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ORIGINAL ARTICLE

An efficient multi-source information fusion approach for dynamic interval-valued data via fuzzy approximate conditional entropy

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Abstract
Information fusion enables the integration and transformation of complimentary data from different sources, providing a unified representation for centralized knowledge discovery, which can contribute to effective decision-making, classification, prediction, and more. Multi-source interval-valued data, represented in the form of intervals to capture uncertainty phenomenons, is a common type of symbolic data that finds extensive applications in the real world. This paper aims to investigate the effective fusion of multi-source interval-valued data and to design dynamic updating algorithms for the situations involving multiple dimensions. The objective is to enhance the efficiency of fusion processes. Firstly, this paper use the Kullback–Leibler divergence to measure the dissimilarity between interval distributions, and construct fuzzy similarity relation. Furthermore, we define a fuzzy information granule structure of interval-valued. Secondly, the concept of fuzzy similarity relations is utilized to construct fuzzy decision-making for objects. Subsequently, based on the aforementioned fuzzy information granule structure and fuzzy decision-making, we propose a novel measure called fuzzy approximate conditional entropy and design a corresponding entropy fusion model. Finally, we discuss various scenarios where dynamic changes occur simultaneously in the attributes and information sources of dynamic multi-source interval-valued data. We design corresponding dynamic update algorithms for these situations. Numerical experiments are conducted on nine UCI datasets to validate our proposed fusion method. The experimental results indicate that our fusion approach exhibits improved classification performance compared to the common fusion methods. The designed dynamic update algorithms are also capable of reducing computation time and enhancing fusion efficiency.

Keywords Information entropy · Interval-valued · Multi-source information fusion · Granular computation

1 Introduction

With the development of advanced information technologies such as big data, cloud computing, and mobile computing, data acquisition is no longer limited to a single source. Data storage and description are now presented in the form of multiple sources, where the relationships between data objects from different sources contain various pieces of information about the knowledge structure. These relationships provide insights into the underlying knowledge and express information about the samples from multiple perspectives. The fundamental principle of multi-source information fusion is to fully utilize various information sources by combining the spatial redundancy, temporal redundancy, or complementary information constraints through specific rules. This process aims to construct a unified representation to obtain information that is more significant and of higher accuracy than that provided by individual sources or single sensors [1]. This approach helps reduce data ambiguity and uncertainty, leading to an improvement in information quality. Currently, multi-source data fusion has become an important research direction and has been successfully applied in various military and civilian domains. In the military field, it has found applications in military command automation systems, strategic warning and defense, multi-target tracking and recognition, and precision-guided weapons. Moreover, its applications have extended to civilian fields such as remote sensing detection,

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medical diagnosis, e-commerce, wireless communication, industrial process monitoring, and fault diagnosis [2–4].

The concept of information granularity was first introduced by Zadeh [5], which subsequently sparked significant attention and research fervor in the field of information granulation. The idea of information granulation has permeated various domains [6], such as the “decomposition” and “partitioning” in automate and systems, operations on interval numbers in interval analysis, the concept of “aggregation” in economics, and the notion of “evidence” in evidence theory. Hobbs [7] proposed a framework for granularity theory, which explores the decomposition and merging of granules and provides methods for constructing granules of different sizes. Lin [8] introduced the concept of domain systems and studied the relationship between domain systems and relational databases. Information granulation encompasses various theories and approaches, including set theory, interval analysis, fuzzy sets, rough sets, probability theory, and the theory of quotient space [9]. Typical examples of information granulation include granulation based on rough set theory [10], granulation based on commercial space theory [11], and granulation based on fuzzy set theory [12]. As one of the effective tools in data mining, information granulation can effectively handle the problem of uncertainty. For instance, Zhang et al. [13] utilized a granular computing model to perform information granulation on information systems, followed by uncertainty measurement using the granulated results. Similarly, Chen et al. [14] employed information granulation methods for uncertainty measurement in domain information systems. Xu et al. [15] carried out information granulation in fuzzy domain information systems and proposed an uncertainty measurement method. Therefore, information granulation can serve as an effective approach to address the challenge of unified modeling in multi-source information systems.

Uncertainty measures in information systems is currently a hot research topic in the field of information science. Rough set theory, proposed by the Polish scholar Pawlak [16], is an information analysis theory that has shown excellent performance in dealing with uncertain data. Rough set theory has been successfully applied in various fields of intelligent information processing, including machine learning [18, 19, 21], decision analysis [17, 25], approximate reasoning [20], data mining [22–24], and more. From the perspective of data analysis, rough set theory possesses numerous advantages. Due to these advantages, many researchers have combined it with multi-source information fusion and achieved outstanding results. For example, Yang et al. [26] explored a multi-granularity-based information fusion method that directly combines multiple sources of information to avoid information loss during the fusion process. Lin et al. [27] studied an information fusion method that combines multi-granularity rough sets with evidence theory. Zhang et al. [28] proposed a data fusion model that utilizes domain rough set models to construct domain granular structures and employs the principles of granular computing to develop uncertainty measurement methods. Zhang et al. [29]also presented a matrix-based multi-granularity fusion method. In the context of multi-label multi-source information systems, Qian et al. [30] proposed a novel conditional entropy and utilized it to fuse multi-label data.

Indeed, the aforementioned information fusion methods mainly focus on single-valued information systems. However, in real-life situations, many phenomena and variables cannot be precisely represented due to the limitations of objective environments and subjective perceptions. In such cases, it is common to use interval representations to describe the uncertainty or imprecision associated with the data. Interval-valued information systems, as a commonly encountered type of information system, are widely present in various fields such as healthcare and finance [31]. Interval-valued data has garnered extensive attention from scholars worldwide, and with the development of rough set theory, significant achievements have been made in the research of interval-valued data. For instance, Dai et al. [32–34] proposed various uncertainty measurement methods for interval-valued information systems using interval rough set models. Their approach involved measuring the roughness of the approximate object set using upper and lower approximations in the interval-valued rough set, and using roughness to represent the uncertainty measurement results of the interval-valued information system. Liu et al. [35]constructed a similarity relation on incomplete interval-valued information systems and proposed an unsupervised attribute reduction method based on α-similarity relation. Dai et al. [36] proposed a fuzzy rough set method based on dominance degree for attribute reduction of incomplete interval-valued data. Xie et al. [37] introduced the definition of interval probability similarity to measure the uncertainty of interval-valued information systems. Xu et al. [38] presented a fault diagnosis method that fuses different diagnostic evidence into interval-valued data. Fumanal-Idocin et al. [39] developed a moderate deviation function to measure the similarity and dissimilarity between a given set of interval-valued data, constructing interval-valued aggregation functions. These functions were applied to two motor imagery-based brain-computer interface (MI-BCI) systems for the classification of EEG signals. The extensive research conducted on interval-valued information systems, as mentioned above, has mainly focused on single-source information systems. However, interval-valued data can also be obtained from multiple diverse sources. For instance, meteorological data collected from various weather stations can be fused to improve weather predictions. Therefore, a key challenge in information fusion is how to effectively integrate and extract useful knowledge from such diverse sources of interval-type data.

Regardless of the fusion methods for single-value information systems mentioned above or the research on single-source interval-valued information systems, they cannot be directly applied to the study of multi-source interval-valued information systems. This paper aims to investigate information fusion methods specifically designed for multi-source interval-valued information systems. In addition, in dynamic multi-source environments, both the sources and attributes undergo continuous changes. For example, when we want to predict the weather conditions of a city, or instance, we can collect data from weather sensors distributed throughout the city. This data includes various weather parameters such as temperature, humidity, and wind speed. Nevertheless, with the advancement of scientific technology, we discover that wind speed is not significant for weather prediction. Therefore, to save the cost of data collection, wind speed is no longer collected. At the same time, due to factors like weather exposure, some sensors may become obsolete and need to be removed from the system. Traditional static fusion methods require recomputing the entire information fusion process, which is costly and even infeasible for large datasets. To reduce computational costs, an incremental fusion method is proposed, which combines newly updated data with accumulated information. For example, Chan et al. [40] put forward an incremental method for fusing unstructured textual data when new text is added. Kotwal et al. introduced a consistent fusion rule for incrementally evaluating the fusion performance of hyperspectral images [41]. Huang et al. [42] presented a dynamic fusion method for interval-valued data that enables fast updates of fusion results when information sources change. However, it is evident that these methods are not applicable for updating fusion results when both attributes and data sources undergo simultaneous changes. Nevertheless, in recent years, researchers have conducted relevant studies in this area. For instance, Zhang et al. [43] proposed dynamic fusion for incomplete interval-valued information systems. Xu et al. [44] developed a dynamic fusion model for multi-source interval-valued ordered information systems. Although these two studies specifically address situations where changes occur simultaneously in attributes and information sources within interval-valued information systems. Although both of these studies address situations where changes occur simultaneously in attributes and information sources within interval-valued information systems, they only consider the endpoint information of interval values when measuring the similarity between two objects. This oversight leads to the loss of effective information contained within the intervals. To address this limitation, this paper proposes a dynamic fusion method based on fuzzy approximate conditional entropy. The main contributions of this paper can be summarized as follows:

1. To address the issue of effective information loss caused by only considering the endpoint information and ignoring the contribution of the internal points in interval-valued data, we propose the following solution. We utilize the principle of statistical distribution to transform interval values into probability distributions. Then, we employ the Kullback–Leibler divergence to measure the dissimilarity between two interval value distributions and construct a fuzzy similarity relation.
2. In order to study uncertainty measures that are more suitable for interval-valued information systems, based on the fuzzy similarity relation, we construct fuzzy decisions for objects and define a fuzzy information structure. A novel entropy measure is proposed by combining fuzzy information structure and fuzzy decision definition. Additionally, we construct a lower bound fusion function to effectively fuse multi-source interval-valued information tables.
3. In order to gain a clearer understanding of the dynamic update mechanism. We have discussed four different situations in which the attributes and information sources of a dynamic multi-source interval-valued information system change simultaneously. Additionally, we have formulated corresponding dynamic update algorithms to eliminate redundant calculations and decrease fusion time during the fusion process.

The rest of this work consists of the following contents: Sect. 2 gives the basic definitions of rough set, and reviews the concepts of information tables. Section 3 introduces fuzzy similarity. An infimum fusion function is provided and the dynamic fusion scenarios where multiple dimensions change simultaneously are discussed in Sect. 4. Section 5 compares the time complexity of dynamic and static fusion algorithms. Section 6 analyzes the results of the experiment and the effectiveness and efficiency of fusion. Finally, in Sect. 7 summarizes the work of this paper and future research. And the framework of the paper can be seen in Fig. 1.

2 Preliminary

In this section, we briefly review some mathematical notions and definitions.

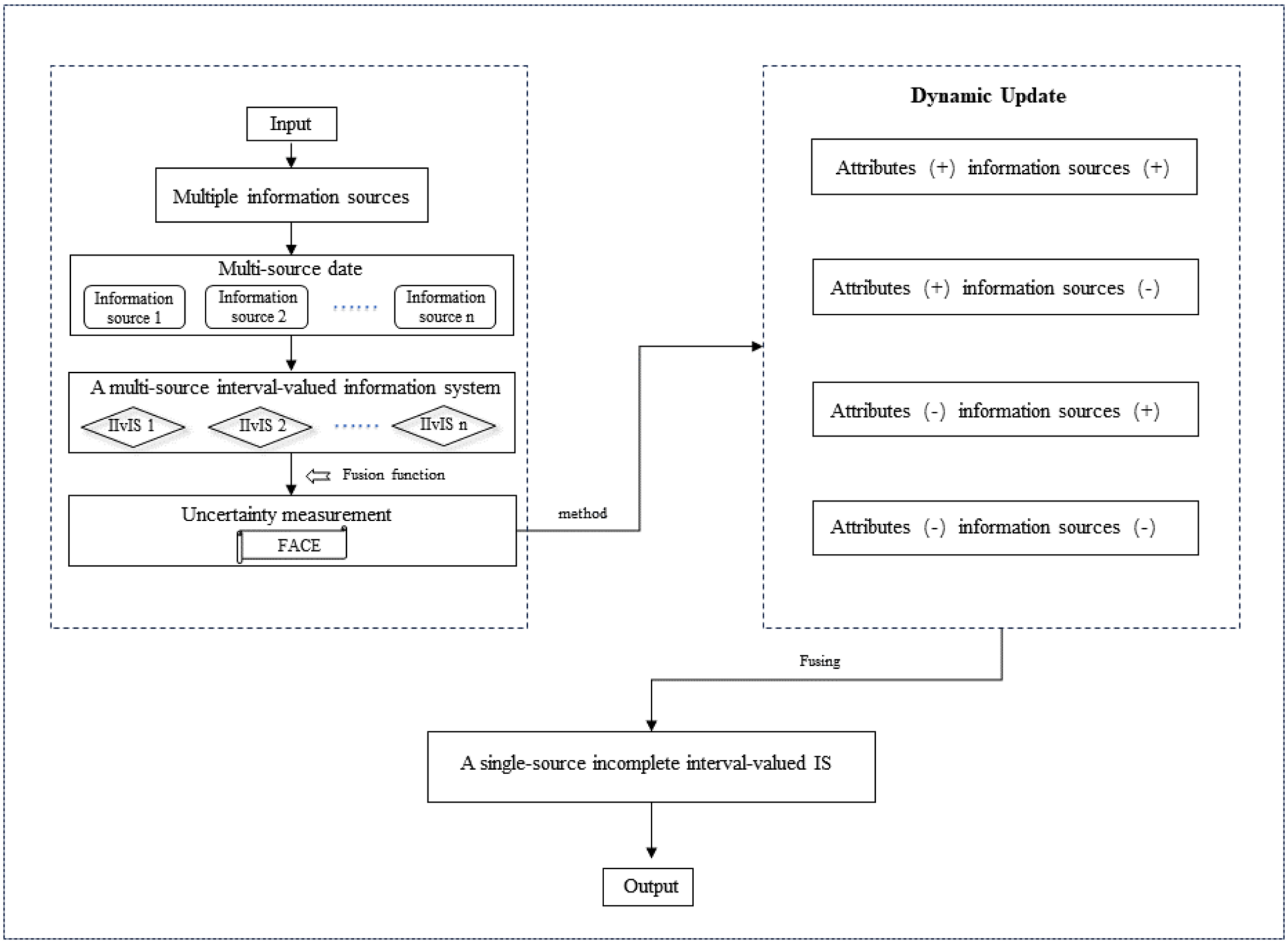

Fig. 1 A framework of the paper

2.1 Rough set theory

Assume that (U, A) is an information system, where U represents a non-empty finite object set, A represents a non-empty finite conditions attribute set. a binary indiscernible relation is determined for any attribute subset $B \subseteq A$ as below:

$$BIR(B) = \left\{(u_i, u_j) \in U | \forall b \in B, b(u_i) = b(u_j)\right\}.$$

According to the binary indiscernible relation $BIR(B)$, we can obtain a partition $U/BIR(B)$ of the object set U. For any attribute subset $B \subseteq A$ and subset $X \subseteq U$, the lower approximation, upper approximation, and boundary region of X with respect to B can be expressed as follows:

$$\underline{BIR(X)} = \left\{u_i \in U | [u_i]_{BIR} \subseteq X\right\},$$

$$\overline{BIR(X)} = \left\{u_i \in U | [u_i]_{BIR} \cap X \neq \varnothing\right\},$$

$$B(X) = \overline{BIR(X)} - \underline{BIR(X)}.$$

Pawlak proposed two numerical measures, accuracy and roughness, for evaluating the uncertainty of a given object set X. These measures can be expressed as follows:

$$Accurac(X) = \frac{\left|\underline{BIR(X)}\right|}{\left|\overline{BIR(X)}\right|}, \tag{1}$$

$$Roughness(X) = 1 - Accurac(X). \tag{2}$$

where |.| denotes the cardinality of elements. The accuracy and roughness concepts are used to characterize the completeness and incompleteness, respectively, of the knowledge about a given set of objects, denoted as X.

2.2 Fuzzy rough set theory

Let the set A on the universe U be a mapping:

$$A : U \to [0,1]$$
$$x \mapsto A(x),$$

for any $x \in U$, then A is called a fuzzy set, $A(x)$ is the membership degree of x to A.

Let $B \subseteq A$ be an attribute subset, and let R_B be a fuzzy binary relation induced by B on U. R_B is referred to as a fuzzy similarity relation if it satisfies the following conditions:

(1) Reflectivity: $R_B(x,x) = 1, \forall x \in U$;
(2) Symmetry: $\forall x, y \in U, R_B(x,y) = R_B(y,x)$.

Given a decision system $(U, A \bigcup D)$, where $U = \{x_1, x_2, \dots, x_n\}, A = \{a_1, a_2, \dots, a_m\}$. Let attribute subset $B \subseteq A$ and R_B is the fuzzy similarity relation on U induced by attribute set A. The decision partition $U/D = \{Y_1, Y_2, \dots, Y_m\}$, then the fuzzy decisions of objects exported by decision attribute D can be expressed as below:

$$\widetilde{FU/D} = \left\{\widetilde{FY}_1, \widetilde{FY}_2, \dots, \widetilde{FY}_m\right\},$$

$$\widetilde{FY}_r(x) = \frac{\left|[x]_A \bigcap Y_r\right|}{[x]_A}, r = 1, 2, \dots, m,$$

where $\widetilde{FY}_r$ is the fuzzy set and $\widetilde{FY}_r(x)$ indicates the membership degree of x to $\widetilde{FY}_r$.

2.3 Multi-source interval-valued information system

Assume that $IvIS = (U, A, V, f)$ is an interval-valued information system, where $U = \{x_1, x_2, \dots, x_n\}$ represents a non-empty and finite object set, $A = \{a_1, a_2, \dots, a_p\}$ represents non-empty and finite attribute set. V is called the range of attribute A. $f : U \times A \to V$ represents information function, $\forall x \in U, a \in A, f(x,a) = [f^-(x,a), f^+(x,a)]$.

Let $IvIS_i = (U, A, V_i, f_i)$ be the i-th $IvIS$, where the meanings of U, A, V_i and f_i as mentioned above. Generally, a multi-source interval-valued information system is defined as below:

$$MsIvIS = \{IIvIS_i \big| IIvIS_i = (U, A, V_i, f_i), i = 1, 2, \dots, N\}.$$

Similarly, $IvDIS = (U, A, V_A, f_A, D, V_D, f_D)$ represents incomplete interval-valued decision information system, where the connotations of U, A, V_A and f_A are in agreement with those mentioned in the $IvIS$, D represents the decision attribute set. V_D represents the range of the decision attribute value. Information function is expressed as $f_D : U \times D \to V_D$. $I = [0,1]$, I^U is called as the family consisted of all fuzzy sets on U. Let $IvDIS = (U, A, V_{A_i}, f_{A_i}, D, V_{D_i}, f_{D_i})$ be the ith $IvDIS$, where the connotations of U, A, V_{A_i}, f_{A_i}, D, V_{D_i} and f_{D_i} as mentioned above. In general, a multi-source interval-valued decision information system($MsIvDIS$) is expressed as follows:

$$\begin{aligned} MsIvDIS &= \{IvDIS_i \big| IvFDIS_i \\ &= (U, A, V_{A_i}, f_{A_i}, D, V_{D_i}, f_{D_i}), i = 1, 2, \cdots, N\}. \end{aligned}$$

For convenience, this article abbreviated the above expression. We use $(U, A \bigcup D)_i$ to represent the decision information system and (U, A_i) to represent the information system (Table 1).

Example 1 So as to better understand the definition of MsIvDIS. We give the example as following. With the awakening of people's health awareness, more and more friends began to develop the habit of regular physical examination. However, because the interval between medical examinations is long, and the specific time and place are not fixed. As a result, many friends have several physical examinations, which are not carried out in the same hospital. After receiving the results of the physical examination, some people will find that the numerical results of the physical examination they did in several hospitals are very different. Tables 2, 3, 4, 5 respectively represent the physical examination results of eight people in four hospitals. Attributes $a_1 - a_6$ indicate hemoglobin counts, leukocyte counts, blood fat, blood sugar, platelet counts, and Hb level, respectively. Suppose that $V_D = \{Leukemia\ patient, Non\ leukemia\ patient\}$, and $U/D = \{Y_1, Y_2\}$, where $Y_1 = \{x_1, x_2, x_6, x_8\}$, $Y_2 = \{x_3, x_4, x_5, x_7\}$.

3 Fuzzy similarity relation for MsIvDIS

Unlike real values, comparing two interval values using traditional methods is challenging. However, inspired by the similarity measure for general interval-valued data proposed in [43], we can define the notion of similarity between two intervals.

3.1 Distance measurement between interval values

Many difference measurements of interval values are associated with distance. Numerous scholars have done a great

Table 1 Abbreviation table

Terminologies	Abbreviations
Interval-valued information system	IvIS
Interval-valued decision information system	IvDIS
Muti-source intervalued decision information system	MsIvDIS
Fuzzy approximate conditional entropy	FACE
Kullback–Leibler divergence	KL divergence

Table 2 Physical examination report of the first hospital $IIvIS_1$

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	[121.56,139.69]	[3.00,7.00]	[115.25,134.96]	[67.23,106.58]	[43.25,186.98]	[70.56,92.98]
x_2	[116.58,124.36]	[4.00,11.00]	[116.96,124.95]	[79.03,117.68]	[80.00,223.98]	[67.34,88.56]
x_3	[105.25,116.06]	[2.00,10.00]	[112.45,141.06]	[118.59,178.45]	[64.25,82.97]	[78.99,99.02]
x_4	[125.19,133.65]	[2.00,9.00]	[111.65,120.98]	[60.45,97.98]	[99.56,239.14]	[66.25,88.29]
x_5	[127.68,135.39]	[4.00,11.00]	[111.62,294.97]	[75.59,108.62]	[138.06,169.45]	[43.00,74.00]
x_6	[127.16,214.15]	[6.00,16.00]	[164.62,295.68]	[81.34,162.58]	[78.37,88.18]	[28.02,61.25]
x_7	[103.69,196.58]	[4.00,9.00]	[177.45,268.36]	[84.58,152.97]	[78.18,98.37]	[30.00,60.02]
x_8	[114.97,196.42]	[10.00,20.00]	[224.12,313.37]	[102.25,162.36]	[67.06,86.86]	[25.06,56.00]
x_9	[119.56,149.28]	[12.00,22.00]	[121.16,167.98]	[90.18,138.28]	[71.68,98.56]	[78.35,88.64]
x_{10}	[138.26,206.05]	[8.00,18.00]	[176.98,265.69]	[96.65,128.46]	[109.98,252.96]	[39.06,70.00]

Table 3 Physical examination report of the second hospital $IIvIS_2$

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	[117.60,148.30]	[1.00,5.00]	[115.56,124.65]	[69.39,108.65]	[45.00,76.25]	[70.00,93.89]
x_2	[109.49,164.37]	[3.00,11.00]	[116.68,124.37]	[79.00,118.65]	[62.59,79.65]	[67.58,82.34]
x_3	[118.26,129.56]	[2.00,10.00]	[112.85,121.94]	[119.03,179.56]	[66.00,84.00]	[78.00,100.05]
x_4	[125.65,153.26]	[1.00,9.00]	[121.36,160.48]	[60.00,98.85]	[83.02,126.58]	[66.25,89.35]
x_5	[126.97,145.34]	[4.00,11.00]	[111.97,294.98]	[81.65,119.75]	[139.28,268.94]	[43.00,74.09]
x_6	[106.26,134.15]	[6.00,16.00]	[116.97,125.68]	[75.45,120]	[68.58,87.79]	[27.34,61.52]
x_7	[123.38,185.96]	[4.00,9.00]	[177.89,245.34]	[84.65,153.88]	[77.89,97.65]	[30.88,60.59]
x_8	[187.68,258.54]	[10.00,20.00]	[149.97,166.65]	[102.00,162.98]	[65.02,84.38]	[26.65,59.34]
x_9	[138.00,249.28]	[11.00,22.00]	[153.26,268.98]	[90.00,150.02]	[88.69,164.35]	[77.85,99.58]
x_{10}	[111.35,214.69]	[8.00,18.00]	[177.39,265.98]	[67.85,118.06]	[109.26,252.97]	[39.00,70.00]

Table 4 Physical examination report of the third hospital $IIvIS_3$

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	[116.85,136.49]	[3.00,.007]	[115.96,124.85]	[72.60,116.98]	[43.00,186.45]	[70.98,93.06]
x_2	[118.79,130.00]	[6.00,12.00]	[103.98,139.85]	[79.69,118.89]	[80.00,223.85]	[67.97,89.65]
x_3	[124.99,163.00]	[2.00,10.00]	[112.30,121.98]	[116.59,159.99]]	[66.58,84.00]	[78.00,100.00]
x_4	[106.69,129.85]	[2.00,9.00]	[111.00,120.85]	[60.00,98.99]	[86.08,168.96]	[66.00,89.99]
x_5	[126.35,135.84]	[4.00,11.00]	[111.98,295.52]	[81.00,119.89]	[139.65,282.59]	[43.00,74.59]
x_6	[156.89,214.98]	[6.00,16.00]	[204.59,295.85]	[80.15,162.38]	[68.85,87.48]	[27.96,61.59]
x_7	[122.98,196.85]	[4.00,9.00]	[177.85,268.35]	[84.69,153.78]	[77.56,97.34]	[30.00,60.00]
x_8	[158.59,233.94]	[10.00,20.00]	[224.96,314.84]	[102.96,142.58]	[65.59,84.98]	[25.00,56.89]
x_9	[118.26,149.35]	[12.00,21.00]	[224.58,268.95]	[94.95,128.58]	[71.98,90.06]	[76.29,89.58]
x_{10}	[161.85,214.95]	[8.00,18.00]	[177.56,267.89]	[67.35,118.86]	[109.38,252.95]	[39.48,71.67]

deal of work to measure the differences between interval values. So far to, there are some familiar distances that can be used to reflect diverseness in interval values. For example:

City-block distance:

$$D_c = |y^- - x^-| + |y^+ - x^+|, \tag{3}$$

Euclid distance:

$$D_E = |y^- - x^-|^2 + |y^+ - x^+|^2, \tag{4}$$

Hausdorf distance etc:

$$D_H = \max(|y^- - x^-|, |y^+ - x^+|), \tag{5}$$

where $x = [x^-, x^+]$ are two interval values.

However, the distance measure mentioned above only considers the endpoints of interval values, but the internal

Table 5 Physical examination report of the fourth hospital $IIvIS_4$

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	[116.48,128.56]	[3.00,7.00]	[115.38,125.97]	[73.97,111.25]	[59.97,139.98]	[70.00,92.39]
x_2	[119.58,131.69]	[4.00,11.00]	[115.25,124.68]	[79.38,118.65]	[82.69,215.48]	[67.00,88.09]
x_3	[109.99,112.98]	[4.00,6.00]	[112.87,121.96]	[119.59,179.98]	[66.78,85.97]	[78.00,99.65]
x_4	[126.43,134.97]	[2.00,9.00]	[111.98,120.99]	[60.0,99.00]	[100.00,260.00]	[66.45,88.9]
x_5	[128.45,136.76]	[4.00,11.00]	[108.97,236.99]	[81.28,120.95]	[138.88,283.69]	[43.00,74.06]
x_6	[127.97,215.96]	[6.00,16.00]	[204.08,296.98]	[81.97,163.65]	[68.06,88.09]	[28.00,61.98]
x_7	[143.96,216.26]	[4.00,9.00]	[177.89,268.38]	[84.56,154.98]	[78.0,98.99]	[30.00,60.00]
x_8	[121.59,169.09]	[10.00,20.00]	[224.96,315.98]	[102.98,163.96]	[65.35,85.97]	[25.65,56.69]
x_9	[119.68,150.98]	[12.00,22.00]	[225.58,269.97]	[90.00,151.98]	[71.96,91.09]	[78.99,88.58]
x_{10}	[142.54,165.86]	[7.00,16.00]	[177.56,268.69]	[68.34,119.98]	[109.36,253.69]	[43.00,76.60]

contribution to information of interval-valued endpoints is ignored, resulting in the loss of interval-valued effective information. Therefore, this paper uses the probability distribution principle to treat an interval as a probability distribution, and uses Kullback–Leibler divergence to calculate the distance between distributions.

Definition 1 For continuous random variables, the KL divergence of the two probability distributions P and Q is defined in the integral form as follows:

$$KL(P\|Q) = \int P(x)\ln\left(\frac{P(x)}{Q(x)}\right)dx, \tag{6}$$

where $P(x)$ and $Q(x)$ is the probability density function of P, Q. In the field of probability statistics, KL divergence can be used to measure the distance between two probability distributions.

Definition 2 Let $x_i = [x_i^-, x_i^+]$, $y_j = [y_j^-, y_j^+]$ are two interval values and $a \in A$,where $x_i, y_j \in U$. The new distance measurement between x_i and y_j is defined as follow:

$$x_i \sim N(\mu_1, \sigma_1^2)\ ,\ y_j \sim N(\mu_2, \sigma_2^2), \tag{7}$$

where

$$\mu_1 = \frac{x_i^+ + x_i^-}{2}, \quad \mu_2 = \frac{y_j^+ + y_j^-}{2}, \tag{8}$$

$$\sigma_1 = \frac{x_i^+ - x_i^-}{2}, \quad \sigma_2 = \frac{y_j^+ - y_j^-}{2}. \tag{9}$$

Based on KL divergence, The interval-valued distance between x_i and y_j is defined as follows:

$$dis_A(x_i, x_j) = \left(\sum_{a\in A} dis_{\{a\}}(x_i, x_j)\right)^{\frac{1}{2}}, \tag{10}$$

where

$$dis_a(x_i, y_j) = \frac{KL\left(x_i \middle\| y_j\right) + KL\left(y_j \middle\| x_i\right)}{2}. \tag{11}$$

3.2 δ-similarity in MsIvDIS

Definition 3 Let $x_i = [x_i^-, x_i^+]$,$y_j = [y_j^-, y_j^+]$ are two interval values and $a \in A$, where $x_i^- < x_i^+, y_j^- < y_j^+$. The similarity of x_i and y_j can be expressed as below:

$$sim_A(x_i, x_j) = e^{-\frac{dis_A(x_i,x_j)}{2\delta^2}},$$

where $\delta > 0$ is a hyperparameter, which is used to control the size of the interval similarity values. Obviously, $0 < Sim_a(x, y) \le 1$ and $Sim_A(x, y) = Sim_A(y, x)$. It should be noted that we set δ to be 0.5 in this paper.

Proposition 1 *Let $x_i = [x_i^-, x_i^+]$,$y_j = [y_j^-, y_j^+]$ are two interval values. Assume attribute subsets B and C, $C \subseteq B \subseteq A$, the similarity of interval samples x_i and y_j under attribute subsets B and C satisfies the following properties*:

$$sim_C(x_i, y_j) \ge sim_B(x_i, y_j).$$

Proof When $C \subseteq B$, we know $|C| \leqslant |B|$, so we have $dis_C(x_i, x_j) \leqslant dis_B(x_i, x_j)$. At the same time $dis_C(x_i, x_j) \leqslant 0$, and because $f(x) = e^x$ increases monotonically on $\mathbb{R}$, it is clear that $g(x) = e^{-x}$ decreases on $\mathbb{R}$, so there is $sim_C(x_i, y_j) \ge sim_B(x_i, y_j)$ □

Definition 4 Suppose that $(U, A \bigcup D)_i$ is the ith $IIvDIS_i$. For condition attribute subset $B \subseteq A$, the fuzzy similarity relation is expressed as follows:

$$R_B^i = (\mathop{\wedge}_{b\in B} Sim_b^i(x_j, x_k))_{n\times n} \; (\forall x_j \in U, (x_j, x_k) \in n \times n), \tag{12}$$

where $Sim_b^i(x_j, x_k)$ denotes the similarity of x_j and x_k under attribute b in the i-th $IIvDIS_i$.

$Sim_B^i(x_j)$ contains the similarity between the object and all other objects in the universe, and can be acknowledged to be a fuzzy information granule. The set of these information granules forms a fuzzy set vector, which is called fuzzy similarity class. The definition is as below:

$$Sim_B^i(x_j) = \frac{R_B^i(x_j, x_1)}{x_1} + \frac{R_B^i(x_j, x_2)}{x_2} + \cdots + \frac{R_B^i(x_j, x_n)}{x_n}. \quad (13)$$

Given an interval-valued decision information system $(U, A \bigcup D)_i$, and attribute subset $B \subseteq A$, interval-valued fuzzy information structure can be expressed as: $Sim_B^i = (Sim_B^i(x_1), Sim_B^i(x_2), \cdots, Sim_B^i(x_n))$.

Example 2 (Continued from Example 1) In accordance with to the above definition, we can figure up the fuzzy similarity class. Let's take attribute a1 of the first information source as an example.

First, according to Definition 2 we can calculate the distance between x_i and $x_j (i, j = 1, 2, \dots, 8)$ w.r.t.a_1.The specific calculation process of distance $d_{a_1}(x_i, x_j)$ is as follows:

$$\begin{aligned} f(x_1, a_1) =& [121.56, 139.69], f(x_2, a_1) = [116.58, 124.36], \\ \mu_1 =& 130.625, \sigma_1 = 9.065, \\ \mu_2 =& 120.47, \sigma_2 = 3.89. \end{aligned}$$

Suppose that:

$$f(x_1, a_1) \sim N(130.625, 9.065^2), f(x_2, a_1) \sim N(120.47, 3.89^2),$$

so according to the Definition 1 we can calculation the KL divergence:

$$KL(f(x_1, a_1) \| f(x_2, a_1)) = 1.0655, KL(f(x_2, a_1) \| f(x_1, a_1)) = 4.7767.$$

Thus, $d_{a_1}(x_1, x_2) = 1.7091$, The same we can also obtain a distance matrix:

$$d_{a_1} = \begin{pmatrix} 0 & 1.70913 & 2.21768 & 0.85281 & 0.97162 & 3.21763 & 2.52382 & 3.23992 & 0.57005 & 2.93864 \\ 1.70913 & 0 & 1.58999 & 1.56513 & 2.02052 & 8.52637 & 6.90912 & 8.19699 & 2.56918 & 7.94978 \\ 2.21768 & 1.58999 & 0 & 2.82746 & 3.34420 & 6.85390 & 5.68189 & 6.97030 & 2.62588 & 6.52132 \\ 0.85281 & 1.56513 & 2.82746 & 0 & 0.38261 & 7.06516 & 5.60540 & 6.50887 & 1.72718 & 6.43973 \\ 0.97162 & 2.02052 & 3.34420 & 0.38261 & 0 & 7.56796 & 5.98007 & 6.83910 & 1.83875 & 6.85172 \\ 3.21763 & 8.52637 & 6.85390 & 7.06516 & 7.56796 & 0 & 0.25895 & 1.24923 & 1.82510 & 0.25353 \\ 2.52382 & 6.90912 & 5.68189 & 5.60540 & 5.98007 & 0.25895 & 0 & 1.18167 & 1.38507 & 0.25279 \\ 3.23929 & 8.19699 & 6.97030 & 6.50887 & 6.83910 & 1.24923 & 1.18167 & 0 & 2.14302 & 0.89719 \\ 0.57005 & 2.56918 & 2.62588 & 1.72718 & 1.83875 & 1.82510 & 1.38507 & 2.14302 & 0 & 1.66454 \\ 2.93864 & 7.94978 & 6.52132 & 6.43973 & 6.85172 & 0.25353 & 0.25279 & 0.89719 & 1.66454 & 0 \end{pmatrix},$$

so the similarity between x_i and x_j w.r.t. a_1 in the first $IIvIS_1$ can be calculated as follows:

$$Sim_{a_1}^1 = \begin{pmatrix} 1 & 0.03277 & 0.01185 & 0.18166 & 0.14324 & 0.00160 & 0.00642 & 0.00153 & 0.31979 & 0.00280 \\ 0.03277 & 1 & 0.04159 & 0.04371 & 0.01758 & 0.00000 & 0.00000 & 0.00000 & 0.00587 & 0.00000 \\ 0.01185 & 0.04159 & 1 & 0.00350 & 0.00125 & 0.00000 & 0.00001 & 0.00000 & 0.00524 & 0.00000 \\ 0.18166 & 0.04371 & 0.00350 & 1 & 0.46523 & 0.00000 & 0.00001 & 0.00000 & 0.03161 & 0.00000 \\ 0.14324 & 0.01758 & 0.00125 & 0.46523 & 1 & 0.00000 & 0.00001 & 0.00000 & 0.02529 & 0.00000 \\ 0.00160 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 1 & 0.59577 & 0.08221 & 0.02599 & 0.60227 \\ 0.00642 & 0.00000 & 0.00001 & 0.00001 & 0.00001 & 0.59577 & 1 & 0.09411 & 0.06265 & 0.60315 \\ 0.00135 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.08221 & 0.09411 & 1 & 0.01376 & 0.16623 \\ 0.31979 & 0.00587 & 0.00524 & 0.03161 & 0.02529 & 0.02599 & 0.06265 & 0.01376 & 1 & 0.03583 \\ 0.00280 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.60227 & 0.60315 & 0.16623 & 0.03583 & 1 \end{pmatrix}.$$

Then the the fuzzy similarity class in the 1-th $IIvIS_1$ can be calculated as follows:

$$Sim^1_{a_1}(x_1)=\frac{1}{x_1}+\frac{0.03277}{x_2}+\frac{0.01185}{x_3}+\frac{0.18166}{x_4}+\frac{0.14324}{x_5}+\frac{0.00160}{x_6}+\frac{0.00642}{x_7}+\frac{0.00153}{x_8}+\frac{0.31979}{x_9}+\frac{0.00280}{x_{10}},$$
$$Sim^1_{a_1}(x_2)=\frac{0.03277}{x_1}+\frac{1}{x_2}+\frac{0.04159}{x_3}+\frac{0.04371}{x_4}+\frac{0.01758}{x_5}+\frac{00587}{x_9},$$
$$Sim^1_{a_1}(x_3)=\frac{0.01185}{x_1}+\frac{0.04159}{x_2}+\frac{1}{x_3}+\frac{0.00350}{x_4}+\frac{0.00125}{x_5}+\frac{0.00001}{x_7}+\frac{0.00524}{x_9},$$
$$Sim^1_{a_1}(x_4)=\frac{0.18166}{x_1}+\frac{0.04371}{x_2}+\frac{0.00350}{x_3}+\frac{1}{x_4}+\frac{0.46523}{x_5}+\frac{0.00001}{x_7}+\frac{0.02529}{x_9},$$
$$Sim^1_{a_1}(x_5)=\frac{0.14324}{x_1}+\frac{0.01758}{x_2}+\frac{0.00125}{x_3}+\frac{0.46523}{x_4}+\frac{1}{x_5}+\frac{0.00001}{x_7}+\frac{0.02529}{x_9},$$
$$Sim^1_{a_1}(x_6)=\frac{0.00160}{x_1}+\frac{1}{x_6}+\frac{0.59577}{x_7}+\frac{0.08221}{x_8}+\frac{0.02599}{x_9}+\frac{0.60227}{x_{10}},$$
$$Sim^1_{a_1}(x_7)=\frac{0.00642}{x_1}+\frac{0.00001}{x_3}+\frac{0.00001}{x_4}+\frac{0.00001}{x_5}+\frac{0.59577}{x_6}+\frac{1}{x_7}+\frac{0.09411}{x_8}+\frac{0.06265}{x_9}+\frac{0.60315}{x_{10}},$$
$$Sim^1_{a_1}(x_8)=\frac{0.00135}{x_1}+\frac{0.08221}{x_6}+\frac{0.09411}{x_7}+\frac{1}{x_8}+\frac{0.01376}{x_9}+\frac{0.16626}{x_{10}},$$
$$Sim^1_{a_1}(x_9)=\frac{0.31979}{x_1}+\frac{0.00587}{x_2}+\frac{0.00524}{x_3}+\frac{0.03161}{x_4}+\frac{0.02529}{x_5}+\frac{0.02599}{x_6}+\frac{0.06265}{x_7}+\frac{0.01376}{x_8}+\frac{1}{x_9}+\frac{0.03583}{x_{10}},$$
$$Sim^1_{a_1}(x_{10})=\frac{0.00280}{x_1}+\frac{0.60227}{x_6}+\frac{0.60315}{x_7}+\frac{0.16623}{x_8}+\frac{0.03583}{x_9}+\frac{1}{x_{10}}.$$

3.3 Uncertainty measurement based on information granularity

In this subsection, we can use the information granular structure mentioned above to define the uncertainty measurement for MsIvDIS.

Definition 5 Given that $(U, A\bigcup D)_i (i = 1, 2, \dots, N)$ is i-th $IvDIS_i$. For $B \subseteq A$, Sim^i_B represents the fuzzy information structure caused by B in ith $IvDIS_i$, $Sim^i_B = (Sim^i_B(x_1), Sim^i_B(x_2), \cdots, Sim^i_B(x_n))$, $\widetilde{FU/D} = \left\{\widetilde{FY}_1, \widetilde{FY}_2, \dots, \widetilde{FY}_m\right\}$ is the fuzzy decision of objects induced by decision attribute D. The fuzzy lower approximation, upper approximation and boundary region of X according to B can be expressed as below:

$$\underline{FSD^i_B}(\widetilde{FY}_r) = \{x_j \in Y_r : Sim^i_B(x_j) \subseteq \widetilde{FY}_r\}, \tag{14}$$

$$\overline{FSD^i_B}(\widetilde{FY}_r) = \{x_j \in U : Sim^i_B(x_j) \bigcap \widetilde{FY}_r \neq \emptyset\}, \tag{15}$$

$$BFSD^i_B(\widetilde{FY}_r) = \overline{FSD^i_B}(\widetilde{FY}_r) - \underline{FSD^i_B}(\widetilde{FY}_r). \tag{16}$$

Definition 6 Assume that $(U, A\bigcup D)_i (i = 1, 2, \dots, N)$ is the ith $IvDIS_i$. $U/D = \{Y_1, Y_2, \dots, Y_m\}$ is the partition of U on the decision attribute set and $\widetilde{FU/D} = \left\{\widetilde{FY}_1, \widetilde{FY}_2, \dots, \widetilde{FY}_m\right\}$ is the fuzzy decision of objects induced by decision attribute D. Sim^i_B represents the fuzzy information structure induced by B in ith $IvDIS_i, Sim^i_B = (Sim^i_B(x_1), Sim^i_B(x_2), \dots, Sim^i_B(x_n))$. The ith fuzzy approximate conditional entropy (FACE) of B is expressed as belows:

$$FACE_i(D|B) = -\sum_{j=1}^{|U|}\sum_{k=1}^{m} \frac{\left|Sim^i_B(X_j)\cap \widetilde{FY}_k\right|}{|U|} \log \frac{\left|Sim^i_B(X_j)\cap \widetilde{FY}_k\right|}{\left|Sim^i_B(X_j)\right|}. \tag{17}$$

Additionally, the δ-approximate conditional entropy $FACE_i(D|B)$ has follow propositions which can be expressed as follows:

(1) $FACE_i(D|B) \geq 0$,
(2) $FACE_i(D|B) < \infty$,
(3) For $x \in U$ and $C \subseteq B \subseteq A$, then $Sim^i_B(x) \leq Sim^i_C(x)$, we have $FACE_i(D|B) \leq FACE_i(D|C)$.

Proof

(1) For $\forall x_j \in U$ and attribute subset $B \subseteq A$, there is $\frac{\left|Sim^i_B(X_j)\cap\widetilde{FY}_k\right|}{\left|Sim^i_B(X_j)\right|} = 1$, then $FACE_i(D|B) = 0$.
(2) If $\exists x_j \in U$, such that $\left|Sim^i_B(X_j)\cap \widetilde{FY}_k\right| = 0$, then $FACE_i(D|B) = \infty$. Thus we have $FACE_i(D|B) < \infty$.
(3) Given $f(x, y) = -x \log_2 \frac{x}{x+y}$, then we have $\frac{\partial f}{\partial x} = \log_2 \frac{x+y}{x} - \frac{y}{(x+y)\ln 2}$. When $x, y > 0$, let $t = \frac{y}{x}$, then we have $\frac{\partial f}{\partial x} = \log_2(1 + t) - \frac{t}{(1+t)\ln 2}$. We let $G(t) = \log_2(1 + t) - \frac{t}{(1+t)\ln 2}$, when $t > 0$, then we know $G'(t) = \frac{1}{(1+t)^2 \ln 2} > 0$, so $G(t)$ monotonically increases

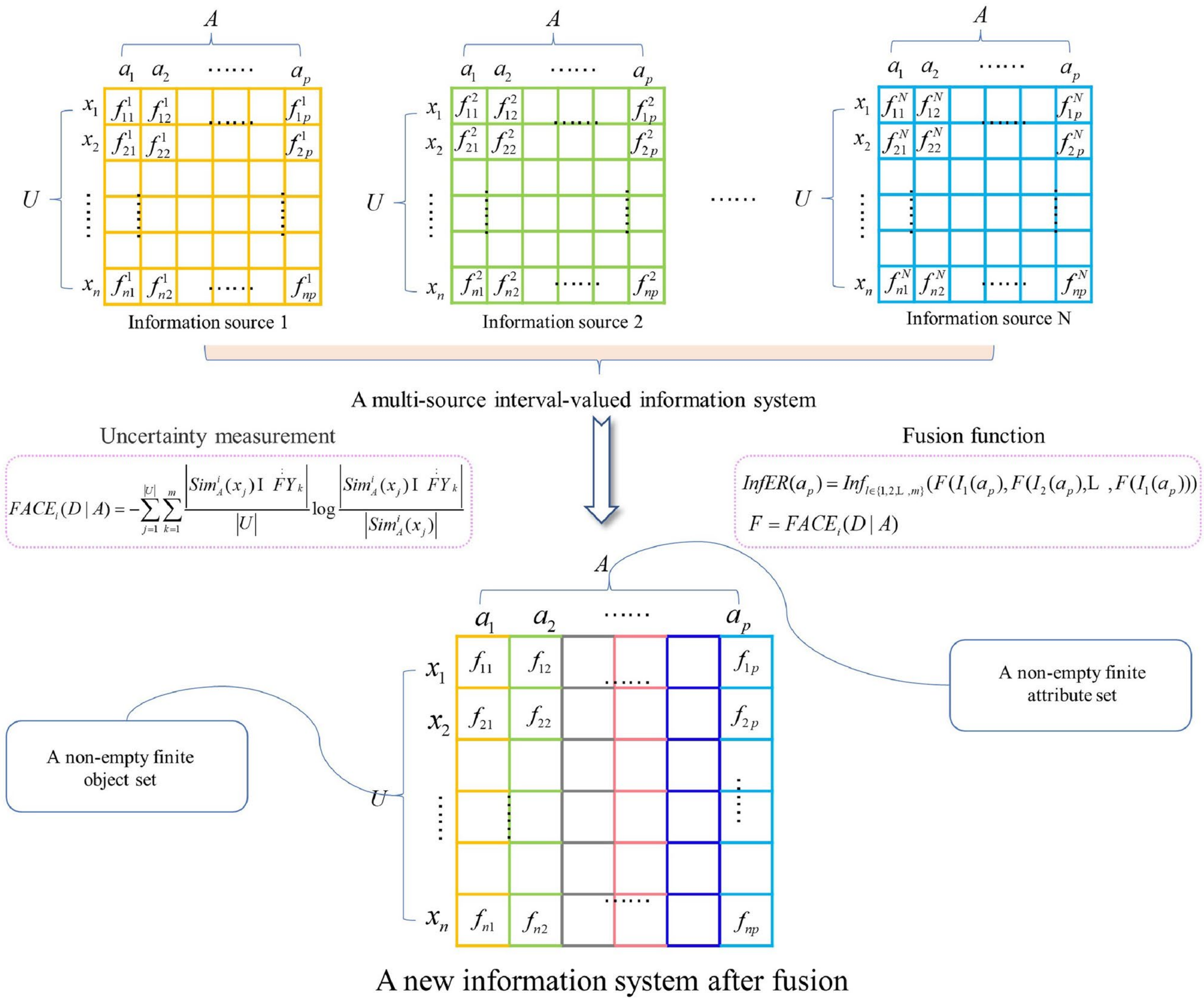

Fig. 2 The fusion process of multi-source incomplete interval-valued information system

Table 6 The fuzzy approximate conditional entropy of information sources for diverse attributes

A	I_1	I_2	I_3	I_4
a_1	0.929975	0.878091	0.956675	**0.736975**
a_2	1.341667	1.258771	**1.195430**	1.205198
a_3	0.941145	**0.891750**	1.001344	1.186731
a_4	**1.091394**	1.449511	1.309411	1.362049
a_5	1.002595	**0.964111**	1.160842	1.256408
a_6	**1.309759**	1.381374	1.349524	1.335237

with respect to t, then $G(t) > G(0) = 0$, thus $G(t) > 0$, i.e. $\frac{\partial f}{\partial x} > 0$. Similarity, when $x, y > 0$, then $\frac{\partial f}{\partial y} = \frac{x}{(x+y)\ln 2} > 0$. And because $\widetilde{FY}_k \in \left\{\widetilde{FY}_1, \widetilde{FY}_2, \ldots, \widetilde{FY}_m\right\}$, so when $x \in U$ and $C \subseteq B \subseteq A$, then according to proposition 1 we can easily know $Sim^i_B(x) \le Sim^i_C(x)$, then we have $\left|Sim^i_B(x) \cap \widetilde{FY}_k\right| \le \left|Sim^i_C(x) \cap \widetilde{FY}_k\right|$ and $\left|Sim^i_B(x) \cap (\widetilde{FY}_k)^C\right| \le \left|Sim^i_C(x) \cap (\widetilde{FY}_k)^C\right|$.

Table 7 The fusion results from fusion function

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	[116.48,128.56]	[3.00,.007]	[115.56,124.65]	[67.23,106.58]	[45.00,76.25]	[70.56,92.98]
x_2	[119.58,131.69]	[6.00,12.00]	[116.68,124.37]	[79.03,117.68]	[62.59,79.65]	[67.34,88.56]
x_3	[109.99,112.98]	[2.00,10.00]	[112.85,121.94]	[118.59,178.45]	[66.00,84.00]	[78.99,99.02]
x_4	[126.43,134.97]	[2.00,9.00]	[121.36,160.48]	[60.45,97.98]	[83.02,126.58]	[66.25,88.29]
x_5	[128.45,136.76]	[4.00,11.00]	[111.97,294.98]	[75.59,108.62]	[139.28,268.94]	[43.00,74.00]
x_6	[127.97,215.96]	[6.00,16.00]	[116.97,125.68]	[81.34,162.58]	[68.58,87.79]	[28.02,61.25]
x_7	[143.96,216.26]	[4.00,9.00]	[177.89,245.34]	[84.58,152.97]	[77.89,97.65]	[30.00,60.02]
x_8	[121.59,169.09]	[10.00,20.00]	[149.97,166.65]	[102.25,162.36]	[65.02,84.38]	[25.06,56.00]
x_9	[119.68,150.98]	[12.00,21.00]	[153.26,268.98]	[90.18,138.28]	[88.69,164.35]	[78.35,88.64]
x_10	[142.54,165.86]	[8.00,18.00]	[177.39,265.98]	[96.65,128.46]	[109.26,252.97]	[39.06,70.00]

$$
\begin{aligned}
&FACE_i(D|B)\\
&= -\sum_{j=1}^{|U|}\sum_{k=1}^{m} \frac{\left|Sim_B^i(X_j)\cap \widetilde{FY_k}\right|}{|U|}\log\frac{\left|Sim_B^i(X_j)\cap \widetilde{FY_k}\right|}{\left|Sim_B^i(X_j)\right|}\\
&= -\sum_{j=1}^{|U|}\sum_{k=1}^{m} \frac{\left|Sim_B^i(X_j)\cap \widetilde{FY_k}\right|}{|U|}\log\frac{\left|Sim_B^i(X_j)\cap \widetilde{FY_k}\right|}{\left|Sim_B^i(X_j\cap \widetilde{FY_k}\right| + \left|Sim_B^i(X_j\cap (\widetilde{FY_k})^C\right|}\\
&\le -\sum_{j=1}^{|U|}\sum_{k=1}^{m} \frac{\left|Sim_C^i(X_j)\cap \widetilde{FY_k}\right|}{|U|}\log\frac{\left|Sim_C^i(X_j)\cap \widetilde{FY_k}\right|}{\left|Sim_C^i(X_j\cap \widetilde{FY_k}\right| + \left|Sim_C^i(X_j\cap (\widetilde{FY_k})^C\right|}\\
&= FACE_i(D|C)
\end{aligned}
$$

.

□

4 Information fusion for dynamic interval-valued data based on FACE

According to the property of fuzzy approximate conditional entropy, we can know the smaller the $FACE_i(D|B)(i = 1, 2, \dots, N)$ is, the more significant the information source is. Hence, we obtain the following fusion function, which can be employed to fuse the MsIvDIS. Then, aiming at the dynamic data of the four situations, we design the corresponding fusion strategy.

Definition 7 For a MsIvDIS $MsIvDIS = \{IIvIIS_i | IIvIIS_i = (U, A, V_i, f_i), i = 1, 2, \dots, N\}$,

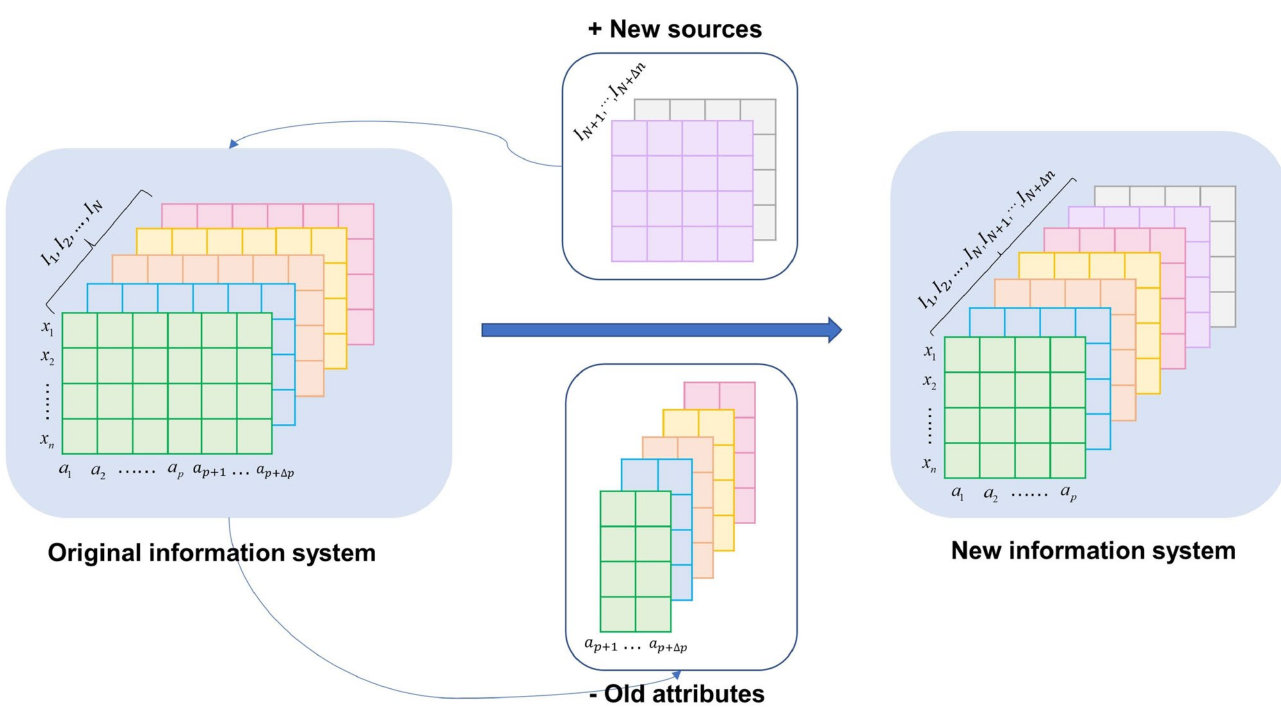

Fig. 3 Dynamic situation 1

where $A = \{a_1, a_2, \dots a_p\}$(For convenience, I_i can represent the ith information source). For all $a_l \in A(m = \{1, 2, \dots, n\})$, the lth attribute of new informtion table after fusion is defined as below:

$$\text{Inf ER}(a_l) = \text{Inf}_{l \in \{1,2,\cdots,m\}}(F(I_1(a_l), F(I_2(a_l), \dots, F(I_N(a_l))), \tag{18}$$

where $F = FACE_i(D|a)$ which can be regard as infimum-measure function.

In this paper, we use Fig. 2 to show the static fusion process more intuitively. The different colors of lines indicate different information sources, and each square represents a corresponding attribute value. Then, according to the fusion function to select the value of attribute, and it is reconstituted a novel information system.

Example 3 (Continued from Examle 2) For the sake of convenience, I_i represent the i-th information system $IIvDIS_i$. Then, the fuzzy approximate condition entropy of information sources for diverse attributes are calculated in Table 6.

The bold values in Table 6 represents the minimum conditional entropy of each attribute under different sources. Then, according to the definition 7, we can obtain the fusion results of Table 7.

4.1 The situation of adding information sources and deleting conditional attributes

Let $MsIvDIS^t$ is the $MSIvDIS$ at time t. The fusion strategy is designed to handle the situations where information sources are added and conditional attributes are deleted. At time $t + 1$, new information sources, denoted as $\{I_{N+1}, I_{N+2}, \dots, I_{N+\Delta n}\}$, are added to the multi-source information system. Additionally, certain conditional attributes, denoted as $\{a_{p+1}, a_{p+2}, \dots, a_{p+\Delta p}\}$, are deleted from the system at time $t + 1$. Specifically as shown in Fig. 3. Then the fusion strategy satisfy the following properties:

Proposition 2 *For this situation and* $a_l \in \{a_1, a_2, \dots, a_p\}$, *we have the following properties*:

(i) If $\min_{i\in[N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l) \geq \min_{i\in\{1,2,\dots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{a_l}^{t}$;

(ii) If $\min_{i\in[N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l) < \min_{i\in\{1,2,\dots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{I_i}(a_l)$, *where* $i = arg\min_{q\in\{N+1,N+2,\dots,N+\Delta n\}} FACE_q(D|a_l)$ *and* $V_{I_i}(a_l)$ *represents the value of* a_l *under* I_i;

Proof

(i) Since $\min_{i\in[N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l) \geq \min_{i\in\{1,2,\dots,N\}} FACE_i(D|a_l)$, then we know $\min_{i\in[1,2,\dots,N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l) = \min_{i\in\{1,2,\dots,N\}} FACE_i(D|a_l)$, so the most information source under attribute a have not changed. Therefore, we can get $V_{a_l}^{t+1} = V_{a_l}^{t}$;

(ii) Since $\min_{i\in[N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l) < \min_{i\in\{1,2,\dots,N\}} FACE_i(D|a_l)$, when Δn information source $\{I_{N+1}, I_{N+2}, \dots, I_{n+\Delta n}\}$ are added, we have $\min_{i\in[1,2,\dots,N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l) = \min_{i\in[N+1,N+2,\dots,N+\Delta n\}} FACE_i(D|a_l)$. Therefore, the most information source under attribute a have changed, then we have $V_{a_l}^{t+1} = V_{I_i}(a_l)$, where $i = arg\min_{q\in\{N+1,N+2,\dots,N+\Delta n\}} FACE_q(D|a_l)$.

□

Based on Property 2, we can perform incremental updates on the fusion information system when new information sources are added and attributes are deleted. To further illustrate this, we can continue studying the previous Example 3 for better understanding.

Example 4 (Continued from Example 3) In the initial scenario, we collected physical examination results of eight patients from two hospitals. However, after consulting with experts, it was determined that the "hemoglobin level" and "platelet count" attributes were not essential for determining leukemia in patients. The experts suggested that patients should undergo testing at multiple hospitals, which aligns with the situation described earlier.

Assuming that at time t, the $MsIvDIS$ has two information sources I_1 and I_2, and six attributes a_1 to a_6. In time $t + 1$, two new information sources I_3 and I_4 are added to $MsIvDIS$, and simultaneously, attributes a_5 and a_6 are removed from the original attribute set. Based on Proposition 2, we only need to compute $FACE_j(D|a_i)$, where $i = 1, 2, 3, 4$ and $j = 3, 4$. The min $FACE_j(D|a_i)$ values for $i = 3, 4$ represent the previous information that does not require recomputation. We then update the min $FACE_j(D|a_i)$ values for $i = 1, 2$ to min $FACE_j(D|a_i)$ for $i = 1, 2$ using the new sources I_3 and I_4. Next, we simulate the dynamic update mechanism in this situation, and the results of the fuzzy conditional entropy are displayed in Table 8.

Table 8 The fuzzy approximate conditional entropy of dynamic situation 1

A	I_1	I_2	I_3	I_4
a_1	0.929975	**0.878091**	0.956675	$\boxed{0.736975}$
a_2	1.341667	**1.258771**	$\boxed{1.195430}$	1.205198
a_3	0.941145	**0.891750**	1.001344	1.186731
a_4	**1.091394**	1.449511	1.309411	1.362049
~~a_5~~	~~1.002595~~	~~**0.964111**~~		
~~a_6~~	~~**1.309759**~~	~~1.381374~~		

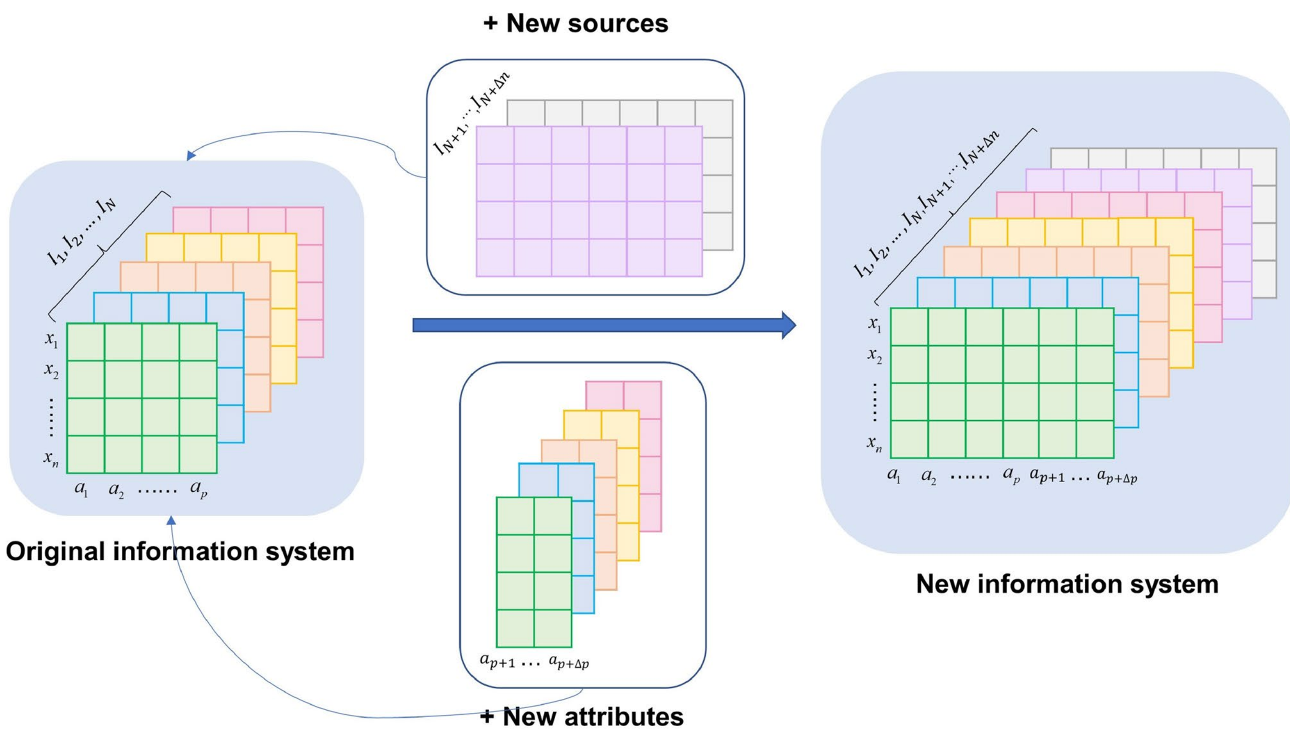

Fig. 4 Dynamic situation 2

Table 9 The fuzzy approximate conditional entropy of dynamic situation 2

A	I_1	I_2	I_3	I_4
a_1	0.929975	**0.878091**	0.956675	0.736975
a_2	1.341667	**1.258771**	1.195430	1.205198
a_3	0.941145	**0.891750**	1.001344	1.186731
a_4	**1.091394**	1.449511	1.309411	1.362049
a_5	1.002595	0.964111	1.160842	1.256408
a_6	1.309759	1.381374	1.349524	1.335237

According to the table shown above, at time $t+1$, we only need to calculate the conditional entropy for the cells highlighted in green. We can compare the entropy values in these cells with the minimum entropy from the previous source. Under attribute a_1, the minimum entropy from the previous source is denoted as $FACE_2(D|a_1)$. After incorporating the new source and updating the minimum entropy, the minimum entropy under attribute a_1 becomes $FACE_4(D|a_4)$, which is highlighted in a red box. For the cells with lower entropy values under attribute a_3 and attribute a_4, there is no need to update the fusion results.

4.2 The situation of information sources and conditional attributes increase at the same time

Let $MsIvDIS^t$ is the MSIvDIS at time t, assume that information sources $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta n}\}$ and conditional attributes $\{a_{p+1}, a_{p+2}, \ldots, a_{p+\Delta p}\}$ are inserted into the $MsIvDIS$ at time $t+1$. Specifically as shown in Fig. 4. Then we can get the following properties:

Proposition 3 *For this situation and $a_l \in \{a_1, a_2, \ldots, a_p\}$, we have the following properties*:

(i) *If* $\min_{i\in\{N+1,N+2,\ldots,N+\Delta n\}} FACE_i(D|a_l) \geq \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{a_l}^{t}$;

(ii) *If* $\min_{i\in\{N+1,\ldots,N+\Delta n\}} FACE_i(D|a_l) < \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{I_i}(a_l)$, *where* $i = arg\min_{q\in\{N+1,\ldots,N+\Delta n\}} FACE_q(D|a_l)$ *and* $V_{I_i}(a_l)$ *represents the value of* a_l *under* I_i;

Proof Same as proposition 2 above. □

Proposition 4 *For this situation and* $\{a_{p+1}, \ldots, a_{p+\Delta p}\}$, *we have the* $V_{a_l} = V_{I_i}(a_l)$, *where*

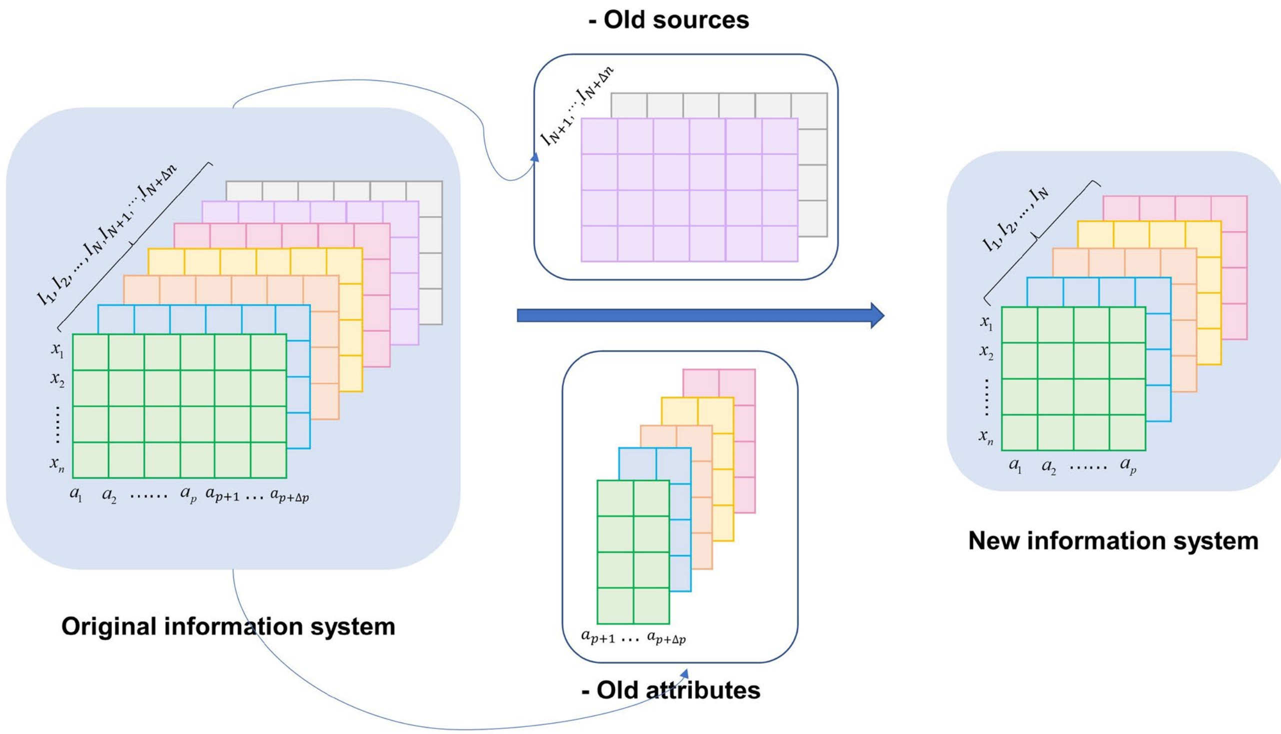

Fig. 5 Dynamic situation 3

$i = arg\min_{q\in\{1,\ldots,N+\Delta n\}} FACE_q(D|a_l)$ and $V_{I_i}(a_l)$ represents the value of a_l under I_i.

Proof According to the proposition of fusion function(definition 9), this proof is obvious. □

Next, we will explore the fusion process of dynamic interval-valued data when there is an increase in both attributes and information sources. This can be accomplished by applying Proposition 3 and Proposition 4. To provide a clearer understanding, we will present an example to illustrate the fusion process.

Example 5 (Continued from Example 3) Similarly, in dynamic case 2, where the original attributes are a_1 to a_4 and there are two existing information sources I_1 and I_2, we can update the fusion results with the aid of Property 3 when new attributes and data sources are added. This situation serves as a simulation of dynamic case 2. Let us now delve into the details of how to update the fusion results in this scenario. The results of fuzzy similarity conditional entropy are presented in Table 9.

At time t, the calculation results of fuzzy similarity conditional entropy are shown in the white background in Table 9. When new attributes and new data sources are added, we only need to calculate $FACE_j(D|a_i)$, where $j = 3, 4$ and $i = 1, 2, 3, 4$, $FACE_j(D|a_i)$, where $j = 1, 2, 3, 4$ and $i = 5, 6$. That is, only the conditional entropy results of the green background in Table 8 need to be calculated. From Table 9, we can intuitively see that $\min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i = 1, 2)$ has changed, and the $\min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i = 3, 4)$ are the former information which do not need to recalculate it. Then update $\min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i = 1, 2)$ to $\min_{j\in\{3,4\}} FACE_j(D|a_i)(i = 1, 2)$ and calculate $\min_{j\in\{,2,3,4\}} FACE_j(D|a_i)(i = 5, 6)$. Finally, the fusion results are updated according to the minimum conditional entropy.

Table 10 The fuzzy approximate conditional entropy of dynamic situation 3

A	I_1	I_2	~~I_3~~	~~I_4~~
a_1	0.929975	$\boxed{0.878091}$	~~0.956675~~	~~**0.736975**~~
a_2	1.341667	$\boxed{1.258771}$	~~**1.195430**~~	~~1.205198~~
a_3	0.941145	**0.891750**	~~1.001344~~	~~1.186731~~
a_4	**1.091394**	1.449511	~~1.309411~~	~~1.362049~~
~~a_5~~	~~1.002595~~	~~**0.964111**~~	~~1.160842~~	~~1.256408~~
~~a_6~~	~~**1.309759**~~	~~1.381374~~	~~1.349524~~	~~1.335237~~

4.3 Situations in which both conditional attributes and information sources decrease at the same time.

Let $MsIvDIS^t$ is the MSIvDIS at time t, at this time, there are $N+\Delta n$ sources in the $MsIvDIs^t$. Assume that information sources $\{I_{N+1}, I_{N+2}, \cdots, I_{N+\Delta n}\}$ and conditional attributes $\{a_{p+1}, a_{p+2}, \ldots, a_{p+\Delta p}\}$ are deleted from the MsIvDIS at time $t+1$. Specifically as shown in Fig. 5 Then we can obtain the following properties:

Proposition 5 *In this case, for* $a_l \in \{a_1, a_2, \ldots, a_l\}$, *we have the following properties*:

(i) *If* $\min_{i\in[N+1,N+2,\ldots,N+\Delta n\}} FACE_i(D|a_l) \geq \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{a_l}^{t}$;

(ii) *If* $\min_{i\in[N+1,N+2,\ldots,N+\Delta n\}} FACE_i(D|a_l) < \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{I_i}(a_l)$, *where* $i = arg\min_{q\in\{1,2,\ldots,N\}} FACE_q(D|a_l)$ *and* $V_{I_i}(a_l)$ *represents the value of* a_l *under* I_i.

Proof

(i) Since $\min_{i\in[N+1,\ldots,N+\Delta n\}} FACE_i(D|a_l) \geq \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, then we konw $\min_{i\in[1,2,N+1,\ldots,N+\Delta n\}} FACE_i(D|a_l) = \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, This means that the information sources with the most relevant information under attribute a_l have not changed. Therefore, Therefore, we can conclude that $V_{a_l}^{t+1} = V_{a_l}^{t}$;

(ii) when Δn information source $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta n}\}$ are deleted from the original information system, we have n information sources in $MsIvDIs^{t+1}$. So, the most information source under attribute a have changed, then we can obtain $V_{a_l}^{t+1} = V_{I_i}(a_l)$, where $i = arg\min_{q\in\{1,2,\ldots,N\}} FACE_q(D|a_l)$.

□

According to Proposition 5, we can fuse dynamic interval-valued data while reducing the number of attributes and information sources. To illustrate this fusion process, we provide an example scenario. The details are as follows:

Table 11 The fuzzy approximate conditional entropy of dynamic situation 4

A	I_1	I_2	I_3	I_4
a_1	0.929975	$\boxed{0.878091}$	~~0.956675~~	~~**0.736975**~~
a_2	1.341667	$\boxed{1.258771}$	~~**1.195430**~~	~~1.205198~~
a_3	0.941145	**0.891750**	~~1.001344~~	~~1.186731~~
a_4	**1.091394**	1.449511	~~1.309411~~	~~1.362049~~
a_5	1.002595	$\boxed{\mathbf{0.964111}}$		
a_6	$\boxed{\mathbf{1.309759}}$	1.381374		

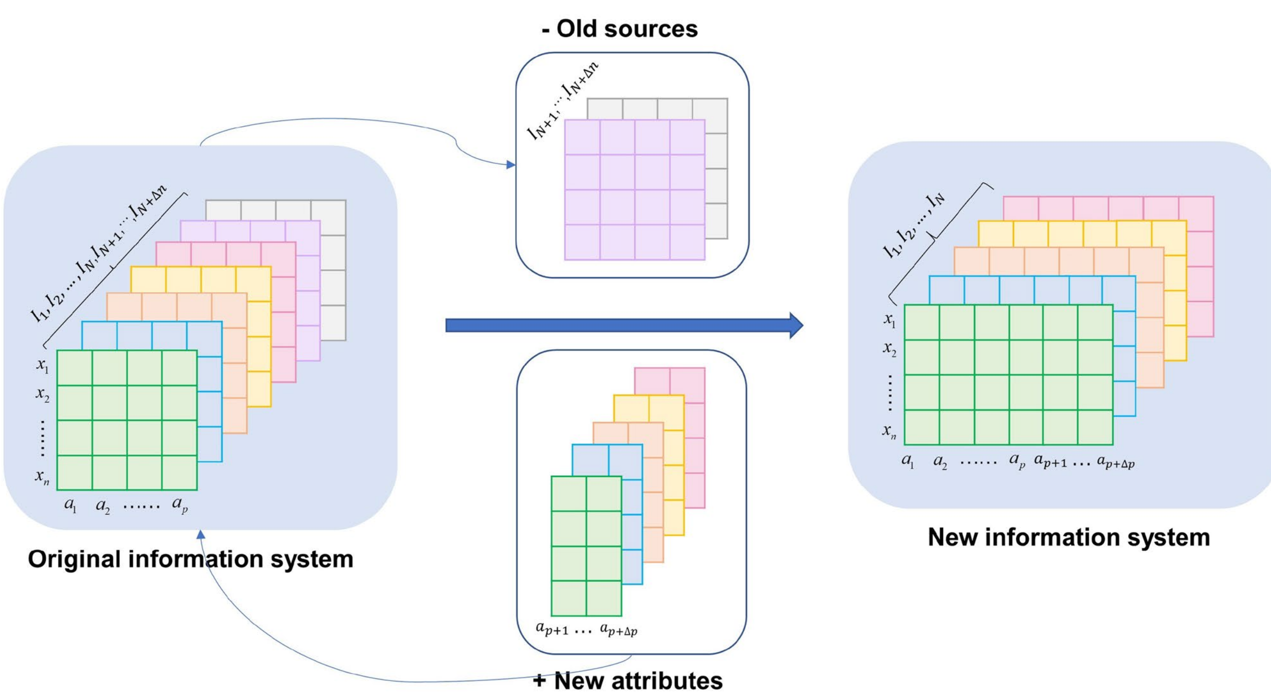

Fig. 6 Dynamic situation 4

Example 6 (Continued from Example 3) In this example, we simulated dynamic situation 3. We have a multi-source information system with six attributes and four existing information sources. In this scenario, two attributes, namely a_5 and a_6, and two existing sources, I_3 and I_4, were removed from the system. According to the property described in the example, when both the attributes and existing data sources are reduced, there is no need to recalculate the fuzzy approximate conditional entropy. Instead, we only need to update the minimum values of $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=1,2,3,4)$ for the remaining attributes and sources.

Table 10 shows the calculation results of fuzzy approximate conditional entropy. The red box marks the updated minimum conditional entropy. It is easy to know that if $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=3,4) = \min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i=3,4)$ has not changed, there is no need to update the values of new table attributes a_3 and a_4. But $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=1,2) \neq \min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i=1,2)$, and it needs to be updated $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=1,2)$, and the corresponding new table fusion results also need to be updated.

4.4 Situations where the number of information sources decreases and the number of conditional attributes increases

Let $MsIvDIS^t$ is the MSIvDIS at time t, at this time, there are $N+\Delta n$ sources in the $MsIvDIS^t$. Assume that information sources $\{I_{N+1}, I_{N+2}, \ldots, I_{N+\Delta n}\}$ are deleted from the MsIvDIS and conditional attributes $\{a_{p+1}, a_{p+2}, \ldots, a_{p+\Delta p}\}$ are added into the MsIvDIS at time $t+1$. Specifically as shown in Fig. 6. Then we can obtain the following properties:

Proposition 6 *In this case, for* $\{a_1, a_2, \ldots, a_p\}$, *we have the following properties*:

(i) *If* $\min_{i\in[N+1,N+2,\ldots,N+\Delta n\}} FACE_i(D|a_l) \geq \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{a_l}^t$;

(ii) *If* $\min_{i\in[N+1,N+2,\ldots,N+\Delta n\}} FACE_i(D|a_l) < \min_{i\in\{1,2,\ldots,N\}} FACE_i(D|a_l)$, *then* $V_{a_l}^{t+1} = V_{I_i}(a_l)$, *where* $i = arg\min_{q\in\{1,2,\ldots,N\}} FACE_q(D|a_l)$ *and* $V_{I_i}(a_l)$ *represents the value of* a_l *under* I_i.

For $\{a_{p+1}, a_{p+2}, \cdots, a_{p+\Delta p}\}$, we have the $V_{a_l} = V_{I_i}(a_l)$, where $i = arg\min_{q\in\{1,2,\ldots,N+\Delta n\}} FACE_q(D|a_l)$ and $V_{I_i}(a_l)$ represents the value of a_l under I_i.

Proof Same as proposition 5 and proposition 6 above. □

According to the property 5, we can fuse the the dynamic interval-valued data with reduced attributes and information sources at the same time. We give an example to illustrate the fusion process of this situation. The details are as follows:

Example 7 (Continued from Example 3) For this example, we simulated the scene of dynamic situation 4. There are four attributes $a_1 - a_4$ and four data sources $I_1 - I_4$ in $MsIvDIS^t$. and two old sources I_3 and I_4 are removed and two new attributes a_5 and a_6 are added at time $t+1$. Table 11 shows the calculation results of FACE. In this case, we just need to calculate $FACE_j(D|a_i)$, where $j=1,2$ and $i=1,2,3,4,5,6$.

From Table 11, we can easily know $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=3,4) = \min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i=3,4)$ has not changed, the corresponding merged new table does not need to be updated. But $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=1,2) \neq \min_{j\in\{1,2,3,4\}} FACE_j(D|a_i)(i=1,2)$, so we require to update $\min_{j\in\{1,2\}} FACE_j(D|a_i)(i=1,2)$,and the attribute values corresponding to the new table.

5 Design of the algorithms

In this section, we present a static fusion algorithm based on fuzzy approximate conditional entropy. This algorithm is designed to fuse the information from multiple sources in a static scenario, where there are no changes in the conditional attributes and information sources. The fuzzy approximate conditional entropy measure is utilized to assess the uncertainty and similarity between the interval-valued data. Additionally, we propose dynamic update algorithms that address dynamic data scenarios where there are simultaneous changes in the conditional attributes and information sources. These update algorithms allow for efficient and accurate fusion of the dynamic data by considering the modified set of attributes and sources. By leveraging the properties of fuzzy approximate conditional entropy, we can effectively update the fusion results and accommodate the changing information landscape.

5.1 FACE-based information fusion algorithm

From the above, we can obtain fusion Algorithm 1 based on fuzzy approximate condition entropy. In Steps 3–6, the computation of the fuzzy similarity class for conditional attribute set can be completed in $O(\left|U^2\right| \times |A| \times N)$. Steps 8–17 are to compute the fuzzy approximate condition entropy, and its the complexity is $O(|U| \times m)$. The time complexity of Steps 1–15 are $O(|U| \times N \times |A| \times (m + |U|))$. The time complexity of Steps 16–24 are $O(|A| \times N)$. Therefore, the total time complexity of algorithm 1 is $O(|U| \times N \times |A| \times (m + |U|) + |A| \times N)$.

Algorithm 1 The static fusion algorithm of MsIvDIS based FACE

Data: $MsIvDIS = \{(U, A, V_{A_i}, f_{A_i}, D, V_{D_i}, f_{D_i}), i = 1, 2, \cdots, N\}$, the decision partition $U/D = \{Y_1, Y_2, ..., Y_m\}$.
Result: A new fusion table.

for $i = 1 : N$ **do**
  # N is the number of information sources;
  for *each* $a \in A$ **do**
    for $\forall x_j, x_k \in U$ **do**
      Calculate the distancce and similarity degree $Sim_a^i(x_j, x_k)$;
      Compute the fuzzy similarity class $Sim_a^i(x_j)$ according to Definition 4;
    end
    $FACE \leftarrow 0$;
    for $r = 1 : m$ **do**
      # m is the cardinality of $|U/D|$;
      Compute the fuzzy decision $\widetilde{FU/D} = \left\{\widetilde{FY}_1, \widetilde{FY}_2, ..., \widetilde{FY}_m\right\}$;
      if $\left|Sim_a^i(x_j) \cap \widetilde{FY}_r\right| > 0$ **then**
        $FACE \leftarrow FACE - \frac{1}{|U|} \log \frac{\left|Sim_a^i(x_j) \cap \widetilde{FY}_r\right|}{\left|Sim_a^i(x_j)\right|}$;
      end
    end
  end
end
for *each* $a \in A$ **do**
  $minFACE \leftarrow \infty$;
  for $i = 1 : N$ **do**
    if $FACE_i(D|a) < minFACE$ **then**
      $minFACE \leftarrow FACE_i(D|a)$;
      $I^a \leftarrow i$;
    end
  end
end
return $\left(V_{a_1}^{I^{a_1}}, V_{a_2}^{I^{a_2}}, ..., V_{a_{|AT|}}^{I^{a_{|AT|}}}\right)$.

5.2 Dynamic fusion algorithm based FACE-IF

In this subsection, we present the dynamic fusion algorithm for the four different situations described in Sect. 4. We also analyze the complexity of the algorithm. Propositions 2 to 6 provide dynamic fusion strategies to update the fusion results when there are changes in both conditional attributes and information sources. Algorithm 2 specifically addresses the scenario where the attribute value decreases and the number of sources increases in the original data. In this algorithm, Steps 1–12 are divided into two parts to update the fusion table:

(1) Calculates the fuzzy approximate conditional entropy of the new source under the remaining attribute sets after deleting the attribute sets from steps 1–5, whose time complexity is $O(\Delta n \times p \times |U| \times (|U| + m))$;
(2) Updates the minimum fuzzy approximate conditional entropy corresponding to each attribute, whose time complexity is $O(p)$.

Thus, the total complexity of dynamic situation 1 is $O(\Delta n \times p \times |U| \times (|U| + m) + p)$

Algorithm 2 The algorithm of dynamic fusion when inserting sources and deleting attributes

Data: The original fusion table $\left(V_{a_1}^t, V_{a_2}^t, ..., V_{a_p}^t, V_{a_{p+1}}^t, ..., V_{a_{p+\Delta p}}^t\right)$, the deleted attributes set $\left\{a_{p+1}, ..., a_{p+\Delta p}\right\}$, the inserted sources set $\{I_{N+1}, I_{N+2}, ..., I_{N+\Delta n}\}$, the decision partition $U/D = \{Y_1, Y_2, ..., Y_m\}$.
Result: Dynamic updated fusion table.

for *i = (N+1) : (N + Δn)* **do**
  for *each* $a \in a_1, a_2, ..., a_p$ **do**
    compute $FACE_i(D|a)$;
  end
end
for *each* $a \in \left\{a_1, a_2, ..., a_p\right\}$ **do**
  if $\min_{i \in \{N+1,N+2,...,N+\Delta n\}} FACE_i(D|a) \geq \min_{i \in \{1,2,...,N\}} FACE_i(D|a)$ **then**
    $V_a^{t+1} = V_a^t$;
  end
  if $\min_{i \in \{N+1,N+2,...,N+\Delta N\}} FACE_i(D|a) < \min_{i \in \{1,2,...,N\}} FACE_i(D|a)$ **then**
    $V_a^{t+1} = V_{I_i}(a), i = \underset{q\{N+1,N+2,...,N+\Delta N\}}{\arg\min} FACE_i(D|a)$;
  end
end
return $\left(V_{a_1}^{t+1}, V_{ainfo_2}^{t+1}, ..., V_{a_p}^{t+1}\right)$.

the Algorithm 3 shows the situation that the number of attribute sets and information sources in the original data table increases at the same time. For Algorithm 3, Steps 1–22 can be divided into three parts: calculating the FACE of new sources under all attributes, and the information entropy of different old sources under new attributes, and updating the minimum FACE of all attributes under different sources. The worst case of total complexity of Algorithm 3 is $O((\Delta n \times (p + \Delta p) + N \times \Delta p) \times |U| \times (|U| + m) + p + \Delta p)$.

Table 13 The time complexity of the dynamic algorithm

Situations	Dynamic algorithm
(1)	$O(\Delta n \times p \times \lvert U \rvert \times (\lvert U \rvert + m) + p)$
(2)	$O((N \times \Delta p + \Delta n \times (p + \Delta p)) \times \lvert U \rvert \times (\lvert U \rvert + m) + p + \Delta p)$
(3)	$O(p)$
(4)	$O(N \times \Delta p \times \lvert U \rvert \times (\lvert U \rvert + m) + p + \Delta p)$

Table 12 Time complexity between of the static algorithm

Situations	Static algorithm
(1)	$O((N + \Delta n) \times p \times \lvert U \rvert \times (\lvert U \rvert + m) + p \times N)$
(2)	$O(((N + \Delta n) \times (p + \Delta p)) \times \lvert U \rvert \times (\lvert U \rvert + m) + (p + \Delta p) \times N)$
(3)	$O(N \times p \times \lvert U \rvert \times (\lvert U \rvert + m) + p \times N)$
(4)	$O(N \times (p + \Delta p) \times \lvert U \rvert \times (\lvert U \rvert + m) + (p + \Delta p) \times N)$

Algorithm 3 The algorithm of dynamic fusion when inserting arrtibutes and data sources

Data: The original fusion table $\left(V_{a_1}^t, V_{a_2}^t, ..., V_{a_p}^t\right)$, the inserted sources set $\{I_{N+1}, I_{N+2}, ..., I_{N+\Delta m}\}$, the inserted attributes set $\left\{a_{p+1}, a_{p+2}, ..., a_{p+\Delta p}\right\}$, the decision partition $U/D = \{Y_1, Y_2, ..., Y_m\}$.
Result: Dynamic updated fusion table.

for $i = (N+1) : (N + \Delta n)$ **do**
  for *each* $a \in \left\{a_1, a_2, ..., a_p, .., a_{p+\Delta p}\right\}$ **do**
    compute $FACE_i(D|a)$; # m is the number of original attributes;
  end
end
for $i = 1 : N$ **do**
  for *each* $a \in \left\{a_{p+1}, a_{p+2}, ..., a_{p+\Delta p}\right\}$ **do**
    compute $FACE_i(D|a)$;
  end
end
for *each* $a \in \left\{a_1, a_2, ..., a_p\right\}$ **do**
  if $\min_{i \in \{N+1, N+2, ..., N+\Delta n\}} FACE_i(D|a) \geq \min_{i \in \{1,2,...,N\}} FACE_i(D|a)$ **then**
    $V_a^{t+1} = V_a^t$;
  end
  if $\min_{i \in \{N+1, N+2, ..., N+\Delta n\}} FACE_i(D|a) < \min_{i \in \{1,2,...,N\}} FACE_i(D|a)$ **then**
    $V_a^{t+1} = V_{I_i}(a), q = \arg\min_{q\{N+1, N+2, ..., N+\Delta n\}} FACE_q(D|a)$;
  end
end
for *each* $a \in \left\{a_{p+1}, a_{p+2}, ..., a_{p+\Delta p}\right\}$ **do**
  compute $i_a = \arg\min_{q \in \{1,2,...,N+\Delta n\}} FACE_q(D|a)$;
  Let $V_a = V_{I_{i_a}}(a)$;
end
return $\left(V_{a_1}^{t+1}, V_{a_2}^{t+1}, ..., V_{a_p}^{t+1}, V_{a_{p+1}}, ..., V_{a_{p+\Delta p}}\right)$.

Table 14 The description of experimental data sets

No	Data set name	Abbreviation	Objects	Attributes	Decision classes
1	Breast cancer Wisconsin	BCW	569	30	2
2	Hill-valley	HV	606	100	2
3	Credit approval	CA	690	16	2
4	Diabetic retinopathy debrecen data set	Diabetic	1151	20	2
5	Wine quality-red	WQR	1599	12	6
6	Car evaluation	CE	1728	7	4
7	Page blocks	PB	5473	10	5
8	Shill bidding	SB	6321	13	2
9	Electrical grid stability simulated data	Ele	10,000	14	2

Table 15 Description of the experimental environment

Name	Model	Parameter
CPU	AMD Ryzen 7 R7-5800H	3.2 GHz
System	Windows11	64 bit
Platform	Python	3.9
Memory	DDR4	16 GB; 3200 MHz
Hard Disk	SKHynix_HFS512GDE9X084N	512G

The Algorithm 4 shows that the attribute values and information sources in the original data table are reduced at the same time. In Algorithm 4, it is not necessary to recalculate the value of FACE, but only to update the minimum value of FACE for different sources in the remaining attributes. Therefore, the time complexity of this algorithm is $O(p)$.

Table 16 Comparison of classification accuracy based on KNN

Datasets	KNN					
	FACE-IF	MixF	MaxF	MeanF	HF	CF
BCW (k = 8)	**61.69±3.44**	59.58 ± 2.49	58.00 ± 3.18	60.47 ± 4.58	59.93 ± 2.99	56.77 ± 4.34
HV (k = 12)	**51.14 ± 5.86**	50.31 ± 5.31	50.81 ± 6.14	50.65 ± 5.89	48.82 ± 5.72	50.64 ± 6.00
CA (k = 9)	**64.06 ± 5.41**	63.77 ± 6.32	60.87 ± 6.11	60.87 ± 5.23	61.30 ± 7.25	61.30 ± 7.25
Diabetic (k = 6)	**52.65 ± 2.82**	51.17 ± 3.65	51.52 ± 4.86	52.47 ± 4.11	49.17 ± 4.03	52.30 ± 2.27
WQR (k = 9)	**43.30 ± 3.01**	42.28 ± 3.83	42.21 ± 3.00	41.53 ± 2.40	41.90 ± 3.51	42.71 ± 3.11
CE (k = 16)	**72.11 ± 2.52**	71.05 ± 2.70	71.36 ± 2.17	71.70 ± 3.25	70.89 ± 2.10	71.29 ± 2.63
PB (k = 3)	**88.42 ± 0.92**	88.38 ± 0.98	88.05 ± 1.13	88.38 ± 1.04	88.03 ± 1.13	88.29 ± 0.90
SB (k = 9)	**89.31 ± 1.24**	89.28 ± 1.24	89.22 ± 1.25	89.25 ± 1.24	89.21 ± 1.24	89.27 ± 1.28
Ele (k = 3)	**56.15 ± 1.26**	56.10 ± 1.47	55.49 ± 1.33	55.50 ± 0.94	54.91 ± 2.25	55.22 ± 1.25

Table 17 Comparison of classification accuracy based on PNN

Datasets	PNN					
	FACE-IF	MixF	MaxF	MeanF	HF	CF
BCW ($\sigma = 0.1$)	**59.75 ± 4.74**	58.17 ± 5.43	56.77 ± 5.58	58.88 ± 5.77	54.83 ± 6.41	55.89 ± 5.09
HV ($\sigma = 0.8$)	**52.49 ± 5.04**	52.33 ± 5.02	51.99 ± 4.88	52.01 ± 5.10	50.16 ± 6.49	52.01 ± 5.10
CA ($\sigma = 0.1$)	**61.88 ± 7.01**	61.88 ± 7.42	58.99 ± 7.53	59.57 ± 8.36	57.39 ± 6.19	60.72 ± 5.96
Diabetic ($\sigma = 0.45$)	**53.25 ± 4.81**	53.17 ± 4.91	53.08 ± 4.81	53.07 ± 4.87	47.86 ± 3.18	52.99 ± 4.61
WQR ($\sigma = 0.4$)	**43.38 ± 3.02**	42.90 ± 3.23	42.46 ± 2.54	42.27 ± 2.28	42.34 ± 3.29	42.52 ± 2.42
CE ($\sigma = 0.21$)	**70.78 ± 1.92**	70.08 ± 2.02	70.26 ± 2.05	70.26 ± 2.05	69.91 ± 2.78	70.60 ± 2.14
PB ($\sigma = 0.35$)	89.80 ± 1.03	**89.84 ± 1.03**	89.76 ± 1.04	89.76 ± 1.04	88.78 ± 1.04	89.40 ± 0.99
SB ($\sigma = 0.18$)	**88.83 ± 1.10**	88.67 ± 1.69	88.05 ± 1.15	88.66 ± 1.13	80.90 ± 1.21	88.28 ± 1.23
Ele ($\sigma = 0.1$)	54.18 ± 0.80	54.41 ± 1.73	**55.35 ± 1.61**	53.97 ± 1.13	53.39 ± 2.10	53.43 ± 1.59

Table 18 Friedman test

Classifiers	Mean ranking						χ_F^2	F_F	P value
	FACE-IF	HF	MinF	MaxF	MeanF	CF			
KNN	1.00	5.28	3.11	4.33	3.33	3.95	27.2293	12.2581	5.15×10^{-5}
PNN	1.33	5.89	2.33	3.89	2.83	3.72	30.8842	17.5033	9.87×10^{-6}

Algorithm 4 The algorithm of dynamic fusion when deleting data sources and attributes

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Data: The original fusion table $\left(V^t_{a_1}, V^t_{a_2}, ..., V^t_{a_p}, V^t_{a_{p+1}}, ..., V^t_{a_{p+\Delta p}}\right)$, the deleted attributes set $\left\{a_{p+1}, ..., a_{p+\Delta p}\right\}$, the deleted sources set $\{I_{N+1}, I_{N+2}, ..., I_{N+\Delta n}\}$, the decision partition $U/D = \left\{Y_1, Y_2, ..., Y_p\right\}$.

Result: Dynamic updated fusion table.

for *each* $a \in \left\{a_1, a_2, ..., a_p\right\}$ **do**

  # n is the number of remained attributes.

  if $\min\limits_{i\in\{N+1,N+2,...,N+\Delta n\}} FACE_i(D|a) \geq \min\limits_{i\in\{1,2,...,N\}} FACE_i(D|a)$ **then**

    $V^{t+1}_a = V^t_a$;

  end

  if $\min\limits_{i\in\{N+1,N+2,...,N+\Delta n\}} FACE_i(D|a) < \min\limits_{q\in\{1,2,...,N\}} FACE_i(D|a)$ **then**

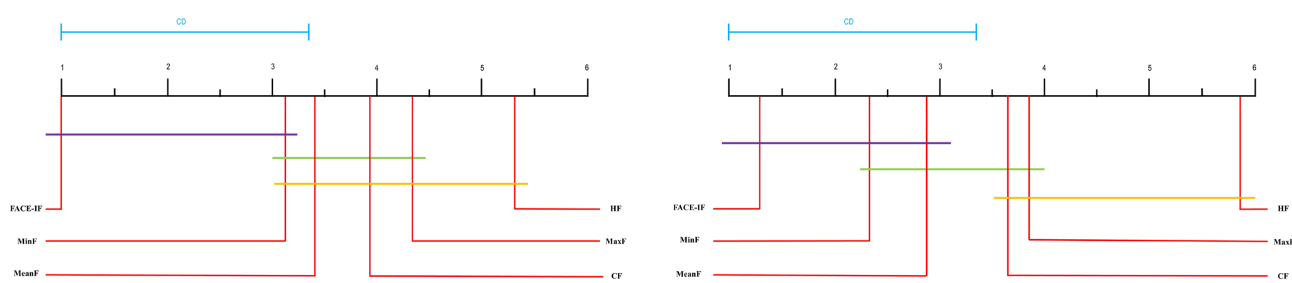
    $V^{t+1}_a = V_{I_i}(a), i = \arg\min\limits_{q\{1,2,...,N\}} FACE_q(D|a)$; #$V_{I_i}(a)$ denotes the value of attribute a under the information source I_i.

  end

end

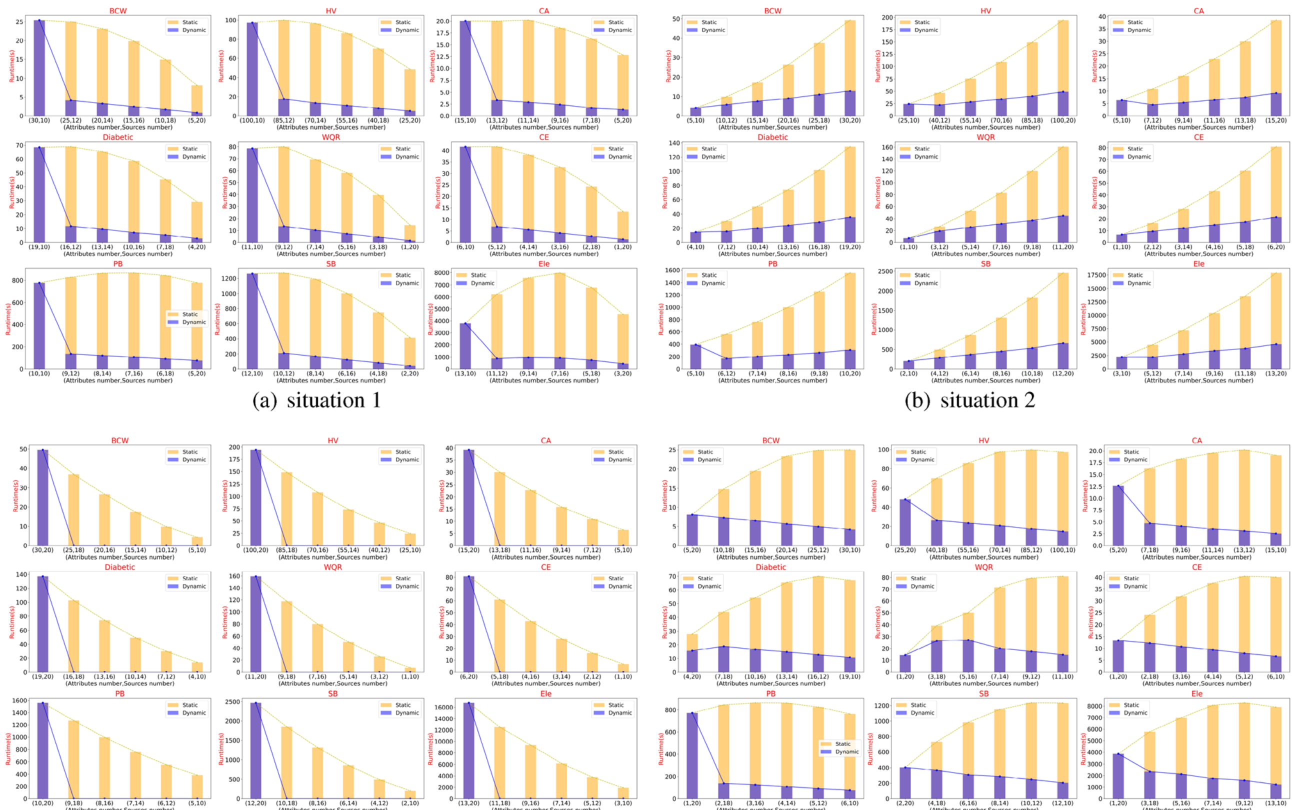
return $\left(V^{t+1}_{a_1}, V^{t+1}_{a_2}, ..., V^{t+1}_{a_p}\right)$.

Algorithm 5 is used to give a description of the process of dynamic fusion of MsIvDIS with the addition of new condition attributes and the deleted of information sources. The Algorithm 5 consists of two parts, namely, calculating the conditional entropy of the remaining sources under the newly added attributes and updating the minimum FACE values of the remaining sources under all attributes. The time complexity of this algorithm is $O(N \times \Delta p \times |U| \times (|U| + m) + p + \Delta p)$.



(a) KNN (b) PNN

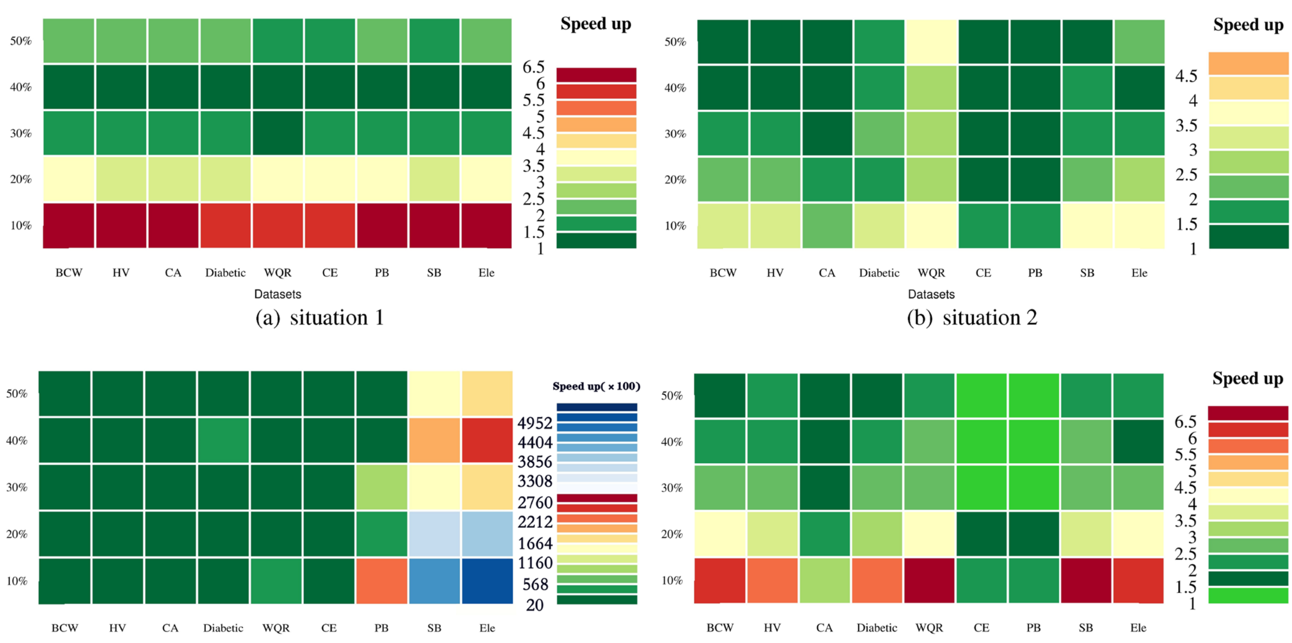
Fig. 7 Accuracy comparison with four distance measures on classifiers KNN and PNN



(a) situation 1 (b) situation 2

(c) situation 3 (d) situation 4

Fig. 8 Running time comparison diagram of four situations



(a) situation 1 (b) situation 2

(c) situation 3 (d) situation 4

Fig. 9 Speed up of four situations

Algorithm 5 The algorithm of dynamic fusion when deleting sources and inserting attributes

Data: The original fusion table $\left(V_{a_1}^t, V_{a_2}^t, ..., V_{a_p}^t\right)$, the deleted sources set $\{I_{N+1}, I_{N+2}, ..., I_{N+\Delta n}\}$, the inserted attributes set $\left\{a_{p+1}, a_{p+2}, ..., a_{p+\Delta p}\right\}$, the decision partition $U/D = \{Y_1, Y_2, ..., Y_m\}$.
Result: Dynamic updated fusion table.

for $q = 1 : N$ **do**
  for *each* $a \in \left\{a_{p+1}, a_{p+2}, ..., a_{p+\Delta p}\right\}$ **do**
    compute $FACE_i(D|a)$;
  end
end
for *each* $a \in \left\{a_1, a_2, ..., a_p\right\}$ **do**
  if $\min\limits_{i\in\{N+1,N+2,...,N+\Delta n\}} FACE_i(D|a) \geq \min\limits_{q\in\{1,2,...,N\}} FACE_i(D|a)$ **then**
    $V_a^{t+1} = V_a^t$;
  end
  if $\min\limits_{q\in\{N+1,N+2,...,N+\Delta n\}} FACE_i(D|a) < \min\limits_{i\in\{1,2,...,N\}} FACE_i(D|a)$ **then**
    $V_a^{t+1} = V_{I_q}(a), i = \arg\min\limits_{q\{1,2,...,N\}} FACE_q(D|a)$;
  end
end
for *each* $a \in \left\{a_{p+1}, a_{p+2}, ..., a_{p+\Delta p}\right\}$ **do**
  compute $i_a = \arg\min\limits_{q\in\{1,2,...,N\}} FACE_q(D|a)$;
  Let $V_a = V_{I_{i_a}}(a)$;
end
return $\left(V_{a_1}^{t+1}, V_{a_2}^{t+1}, ..., V_{a_p}^{t+1}, V_{a_{p+1}}, ..., V_{a_{p+\Delta p}}\right)$.

To sum up, in order to compare the efficiency of dynamic fusion algorithm with that of static fusion algorithm of FACE-IF more intuitively, we use a stable to show it. Specific as the Tables 12 and 13.

6 Experimental analysis

In this section, we evaluate the effectiveness and efficiency of the proposed approach by conducting comparative experiments on nine data sets obtained from the UCI database (https://archive.ics.uci.edu/ml/index.php). The details of these data sets are provided in Table 14. The experimental programs were executed on a personal computer with specific hardware and software configurations, as described in Table 15.

As is known to all, the MsIIvDIS cannot be obtained directly from any common databases. So we can use the method in [42] to generate MsIvDIS. The detailed steps are as below:

(1) Convert single-valued data in the original dataset to interval-valued data: Let $V(x, a)$ represents the value of x under attribute a, $\forall x \in U, a \in A$, $f^-(x,a) = V(x,a)$-$2\sigma_a$, $f^-(x,a) = V(x,a) + 2\sigma_a$, where σ_a denotes the standard deviation of the attribute a in the same decision class.

(2) Generate MsIIvDIS: First of all, m random numbers $\{r_1, r_2, r_3, \ldots, r_m\}$ that obey Gaussian distribution $N(0, 0.1)$ are generated randomly. If $r_i > 0$, then $f_i^-(x,a) = f^-(x,a)(1-r)$and $f_i^+(x,a) = f^+(x,a)(1+r)$, otherwise $f_i^-(x,a) = f^-(x,a)(1+r)$ and $f_i^+(x,a) = f^+(x,a)(1-r)$.

We evaluate the effectiveness of the proposed fusion method by comparing with the existing methods. As shown below respectively.

We generate $n = 20$ sources, then the other three fusion approaches are expressed as below:

(i) Max fusion approach can be written as MaxF: MaxF $f^-(x,a) = min\left\{f_1^-(x,a), f_2^-(x,a), f_3^-(x,a), \ldots, f_n^-(x,a)\right\}$, MaxF $f^+(x,a) = max\left\{f_1^+(x,a), f_2^+(x,a), f_3^+(x,a), \ldots, f_n^+(x,a)\right\}$, where $f^-(x,a)$ and $f^+(x,a)$ are the left and right endpoints of max fusion result, respectively;

(ii) Min fusion method can be written as MinF: MinF $f^{-}(x,a) = max\{f_1^{-}(x,a), f_2^{-}(x,a), f_3^{-}(x,a), \dots, f_n^{-}(x,a)\}$, MinF $f^{+}(x,a) = min\{f_1^{+}(x,a), f_2^{+}(x,a), f_3^{+}(x,a), \dots, f_n^{+}(x,a)\}$, where $f^{-}(x,a)$ and $f^{+}(x,a)$ are the left and right endpoints of min fusion result, respectively;

(iii) Mean fusion method can be written as MeanF: MeanF $f^{-}(x,a) = mean\{f_1^{-}(x,a), f_2^{-}(x,a), f_3^{-}(x,a), \dots, f_n^{-}(x,a)\}$, MeanF $f^{+}(x,a) = mean\{f_1^{+}(x,a), f_2^{+}(x,a), f_3^{+}(x,a), \dots, f_n^{+}(x,a)\}$, where $f^{-}(x,a)$ and $f^{+}(x,a)$ are the left and right endpoints of mean fusion result, respectively;

(iv) The method of information are introduced by Huang et al. [42] (written as HF);

(v) The fusion approach are inroduced by Zhang et al. [43] (written as CF).

6.1 Analysis of classification effect based on FACE-IF fusion method

In this study, we evaluate the performance of the fusion method based on FACE-IF, proposed in this paper, in comparison to other existing fusion methods, using classification accuracy as the evaluation metric. We employ two popular classification learning algorithms, namely the K-nearest neighbor (KNN) classifier and the probabilistic neural network (PNN) classifier, to assess the effectiveness of different fusion algorithms.

To ensure the reliability of our results, we utilize ten-fold cross-validation, which involves dividing the dataset into ten subsets, performing the classification on nine subsets, and evaluating the accuracy on the remaining subset. This process is repeated ten times to obtain robust and representative performance measures. The mean classification precision and standard deviation are calculated based on the ten repetitions.

Tables 16 and 17 present the mean classification precision and standard deviation, respectively, obtained through the ten-times ten-fold cross-validation. In these tables, the bold values indicate the highest classification effectiveness among the different fusion methods. It is important to note that the performance of the KNN and PNN classifiers can be influenced by their respective parameters, such as the value of k for KNN and the parameter σ for PNN. These parameters can be adjusted to achieve optimal results, making these classifiers more flexible and adaptable to different situations.

The results presented in Tables 16 and 17 clearly demonstrate that, in most scenarios, the fusion approach based on FACE-IF outperforms the other five fusion methods, namely MaxF, MinF, MeanF, HF, and CF, in terms of classification accuracy. This indicates the superior effectiveness of the proposed approach in fusing interval-valued data for classification tasks.

Based on the results presented in Tables 16 and 17, it is evident that the fusion method based on FACE-IF achieves superior classification performance compared to other existing approaches, particularly when employing the K-nearest neighbor (KNN) classifier. Across almost all datasets, FACE-IF consistently outperforms the other fusion methods in terms of classification accuracy.

When considering the probabilistic neural network (PNN) classifier, it is worth noting that the fusion algorithm based on FACE-IF demonstrates excellent classification performance in seven out of the nine datasets, with the exceptions of the PB dataset and the Ele dataset. Although in the PB dataset, the classification accuracy of FACE-IF is slightly lower compared to the MinF method, it still outperforms the MaxF, MeanF, HF, and CF algorithms. Similarly, in the Ele dataset, FACE-IF demonstrates superior classification performance compared to the MeanF, CF, and HF methods. These nuanced observations contribute to a more comprehensive understanding, reinforcing the overall dominance of FACE-IF, especially when coupled with the KNN classifier.

The findings clearly highlight the superiority of the FACE-IF algorithm in achieving high classification effectiveness compared to other fusion methods, particularly when combined with the KNN classifier. Even in scenarios where FACE-IF may exhibit slightly lower performance compared to certain methods, it still maintains competitive performance and showcases its ability to effectively handle interval-valued data fusion for classification tasks.

6.2 Statistical analysis

In this subsection, we conduct a systematic investigation of the statistical performance of different fusion algorithms in terms of classification accuracy. To achieve this, we employ the Friedman test followed by a post hoc test to compare the performance of the fusion methods. The Friedman statistic is described [47] as:

$$\chi_F^2 = \frac{12N}{k(k+1)}\left(\sum_{j=1}^{k} R_j^2 - \frac{k(k+1)^2}{4}\right),$$
$$F_F = \frac{(N-1)\chi_F^2}{N(k-1) - \chi_F^2},$$

where N represents the number of data sets, while k represents the number of methods; $R_j (j = 1, 2, \dots, k)$ represents the Average ranking of a certain approach on all data sets. and F_F represents an F-distribution with $(k-1)$ and $(k-1)(N-1)$ degrees of freedom. Then the critical difference is expressed [46, 47] as:

$$CD_\alpha = q_\alpha \sqrt{\frac{k(k+1)}{6N}},$$

where α expresses the significance level and q_α represents a critical value [47].

For all data sets, we use following the statistical test. The values of average ranking are obtained by averaging the sorting of classification accuracy. The best level value for accuracy measurement is set to 1; the second is set to 2, and so on. Tables 16 and 17 shows the changes of fusion classification accuracy of nine data sets under six different fusion algorithm, the Friedman tests are accomplished by the comparison of this paper's method based FACE-IF with MinF, MaxF, MeanF, HF and CF. When all algorithms are equal in measures of classification accuracy, the null hypothesis of Friedman's test can be established. Then, the rankings of the six models can be lightly computed and their average order are acquired under the KNN and PNN. Thus, the values of χ_F^2 and F_F can be calculated. Table 17 shows the average sort results of the six models and the values of χ_F^2 and F_F under the classifier KNN and PNN. When the significance level is equal 10%. It follows from [45],by calculation, one has the critical point of $F(6-1,(6-1)\times(9-1))$ in the F-distance is calculated to be 1.997, and the critical point $q_{0.1}$ in the Nemenyi test is 2.589, the critical difference is 2.829, that is, $CD = 2.829$. So, all null hypotheses are refused, and the six fusion algorithm are various under KNN and PNN (Table 18).

So as to compare the differences between the fusion results under different measures more intuitively, we use CD critical charts [46] to connect methods that do not differ significantly from each other, and then in these graphs the critical values between all models can be clearly illustrated. Figure 7 is the CD critical diagram which shows the comparison of the fusion result based FACE-IF with the other five fusion algorithm. As can be seen from Fig. 7, we can know the significant differences in fusion results under six fusion methods. In Fig. 7a, under the classifier KNN, The average ranking of fusion approach based on FACE-IFis the lowest, and the performance of fusion method proposed in this paper is clearly better than HF, MeanF, CF and MaxF. Likewise, as shown in Fig. 7b on the PNN, we can find the fusion result based on FACE-IF outperforms the HF, MaxF and HF, and is similar to MinF and MeanF. In conclusion, the fusion effect using FACE-IF really outperforms the other five approaches under the outcomes of the Friedman statistic test.

6.3 Efficiency analysis

In this subsection, we aim to evaluate the efficiency of the dynamic updating algorithms by comparing the running time of the static fusion algorithm with that of the dynamic fusion algorithm. Figure 8 illustrates the comparison of running times between the static algorithm and the dynamic fusion algorithm across nine datasets. In the presented graph, the x-axis represents the combination of attributes and sources, while the y-axis indicates the corresponding running time. Orange bars represent the runtime of the static fusion algorithm at each coordinate, and purple bars illustrate the dynamic fusion algorithm's runtime as the number of attributes and sources changes between the previous and current coordinates. The results depicted in the graph clearly demonstrate the efficiency of the proposed dynamic updating algorithm, specifically in reducing the runtime required for updating fusion results when both attributes and sources undergo simultaneous changes. This provides robust evidence supporting the efficacy of the designed dynamic fusion algorithm.

The above experiments were conducted to verify that our proposed dynamic update algorithm can reduce the time required for fusion and improve fusion efficiency. To further validate the effectiveness of the aforementioned dynamic updating algorithm, we increased the number of sources in the corresponding datasets to 100. To analyze the computational efficiency under the four dynamic situations. We calculate the acceleration ratio by simultaneously varying the attributes and sources by 10%, 20%, 30%, 40%, and 50% in each of the four dynamic scenarios, where $Ratio = T_{staticalgorithm}/T_{incrementalalgorithm}$. When augmenting the number of sources, we employ a strategy wherein 50% of the initial number of sources serves as the baseline. For instance, in a scenario with 100 sources, the baseline is set at 50, incremented subsequently by 10%, 20%,..., up to 50%. Essentially, this translates to an increase from 50 to 100. Conversely, when reducing the number of sources, we decrease by 10% from the original count, diminishing it from 100 to 90. Continuing this pattern, reductions progress in 10% increments, ultimately reaching a 50% reduction from the original count, bringing it down from 100 to 50.Similarly, for variations in the number of attributes, a comparable methodology is applied. However, since some datasets have a small number of attributes, our approach may vary slightly. Take the BCW dataset, for example, featuring 30 conditional attributes. When increasing attributes, 50% of the original count serves as the base, increasing by 10%, resulting in a rise from 15 to 18. Subsequent increments of 20% lead to an increase from 15 to 21, while a 50% increment results in an increase from 15 to 30. Conversely, when reducing attributes, a 10% decrement from 30 brings it down to 27, and so forth. For datasets with a smaller number of conditional attributes like WQR, CE, PB, SB, and Ele, attribute count adjustments are made in 10% increments, using a step size of 1. However, for Diabetic and CA datasets, a step size of 2 is employed when increasing or decreasing attributes. Following the computation of the speedup ratio, the results are visualized as a heatmap in Fig. 9, allowing for the observation of changing trends through variations in color intensity.

From Fig. 9, in all four dynamic scenarios, the speedup ratio of the dynamic algorithm compared to the static algorithm is greater than 1. This indicates that our proposed dynamic update algorithm is effective in reducing runtime and improving fusion efficiency. It is noticeable that the speedup ratio is particularly significant in the third dynamic scenario, where both the number of attributes and the number of sources are reduced simultaneously. This indicates that the dynamic update algorithm proposed in this paper has a particularly significant effect in the third dynamic scenario. Furthermore, we observe that for the same dataset, the highest speedup ratio is achieved when both the attributes and sources vary by 10% simultaneously in all four scenarios.

7 Conclusion

In this article, we propose an information fusion method based on fuzzy approximate conditional entropy to enhance the classification performance of multi-source interval-valued information systems. Firstly, we apply the principle of statistical distribution to treat interval values as probability distributions. Subsequently, we utilize the Kullback–Leibler divergence to quantify the disparities between two interval value distributions, constructing a fuzzy similarity relation. This relation is then employed to establish fuzzy decisions. Secondly, based on the fuzzy similarity relation, we define the interval fuzzy information granular structure. Furthermore, by combining this information granular structure with fuzzy decision, we proposed a novel entropy measure named as fuzzy approximate conditional entropy and constructed a information fusion approach based FACE. Thirdly, based on the properties of the proposed entropy measure, we analyze four dynamic update mechanisms in different dynamic environments. These mechanisms aim to efficiently reduce computation time and improve fusion efficiency. Finally, the results of extensive experiments demonstrate that our proposed fusion method significantly enhances fusion performance to a considerable extent. Moreover, the introduced dynamic update algorithms prove effective in reducing computation time and avoiding redundant calculations, thereby enhancing the overall efficiency of the fusion process. Due to the inherently high computational complexity of entropy fusion, in future work, we will continue to explore more efficient fusion strategies and consider dynamic fusion methods for heterogeneous information systems where three dimensions change simultaneously.

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Data availability No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Dubois D, Liu WR, Ma JB (2016) The basic principles of uncertain information fusion. Inf Fusion 32:12–39
2. Han CZ, Zhu HY, Duan ZS (2006) Multi-source information fusion. Tsinghua University Press, Beijing
3. Pan Q (2013) Multi-source information fusion theory and its applications. Tsinghua University Press, Beijing
4. Hall DL (2001) Mathematical techniques in multisensory data fusion. CRC Press, Boca Raton
5. Zadeh LA (1996) Fuzzy sets and information granularity. In: Fuzzy sets, Fuzzy logic, and Fuzzy systems, pp 449–463
6. Ba J, Liu KY, Yang XB et al (2023) Gift: granularity over specific-class for feature selection. Artif Intell Rev 56:12201–12232
7. Hobbs JR (1985) Granularity. In: Proceedings of the 9th international joint conference on Artificial intelligence, vol 1, pp 432–435
8. Lin TY (1988) Neighborhood systems and relational databases. In: Proceedings of the 1988 ACM sixteenth annual conference on Computer science (CSC '88). Association for Computing Machinery, New York, p 725. https://doi.org/10.1145/322609.323183
9. Zadeh LA (1997) Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic. Fuzzy Sets Syst 90(2):111–127
10. Yao YY (2001) Information granulation and rough set approximation. Int J Intell Syst 16(1):87–104
11. Zhang L, Zhang B (2005) A quotient space approximation model of multiresolution signal analysis. J Comput Sci Technol 20:90–94
12. Pedrycz W, Homenda W (2013) Building the fundamentals of granular computing: a principle of justifiable granularity. Appl Soft Comput 13(10):4209–4218
13. Zhang GQ, Li ZW et al (2018) Information structures and uncertainty measures in a fully fuzzy information system. Int J Approx Reason 101:119–149
14. Chen YM, Qin N, Li W et al (2019) Granule structures, distances and measures in neighborhood systems. Knowl Based Syst 165(1):268–281
15. Qin B, Zeng FP, Yan KS (2018) Knowledge structures in a tolerance knowledge base and their uncertainty measures. Knowl Based Syst 151(1):198–215
16. Pawlak Z (1998) Rough set theory and its applications to data analysis. Cybern Syst 29:661–688
17. Guo DD, Jiang CM, Sheng RX, Liu SS (2022) A novel outcome evaluation model of tree-way decision: a change viewpoint. Inf Sci 607:1089–1110
18. Guo DD, Xu W, Qian YH, Ding WP (2023) Fuzzy-granular concept-cognitive learning via three-way decision: performance

evaluation on dynamic knowledge discovery. In: IEEE transactions on fuzzy systems, pp 1–12
19. Yan C, Yang XB, Li JH et al (2022) Fusing attribute reduction accelerators. Inf Sci 587:354–370
20. Guo DD, Jiang CM, Wu P (2022) Three-way decision based on confidence level change in rough set. Int J Appprox Reason 143:57–77
21. Xu WH, Guo DD, Qian YH, Ding WP (2023) Two-way concept-cognitive learning method: a fuzzy-based progressive learning. IEEE Trans Fuzzy Syst 31(6):1885–1899
22. Han J, Cai Y, Cercone N (1993) Data-driven discovery of quantitative rules in relational databases. IEEE Trans Knowl Data Eng 5(1):29–40
23. Chan CC (1998) A rough set approach to attribute generalization in data mining. Inf Sci 107(1–4):169–176
24. Liu KY, Yang XB, Yu HL et al (2019) Rough set based semi-supervised feature selection via ensemble selector. Knowl Based Syst 165(1):282–296
25. Fan TF, Liu DR, Tzeng GH (2007) Rough set-based logics for multicriteria decision analysis. Eur J Oper Res 182(1):340–355
26. Yang L, Xu WH, Zhang XY, Sang BB (2020) Multi-granulation method for information fusion in multi-source decision information system. Int J Approx Reason 122:47–65
27. Lin GP, Liang JY, Qian YH (2015) An information fusion approach by combining multi-granulation rough sets and evidence theory. Inf Sci 314:184–199
28. Zhang PF, Li TR, Zhong Y et al (2022) A data-level fusion model for unsupervised attribute selection in multi-source homogeneous data. Inf Fusion 80:87–103
29. Zhang XY, Huang XD, Xu WH (2023) Matrix-based multi-granulation fusion approach for dynamic updating of knowledge in multi-source information. Knowl Based Syst 262:110257
30. Qian WB, Yu SD, Yang J, Wang YL, Zhang JH (2020) Multi-label feature selection based on information entropy fusion in multi-source decision system. Evol Intell 13(2):255–268
31. Shu B, Kung JY, Shu MH (2013) Interval-valued process data monitoring and controlling. Artif Intell Res 2(3):90–101
32. Dai JH, Wang WT, Xu Q et al (2012) Uncertainty measurement for interval-valued decision systems based on extended conditional entropy. Knowl Based Syst 27:443–450
33. Dai JH, Wang WT, Mi JS (2013) Uncertainty measurement for interval-valued information systems. Inf Sci 251(1):63–78
34. Dai JH, Wei BJ, Zhang XH et al (2017) Uncertainty measurement for incomplete interval-valued information systems based on α -weak similarity. Knowl Based Syst 136(15):159–171
35. Liu XF, Dai JH, Chen JL, Zhang CC (2020) Unsupervised attribute reduction based on α -approximate equal relation in interval-valued information systems. Int J Mach Learn Cybern 11:2021–2038
36. Hua JH, Yan YJ, Li ZW, Liao BS (2018) Dominance-based fuzzy rough set approach for incomplete interval-valued data. J Intell Fuzzy Syst 34(1):423–436
37. Xie NX, Liu M, Liu ZW et al (2019) New measures of uncertainty for an interval-valued information system. Inf Sci 470:156–174
38. Xu XB, Zhang Z, Xu DL, Chen YW (2016) Interval-valued evidence updating with reliability and sensitivity analysis for fault diagnosis. Int J Comput Intell Syst 9(3):396–415
39. Fumanal-Idocin J et al (2022) Interval-valued aggregation functions based on moderate deviations applied to motor-imagery-based brain-computer interface. IEEE Trans Fuzzy Syst 30(7):2706–2720
40. Chan HK, Hu ZX, Beng LH (2011) Fusion of simplified entity networks from unstructured text. In: 14th international conference on information fusion, pp 1–7
41. Kotwal K, Chaudhuri S (2013) A novel approach to quantitative evaluation of hyperspectral image fusion techniques. Inf Fusion 14(1):5–18
42. Huang YY, Li TR, Luo C, Fujita H, Horng SJ (2018) Dynamic fusion of multisource interval-valued data by fuzzy granulation. IEEE Trans Fuzzy Syst 26(6):3403–3417
43. Zhang XY, Chen XW, Xu WH, Ding WP (2022) Dynamic information fusion in multi-source incomplete interval-valued information system with variation of information sources and attributes. Inf Sci 608:1–27
44. Xu WH, Pan YZ, Chen XW, Qian YH, Ding WP (2023) A novel dynamic fusion approach using information entropy for interval-valued ordered datasets. IEEE Trans Big Data 9(03):845–859
45. Friedman M (1940) A comparison of alternative tests of significance for the problem of m rankings. Ann Math Stat 11(1):86–92
46. Demsar J, Schuurmans D (2006) Statistical comparison of classifiers over multiple data sets. J Mach Learn Res 7:1–30
47. Sun L, Wang LY, Ding WP, Qian YH, Xu JC (2021) Feature selection using fuzzy neighborhood entropy-Based uncertainty measures for fuzzy neighborhood multigranulation rough sets. IEEE Trans Fuzzy Syst 29(1):19–33