	2.7±0.5	2.0±0.0	2.0±0.0	3.5±1.1	2.0±0.0
	0.8	2.3±0.5	2.8±0.4	2.2±0.4	2.0±0.0	2.6±0.5	2.0±0.0	2.0±0.0	2.4±0.5	2.0±0.0	2.0±0.0	2.6±0.5	2.2±0.4	2.0±0.0	2.8±0.4	2.0±0.0
	0.9	2.1±0.3	2.6±0.5	2.1±0.3	2.0±0.0	3.0±0.8	2.1±0.3	2.1±0.3	2.7±0.6	2.0±0.0	2.1±0.3	2.2±0.4	2.0±0.0	2.0±0.0	2.6±0.5	2.0±0.0
12	0.5	2.0±0.0	2.0±0.0	2.3±0.5	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0
	0.6	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.2±0.4	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0
	0.7	2.1±0.3	2.0±0.0	2.0±0.0	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.1±0.3	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0
	0.8	2.3±0.5	2.0±0.0	2.0±0.0	2.1±0.3	2.1±0.3	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0
	0.9	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0	2.0±0.0

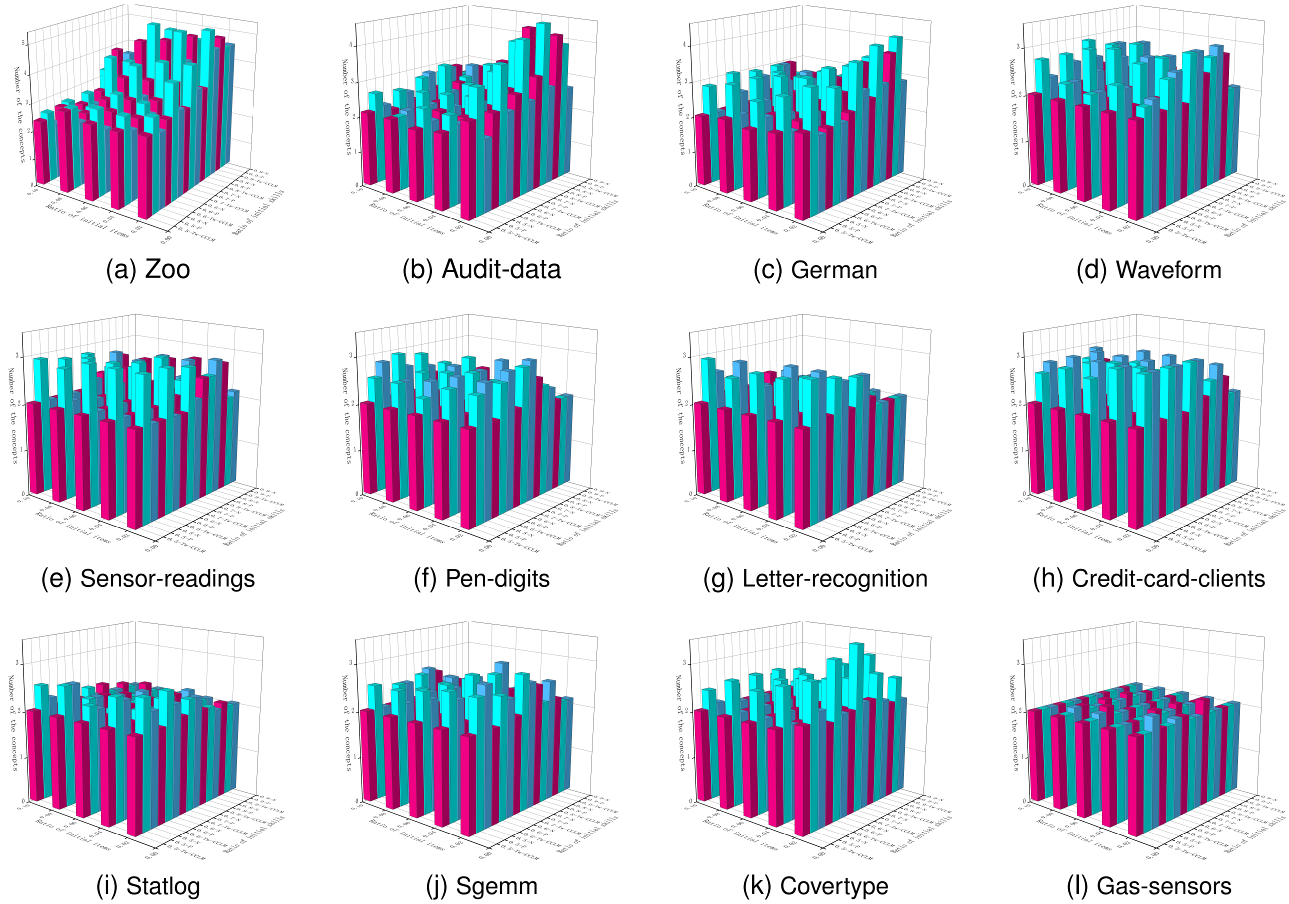


Fig. 4. Number of the acquired concepts.

of Tw-CCLM, the three-way concepts learned by this model can describe more accurate knowledge. However, in practical scenarios, the items and skills may dynamically increase or decrease. Therefore, in future study, the focus of our research will be on how to apply this model in dynamic environment. In addition, the shortcomings of this approach in practical applications will be the focus of attention next.

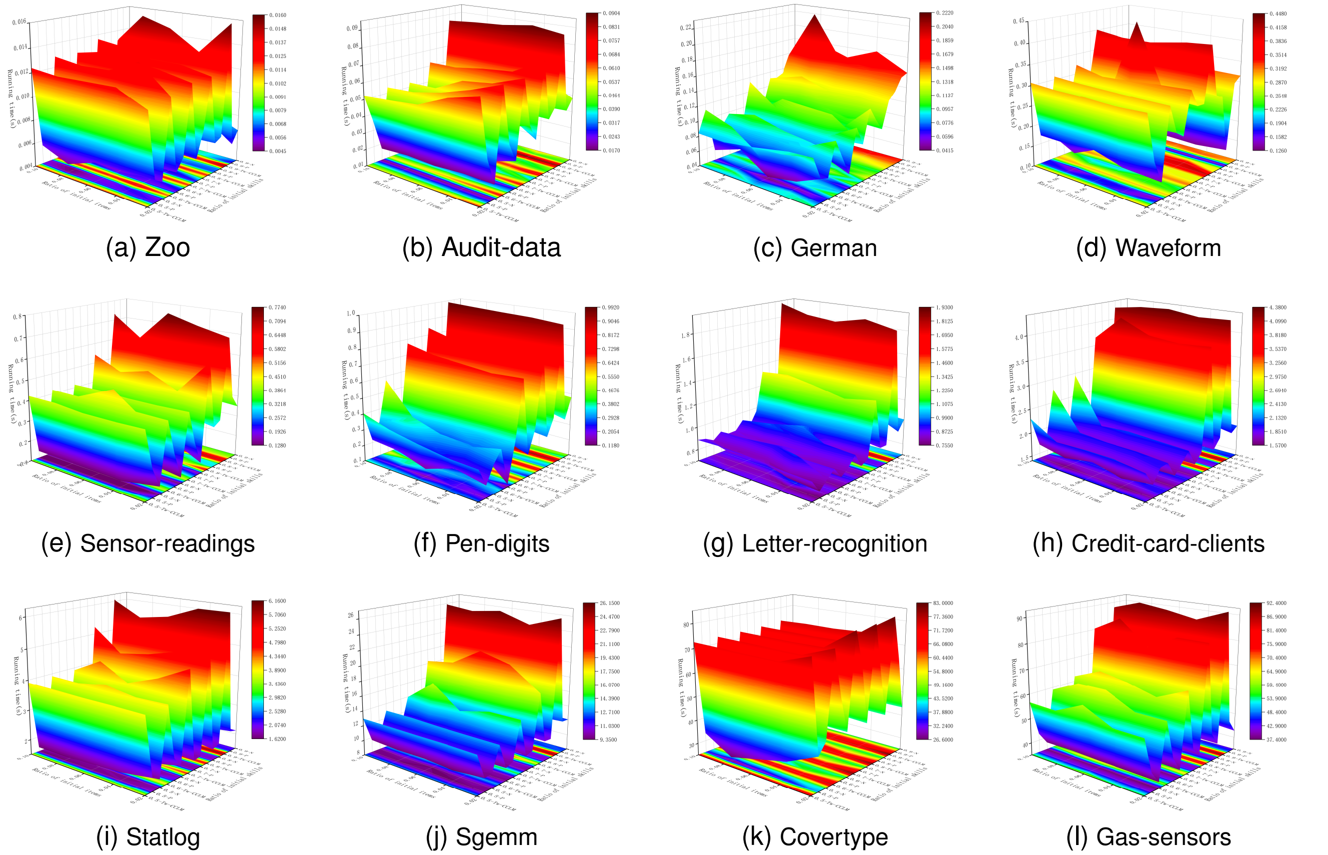


Fig. 5. Running time(s) of the acquired concepts.

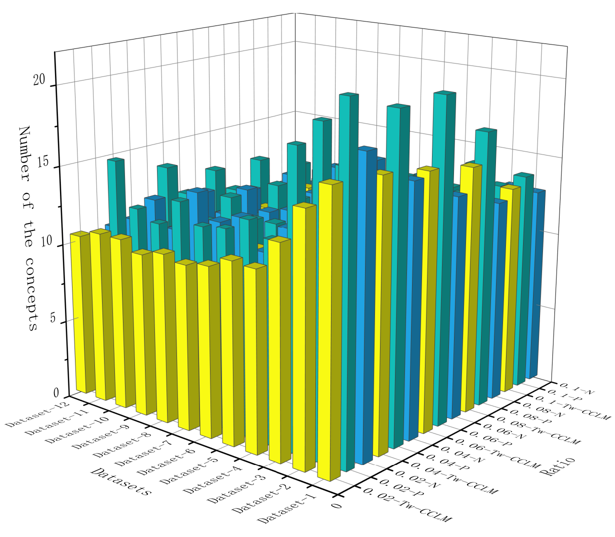


Fig. 6. Number of the concepts under different ratios

REFERENCES

[1] Y. Zhao, J. Li, W. Liu, and W. Xu, "Cognitive concept learning from incomplete information," *International Journal of Machine Learning and Cybernetics*, vol. 8, pp. 159–170, 2017.

[2] R. Wille, "Restructuring lattice theory: an approach based on hierarchies of concepts," in *Formal Concept Analysis: 7th International Conference, ICFCA 2009 Darmstadt, Germany, May 21-24, 2009 Proceedings 7*. Springer, 2009, pp. 314–339.

[3] C. Zhang, E. C. Tsang, W. Xu, Y. Lin, L. Yang, and J. Wu, "Dynamic updating variable precision three-way concept method based on two-way concept-cognitive learning in fuzzy formal contexts," *Information Sciences*, vol. 655, p. 119818, 2024.

[4] D. Guo, W. Xu, Y. Qian, and W. Ding, "Fuzzy-granular concept-cognitive learning via three-way decision: Performance evaluation on dynamic knowledge discovery," *IEEE Transactions on Fuzzy Systems*, 2023.

[5] A. Burusco Juandeaburre and R. Fuentes-González, "The study of the l-fuzzy concept lattice," *Mathware & Soft Computing*, vol. 1, no. 3, pp. 209–218, 1994.

[6] W. Zhang, J. Ma, and S. Fan, "Variable threshold concept lattices," *Information Sciences*, vol. 177, no. 22, pp. 4883–4892, 2007.

[7] C. Zhang, E. C. Tsang, W. Xu, Y. Lin, and L. Yang, "Incremental concept-cognitive learning approach for concept classification oriented to weighted fuzzy concepts," *Knowledge-Based Systems*, vol. 260, p. 110093, 2023.

[8] J. Li, L. Liu, J. Sun, H. Mo, J.-J. Yang, S. Chen, H. Liu, Q. Wang, and H. Pan, "Comparison of different machine learning approaches to predict small for gestational age infants," *IEEE Transactions on Big Data*, vol. 6, no. 2, pp. 334–346, 2016.

[9] J. Ding, X.-H. Hu, and V. Gudivada, "A machine learning based framework for verification and validation of massive scale image data," *IEEE Transactions on Big Data*, vol. 7, no. 2, pp. 451–467, 2017.

[10] K. Yuan, D. Miao, W. Pedrycz, H. Zhang, and L. Hu, "Multigranularity data analysis with zentropy uncertainty measure for efficient and robust feature selection," *IEEE Transactions on Cybernetics*, 2024.

[11] W. Xu, Y. Pan, X. Chen, W. Ding, and Y. Qian, "A novel dynamic fusion approach using information entropy for interval-valued ordered datasets," *IEEE Transactions on Big Data*, vol. 9, no. 3, pp. 845–859, 2023.

[12] L. A. Zadeh, "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*, vol. 90, no. 2, pp. 111–127, 1997.

[13] K. Yuan, D. Miao, W. Pedrycz, W. Ding, and H. Zhang, "Ze-hfs: Zentropy-based uncertainty measure for heterogeneous feature selection

and knowledge discovery," *IEEE Transactions on Knowledge and Data Engineering*, 2024.

[14] X. Deng, J. Li, Y. Qian, and J. Liu, "An emerging incremental fuzzy concept-cognitive learning model based on granular computing and conceptual knowledge clustering," *IEEE Transactions on Emerging Topics in Computational Intelligence*, 2024.

[15] X. Zhang and D. Miao, "Two basic double-quantitative rough set models of precision and grade and their investigation using granular computing," *International Journal of Approximate Reasoning*, vol. 54, no. 8, pp. 1130–1148, 2013.

[16] W. Wu, Y. Leung, and J. Mi, "Granular computing and knowledge reduction in formal contexts," *IEEE Transactions on Knowledge and Data Engineering*, vol. 21, no. 10, pp. 1461–1474, 2008.

[17] W. Xu and W. Li, "Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets," *IEEE Transactions on Cybernetics*, vol. 46, no. 2, pp. 366–379, 2016.

[18] F. J. Cabrerizo, R. Al-Hmouz, A. Morfeq, M. Á. Martínez, W. Pedrycz, and E. Herrera-Viedma, "Estimating incomplete information in group decision making: A framework of granular computing," *Applied Soft Computing*, vol. 86, p. 105930, 2020.

[19] D. Guo and W. Xu, "Fuzzy-based concept-cognitive learning: An investigation of novel approach to tumor diagnosis analysis," *Information Sciences*, vol. 639, p. 118998, 2023.

[20] D. Guo, W. Xu, Y. Qian, and W. Ding, "M-fccl: Memory-based concept-cognitive learning for dynamic fuzzy data classification and knowledge fusion," *Information Fusion*, vol. 100, p. 101962, 2023.

[21] Y. Shi, Y. Mi, J. Li, and W. Liu, "Concept-cognitive learning model for incremental concept learning," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 2, pp. 809–821, 2018.

[22] Y. Mi, W. Liu, Y. Shi, and J. Li, "Semi-supervised concept learning by concept-cognitive learning and concept space," *IEEE Transactions on Knowledge and Data Engineering*, vol. 34, no. 5, pp. 2429–2442, 2020.

[23] K. Yuan, W. Xu, W. Li, and W. Ding, "An incremental learning mechanism for object classification based on progressive fuzzy three-way concept," *Information Sciences*, vol. 584, pp. 127–147, 2022.

[24] J. Wang, W. Xu, W. Ding, and Y. Qian, "Multi-view fuzzy concept-cognitive learning with high-order information fusion of fuzzy attributes," *IEEE Transactions on Fuzzy Systems*, 2024.

[25] W. Xu and Y. Chen, "Multi-attention concept-cognitive learning model: A perspective from conceptual clustering," *Knowledge-Based Systems*, vol. 252, p. 109472, 2022.

[26] J. Wu, E. C. Tsang, W. Xu, C. Zhang, and L. Yang, "Correlation concept-cognitive learning model for multi-label classification," *Knowledge-Based Systems*, vol. 290, p. 111566, 2024.

[27] M. Tao, F. Hao, L. Wei, H. Zhi, S. O. Kuznetsov, and G. Min, "Fairness-aware maximal cliques identification in attributed social networks with concept-cognitive learning," *IEEE Transactions on Computational Social Systems*, 2024.

[28] Y. Ren, Y. Zhang, and W. Hong, "Incremental concept cognitive learning in dynamic formal contexts based on attribute partial order structure diagram," *Computational and Applied Mathematics*, vol. 43, no. 6, p. 319, 2024.

[29] W. Xu, J. Pang, and S. Luo, "A novel cognitive system model and approach to transformation of information granules," *International Journal of Approximate Reasoning*, vol. 55, no. 3, pp. 853–866, 2014.

[30] B. Schölkopf, F. Locatello, S. Bauer, N. R. Ke, N. Kalchbrenner, A. Goyal, and Y. Bengio, "Toward causal representation learning," *Proceedings of the IEEE*, vol. 109, no. 5, pp. 612–634, 2021.

[31] C. Chaccour, W. Saad, M. Debbah, Z. Han, and H. V. Poor, "Less data, more knowledge: Building next generation semantic communication networks," *arXiv preprint arXiv:2211.14343*, 2022.

[32] M. Hu, E. C. Tsang, Y. Guo, Q. Zhang, D. Chen, and W. Xu, "A novel approach to concept-cognitive learning in interval-valued formal contexts: a granular computing viewpoint," *International Journal of Machine Learning and Cybernetics*, vol. 13, no. 4, pp. 1049–1064, 2022.

[33] W. Xu, D. Guo, J. Mi, Y. Qian, K. Zheng, and W. Ding, "Two-way concept-cognitive learning via concept movement viewpoint," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 34, no. 10, pp. 6798–6812, 2023.

[34] W. Xu, D. Guo, Y. Qian, and W. Ding, "Two-way concept-cognitive learning method: a fuzzy-based progressive learning," *IEEE Transactions on Fuzzy Systems*, vol. 31, no. 6, pp. 1885–1899, 2022.

[35] X. Zhang, D. Guo, and W. Xu, "Two-way concept-cognitive learning with multi-source fuzzy context," *Cognitive Computation*, vol. 15, no. 5, pp. 1526–1548, 2023.

[36] X. Xie, J. Li, and B. Wang, "Two-way concept-cognitive learning method: a perspective from progressive learning of fuzzy skills," *International Journal of General Systems*, pp. 1–35, 2024.

[37] J. Pang, B. Zhang, and M. Chen, "A novel l-fuzzy concept learning via two-way concept-cognitive learning and residuated implication," *International Journal of Fuzzy Systems*, vol. 26, no. 8, pp. 2783–2804, 2024.

[38] X. Xie, W. Xu, and J. Li, "A novel concept-cognitive learning method: A perspective from competences," *Knowledge-Based Systems*, vol. 265, p. 110382, 2023.

[39] D. Guo, W. Xu, W. Ding, Y. Yao, X. Wang, W. Pedrycz, and Y. Qian, "Concept-cognitive learning survey: Mining and fusing knowledge from data," *Information Fusion*, vol. 109, p. 102426, 2024.

[40] D. Guo and W. Xu, "R-fccl: A novel fuzzy-based concept-cognitive learning approach for high- dimensional data," *Journal of Computer Research and Development*, 2024.

[41] J. P. Doignon and J. C. Falmagne, "Spaces for the assessment of knowledge," *International Journal of Man-Machine Studies*, vol. 23, no. 2, pp. 175–196, 1985.

[42] W. Yin, "Adaptive testing based on knowledge space theory," in *Cyber Security Intelligence and Analytics: 2021 International Conference on Cyber Security Intelligence and Analytics (CSIA2021)*, vol. 1. Springer, 2021, pp. 416–421.

[43] D. J. Weiss, "Better data from better measurements using computerized adaptive testing," *Journal of Methods and Measurement in the Social Sciences*, vol. 2, no. 1, pp. 1–27, 2011.

[44] A. Rusch and R. Wille, "Knowledge spaces and formal concept analysis," in *Data Analysis and Information Systems: Statistical and Conceptual Approaches Proceedings of the 19th Annual Conference of the Gesellschaft für Klassifikation eV University of Basel, March 8–10, 1995*. Springer, 1996, pp. 427–436.

[45] J. Li. and W. Sun, "Knowledge space,formal context and knowledge base," *Journal of Northwest University(Natural Science Edition)*, vol. 49, no. 4, pp. 517–526, 2019.

[46] C. Doble, J. Matayoshi, E. Cosyn, H. Uzun, and A. Karami, "A data-based simulation study of reliability for an adaptive assessment based on knowledge space theory," *International Journal of Artificial Intelligence in Education*, vol. 29, no. 2, pp. 258–282, 2019.

[47] I. Düntsch and G. Gediga, "Skills and knowledge structures," *British Journal of Mathematical and Statistical Psychology*, vol. 48, no. 1, pp. 9–27, 1995.

[48] J. Heller, A. Ünlü, and D. Albert, "Skills, competencies and knowledge structures," *Knowledge Spaces: Applications in Education*, pp. 229–242, 2013.

[49] Y. Zhou and J. Li, "Skill reduction and assessment in formal context," *Journal of Frontiers of Computer Science and Technology*, vol. 16, no. 3, pp. 692–702, 2022.

[50] D. F. Yinfeng Zhou, Jinjin Li and T. Yang, "Learning paths and skills assessment in formal context," *Pattern Recognition and Artificial Intelligence*, vol. 34, no. 12, pp. 1069–1084, 2021.

[51] W. Sun, J. Li, X. Ge, and Y. Lin, "Knowledge structures delineated by fuzzy skill maps," *Fuzzy Sets and Systems*, vol. 407, pp. 50–66, 2021.

[52] B. Long, W. Xu, X. Zhang, and L. Yang, "The dynamic update method of attribute-induced three-way granular concept in formal contexts," *International Journal of Approximate Reasoning*, vol. 126, pp. 228–248, 2020.

[53] N. Duntsch and G. Gediga, "Modal-style operators in qualitative data analysis," in *2002 IEEE International Conference on Data Mining, 2002. Proceedings*. IEEE, 2002, pp. 155–162.

[54] L. Wei and T. Qian, "The three-way object oriented concept lattice and the three-way property oriented concept lattice," in *2015 International Conference on Machine Learning and Cybernetics (ICMLC)*, vol. 2. IEEE, 2015, pp. 854–859.

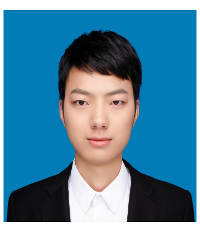
[55] H. Zhi, J. Qi, T. Qian, and L. Wei, "Three-way dual concept analysis," *International Journal of Approximate Reasoning*, vol. 114, pp. 151–165, 2019.

[56] Y. Yao, "A comparative study of formal concept analysis and rough set theory in data analysis," in *Rough Sets and Current Trends in Computing: 4th International Conference, RSCTC 2004, Uppsala, Sweden, June 1-5, 2004. Proceedings 4*. Springer, 2004, pp. 59–68.

[57] X. Zhao, D. Miao, and B. Q. Hu, "On relationship between three-way concept lattices," *Information Sciences*, vol. 538, pp. 396–414, 2020.

[58] J.-C. Falmagne and J.-P. Doignon, *Learning spaces: Interdisciplinary applied mathematics*. Springer Science & Business Media, 2010.

Weihua Xu received the Ph.D. degree from School of Sciences, Xi'an Jiaotong University, Xi'an, China, in 2007, and the M.Sc. degree from School of Mathematics and Information Sciences, Guangxi University, Nanning, China, in 2004. He is currently a professor with the College of Artificial Intelligence, Southwest University, Chongqing, China. He serves in the editorial boards of several international journals, and has published 3 monographs and over 120 articles in international journals. His current research interests include granular computing, approximate reasoning, fuzzy sets, rough sets, concept lattices, cognitive computing, and evolutionary computing.



Di Jiang received the B.E. degree from the Yancheng Institute Of Technology, Yancheng, China, in 2022. Currently, he is pursuing a Master's degree at the School of Artificial Intelligence, Southwest University, Chongqing, China. His current research interests include concept learning, cognitive computing, granular computing.