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Matrix-based approximation dynamic update approach to multi-granulation neighborhood rough sets for intuitionistic fuzzy ordered datasets

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ABSTRACT

With the explosive growth and rapid changes in the use of data, information systems are constantly evolving. Timely dynamic updates have become imperative with real-time monitoring of data increasingly common. Effectively characterizing the approximation space in dynamic environments is of significant concern. This paper investigates a dynamic update mechanism for generalized multi-granulation neighborhood dominant rough sets, based on a matrix form, in an intuitionistic fuzzy ordered information table. We first define support and inclusion functions to construct the model of generalized multi-granulation neighborhood dominant rough sets. Additionally, we analyze the dynamic update process in which objects are added or removed in matrix form. Corresponding dynamic update algorithms are proposed based on generalized multi-granulation neighborhood dominant rough sets. Finally, to validate the effectiveness of the matrix-based dynamic approximation update algorithm, eight UCI datasets are used to perform experiments. The results verify that our matrix-based dynamic update algorithm is effective in approximating updates for dynamic intuitionistic fuzzy ordered information datasets.

1. Introduction

As a fundamental method, granular computing can effectively deal with problems of uncertainty and incompleteness and can simplify complex phenomena. It has been widely applied and developed in various modern information fields. Granular computing mainly involves granulating data to form abstract concepts [1], which can provide useful knowledge. Especially in this age of explosive increases in volumes of data, granular computing has emerged as an efficient method for knowledge acquisition. Exploring uncertain information has gained significant attention as a trending research area. To address the uncertainty of information in real-world scenarios, researchers have introduced rough sets, fuzzy sets, and intuitionistic fuzzy sets. Classical rough set theory (RST) is a data mining technique used to handle uncertainty and fuzziness. Introduced by mathematician Z. Pawlak [2] in 1982, rough sets have become an important mathematical tool for knowledge discovery and representation through approximate reasoning. RST does not rely on prior knowledge but mainly depends on existing data to approximate knowledge, which helps people better understand and analyze data. RST has found widespread applications in diverse fields, including data mining[3–5], decision analysis [6], and pattern recognition[7]. In classical RST, Pawlak used equivalence relations to construct equivalence classes and used these classes as elements for set operations. However, the classical RST initially applied only to discrete datasets. To extend the applicability of RST to other types of datasets, many extension models of RST have been proposed successively by researchers [8–12].

Atanassov [13] proposed intuitionistic fuzzy sets (IFSs) as an extension of Zadeh's fuzzy sets [14] in 1986. An IFS is an extension of a fuzzy set that considers the degree of membership and non-membership of an object to a target set. In recent years, have seen widespread attention on combining IFS theory with RST[15]. For instance, Singh et al. developed an attribute reduction model designed explicitly for the IFRS model[16]. Huang et al. developed an intuitive fuzzy rough set model for multi-granular data. They discussed the hierarchical structure of the intuitive fuzzy rough set and explored uncertainty measures associated with it [17].

In practical applications, describing objects from multiple perspectives is essential. However, classical single-granulation models can only provide descriptions from a single perspective. To apply RST more broadly to complex description processing, Qian et al. introduced the concept of multi-granulation rough sets [18,19], which use knowledge from spaces with different granularities to approach concepts approximately. Subsequently, numerous scholars have dedicated their efforts to researching multi-granulation rough set models, such as approximate concepts [20] and granularity selection [21]. Researchers have developed two models: optimistic and pessimistic multi-granulation rough

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Table 1
The review of some extended rough set models for IFIS.

Year	Authors	Research content	Reference
2012	Huang et al.	Dominance-based rough set model in IFIS	[23]
2014	Gong et al.	Variable precision intuitionistic fuzzy rough sets model	[24]
2014	Huang et al.	Intuitionistic fuzzy multigranulation rough sets	[25]
2015	Roy et al.	Neighborhood rough set model in IFIS	[26]
2017	Zhang et al.	Generalized dominance rough set models for dominance IFIS	[27]
2018	Shreevastava et al.	Intuitionistic fuzzy neighborhood rough set model for feature selection	[28]
2022	Zhang et al.	Multigranulation rough set methods in intuitionistic fuzzy datasets	[29]
2023	Zhang et al.	SPRS model and DPRS model for ordered IFIS	[30]

sets. However, the existing multi-granulation rough set models often face challenges in achieving a balanced level of granularity approximation, leading to either overly strict or loose descriptions. This restricts their flexibility and effectiveness in practical applications. To address this challenge, Xu et al. introduced the generalized multi-granulation rough set model [22], which extends the concept by incorporating a principle of majority rule based on real-world scenarios. In recent years, researchers have successively proposed various extended rough set models for IFSs, as shown in Table 1.

However, there are two insufficiencies. On the one hand, they did not consider the generalized multi-granulation neighborhood dominant rough set (GMNDRS) simultaneously. On the other hand, the dynamic nature of intuitive fuzzy data has not been considered.

In recent years, incremental approximation updates have attracted the interest of scholars. Some approximation update methods require recomputation when new data are added or old data are removed. This undoubtedly leads to a significant amount of redundant computation. Therefore, finding suitable approximation update techniques for dynamic intuitionistic fuzzy data is crucial. Zhang et al. proposed a dynamic updating approximation method for multi-granulation interval-valued hesitant fuzzy information systems with time-evolving attributes [31]. Yang et al. proposed an incremental fuzzy probability decision theory method with dynamic three-way approximations [32]. Guo et al. proposed a method called M-FCCL for handling dynamic fuzzy data classification and knowledge fusion [33]. Zhang et al. introduced a matrix-based approach for updating dynamic knowledge from multi-source information, which leverages multi-granulation fusion and provides a method for dynamically updating knowledge using matrices [34]. Inspired by this, to improve computation time and efficiency, we will employ a matrix-based approach for dynamic approximation updates in this paper.

This paper proposes an approximation update algorithm based on the GMNDRS model for dynamic intuitionistic fuzzy ordered datasets with temporal evolution characteristics. The main contributions of this paper are as follows:

(1) A method is presented for studying generalized multi-granulation neighborhood dominant rough sets using a matrix representation of generalized neighborhood dominant relations and feature functions.
(2) A matrix-based dynamic update mechanism adds the latest information, deletes redundant information, and effectively processes the approximate values of GMNDRS.
(3) Experiments were conducted using eight datasets downloaded from the UCI repository. The experimental results demonstrate that the proposed matrix dynamic method improves computational efficiency when the objects in an intuitionistic fuzzy ordered information system (IFOIS) change.

The remainder of the paper is structured as follows. Section 2 reviews some basic concepts of GMNDRS in IFOISs. Section 3 defines the GMNDRS based on a matrix representation. Section 4 introduces the dynamic update mechanism of a GMNDRS, including two cases of adding and removing objects. Section 5 presents a dynamic algorithm for updating GMNDRS approximation sets from a matrix perspective. Section 6 describes experimental analyses that were conducted using eight datasets. Finally, Section 7 briefly summarizes the paper's content and discusses prospects for future research directions. Fig. 1 provides a concise depiction of the framework employed in this paper.

2. Related work

This section reviews the basic concepts of IFOISs, neighborhood dominant rough sets, and generalized multi-granulation rough sets based on IFOISs.

2.1. Intuitionistic fuzzy ordered information systems [35]

An intuitionistic fuzzy information system (IFIS) is an extension of a classical fuzzy information system. It is designed to address the problems of fuzziness and uncertainty. Unlike traditional fuzzy sets, intuitionistic fuzzy sets allow elements to have diverse levels of membership and non-membership of the set. Intuitionistic fuzzy data are called an intuitionistic fuzzy information system in RST.

Let $\widetilde{I} = (U, AT, G)$ be an information system, where U represents a finite set of objects and AT denotes a finite set of conditional attributes. For all $g \in G$, $x_i \in U$, and $a \in AT$, it holds that $g(a, x_i) = (\mu_{x_i}(a), \nu_{x_i}(a))$. Here, $\mu_{x_i}(a)$ and $\nu_{x_i}(a)$ are functions mapping U to the interval $[0,1]$, satisfying $0 \le \mu_{x_i}(a)+\nu_{x_i}(a) \le 1$. They indicate the degree of membership and non-membership of the object x_i under attribute a. The hesitation degree $\omega_{x_i}(a) = 1 - \mu_{x_i}(a) - \nu_{x_i}(a)$ indicates the level of uncertainty for x under a, with $\omega_{x_i}(a) \in [0,1]$. When $\omega_{x_i}(a) = 0$, the IFIS reduces to a classical fuzzy set.

Let $\widetilde{I} = (U, AT, G)$ be an IFIS, $\forall a \in AT$, attribute values can be compared in the IFIS, and we define

$$\begin{aligned} &g(x_i, a) \le g(x_j, a) \Leftrightarrow (\forall a \in AT)[\mu_{x_i}(a) \le \mu_{x_j}(a), \nu_{x_i}(a) \ge \nu_{x_j}(a)],\\ &g(x_i, a) \ge g(x_j, a) \Leftrightarrow (\forall a \in AT)[\mu_{x_i}(a) \ge \mu_{x_j}(a), \nu_{x_i}(a) \le \nu_{x_j}(a)]. \end{aligned} \tag{1}$$

There are decreasing and increasing partial orders under the IFIS. In the IFIS, if $a \in AT$ is a criterion, then there exists a relation $\ge$. The statement $x_i \ge_a x_j$ signifies that x_i exhibits dominance over x_j in relation to the criterion a. If all attributes in the table are criteria, then $\widetilde{I^{\ge}}$ is an IFOIS and is denoted as $\widetilde{I^{\ge}} = (U, AT, G)$. Furthermore, the intuitionistic fuzzy ordered decision information table is $\widetilde{I^{\ge}} = (U, AT \cup \{d\}, G)$, where $R_d = \{x_j, x_i \in U \mid g(x_i, d) = g(x_j, d)\}$ is an equivalence relation.

The increase in neighborhood dominant classes settles the limitation of the equivalence relation. Based on this, the neighborhood dominant RST will be introduced.

2.2. Dominance-based neighborhood rough sets [29]

The degree of dissimilarity between different intuitionistic fuzzy objects varies due to the differences in values within intuitionistic fuzzy datasets. To address this concern, we propose a distance function to measure the dissimilarity between two intuitionistic fuzzy objects. This distance function considers the impact of value differences in intuitionistic fuzzy datasets, ensuring that it accurately reflects the extent of dissimilarity between intuitionistic fuzzy objects. In practical problems, objects with different levels of dominance may exist. Hence,

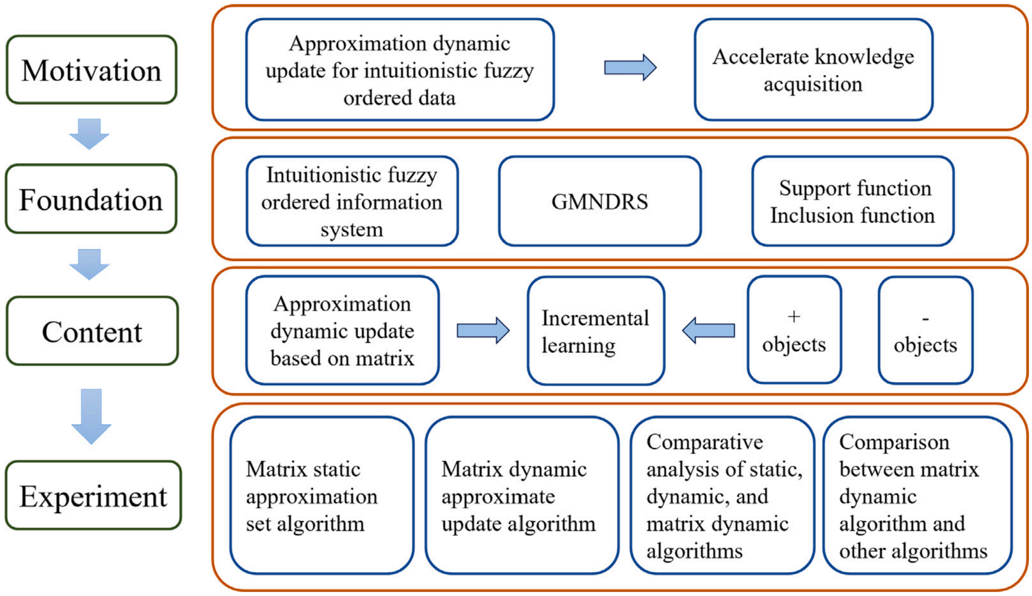


Fig. 1. The framework of our work.

introducing dominant rough sets based on neighborhood rough sets allows for a more comprehensive consideration of object relationships. This facilitates the division of objects into subsets characterized by neighborhood dominant relationships.

Letting $\widetilde{I}^{\ge} = (U, AT, G)$ be an IFOIS, $\forall x_m, x_n \in U$, $\forall A_t \subseteq AT$, and $\forall t \in \{1, 2, \dots, s\}$, the definitions of the neighborhood rough set, dominance rough set, and neighborhood dominance rough set for objects x_m and x_n under the attribute set A_t are as follows:

$$\begin{aligned} \widetilde{R}^{\delta}_{A_t} &= \left\{ (x_m, x_n) \in U \;\middle|\; \widehat{D}(x_m, x_n) \ge \delta \right\}, \\ \widetilde{R}^{\ge}_{A_t} &= \left\{ (x_m, x_n) \in U \;\middle|\; [\mu_{A_t}(x_m) \le \mu_{A_t}(x_n), \nu_{A_t}(x_m) \ge \nu_{A_t}(x_n)] \right\}, \\ \widetilde{R}^{\delta\ge}_{A_t} &= \Big\{ (x_m, x_n) \in U \;\Big|\; [\widehat{D}(x_m, x_n) \ge \delta] \wedge [\mu_{A_t}(x_m) \\ &\quad \le \mu_{A_t}(x_n), \nu_{A_t}(x_m) \ge \nu_{A_t}(x_n)] \Big\}. \end{aligned} \tag{2}$$

Here, $\widehat{D}$ is a distance function between x_i and x_j over all attributes, defined as

$$\widehat{D}(x_i, x_j) = \left(\sum_{t=1}^{|A_t|} \left(\left| \mu_{A_t}(x_i) - \mu_{A_t}(x_j) \right|^q + \left| \nu_{A_t}(x_i) - \nu_{A_t}(x_j) \right|^q \right) \right)^{\frac{1}{q}}, \tag{3}$$

where $\mu_{A_t}(x_i)$ and $\mu_{A_t}(x_j)$ represent the fuzzy membership degrees of x_i and x_j under attribute set A_t, respectively, and $\nu_{A_t}(x_i)$ and $\nu_{A_t}(x_j)$ are the non-fuzzy membership degrees of x_i and x_j under attribute set A_t. The neighborhood radius δ needs to satisfy $\delta \ge 0$. The distance function $\widehat{D}$ can use either the Manhattan or Euclidean distance. When $q = 1$, this indicates the Manhattan distance, and $q = 2$ the Euclidean distance.

Letting $\widetilde{I}^{\ge} = (U, AT, G)$ be an IFOIS, then $\forall x_m, x_n \in U$, $\forall A_t \subseteq AT$, and $\forall t \in \{1, 2, \dots, s\}$, the corresponding neighborhood class, dominant class, and neighborhood dominant class of x_m and x_n under the attribute set A_t, are defined as follows:

$$\begin{aligned} \widetilde{[x_m]}^{\delta}_{A_t} &= \left\{ x_n \in U \;\middle|\; (x_m, x_n) \in \widetilde{R}^{\delta}_{A_t} \right\}, \\ \widetilde{[x_m]}^{\ge}_{A_t} &= \left\{ x_n \in U \;\middle|\; (x_m, x_n) \in \widetilde{R}^{\ge}_{A_t} \right\}, \\ \widetilde{[x_m]}^{\delta\ge}_{A_t} &= \left\{ x_n \in U \;\middle|\; (x_m, x_n) \in \widetilde{R}^{\delta\ge}_{A_t} \right\}, \end{aligned} \tag{4}$$

where the neighborhood dominant class $\widetilde{[x_i]}^{\delta\ge}_{A_t}$ is a set of objects. For each pair of objects x_m and x_n, the distance between them is required to be less than δ, and all the domains under attributes are completely pre-ordered according to A_t.

Let denote

$$U/\widetilde{R}^{\ge}_{A_t} = \left\{ \widetilde{[x_i]}^{\ge}_{A_t} \;\middle|\; x_i \in U \right\}. \tag{5}$$

In which $i \in \{1, 2, 3, \dots, |U|\}$, and $\widetilde{[x_i]}^{\ge}_{A_t}$ represents the class of the IFODIS on attribute A_t with respect to the object set containing x_m, where $U/\widetilde{R}^{\ge}_{A_t}$ is a coverage of U in the IFODIS over the attribute set A_t.

Suppose we are given an IFOIS $\widetilde{I} = (U, AT, G)$, where A_t is a subset of attributes. For all $t \in \{1, 2, \dots, s\}$, $X \subseteq U$, the upper and lower approximations of X with respect to the neighborhood dominant relation $\widetilde{R}^{\delta\ge}_{A_t}$ can be defined as follows:

$$\begin{aligned} \underline{\widetilde{R}^{\delta\ge}_{A_t}} &= \left\{ x \in U \;\middle|\; \widetilde{[x_i]}^{\delta\ge}_{A_t} \subseteq X \right\}, \\ \overline{\widetilde{R}^{\delta\ge}_{A_t}} &= \left\{ x \in U \;\middle|\; \widetilde{[x_i]}^{\delta\ge}_{A_t} \cap X \ne \varnothing \right\}. \end{aligned} \tag{6}$$

Here, $\overline{\widetilde{R}^{\delta\ge}_{A_t}}$ and $\underline{\widetilde{R}^{\delta\ge}_{A_t}}$ represent a pair of rough set approximation operators. When $\overline{\widetilde{R}^{\delta\ge}_{A_t}} = \underline{\widetilde{R}^{\delta\ge}_{A_t}}$, X is called a definable set. Otherwise, X is a rough set.

In terms of the attribute set A_t and the neighborhood dominant relation $\widetilde{R}^{\delta\ge}_{A_t}$, the negative domain, boundary domain, and positive domain of set X can be defined as follows:

$$\begin{aligned} NEG^{\delta\ge}_{A_t}(X) &= U - \overline{\widetilde{R}^{\delta\ge}_{A_t}}, \\ BND^{\delta\ge}_{A_t}(X) &= \overline{\widetilde{R}^{\delta\ge}_{A_t}} - \underline{\widetilde{R}^{\delta\ge}_{A_t}}, \\ POS^{\delta\ge}_{A_t}(X) &= \underline{\widetilde{R}^{\delta\ge}_{A_t}}. \end{aligned} \tag{7}$$

2.3. Optimistic and pessimistic multi-granulation neighborhood dominant rough sets [36]

In complex practical problems, it is not accurate to characterize objects only from a single perspective. A domain U may not be divided by a single relation, but often by multiple relations. Considering this, Qian et al. [24] were the first to propose the notion of multi-granulation rough sets, aiming to offer a more comprehensive depiction of objects.

Letting $\widetilde{I}^{\ge}$ be an IFOIS, then for $A_t \subseteq AT$, $\forall t \in \{1, 2, \dots, s\}$, and $X \subseteq U$, the optimistic multi-granulation lower and upper approximations of set X based on the neighborhood dominant relation $\widetilde{R}^{\delta\ge}_{A_t}$ can be defined as follows:

$$\overline{OM_{\sum_{t=1}^{s}} \widetilde{R}_{A_t}^{\delta\geq}}(X) = \left\{ x \in U \mid \bigwedge_{t=1}^{s} \left([x]_{\widetilde{R}_{A_t}^{\delta\geq}} \cap X \neq \emptyset \right) \right\},$$
$$\underline{OM_{\sum_{t=1}^{s}} \widetilde{R}_{A_t}^{\delta\geq}}(X) = \left\{ x \in U \mid \bigvee_{t=1}^{s} \left([x]_{\widetilde{R}_{A_t}^{\delta\geq}} \subseteq X \right) \right\}. \tag{8}$$

Here, the symbol $\bigwedge$ represents "or" and $\bigvee$ represents "and." Furthermore, if $\overline{OM}(X) \neq \underline{OM}(X)$, then X is considered a definable set to the neighborhood dominant relation $\widetilde{R}_{A_t}^{\delta\geq}$; otherwise, X is classified as a rough set.

Continuing from the above, the pessimistic multi-granulation lower and upper approximations of set X based on the neighborhood dominant relation $\widetilde{R}_{A_t}^{\delta\geq}$ can be defined as follows:

$$\overline{PM_{\sum_{t=1}^{s}} \widetilde{R}_{A_t}^{\delta\geq}}(X) = \left\{ x \in U \mid \bigvee_{t=1}^{s} \left([x]_{\widetilde{R}_{A_t}^{\delta\geq}} \cap X \neq \emptyset \right) \right\},$$
$$\underline{PM_{\sum_{t=1}^{s}} \widetilde{R}_{A_t}^{\delta\geq}}(X) = \left\{ x \in U \mid \bigwedge_{t=1}^{s} \left([x]_{\widetilde{R}_{A_t}^{\delta\geq}} \subseteq X \right) \right\}. \tag{9}$$

If $\overline{PM}(X) \neq \underline{PM}(X)$, then X is considered a definable set to the neighborhood dominant relation $\widetilde{R}_{A_t}^{\delta\geq}$. Otherwise, X is classified as a rough set.

2.4. Generalized multi-granulation neighborhood dominant rough sets [22]

The concept of a multi-granulation neighborhood dominant rough set is an emerging research direction in RST. It involves estimating upper and lower approximations using multiple granular structures. For an object $x \in U$, if all the neighborhood dominant classes of an element x are encompassed within the given concept, then x is considered an element in the optimistic lower approximation. Suppose there is a non-empty intersection between any of the neighborhood dominant classes of element x and the given concept. In that case, x is considered an element in the pessimistic upper approximation. However, classical multi-granulation RST, founded on both pessimistic and optimistic approximations, is limited in practical applications due to the overly restrictive nature of optimistic and pessimistic lower approximations and the overly relaxed nature of the upper approximations. To address this problem and better suit real-life applications, Xu et al. [34] introduced the generalized multi-granulation rough set model and investigated the selection of the optimal granulation within this framework. Compared with traditional multi-granulation RST, the generalized multi-granulation rough set model is more flexible and practical. Taking inspiration from this, the generalized multi-granulation neighborhood dominant RST will be investigated. First, we define a supporting function, which is defined as follows:

Letting $\widetilde{I}^{\geq}$ be an IFOIS, then for $A_t \subseteq AT$, $\forall t \in \{1,2,3,\ldots,s\}$, and $X \subseteq U$, the characteristic function $S_X^{\widetilde{R}_{A_t}^{\delta\geq}}(x)$ of x denotes the inclusion relation between the neighborhood dominant class $[x]\widetilde{R}A_t{}^{\delta\geq}$ and the set X. Its definition is as follows:

$$S_X^{\widetilde{R}_{A_t}^{\delta\geq}}(x) = \begin{cases} 1, & [x]_{\widetilde{R}_{A_t}^{\delta\geq}} \subseteq X \\ 0, & \text{otherwise} \end{cases}. \tag{10}$$

We refer to $S^{\widetilde{R}_{A_t}^{\delta\geq}}X(x)$ as the supporting characteristic function of $x \in U$. It indicates whether the object x precisely supports the concept X with respect to $\widetilde{R}A_t{}^{\delta\geq}$.

Property 1. *For x in U, $A_t \subseteq AT$, and $\forall t \in \{1,2,3,\ldots,s\}$, $S_X^{\widetilde{R}_{A_t}^{\delta\geq}}(x)$ has the following properties:*

(1) $S_{\sim X}^{\widetilde{R}_{A_t}^{\delta\geq}}(x) = \begin{cases} 1, & [x]_{\widetilde{R}_{A_t}^{\delta\geq}} \cap X \neq \emptyset \\ 0, & [x]_{\widetilde{R}_{A_t}^{\delta\geq}} \cap X = \emptyset \end{cases}$

(2) $S_{\emptyset}^{\widetilde{R}_{A_t}^{\delta\geq}}(x) = 0, S_U^{\widetilde{R}_{A_t}^{\delta\geq}}(x) = 1.$

Let $\widetilde{I}^{\geq}$ be an IFOIS, $A_t \subseteq AT$, $\forall t \in \{1,2,3,\ldots,s\}$, and $X \subseteq U$. The characteristic function is $S_X^{\widetilde{R}_{A_t}^{\delta\geq}}(x)$ of x. For all $\beta \in (0.5, 1]$, the lower and upper approximations of X concerning $[x]_{\widetilde{R}_{A_t}^{\delta\geq}}$ can be defined as follows:

$$\underline{GM}(X)_\beta^{\delta\geq} = \left\{ x \in U \mid \frac{\sum_{t=1}^{s} S_X^{\widetilde{R}_{A_t}^{\delta\geq}}(x)}{s} \geq \beta \right\},$$
$$\overline{GM}(X)_\beta^{\delta\geq} = \left\{ x \in U \mid \frac{\sum_{t=1}^{s} 1 - S_{\sim X}^{\widetilde{R}_{A_t}^{\delta\geq}}(x)}{s} > 1 - \beta \right\}. \tag{11}$$

The set X is referred to as definable if and only if $\underline{GM}(X)\beta^{\delta\geq} = \overline{GM}(X)\beta^{\delta\geq}$. Otherwise, if the lower and upper approximation sets are distinct, X is considered rough. We denote this model as the generalized multi-granulation rough set model, with β referred to as the information level with respect to $[x]_{\widetilde{R}A_t{}^{\delta\geq}}$.

3. Generalized multi-granulation neighborhood dominant rough sets based on matrix representation

This section focuses on a novel matrix representation of the neighborhood dominant relation and feature function for studying the GM-NDRS. The framework of the approach can be seen in Fig. 2.

Definition 1. Let $\widetilde{I}^{\geq} = (U, AT, G)$ be an IFOIS, for $U = x_1, x_2, \ldots, x_n$, $A_t \subseteq AT$, $\forall t \in \{1,2,\ldots,s\}$. Then $\widetilde{R}_{A_t}^{\delta}$, $\widetilde{R}_{A_t}^{\geq}$, $\widetilde{R}_{A_t}^{\delta\geq}$ represent the neighborhood relation, dominant relation, and neighborhood dominant relation of a single granularity A_t, respectively. The neighborhood dominant relation matrix $M_{A_t}^{\delta\geq} = [m_{ij,A_t}^{\delta\geq}]_{n\times n}$ and $m_{ij,A_t}^{\delta\geq}$ with respect to A_t can be defined as follows:

$$M_{A_t}^{\delta\geq} = \begin{pmatrix} m_{11,A_t}^{\delta\geq} & m_{12,A_t}^{\delta\geq} & m_{13,A_t}^{\delta\geq} & \cdots & m_{1n,A_t}^{\delta\geq} \\ m_{21,A_t}^{\delta\geq} & m_{22,A_t}^{\delta\geq} & m_{23,A_t}^{\delta\geq} & \cdots & m_{2n,A_t}^{\delta\geq} \\ m_{31,A_t}^{\delta\geq} & m_{32,A_t}^{\delta\geq} & m_{33,A_t}^{\delta\geq} & \cdots & m_{3n,A_t}^{\delta\geq} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n1,A_t}^{\delta\geq} & m_{n2,A_t}^{\delta\geq} & m_{n3,A_t}^{\delta\geq} & \cdots & m_{nn,A_t}^{\delta\geq} \end{pmatrix}, \tag{12}$$

and

$$m_{ij,A_t}^{\delta\geq} = \begin{cases} 1, (\widehat{D}(x_i, x_j) \leq \delta) \wedge [\mu_{A_t}(x_i) \leq \mu_{A_t}(x_j), \nu_{A_t}(x_i) \geq \nu_{A_t}(x_j)] \\ 0, otherwise \end{cases}.$$

where $\forall x_i, x_j \in U$, $\delta \geq 0, \delta$ is the neighborhood radius, $\widehat{D}(x_i, x_j)$ denotes the distance function with respect to A_t, and $m_{ij,A_t}^{\delta\geq}$ is the basic element of the neighborhood dominant relation matrix $M_{A_t}^{\delta\geq}$.

Definition 2. Let $\widetilde{I}^{\geq} = (U, AT, G)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$, $A_t \subseteq AT$, $\forall t \in \{1,2,\ldots,s\}$. For all $x_i \in U$, $[x_i]_{\widetilde{R}_{A_t}^{\delta\geq}}$ represents the neighborhood dominant class of x_i with respect to A_t. Let $X \subseteq U$, and let x_i be represented by the feature column vector with respect to $A_t (t = 1,2,\ldots,s)$, which is denoted as $H_{x_i}(A_t) = [h_{A_1}(x_i), h_{A_2}(x_i), \ldots, h_{A_s}(x_i)]^T$ and $L_{x_i}(A_t) = [l_{A_1}(x_i), l_{A_2}(x_i), \ldots, l_{A_s}(x_i)]^T$, where the feature elements

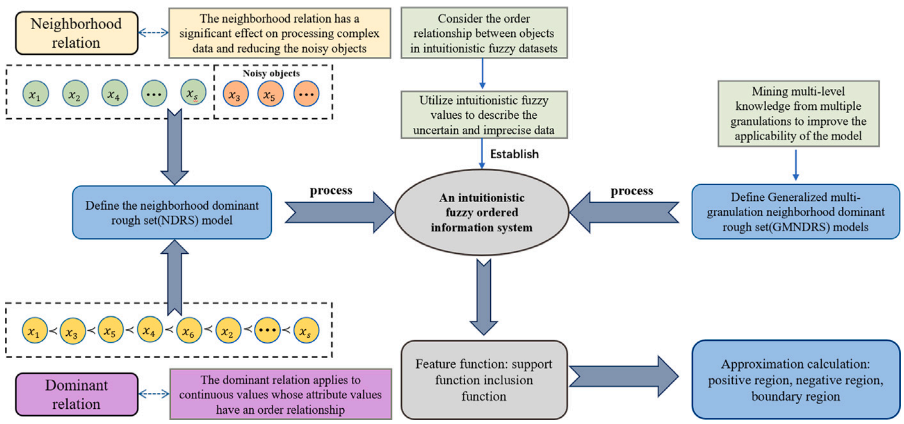


Fig. 2. The brief framework of this section.

Table 2
An IFOIS about housing evaluation.

U	b_1	b_2	b_3	b_4	b_5
x_1	(0.3,0.5)	(0.6,0.4)	(0.5,0.2)	(0.7,0.1)	(0.5,0.4)
x_2	(0.2,0.7)	(0.1,0.8)	(0.4,0.5)	(0.7,0.1)	(0.2,0.8)
x_3	(0.2,0.7)	(0.1,0.8)	(0.4,0.5)	(0.7,0.1)	(0.2,0.8)
x_4	(0.1,0.8)	(0.1,0.8)	(0.2,0.7)	(0.1,0.8)	(0.2,0.8)
x_5	(0.9,0.1)	(0.8,0.1)	(0.8,0.1)	(0.9,0.0)	(0.7,0.1)
x_6	(0.4,0.6)	(0.8,0.1)	(0.6,0.3)	(0.9,0.0)	(0.7,0.1)
x_7	(0.3,0.5)	(0.7,0.3)	(0.5,0.1)	(0.7,0.1)	(0.6,0.3)
x_8	(0.8,0.2)	(0.8,0.1)	(0.7,0.1)	(1.0,0.0)	(0.7,0.1)
x_9	(0.8,0.2)	(0.9,0.0)	(0.7,0.1)	(0.8,0.2)	(1.0,0.0)
x_{10}	(0.9,0.1)	(0.9,0.0)	(0.8,0.1)	(0.6,0.3)	(1.0,0.0)

are as follows:

$$h_{A_t}^X(x_i) = \begin{cases} 1, & [x_i]_{\widetilde{R}_{A_t}^{\delta\geq}} \subseteq X \\ 0, & \text{otherwise} \end{cases} \quad t \in \{1,2,\dots,s\}, x_i \in U,$$
$$l_{A_t}^{\sim X}(x_i) = \begin{cases} 1, & [x_i]_{\widetilde{R}_{A_t}^{\delta\geq}} \cap X = \varnothing \\ 0, & [x_i]_{\widetilde{R}_{A_t}^{\delta\geq}} \cap X \neq \varnothing \end{cases} \quad t \in \{1,2,\dots,s\}, x_i \in U. \tag{13}$$

According to Definition 2, we assign a value of 1 to the feature element $h_{A_t}^X(x_i)$ of object x_i if its neighborhood dominant class to $A_t(t = 1,2,\dots,s)$ is contained in the target concept X, and 0 if not. Similarly, we assign a value of 1 to the feature element $l_{A_t}^X(x_i)$ of object x_i if the intersection of the neighborhood dominant class of x_i under attribute A_t and $\sim X$ is empty, and 0 if not.

Definition 3. Let $\widetilde{I}^{\geq} = (U, AT, G)$ be an IFOIS, where $U = \{x_1, x_2, \dots, x_n\}$ and $A_t \subseteq AT$, $\forall t \in \{1,2,\dots,s\}$. For all $x_i \in U$, we can define the two characteristic functions of x_i under A:

$$H_{x_i}^X(A) = \frac{\sum h_{A_t}^X(x_i)}{s} \quad t \in \{1,2,\dots,s\},$$
$$L_{x_i}^{\sim X}(A) = \frac{\sum l_{A_t}^{\sim X}(x_i)}{s} \quad t \in \{1,2,\dots,s\}. \tag{14}$$

The characteristic function $H_{x_i}^X(A)$ represents the average degree of inclusion of object x_i with respect to X across all attributes A_t. On the other hand, $H_{x_i}^{\sim X}(A)$ represents the average degree of inclusion of object x_i with respect to the complement of the target set X across attributes A_t.

Example 1. Consider an IFOIS for housing evaluation given by Table 2, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ is the domain, consisting of 10 houses and $AT = \{b_1, b_2, b_3, b_4, b_5\}$ denotes the set of condition attributes associated with the system, including geographic location, floor, price, area, and layout. Let the target set $X = \{x_5, x_6, x_8, x_9\}$. From Table 2, we can obtain the neighborhood dominant relation matrices $M_{A_1}^{\delta\geq}$, $M_{A_2}^{\delta\geq}$, $M_{A_3}^{\delta\geq}$, and $M_{A_4}^{\delta\geq}$ with respect to $A_1 = \{b_1, b_2\}$, $A_2 = \{b_3, b_4, b_5\}$, $A_3 = \{b_1, b_2, b_3, b_4\}$, and $A_4 = \{b_2, b_3, b_4, b_5\}$, letting $\delta = 1.0$.

According to Definition 2, we can calculate the feature column vectors for each object.

$$\begin{aligned}
&H_{x_1}(A) = [0,0,0,0]^T, H_{x_2}(A) = [0,0,0,0]^T,\\
&H_{x_3}(A) = [0,0,0,0]^T, H_{x_4}(A) = [0,0,0,0]^T,\\
&H_{x_5}(A) = [0,0,1,1]^T, H_{x_6}(A) = [1,1,1,1]^T,\\
&H_{x_7}(A) = [0,0,0,0]^T, H_{x_8}(A) = [0,0,1,1]^T,\\
&H_{x_9}(A) = [0,0,1,1]^T, H_{x_{10}}(A) = [0,0,0,0]^T,\\
&L_{x_1}(A) = [1,1,1,1]^T, L_{x_2}(A) = [1,1,1,1]^T,\\
&L_{x_3}(A) = [1,1,1,1]^T, L_{x_4}(A) = [1,1,1,1]^T,\\
&L_{x_5}(A) = [0,0,0,0]^T, L_{x_6}(A) = [0,0,0,0]^T,\\
&L_{x_7}(A) = [1,1,1,1]^T, L_{x_8}(A) = [0,0,0,0]^T,\\
&L_{x_9}(A) = [0,0,0,0]^T, L_{x_{10}}(A) = [1,1,1,1]^T.
\end{aligned}$$

According to Definition 3, we can get the corresponding characteristic functions for x_i.

$$\begin{aligned}
&H_{x_1}^X(A) = 0, H_{x_2}^X(A) = 0, H_{x_3}^X(A) = 0, H_{x_4}^X(A) = 0, H_{x_5}^X(A) = \tfrac{1}{2},\\
&H_{x_6}^X(A) = 1, H_{x_7}^X(A) = 0, H_{x_8}^X(A) = \tfrac{1}{2}, H_{x_9}^X(A) = \tfrac{1}{2}, H_{x_{10}}^X(A) = 0,\\
&L_{x_1}^{\sim X}(A) = 1, L_{x_2}^{\sim X}(A) = 1, L_{x_3}^{\sim X}(A) = 1, L_{x_4}^{\sim X}(A) = 1, L_{x_5}^{\sim X}(A) = 0,\\
&L_{x_6}^{\sim X}(A) = 0, L_{x_7}^{\sim X}(A) = 1, L_{x_8}^{\sim X}(A) = 0, L_{x_9}^{\sim X}(A) = 0, L_{x_{10}}^{\sim X}(A) = 1.
\end{aligned}$$

$$M_{A_1}^{\delta\geq} = \begin{pmatrix}
1&0&0&0&0&0&1&0&0&0\\
0&1&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0\\
0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&1&1&0&1&1&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1
\end{pmatrix},$$

$$M_{A_2}^{\delta\geq} = \begin{pmatrix}
1&0&0&0&0&0&1&0&0&0\\
0&1&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0\\
0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&1&1&1\\
0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1
\end{pmatrix},$$

$$M_{A_3}^{\delta\geq} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{A_4}^{\delta\geq} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Based on the dominant relation matrices of each granular structure, we define feature column vectors and characteristic functions for all such structures, which can be further used to construct the positive, negative, and boundary domain vectors of generalized multi-granulation rough sets. It is also possible to analyze the optimistic and pessimistic positive, negative, and boundary domain vectors.

Definition 4. Let $\widetilde{I}^{\geq} = (U, AT, G)$ be an IFOIS, where $A_t \subseteq AT$ for all $t \in \{1, 2, \ldots, s\}$ and $U = \{x_1, x_2, \ldots, x_n\}$. For any $x_i \in U$, let $\alpha \in [0, 0.5)$, $\beta \in (0.5, 1]$, and let $A = A_1 \cup A_2 \cup \cdots \cup A_s$ be the granularity. Further, let $G_A^{POS}(X) = [g_A^{POS}(x_i)]_{n\times 1}$, $G_A^{BND}(X) = [g_A^{BND}(x_i)]_{n\times 1}$, and let $G_A^{NEG}(X) = [g_A^{NEG}(x_i)]_{n\times 1}$ represent the boundary domain and negative domain, respectively.

$$g_A^{POS}(x_i) = \begin{cases} 1, & If \quad H_{x_i}^{X}(A) \geq \beta \\ 0, & otherwise \end{cases},$$
$$g_A^{BND}(x_i) = \begin{cases} 1, & If \quad H_{x_i}^{X}(A) < \beta, L_{x_i}^{\sim X}(A) < \alpha \\ 0, & otherwise \end{cases}, \tag{15}$$
$$g_A^{NEG}(x_i) = \begin{cases} 1, & If \quad L_{x_i}^{\sim X}(A) \geq \beta \\ 0, & otherwise \end{cases}.$$

Here, x_i can be seen to belong to the positive domain by comparing its characteristic function $H_{x_i}^{X}(A)$ under attribute A with the parameter β. The positive domain vector $G_A^{POS}(X)$ is obtained from the functions $g_A^{POS}(x_i)(x_i \in U)$ for all objects. Similarly, we can calculate the boundary domain vector $G_A^{BND}(X)$ and the negative domain vector $G_A^{NEG}(X)$.

Example 2 (*Continuing Example 1*)**.** Based on Definition 4, we define $A = A_1 \cup A_2 \cup A_3 \cup A_4$. We can calculate the basic positive domain vector $G_A^{POS}(X)$, the basic boundary domain vector $G_A^{BND}(X)$, and the basic negative domain vector $G_A^{NEG}(X)$ under granularity A.

Let $\alpha = \frac{1}{5}, \beta = 1$, we can obtain

$$G_A^{POS}(X) = [0, 0, 0, 0, 0, 1, 0, 0, 0, 0],$$
$$G_A^{BND}(X) = [0, 0, 0, 0, 1, 0, 0, 1, 1, 0],$$
$$G_A^{NEG}(X) = [1, 1, 1, 1, 0, 0, 1, 0, 0, 1].$$

Let $\alpha = \frac{1}{4}, \beta = \frac{2}{3}$, we can obtain

$$G_A^{POS}(X) = [0, 0, 0, 0, 0, 1, 0, 0, 0, 0],$$
$$G_A^{BND}(X) = [0, 0, 0, 0, 1, 0, 0, 1, 1, 0],$$
$$G_A^{NEG}(X) = [1, 1, 1, 1, 0, 0, 1, 0, 0, 1].$$

Let $\alpha = \frac{1}{3}, \beta = \frac{1}{2}$, we can obtain

$$G_A^{POS}(X) = [0, 0, 0, 0, 1, 1, 0, 1, 1, 0],$$
$$G_A^{BND}(X) = [0, 0, 0, 0, 1, 0, 0, 1, 1, 0],$$
$$G_A^{NEG}(X) = [1, 1, 1, 1, 0, 0, 1, 0, 0, 1].$$

4. Theory of matrix approximation dynamic update multi-granulation neighborhood dominant rough sets

In an age characterized by the abundance of big data, information is constantly changing in a complex and interconnected manner. It is, therefore, crucial to promptly incorporate the latest information while eliminating redundant data. Matrix dynamic techniques, being a significant strategy in the field of data mining, are employed to effectively manage data updates. This section introduces dynamic matrix-based mechanisms to update the approximate values of multi-granulation neighborhood dominant rough sets when adding or deleting data structures.

As the number of objects in an IFOIS increases, the neighborhood dominant relation matrices need to be updated for each granularity structure. We first discuss the update mechanism for the neighborhood dominant relation matrix when objects are added.

Let $\widetilde{I}^{\geq} = (U, AT, G)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$, $A_t \subseteq AT$, $\forall t \in \{1, 2, \ldots, s\}$, and the neighborhood dominant relation matrix with respect to A_t is $M_{A_t}^{\delta\geq} = [m_{ij,A_t}^{\delta\geq}]_{n\times n}$. After the addition of n objects, the updated positive, negative, and boundary domain vectors are defined as follows:

$$G_A^{POS}(X) = [g_A^{POS}(x_1), g_A^{POS}(x_2), \ldots, g_A^{POS}(x_n), g_A^{POS}(x_{n+1'}), \ldots, g_A^{POS}(x_{n+n'})],$$
$$G_A^{BND}(X) = [g_A^{BND}(x_1), g_A^{BND}(x_2), \ldots, g_A^{BND}(x_n), g_A^{BND}(x_{n+1'}), \ldots, g_A^{BND}(x_{n+n'})],$$
$$G_A^{NEG}(X) = [g_A^{NEG}(x_1), g_A^{NEG}(x_2), \ldots, g_A^{NEG}(x_n), g_A^{NEG}(x_{n+1'}), \ldots, g_A^{NEG}(x_{n+n'})]. \tag{16}$$

In addition to objects being added to the IFOIS, there may also be cases in which some objects are removed.Removing objects also requires updating the neighborhood dominant relation matrices for each granularity structure. Next, we discuss the mechanism for performing these updates.

Definition 5. Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A_t \subseteq AT$, $\forall\ t \in \{1, 2, \ldots, s\}$. The neighborhood dominant relation matrix $M_{A_t}^{\delta\geq} = [m_{ij,A_t}^{\delta\geq}]_{n\times n}$ for A_t. After adding n objects, the new neighborhood dominant relation matrix $M_{A_t}^{\delta\geq'}$ for attribute A_t is defined as follows:

$$M_{A_t}^{\delta\geq'} = \left(\begin{array}{cccc|ccc} m_{11,A_t}^{\delta\geq} & m_{12,A_t}^{\delta\geq} & \cdots & m_{1,n,A_t}^{\delta\geq} & m_{1,n+1',A_t}^{\delta\geq'} & \cdots & m_{1,n+n',A_t}^{\delta\geq'} \\ m_{21,A_t}^{\delta\geq} & m_{22,A_t}^{\delta\geq} & \cdots & m_{2,n,A_t}^{\delta\geq} & m_{1\ 2,n+1',A_t}^{\delta\geq'} & \cdots & m_{2,n+n',A_t}^{\delta\geq'} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ m_{n,1,A_t}^{\delta\geq} & m_{n,2,A_t}^{\delta\geq} & \cdots & m_{n,n,A_t}^{\delta\geq} & m_{n',n+1',A_t}^{\delta\geq'} & \cdots & m_{n,n+n'\,A_t}^{\delta\geq'} \\ \hline m_{n+1',1,A_t}^{\delta\geq'} & m_{n+1',2,A_t}^{\delta\geq'} & \cdots & m_{n+1',n,A_t}^{\delta\geq'} & m_{n+1',n+1',A_t}^{\delta\geq'} & \cdots & m_{n+1',n+n',A_t}^{\delta\geq'} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ m_{n+n',1,A_t}^{\delta\geq'} & m_{n+n',2,A_t}^{\delta\geq'} & \cdots & m_{n+n',n,A_t}^{\delta\geq'} & m_{n+n',n+1',A_t}^{\delta\geq'} & \cdots & m_{n+n',n+n',A_t}^{\delta\geq'} \end{array}\right).$$

For all $i, j = \{1, 2, \ldots, n\}$, we can obtain

$$
\begin{aligned}
m^{\delta\geq'}_{i,n+j} &= \begin{cases} 1, & If \ \ x_{n+j} \in \widetilde{[x_i]}^{\delta\geq}_{A_t} \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n, j = 1, 2, \ldots, n', \\
m^{\delta\geq'}_{n+i,j} &= \begin{cases} 1, & If \ \ x_j \in \widetilde{[x_{n+i}]}^{\delta\geq}_{A_t} \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n', j = 1, 2, \ldots, n, \\
m^{\delta\geq'}_{n+i,n+j} &= \begin{cases} 1, & If \ \ x_{n+j} \in \widetilde{[x_{n+i}]}^{\delta\geq}_{A_t} \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n', j = 1, 2, \ldots, n'.
\end{aligned} \tag{17}
$$

As mentioned above,each granularity structure's neighborhood dominant relation matrix plays a fundamental role in calculating the multi-granulation neighborhood dominant rough sets. When adding objects, the key step in performing these calculations is to update each granularity's neighborhood dominant relation matrix. On the basis of the updated neighborhood dominant relation matrix, we can further discuss the inclusion relationship between the updated neighborhood classes and the target set. Therefore, the following theorem elaborates the dynamic update mechanism of the global characteristic matrix after adding objects.

Definition 6. Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A_t \subseteq AT$, $\forall\ t \in \{1, 2, \ldots, s\}$. After adding n objects, the domain of discourse becomes U^+, and the global characteristic matrix of U^+ is updated as follows:

$$
H_{U^+}(A_t) = \left(\begin{array}{ccccc}
h_{A_1}(x_1) & h_{A_2}(x_1) & h_{A_3}(x_1) & \ldots & h_{A_s}(x_1) \\
h_{A_1}(x_2) & h_{A_2}(x_2) & h_{A_3}(x_2) & \ldots & h_{A_s}(x_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h_{A_1}(x_n) & h_{A_2}(x_n) & h_{A_3}(x_n) & \ldots & h_{A_s}(x_n) \\
\hline
h_{A_1}(x_{n+1'}) & h_{A_2}(x_{n+1'}) & h_{A_3}(x_{n+1'}) & \ldots & h_{A_s}(x_{n+1'}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h_{A_1}(x_{n+n'}) & h_{A_2}(x_{n+n'}) & h_{A_3}(x_{n+n'}) & \ldots & h_{A_1}(x_{n+n'})
\end{array} \right),
$$

$$
L_{U^+}(A_t) = \left(\begin{array}{ccccc}
l_{A_1}(x_1) & l_{A_2}(x_1) & l_{A_3}(x_1) & \ldots & l_{A_s}(x_1) \\
l_{A_1}(x_2) & l_{A_2}(x_2) & l_{A_3}(x_2) & \ldots & l_{A_s}(x_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
l_{A_1}(x_n) & l_{A_2}(x_n) & l_{A_3}(x_n) & \ldots & h_{A_s}(x_n) \\
\hline
l_{A_1}(x_{n+1'}) & l_{A_2}(x_{n+1'}) & l_{A_3}(x_{n+1'}) & \ldots & l_{A_s}(x_{n+1'}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
l_{A_1}(x_{n+n'}) & l_{A_2}(x_{n+n'}) & l_{A_3}(x_{n+n'}) & \ldots & l_{A_1}(x_{n+n'})
\end{array} \right).
$$

For all $i, j = \{1, 2, \ldots, n\}$, we can obtain

$$
\begin{aligned}
h^X_{A_t}(x_{n+i}) &= \begin{cases} 1, & \widetilde{[x_{n+i}]}^{\delta\geq}_{A_t} \subseteq X \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n', t \in \{1, 2, \ldots, s\}, \\
l^{\sim X}_{A_t}(x_{n+i}) &= \begin{cases} 1, & \widetilde{[x_{n+i}]}^{\delta\geq}_{A_t} \cap X = \emptyset \\ 0, & \widetilde{[x_{n+i}]}^{\delta\geq}_{A_t} \cap X \neq \emptyset \end{cases} \quad i = 1, 2, \ldots, n', t \in \{1, 2, \ldots, s\}.
\end{aligned} \tag{18}
$$

Based on Definition 6, when adding a set of objects U^+, we can update the characteristic matrix directly without recalculating the previously retained results. Based on the updated characteristic matrix, we can calculate the basic positive domain vector $G^{POS}_A(X)$, the basic negative domain vector $G^{NEG}_A(X)$, and the basic boundary domain vector $G^{BND}_A(X)$ of the multi-granulation neighborhood dominant rough sets after adding objects.

Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A_t \subseteq AT$, $\forall\ t \in \{1, 2, \ldots, s\}$. After adding n objects, the updated basic positive, basic negative, and basic boundary domain vectors are as follows:

$$
\begin{aligned}
G^{POS}_A(X) &= [g^{POS}_A(x_1), g^{POS}_A(x_2), \ldots, g^{POS}_A(x_n), g^{POS}_A(x_{n+1'}), \ldots, g^{POS}_A(x_{n+n'})], \\
G^{BND}_A(X) &= [g^{BND}_A(x_1), g^{BND}_A(x_2), \ldots, g^{BND}_A(x_n), g^{BND}_A(x_{n+1'}), \ldots, g^{BND}_A(x_{n+n'})], \\
G^{NEG}_A(X) &= [g^{NEG}_A(x_1), g^{NEG}_A(x_2), \ldots, g^{NEG}_A(x_n), g^{NEG}_A(x_{n+1'}), \ldots, g^{NEG}_A(x_{n+n'})].
\end{aligned} \tag{19}
$$

In addition to adding objects, there may also be situations in which some objects are removed from the IFOIS.Removing objects also requires updating the neighborhood dominant relation matrices for each granularity structure. Next, we discuss the mechanism by which these updates are performed.

Definition 7. Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A_t \subseteq AT$, $\forall\ t \in \{1, 2, \ldots, s\}$. The neighborhood dominant relation matrix $M^{\delta\geq}_{A_t} = [m^{\delta\geq}_{ij,A_t}]_{n\times n}$ for A_t, after removing n objects, the new neighborhood dominant relation matrix $M^{\delta\geq'}_{A_t}$ for attribute A_t is defined as follows:

$$
M^{\delta\geq'}_{A_t} = \left(\begin{array}{ccc|ccc}
m^{\delta\geq}_{11,A_t} & \ldots & m^{\delta\geq}_{1,n-n'-1,A_t} & m^{\delta\geq'}_{1,n-n',A_t} & \ldots & m^{\delta\geq'}_{1,n,A_t} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
m^{\delta\geq}_{n-n'-1,1,A_t} & \ldots & m^{\delta\geq}_{n-n'-1,n-n'-1,A_t} & m^{\delta\geq'}_{n-n'-1,n-n',A_t} & \ldots & m^{\delta\geq'}_{n-n'-1,n,A_t} \\
\hline
m^{\delta\geq'}_{n-n',1,A_t} & \ldots & m^{\delta\geq'}_{n-n',n-n'-1,A_t} & m^{\delta\geq'}_{n-n',n-n',A_t} & \ldots & m^{\delta\geq'}_{n-n',n,A_t} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
m^{\delta\geq'}_{n,1,A_t} & \ldots & m^{\delta\geq'}_{n,n-n'-1,A_t} & m^{\delta\geq'}_{n,n-n',A_t} & \ldots & m^{\delta\geq'}_{n,n,A_t}
\end{array} \right).
$$

For all $i, j = \{1, 2, \ldots, n\}$, we can obtain

$$
\begin{aligned}
m^{\delta\geq'}_{i,n-j} &= \begin{cases} 1, & If \ \ x_{n-j} \in \widetilde{[x_i]}^{\delta\geq}_{A_t} \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n, j = 1, 2, \ldots, n', \\
m^{\delta\geq'}_{n-i,j} &= \begin{cases} 1, & If \ \ x_j \in \widetilde{[x_{n-i}]}^{\delta\geq}_{A_t} \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n', j = 1, 2, \ldots, n, \\
m^{\delta\geq'}_{n-i,n-j} &= \begin{cases} 1, & If \ \ x_{n-j} \in \widetilde{[x_{n-i}]}^{\delta\geq}_{A_t} \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n', j = 1, 2, \ldots, n'.
\end{aligned} \tag{20}
$$

As mentioned above, the neighborhood dominant relation matrix of each granular structure Definition 7 explains the changes in neighborhood dominant relations at each granularity structure in the matrix form when removing objects. To update the global characteristic matrix, we use the updated results for the neighborhood dominant relations in matrix form and their inclusion relationship with the target concept. Then, we obtain the global feature in matrix form when removing objects.

Definition 8. Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A_t \subseteq AT$, $\forall t \in \{1, 2, \ldots, s\}$. After removing n objects, the universe of discourse becomes U^-, and the updated global characteristic matrix of U^- is as follows:

$$
H_{U^-}(A_t) = \left(\begin{array}{ccccc}
h_{A_1}(x_1) & h_{A_2}(x_1) & h_{A_3}(x_1) & \ldots & h_{A_s}(x_1) \\
h_{A_1}(x_2) & h_{A_2}(x_2) & h_{A_3}(x_2) & \ldots & h_{A_s}(x_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h_{A_1}(x_{n-n'-1}) & h_{A_2}(x_{n-n'-1}) & h_{A_3}(x_{n-n'-1}) & \ldots & h_{A_s}(x_{n-n'-1}) \\
\hline
h_{A_1}(x_{n-n'}) & h_{A_2}(x_{n-n'}) & h_{A_3}(x_{n-n'}) & \ldots & h_{A_s}(x_{n-n'}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h_{A_1}(x_n) & h_{A_2}(x_n) & h_{A_3}(x_n) & \ldots & h_{A_1}(x_n)
\end{array} \right),
$$

$$
L_{U^-}(A_t) = \left(\begin{array}{ccccc}
l_{A_1}(x_1) & l_{A_2}(x_1) & l_{A_3}(x_1) & \ldots & l_{A_s}(x_1) \\
l_{A_1}(x_2) & l_{A_2}(x_2) & l_{A_3}(x_2) & \ldots & l_{A_s}(x_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
l_{A_1}(x_{n-n'-1}) & l_{A_2}(x_{n-n'-1}) & l_{A_3}(x_{n-n'-1}) & \ldots & l_{A_s}(x_{n-n'-1}) \\
\hline
l_{A_1}(x_{n-n'}) & l_{A_2}(x_{n-n'}) & l_{A_3}(x_{n-n'}) & \ldots & l_{A_s}(x_{n-n'}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
l_{A_1}(x_n) & l_{A_2}(x_n) & l_{A_3}(x_n) & \ldots & l_{A_1}(x_n)
\end{array} \right).
$$

For all $i, j = \{1, 2, \ldots, n\}$, we can obtain

$$h_{A_t}^{X}(x_{n+i}) = \begin{cases} 1, & \widetilde{[x_{n-i}]}_{A_t}^{\delta\geq} \subseteq X \\ 0, & otherwise \end{cases} \quad i = 1, 2, \ldots, n', t \in \{1, 2, \ldots, s\},$$

$$l_{A_t}^{\sim X}(x_{n+i}) = \begin{cases} 1, & \widetilde{[x_{n-i}]}_{A_t}^{\delta\geq} \cap X = \emptyset \\ 0, & \widetilde{[x_{n-i}]}_{A_t}^{\delta\geq} \cap X \neq \emptyset \end{cases} \quad i = 1, 2, \ldots, n', t \in \{1, 2, \ldots, s\}. \tag{21}$$

Definition 8 explains the dynamic update process for the global characteristic matrix when objects are removed from the set U^{-}. Based on the updated characteristic matrix, we can calculate the positive domain vector $G_A^{POS}(X)$, the negative domain vector $G_A^{NEG}(X)$, and the boundary domain vector $G_A^{BND}(X)$ of the multi-granulation neighborhood dominant rough set after removing objects.

Let $\widetilde{I}^{\geq} = (U, AT, F)$ be an IFOIS, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A_t \subseteq AT$, $\forall t \in \{1, 2, \ldots, s\}$. After removing n objects, the updated positive domain vector, negative domain vector, and boundary domain vector are as follows:

$$\begin{aligned} G_A^{POS}(X) &= [g_A^{POS}(x_1), g_A^{POS}(x_2), \ldots, g_A^{POS}(x_{n-n'-1}), g_A^{POS}(x_{n-n'}), \ldots, g_A^{POS}(x_n)], \\ G_A^{BND}(X) &= [g_A^{BND}(x_1), g_A^{BND}(x_2), \ldots, g_A^{BND}(x_{n-n'-1}), g_A^{BND}(x_{n-n'}), \ldots, g_A^{BND}(x_n)], \\ G_A^{NEG}(X) &= [g_A^{NEG}(x_1), g_A^{NEG}(x_2), \ldots, g_A^{NEG}(x_{n-n'-1}), g_A^{NEG}(x_{n-n'}), \ldots, g_A^{NEG}(x_n)]. \end{aligned} \tag{22}$$

5. Algorithm design

On the basis of the matrix dynamic update mechanism proposed in the previous section, this section discusses the design of dynamic algorithms for updating the neighborhood dominant rough set approximation from a matrix perspective when objects are added or removed. The tests were performed on a personal computer running Windows 11 with an AMD Ryzen 7 5800H 3.2 GHz processor and 16 GB of memory. All the algorithms were implemented using Python in the Anaconda Navigator environment, and the computation time was the primary evaluation metric.

5.1. Matrix static approximation set algorithm

Here we present the matrix static approximation algorithm, which plays two crucial roles. First, it establishes the groundwork and provides initial computation results for the subsequent matrix dynamic approximation update algorithm. Second, by employing the computation results of the matrix static approximation algorithm as a reference benchmark, the feasibility of the dynamic algorithm is demonstrated. Algorithm 1 represents a matrix static approach for calculating the approximation sets of the multi-granulation neighborhood dominant rough set. It consists of several parts: first, an IFOIS is input as the test system together with the target concept X. Steps 1–3 calculate the neighborhood dominant matrix of the IFOIS based on the neighborhood dominant relation. Steps 4–10 calculate the characteristic elements of a single object, and steps 11–12 provide the characteristic column vector of the object for all granularities. Steps 13–21 determine whether an object belongs to the negative, positive, or boundary domains. Step 22 returns the positive, negative, and boundary domain vectors of the neighborhood dominant rough set approximation when the object set remains unchanged.

5.2. Matrix dynamic approximate update algorithms

In Algorithms 2 and 3, we introduced a dynamic update mechanism for the multi-granulation neighborhood dominant rough set matrix framework. Algorithm 2 is for the scenario in which objects are added.

Algorithm 1: The matrix static algorithm of computing approximation sets for the neighborhood dominant multi-granulation rough set in an IFOIS.

Input: An IFOIS $\widetilde{I}^{\geq} = (U, AT, G)$, $A_t \subseteq AT (t = 1, \ldots, s)$, $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$.
Output: The positive domain, negative domain, and boundary domain of multi-granulation neighborhood dominant rough sets

1 **while** $t > 0$ **do**
2 | Computing neighborhood dominant matrix $M_{A_t}^{\delta\geq}$;
3 **end**
4 **while** $t > 0$ **do**
5 | **if** $\widetilde{[x_i]}_{A_t}^{\delta\geq} \subseteq X$ **then**
6 | | $h_{A_t}^{X}(x_i) = 1$;
7 | **else**
8 | | $l_{A_t}^{\sim X}(x_i) = 1$;
9 | **end**
10 **end**
11 $H_{x_i}(A_t) = [h_{A_1}(x_i), h_{A_2}(x_i), \ldots, h_{A_s}(x_i)]^T$,calculation $H_{x_i}^{X}(A)$;
12 $L_{x_i}(A_t) = [l_{A_1}(x_i), l_{A_2}(x_i), \ldots, l_{A_s}(x_i)]^T$,calculation$L_{x_i}^{\sim X}(A)$;
13 **while** $i > 0$ **do**
14 | **if** $H_{x_i}^{X}(A) \geq \beta$ **then**
15 | | $g_A^{POS}(x_i) = 1$;
16 | **else**
17 | | **if** $H_{x_i}^{X}(A) < \beta, L_{x_i}^{\sim X}(A) < \alpha$ **then**
18 | | | $g_A^{BND}(x_i) = 1$;
19 | | **else**
20 | | | $g_A^{NEG}(x_i) = 1$;
21 | | **end**
22 | **end**
23 **end**
24 **return** $G_A^{POS}(X)$,$G_A^{BND}(X)$,$G_A^{NEG}(X)$.

The input of Algorithm 2 includes an IFOIS, the neighborhood dominant relation matrix $M_{A_t}^{\delta\geq}$ before adding objects, the feature column vectors $H_{x_i}(A_t)$ and $L_{x_i}(A_t)$, the threshold values α and β, and the added objects $U^{+} = \{x_i | i = n+1, n+2, \ldots, n+n'\}$. The algorithm's output comprises the updated positive, negative, and boundary domains of the multi-granulation neighborhood dominant rough set. Steps 1–3 analyze the target set X after adding objects. Steps 4–23 calculate the elements of the neighborhood dominant matrix after adding objects. Step 24 calculates the updated neighborhood dominant relation matrix $M_{A_t}^{\delta\geq'}$, and steps 25–34 calculate the global feature matrices $H_{U^{+}}(A_t)$ and $L_{U^{+}}(A_t)$ based on the updated neighborhood dominant relation matrix $M_{A_t}^{\delta\geq'}$ and the target concept X. Step 35 determines the positive, negative, and boundary domains based on the global feature matrices and the threshold values α and β. Finally, step 36 returns the updated positive, negative, and boundary domains of the multi-granulation neighborhood dominant rough set.

Algorithm 3 is a dynamic algorithm to compute approximation sets when objects are removed. The input includes an IFOIS, the target concept X, the neighborhood dominant relation matrix $M_{A_t}^{\delta\geq}$ before removing objects, the threshold values α and β, and the removed objects $U^{-} = \{x_i | i = n - n' + 1, n - n' + 2, \ldots, n\}$. The output consists of the updated positive, negative, and boundary domains of the multi-granulation neighborhood dominant rough set. Steps 1–3 calculate the changes in target concept X after removing objects. Steps 4–12 update the neighborhood dominant relation matrix $M_{A_t}^{\delta\geq'}$ after removing objects. In steps 13–21, we update the global feature

Algorithm 2: Matrix dynamic algorithm of computing approximation sets for adding objects to the multi-granulation neighborhood dominant rough set in an IFOIS.

Input: An IFOIS $\widetilde{I}^{\geq} = (U, AT, F)$, $A_t \in AT (t = 1, \dots, s)$, $U^{+} = \{ x_i \mid i = n+1, n+2, \dots, n+n' \}$,target concept set $X \subseteq U$, neighborhood dominant relation matrix $M_{A_t}^{\delta\geq}$, threshold α, β, feature column vector $H_{x_i}(A_t)$, $L_{x_i}(A_t)$

Output: The updated positive domain, negative domain, and boundary domain of the multi-granulation neighborhood dominant rough set.

Add objects $U^{+} = \{ x_i \mid i = n+1, n+2, \dots, n+n' \}$;
if $X^{+} \subseteq U^{+}$ **then**
  $X' = X \cup X^{+}$;
end
while $t > 0$ **do**
  while $i > 0$ **do**
    while $j \geq n+1$ **do**
      if $x_j \in \widetilde{[x_i]}_{A_t}^{\delta\geq}$ **then**
        $m_{ij,A_t}^{\delta\geq} = 1$;
      else
        $m_{ij,A_t}^{\delta\geq} = 0$;
      end
    end
  end
  while $i \geq n+1$ **do**
    while $j > 0$ **do**
      if $x_j \in \widetilde{[x_i]}_{A_t}^{\delta\geq}$ **then**
        $m_{ij,A_t}^{\delta\geq} = 1$;
      else
        $m_{ij,A_t}^{\delta\geq} = 0$;
      end
    end
  end
end
Update neighborhood dominant relation matrix $M_{A_t}^{\delta\geq'}$;
Update global feature matrix updated for U^{+}: $H_{U^{+}}(A_t)$ and $L_{U^{+}}(A_t)$;
Return to steps 13-21 in Algorithm 1, and calculate the updated $G_A^{POS}(X)$, $G_A^{BND}(X)$, $G_A^{NEG}(X)$;
return $G_A^{POS}(X)$, $G_A^{BND}(X)$, $G_A^{NEG}(X)$.

Algorithm 3: Matrix dynamic algorithm of computing approximation sets for removing objects from the multi-granulation neighborhood dominant rough set in an IFOIS.

Input: An IFOIS $\widetilde{I}^{\geq} = (U, AT, F)$, $U^{-} = \{ x_i \mid i = n-n'+1, n-n'+2, \dots, n \}$, $A_t \in AT (t = 1, \dots, s)$, target concept set $X \subseteq U$, neighborhood dominant relation matrix $M_{A_t}^{\delta\geq}$, threshold α, β, feature column vector $H_{x_i}(A_t)$, $L_{x_i}(A_t)$

Output: Updated positive domain, negative domain, and boundary domain of the multi-granulation neighborhood dominant rough set.

1 Removing objects $U^{-} = \{ x_i \mid i = n-n'+1, n-n'+2, \dots, n \}$;
2 **if** $X^{-} \subseteq U^{-}$ **then**
3   $X' = X - X^{+}$;
4 **end**
5 **while** $t > 0$ **do**
6   **while** $n - n' \geq i > 0$ **do**
7   **while** $n - n' \geq i = j > 0$ **do**
8       $m_{ij,A_t}^{\delta\geq'} = m_{ij,A_t}^{\delta\geq}$;
9       $m_{ji,A_t}^{\delta\geq'} = m_{ji,A_t}^{\delta\geq}$;
10   **end**
11   **end**
12 **end**
13 Update the neighborhood dominant relation matrix $M_{A_t}^{\delta\geq'}$;
14 **while** $i > 0$ **do**
15   **while** $t > 0$ **do**
16   **if** $\widetilde{[x_i]}_{A_t}^{\delta\geq'} \subseteq X'$ **then**
17       $h_{A_t}^{X}(x_i) = 1$;
18   **end**
19   **else if** $\widetilde{[x_i]}_{A_t}^{\delta\geq'} \cap X' = \emptyset$ **then**
20       $l_{A_t}^{\sim X}(x_i) = 1$;
21   **end**
22   **end**
23 **end**
24 Update the global feature matrix for U^{+}: $H_{U^{-}}(A_t)$ and $L_{U^{-}}(A_t)$;
25 Return to steps 13-21 in Algorithm 1, and calculate the updated $G_A^{POS}(X)$, $G_A^{BND}(X)$, $G_A^{NEG}(X)$;
26 **return** $G_A^{POS}(X)$, $G_A^{BND}(X)$, $G_A^{NEG}(X)$.

matrices $H_{U^{-}}(A_t)$ and $L_{U^{-}}(A_t)$ after removing objects U^{-}. Then step 23 calculates the updated positive, negative, and boundary domains. Step 24 returns the updated positive, negative, and boundary domains of the multi-granulation neighborhood dominant rough set.

6. Experiments and analysis

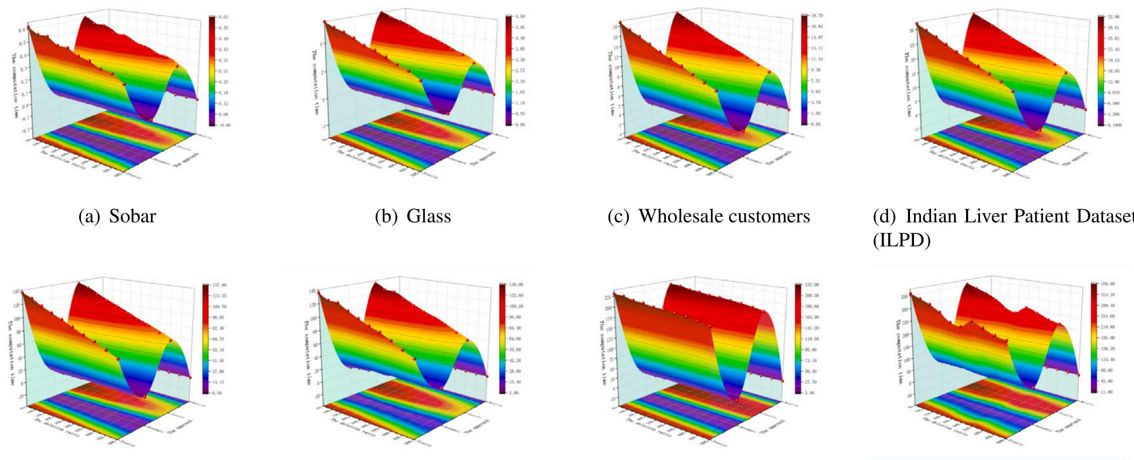
In this section, we describe a series of experiments that were conducted to validate the proposed matrix dynamic algorithm's computational performance against general dynamic and static methods. We evaluated its efficiency on the eight datasets shown in Table 3. During the experiments, we compared the computational performance during the addition and removal of objects in the approximate updating process.

6.1. Comparative analysis of methods

Our study is grounded in intuitionistic fuzzy datasets, which we constructed by normalizing the numerical values in the downloaded UCI datasets. Moreover, considering the total number of objects, we partitioned each dataset into two equal segments. One portion remained unaltered, preserving the original dataset, while the other was further divided into ten parts and treated as the added or removed datasets. We selected the first decision class as the target concept. To ensure generality, we chose two pairs of parameters, ($\alpha = 0.5, \beta = 0.2$) and ($\alpha = 0.6, \beta = 0.25$). Moreover, for ease of computation and understanding, we selected attribute sets $A_1 = \{b_1, b_2\}$, $A_2 = \{b_2, b_3\}$, and $A_3 = \{b_3, b_4\}$.

Table 3
Summary of the experimental datasets.

Data set	Samples	Attributes	Classes
Sobar	72	19	2
Glass	214	9	6
Wholesale customers	440	3	6
Indian Liver Patient Dataset	583	9	2
Banknote authentication	1372	4	2
Wireless	2000	7	4
Customer Churn	3334	10	2
Page blocks	5473	10	5



(a) Sobar (b) Glass (c) Wholesale customers (d) Indian Liver Patient Dataset (ILPD)

(e) Banknote authentication (f) Wireless (g) Customer Churn (h) Page blocks

Fig. 3. The calculation chart of our proposed when deleting a certain proportion of objects for various experimental datasets ($\alpha = 0.5, \beta = 0.2$).

The final experimental results are shown in Tables 4–7, where the computation time is measured in seconds. "Static," "Dynamic," and "Matrix" refer to the static, dynamic, and matrix dynamic algorithms, respectively. "Ratio" represents the ratio of removed or added objects to initial objects. The experimental section includes time comparisons for dynamic updates under two threshold values, α and β. To better compare the effect of object removal between the matrix dynamic, general dynamic, and classical methods, we plotted three-dimensional surface graphs based on Tables 4 and 5. Detailed information is shown in Figs. 2 and 3. In the subgraph depicted in Fig. 2, the x-axis corresponds to the ratio of removed objects to initial objects. The y-axis represents the static, dynamic, and matrix dynamic methods, while the z-axis illustrates the computation time for these three methods. As the number of removed objects increases, the classical method becomes time-consuming but shows a decreasing trend. The dynamic method exhibits a notable reduction in computation time compared with the classical method, and the time variation remains minimal as the number of removed objects increases. There is no significant difference between the general dynamic and matrix dynamic methods when computing small datasets. However, for larger datasets, the time cost for the general dynamic method increases significantly, while the rate of increase for the matrix dynamic method remains stable. Hence, the cardinality of the object set within the dataset emerges as the crucial factor influencing the performance of the three methods. Notably, the matrix dynamic method is more efficient than the general dynamic and static methods for computations involving large datasets.

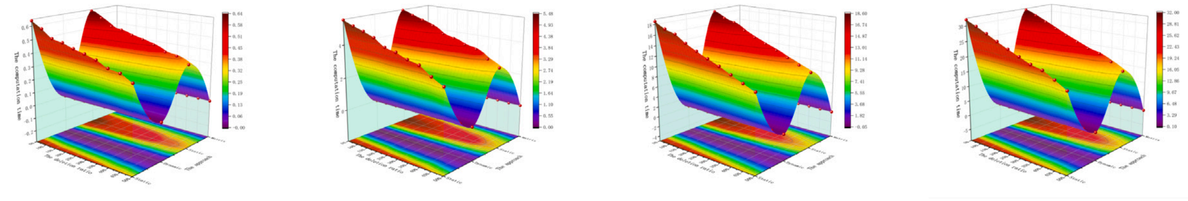
Tables 6 and 7 show a comparison of the time taken by the matrix dynamic, general dynamic, and static methods during object addition in an IFOIS. We divided each dataset into two parts. One was used as the initial dataset, while the other part served as the added dataset. The original object set of the added dataset is denoted as K. The added dataset was split into ten parts, which were added to the initial dataset sequentially, up to a maximum of ten times. The experimental results are shown in Figs. 4 and 5, where we have also plotted the computation time as a three-dimensional surface. The x-axis corresponds to the ratio of added objects to the initial dataset, the y-axis represents the static, dynamic, and matrix dynamic methods, and the z-axis corresponds to the computation time. As the number of added objects increases, it is evident from the results in the figures that the computation time for each method increases. Still, the matrix dynamic method maintains a significant advantage in terms of time cost. By comparing the computation time for each dataset, we conclude that the larger the object set in the dataset, the longer the time required to update the approximate set, and the matrix dynamic method performs better in object addition (see Fig. 6).

6.2. Comparative analysis of algorithms

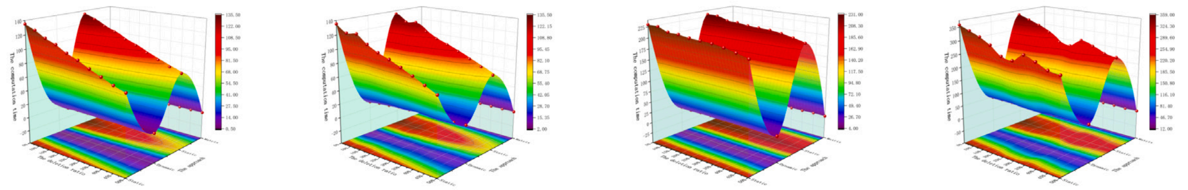
In this section, we describe comparative experiments conducted using the same eight datasets shown in Table 3. First, for the algorithms that involve adding new objects, we selected 50% of the datasets as the base set and then sequentially added the remaining objects in five iterations, with each iteration adding 10% of the objects to the base set. Similarly, we used all the datasets as the base set for the algorithms involving the removal of objects. We performed five iterations of object removal, with each iteration randomly removing 10% of the objects. Finally, we compared this algorithm with the other four to assess its effectiveness. All times are measured in seconds. We used $\alpha = 0.5$ and $\beta = 0.2$ because the GMNDRS exhibited a higher computational time than the other threshold groups on most datasets. The details of the four selected comparative algorithms are as follows:

(1) Infinite feature selection (INF-FS) [37] is a feature selection framework that considers the feature selection problem as a regularization problem, in which features are represented as nodes in a weighted fully connected graph and selected based on the length of paths.
(2) Hybrid kernel-based fuzzy complementary mutual information (HKCMI) [38] is a novel approach to unsupervised mixed-attribute reduction that uses a hybrid kernel function to define a fuzzy complementary entropy.
(3) The k-nearest neighborhood conditional mutual information method (KNCMI) [39] is a new feature selection method that effectively integrates the advantages of k-neighborhood and k-nearest neighbors while considering both heterogeneous data and feature interaction.
(4) Weighted dominance-based neighborhood conditional entropy (WDNCE) [40] is a weighted dominance-based neighborhood rough set method that assigns different weights to conditional attributes and evaluates attribute significance using a matrix-based conditional entropy. Moreover, heuristic algorithms and corresponding incremental mechanisms are introduced based on entropy to handle object addition.

Tables 8–11 compare the approximate update times for the GMNDRS method with INF-FS, KNCMI, HKCMI, and WDNCE when deleting objects at different proportions (10%, 20%, 30%, 40%, 50%). Similarly, Tables 12–15 compare the approximate update times when adding

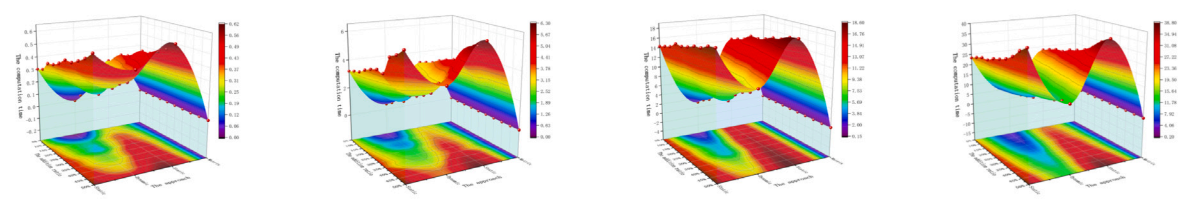


(a) Sobar (b) Glass (c) Wholesale customers (d) Indian Liver Patient Dataset

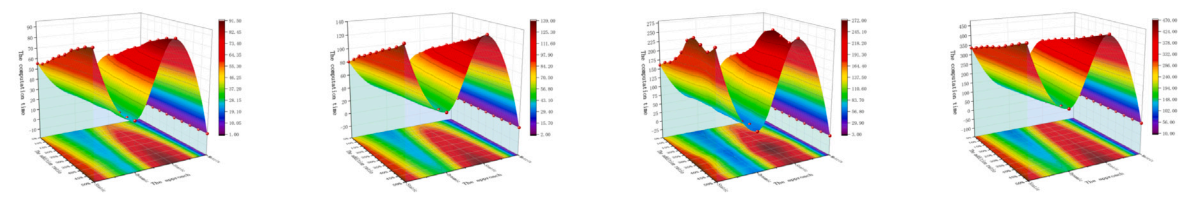


(e) Banknote authentication (f) Wireless (g) Customer Churn (h) Page blocks

Fig. 4. The calculation chart of our proposed when deleting a certain proportion of objects for various experimental datasets ($\alpha = 0.6, \beta = 0.25$).

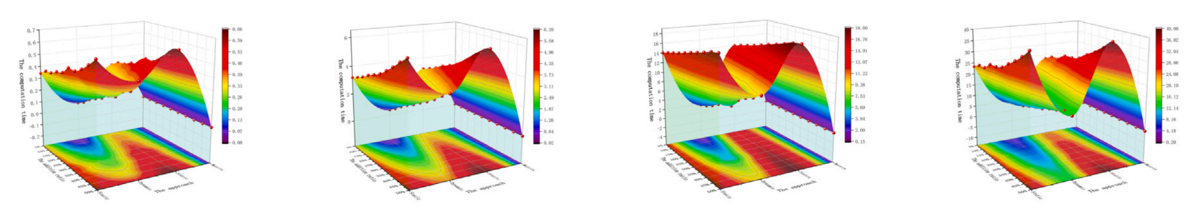


(a) Sobar (b) Glass (c) Wholesale customers (d) Indian Liver Patient Dataset

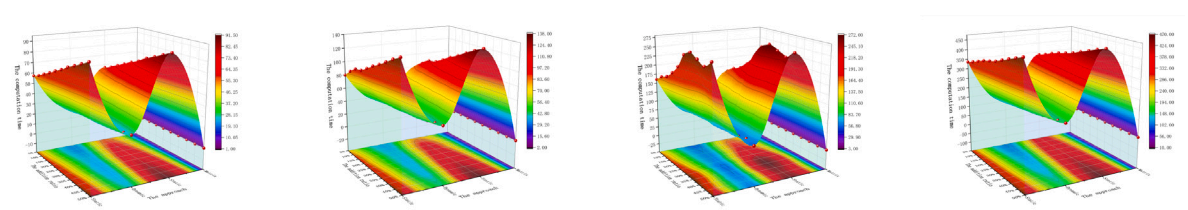


(e) Banknote authentication (f) Wireless (g) Customer Churn (h) Page blocks

Fig. 5. The calculation chart of our proposed when adding a certain proportion of objects for various experimental datasets ($\alpha = 0.5, \beta = 0.2$).



(a) Sobar (b) Glass (c) Wholesale customers (d) Indian Liver Patient Dataset



(e) Banknote authentication (f) Wireless (g) Customer Churn (h) Page blocks

Fig. 6. The calculation chart of our proposed when adding a certain proportion of objects for various experimental datasets ($\alpha = 0.6, \beta = 0.25$).

Table 4
Computational time of algorithms for deleting a certain proportion of objects in GMNDRS ($\alpha = 0.5, \beta = 0.2$).

Ratio	Method	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
	Static	0.616	0.571	0.569	0.517	0.512	0.469	0.449	0.410	0.396	0.342
S	General dynamic	0.0	0.0	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.003
	Matrix dynamic	0.0	0.0	0.001	0.017	0.018	0.0	0.017	0.002	0.005	0.006
	Static	5.498	5.216	5.020	4.744	4.389	4.186	3.909	3.574	3.331	2.909
G	General dynamic	0.0	0.002	0.002	0.019	0.025	0.030	0.019	0.036	0.037	0.044
	Matrix dynamic	0.024	0.033	0.018	0.019	0.018	0.028	0.018	0.037	0.018	0.026
	Static	18.692	17.601	16.673	15.591	14.533	13.625	12.736	11.704	10.710	9.793
Wc	General dynamic	0.132	0.199	0.140	0.098	0.163	0.126	0.117	0.159	0.164	0.108
	Matrix dynamic	0.087	0.088	0.100	0.088	0.098	0.096	0.092	0.105	0.090	0.086
	Static	31.974	30.328	28.704	26.886	25.152	23.594	21.992	20.424	18.743	16.677
ILPD	General dynamic	0.395	0.353	0.298	0.326	0.318	0.259	0.196	0.293	0.315	0.401
	Matrix dynamic	0.174	0.162	0.163	0.154	0.157	0.154	0.152	0.162	0.159	0.192
	Static	136.646	130.964	122.617	115.277	108.859	101.302	93.911	88.059	80.598	72.518
Ba	General dynamic	4.952	3.361	4.206	4.040	4.989	3.901	3.597	3.845	3.471	3.830
	Matrix dynamic	0.844	0.857	0.850	0.830	0.828	0.860	0.831	0.846	0.857	0.860
	Static	135.955	127.555	126.186	117.444	109.618	102.443	96.643	88.695	80.279	72.330
W	General dynamic	8.212	11.216	12.036	14.502	13.711	15.652	14.252	14.450	13.945	15.920
	Matrix dynamic	2.159	2.496	2.171	2.259	2.303	2.311	2.517	2.290	2.270	2.289
	Static	231.218	229.539	223.815	221.109	217.003	215.412	212.051	208.761	204.296	198.467
CC	General dynamic	13.445	12.602	13.296	14.031	13.002	13.463	13.519	13.288	12.433	11.104
	Matrix dynamic	5.410	5.032	5.116	5.252	5.454	5.235	5.700	5.180	5.316	5.579
	Static	358.095	344.173	329.658	307.653	277.117	269.305	297.084	281.352	263.891	255.296
Pb	General dynamic	46.781	47.906	48.074	48.434	45.155	43.226	43.150	45.118	44.585	45.027
	Matrix dynamic	12.582	13.163	12.091	11.664	12.295	12.746	12.095	13.015	12.421	12.090

Table 5
Computational time of algorithms for deleting a certain proportion of objects in GMNDRS ($\alpha = 0.6, \beta = 0.25$).

Ratio	Method	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
	Static	0.640	0.593	0.537	0.522	0.495	0.474	0.450	0.410	0.396	0.350
S	General dynamic	0.0	0.0	0.0	0.001	0.001	0.002	0.002	0.001	0.001	0.003
	Matrix dynamic	0.0	0.0	0.002	0.003	0.005	0.005	0.005	0.004	0.004	0.003
	Static	5.462	5.215	5.012	4.756	4.340	4.183	3.916	3.751	3.284	2.910
G	General dynamic	0.0	0.002	0.002	0.014	0.025	0.033	0.019	0.034	0.035	0.042
	Matrix dynamic	0.021	0.023	0.018	0.019	0.018	0.024	0.019	0.033	0.020	0.026
	Static	18.551	17.470	16.873	15.581	14.533	13.625	12.736	11.704	10.710	9.780
Wc	General dynamic	0.130	0.198	0.138	0.095	0.165	0.122	0.121	0.160	0.162	0.110
	Matrix dynamic	0.083	0.078	0.001	0.015	0.068	0.093	0.092	0.104	0.090	0.094
	Static	32.000	30.415	28.710	26.880	25.160	23.580	22.001	21.013	19.012	16.750
ILPD	General dynamic	0.395	0.349	0.290	0.330	0.312	0.264	0.231	0.312	0.297	0.311
	Matrix dynamic	0.168	0.162	0.163	0.149	0.167	0.149	0.155	0.166	0.159	0.155
	Static	135.131	128.934	120.683	114.710	106.210	100.003	92.051	85.019	78.046	72.518
Ba	General dynamic	4.829	3.318	4.209	4.039	4.089	3.513	3.759	3.335	3.174	3.380
	Matrix dynamic	0.804	0.843	0.816	0.830	0.828	0.812	0.839	0.836	0.829	0.858
	Static	135.315	125.018	126.495	115.058	107.006	101.030	94.034	86.345	78.279	70.083
W	General dynamic	8.833	10.941	12.161	13.882	14.091	15.553	14.612	13.230	13.813	14.782
	Matrix dynamic	2.161	2.367	2.217	2.224	2.343	2.309	2.357	2.303	2.210	2.231
	Static	230.113	229.443	221.024	220.199	216.763	215.414	213.937	210.115	204.513	197.259
CC	General dynamic	13.430	12.790	13.487	13.893	13.114	13.706	13.232	13.281	12.844	11.964
	Matrix dynamic	5.410	5.032	5.030	5.335	5.655	5.625	5.024	5.082	5.732	5.179
	Static	358.547	343.718	328.189	306.293	274.273	267.371	295.331	280.305	260.095	254.007
Pb	General dynamic	46.880	46.995	47.314	46.309	45.045	43.894	43.225	45.090	44.310	45.620
	Matrix dynamic	12.418	12.403	12.916	12.568	12.593	12.077	12.697	13.254	12.031	12.882

objects. The unit of time consumption is seconds. The eight tables show that the proposed GMNDRS algorithm requires less time in most datasets. These time results reflect the effectiveness of the GMNDRS and the matrix dynamic update mechanism.

To facilitate a more concise and intuitive comparison between GMNDRS and the other four algorithms, we have generated a 3D bar chart based on the data in Tables 8–15. Further details can be found in Figs. 7 and 8. In one subplot of Fig. 7, the x-axis represents the proportion of removed objects, taking values from 10% to 50% with a step of 10%, and on the z-axis are GMNDRS and the four comparison algorithms sorted in order of increasing approximate update times, from left to right. The y-axis displays the computation time for these five algorithms. Figs. 7 and 8 provide a visual representation highlighting the effectiveness of the GMNDRS dynamic update algorithm.

7. Conclusions and future work

With the rapid development of technology, dynamic changes in data have led to increased time consumption in approximate updates. To address this challenge, GMNDRS offers an efficient approach by leveraging multiple levels and reducing redundant computation. Obtaining the most recent knowledge, built on previous knowledge, renders the dynamically updating approximations set remarkably efficient in a time-evolving information system. This paper discussed the basic concepts of generalized multi-granulation neighborhood dominant

Table 6
Computational time of algorithms for adding a certain proportion of objects in GMNDRS ($\alpha = 0.5, \beta = 0.2$).

Ratio	Method	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
S	Static	0.301	0.323	0.389	0.402	0.433	0.477	0.480	0.511	0.549	0.602
	General dynamic	0.000	0.073	0.170	0.169	0.159	0.245	0.257	0.314	0.349	0.460
	Matrix dynamic	0.003	0.004	0.005	0.006	0.006	0.005	0.004	0.002	0.004	0.005
G	Static	3.192	3.394	3.547	3.875	4.126	4.322	4.254	5.451	5.757	6.218
	General dynamic	0.380	0.728	1.362	1.328	1.968	2.182	2.483	2.854	3.710	3.773
	Matrix dynamic	0.028	0.025	0.025	0.025	0.017	0.020	0.022	0.020	0.020	0.022
Wc	Static	14.276	14.759	15.183	16.246	16.182	16.585	17.211	17.415	18.170	18.410
	General dynamic	1.122	2.358	3.219	4.515	4.820	5.137	7.063	8.261	8.967	10.284
	Matrix dynamic	0.168	0.170	0.174	0.174	0.175	0.166	0.168	0.168	0.162	0.178
ILPD	Static	23.281	24.750	25.921	26.130	28.004	29.010	30.507	32.801	36.265	38.704
	General dynamic	3.908	4.490	5.162	6.290	7.357	9.903	10.577	11.315	12.587	12.988
	Matrix dynamic	0.218	0.229	0.223	0.222	0.217	0.226	0.225	0.226	0.223	0.225
Ba	Static	55.203	57.450	62.033	67.715	72.097	75.898	81.955	85.414	88.990	91.105
	General dynamic	21.256	21.805	22.761	23.156	23.801	24.010	24.993	25.898	25.298	26.484
	Matrix dynamic	1.315	1.256	1.279	1.344	1.325	1.320	1.380	1.373	1.364	1.383
W	Static	80.042	88.014	96.330	101.224	107.794	113.261	119.926	124.940	130.736	138.690
	General dynamic	31.903	32.666	34.150	35.587	36.924	37.525	38.487	39.685	41.808	43.730
	Matrix dynamic	2.104	2.180	2.182	2.212	2.209	2.130	2.133	2.280	2.219	2.178
CC	Static	160.308	175.082	183.475	200.925	223.270	259.180	271.310	261.074	250.684	268.189
	General dynamic	50.090	55.078	57.017	57.980	59.700	53.974	56.074	57.062	61.989	55.648
	Matrix dynamic	3.946	4.770	3.996	4.080	4.039	4.076	3.890	3.780	3.557	3.914
Pb	Static	335.452	349.757	365.311	380.190	391.120	402.805	420.070	436.729	453.292	469.064
	General dynamic	99.830	102.904	109.037	114.702	119.910	126.322	130.421	139.202	145.460	151.900
	Matrix dynamic	11.651	11.501	11.591	10.643	11.905	11.313	11.577	11.949	11.778	11.846

Table 7
Computational time of algorithms for adding a certain proportion of objects in GMNDRS ($\alpha = 0.6, \beta = 0.25$).

Ratio	Method	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
S	Static	0.342	0.383	0.392	0.423	0.440	0.498	0.496	0.531	0.584	0.644
	General dynamic	0.041	0.075	0.126	0.162	0.195	0.241	0.260	0.326	0.355	0.456
	Matrix dynamic	0.005	0.005	0.006	0.006	0.006	0.005	0.004	0.001	0.004	0.005
G	Static	3.183	3.403	3.637	3.951	4.178	4.437	4.816	5.231	5.694	6.110
	General dynamic	0.398	0.743	1.125	1.481	1.768	2.137	2.473	2.854	3.310	3.670
	Matrix dynamic	0.028	0.025	0.025	0.025	0.027	0.022	0.025	0.023	0.022	0.024
Wc	Static	14.051	14.570	15.013	15.463	15.988	16.544	17.031	17.561	18.149	18.421
	General dynamic	1.681	2.238	3.010	4.015	4.930	5.178	7.006	8.122	8.908	10.014
	Matrix dynamic	0.168	0.174	0.171	0.174	0.175	0.166	0.163	0.168	0.167	0.170
ILPD	Static	23.322	25.137	25.282	27.836	28.074	29.039	31.568	33.799	36.515	39.903
	General dynamic	3.845	4.980	5.583	6.950	8.117	9.139	10.735	11.810	13.005	12.665
	Matrix dynamic	0.220	0.230	0.227	0.225	0.227	0.230	0.219	0.226	0.223	0.228
Ba	Static	57.202	60.125	64.062	67.751	71.812	75.325	79.874	83.359	87.190	91.105
	General dynamic	20.350	20.095	21.060	22.542	22.831	23.020	23.390	24.033	25.093	26.648
	Matrix dynamic	1.328	1.315	1.372	1.344	1.405	1.326	1.397	1.352	1.359	1.370
W	Static	79.712	89.322	96.695	98.679	105.098	111.959	117.922	124.821	131.736	137.568
	General dynamic	33.153	33.810	35.426	36.475	36.315	37.730	38.411	39.159	41.832	43.805
	Matrix dynamic	2.202	2.240	2.266	2.129	2.132	2.129	2.240	2.107	2.132	2.093
CC	Static	162.440	177.565	189.742	203.956	230.137	265.910	270.505	263.098	255.191	273.163
	General dynamic	54.133	56.917	55.212	57.036	58.109	53.006	54.890	58.083	60.025	53.344
	Matrix dynamic	4.050	4.057	3.913	4.091	4.131	4.159	4.140	3.982	4.078	4.294
Pb	Static	338.082	351.791	364.417	380.050	392.178	405.130	422.073	439.129	456.015	473.755
	General dynamic	98.730	102.650	108.047	115.595	120.480	126.700	133.016	139.220	145.312	152.436
	Matrix dynamic	10.601	11.305	11.669	10.803	11.075	12.013	11.406	11.746	11.527	11.830

Table 8
Computation time of different algorithms for deleting a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Datasets	10%		20%		30%		40%		50%	
	GMNDRS	INF-FS	GMNDRS	INF-FS	GMNDRS	INF-FS	GMNDRS	INF-FS	GMNDRS	INF-FS
S	0	0.375	0.017	0.394	0	0.403	0.002	0.429	0.006	0.583
G	0.033	0.119	0.019	0.098	0.028	0.173	0.037	0.24	0.026	0.218
Wc	0.088	0.324	0.088	0.352	0.096	0.337	0.105	0.2667	0.086	0.283
ILPD	0.162	0.2582	0.154	0.216	0.154	0.246	0.162	0.219	0.192	0.221
Ba	0.357	0.241	0.43	0.221	0.36	0.189	0.426	0.198	0.396	0.236
W	2.496	2.398	2.259	2.927	2.311	2.872	2.29	2.205	2.289	2.617
CC	5.032	5.678	5.252	5.768	5.235	6.426	5.18	6.503	5.579	6.883
Pb	13.163	13.187	11.664	14.278	12.746	12.293	13.015	14.294	12.09	13.498

Table 9
Computation time of different algorithms for deleting a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

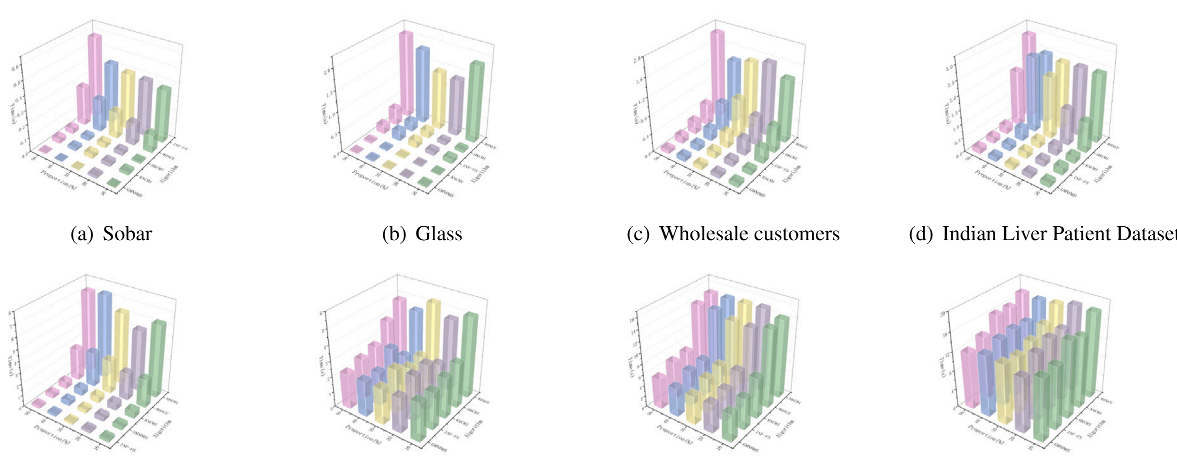
Datasets	10%		20%		30%		40%		50%	
	GMNDRS	KNCMI	GMNDRS	KNCMI	GMNDRS	KNCMI	GMNDRS	KNCMI	GMNDRS	KNCMI
S	0	0.021	0.017	0.026	0	0.034	0.002	0.023	0.006	0.036
G	0.033	0.023	0.019	0.027	0.028	0.042	0.037	0.04	0.026	0.048
Wc	0.088	0.101	0.088	0.132	0.096	0.157	0.105	0.136	0.086	0.163
ILPD	0.162	0.242	0.154	0.335	0.154	0.272	0.162	0.448	0.192	0.297
Ba	0.357	0.591	0.43	0.630	0.36	0.587	0.426	0.612	0.396	0.608
W	2.496	2.63	2.259	3.124	2.311	2.35	2.29	3.124	2.289	2.863
CC	5.032	7.387	5.252	7.294	5.235	6.982	5.18	6.695	5.579	7.092
Pb	13.163	15.109	11.664	15.284	12.746	14.982	13.015	14.294	12.09	16.233

Table 10
Computation time of different algorithms for deleting a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Datasets	10%		20%		30%		40%		50%	
	GMNDRS	HKCMI	GMNDRS	HKCMI	GMNDRS	HKCMI	GMNDRS	HKCMI	GMNDRS	HKCMI
S	0	0.011	0.017	0.03	0	0.036	0.002	0.027	0.006	0.04
G	0.033	0.068	0.019	0.092	0.028	0.178	0.037	0.239	0.026	0.41
Wc	0.088	0.571	0.088	0.63	0.096	0.873	0.105	0.638	0.086	0.472
ILPD	0.162	1.087	0.154	1.2821	0.154	2.198	0.162	2.671	0.192	1.972
Ba	0.357	6.298	0.43	5.293	0.36	6.293	0.426	7.342	0.396	7.193
W	2.496	2.879	2.259	2.382	2.311	2.471	2.29	2.084	2.289	3.983
CC	5.032	14.245	5.252	15.134	5.235	15.031	5.18	14.928	5.579	14.963
Pb	13.163	16.234	11.664	15.294	12.746	13.435	13.015	14.602	12.09	16.293

Table 11
Computation time of different algorithms for deleting a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Datasets	10%		20%		30%		40%		50%	
	GMNDRS	WDNCE	GMNDRS	WDNCE	GMNDRS	WDNCE	GMNDRS	WDNCE	GMNDRS	WDNCE
S	0	0.125	0.017	0.157	0	0.19	0.002	0.231	0.006	0.279
G	0.033	1.872	0.019	1.382	0.028	1.423	0.037	1.83	0.026	2.08
Wc	0.088	1.372	0.088	1.583	0.096	1.489	0.105	1.393	0.086	1.872
ILPD	0.162	2.414	0.154	2.418	0.154	2.3981	0.162	2.491	0.192	2.984
Ba	0.357	2.452	0.43	2.376	0.36	2.9163	0.426	3.0291	0.396	2.7831
W	2.496	5.184	2.259	4.652	2.311	5.33	2.29	4.483	2.289	4.835
CC	5.032	14.138	5.252	13.235	5.235	13.392	5.18	14.284	5.579	14.284
Pb	13.163	18.235	11.664	18.421	12.746	17.583	13.015	17.284	12.09	17.683



(a) Sobar (b) Glass (c) Wholesale customers (d) Indian Liver Patient Dataset

(e) Banknote authentication (f) Wireless (g) Customer Churn (h) Page blocks

Fig. 7. The computation time of different algorithms when deleting a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Table 12
Computation time of different algorithms for adding a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Datasets	10%		20%		30%		40%		50%	
	GMNDRS	INF-FS	GMNDRS	INF-FS	GMNDRS	INF-FS	GMNDRS	INF-FS	GMNDRS	INF-FS
S	0.004	0.484	0.006	0.5	0.005	0.479	0.002	0.422	0.005	0.355
G	0.025	0.102	0.025	0.006	0.02	0.097	0.02	0.097	0.022	0.098
Wc	0.17	0.306	0.174	0.32	0.166	0.313	0.168	0.294	0.178	0.292
ILPD	0.229	0.222	0.222	0.263	0.226	0.247	0.226	0.276	0.225	0.366
Ba	1.256	0.322	1.344	0.321	1.32	0.322	1.373	0.323	1.383	0.324
W	2.18	2.077	2.212	2.683	2.13	3.194	2.28	3.392	2.178	4.11
CC	4.77	5.224	4.08	5.275	4.07	6.218	3.78	6.276	3.91	7.273
Pb	11.501	11.269	10.643	12.348	11.313	12.01	11.949	14.401	11.846	13.441

Table 13
Computation time of different algorithms for adding a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Datasets	10%		20%		30%		40%		50%	
	GMNDRS	KNCMI	GMNDRS	KNCMI	GMNDRS	KNCMI	GMNDRS	KNCMI	GMNDRS	KNCMI
S	0.004	0.006	0.006	0.006	0.005	0.007	0.002	0.009	0.005	0.012
G	0.025	0.02	0.025	0.027	0.02	0.035	0.02	0.044	0.022	0.055
Wc	0.17	0.264	0.174	0.278	0.166	0.264	0.168	0.281	0.178	0.281
ILPD	0.229	0.145	0.222	0.262	0.226	0.273	0.226	0.325	0.225	0.503
Ba	1.256	0.438	1.344	0.555	1.32	0.73	1.373	0.92	1.383	1.137
W	2.18	2.376	2.212	2.864	2.13	2.433	2.28	3.082	2.178	3.809
CC	4.77	5.235	4.08	7.084	4.07	9.283	3.78	11.686	3.91	14.471
Pb	11.501	13.106	10.643	13.216	11.313	15.991	11.949	17.782	11.846	18.14

Table 14
Computation time of different algorithms for adding a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

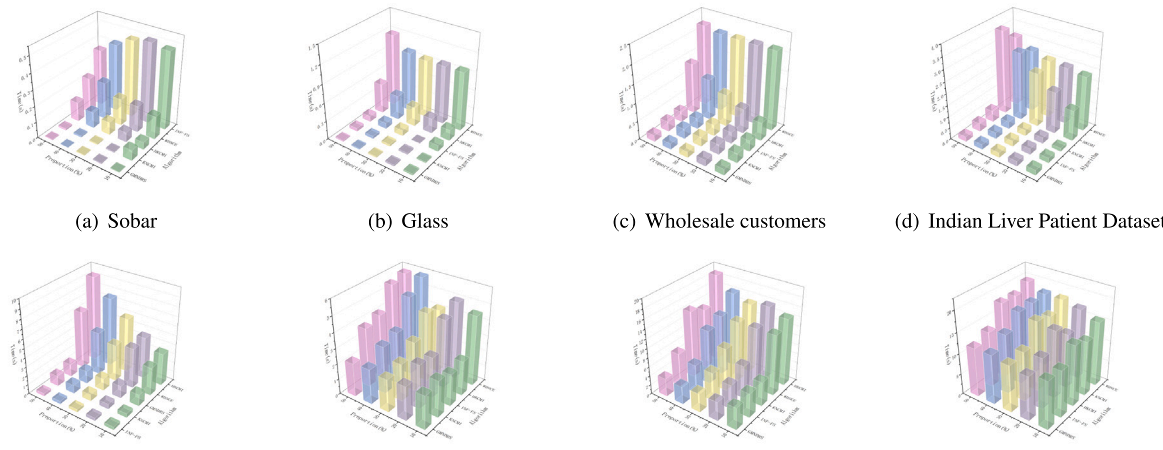
Datasets	10%		20%		30%		40%		50%	
	GMNDRS	HKCMI	GMNDRS	HKCMI	GMNDRS	HKCMI	GMNDRS	HKCMI	GMNDRS	HKCMI
S	0.004	0.046	0.006	0.061	0.005	0.081	0.002	0.101	0.005	0.122
G	0.025	0.181	0.025	0.247	0.02	0.318	0.02	0.404	0.022	0.493
Wc	0.17	0.5	0.174	0.683	0.166	0.889	0.168	1.130	0.178	1.32
ILPD	0.229	1.331	0.222	1.819	0.226	2.350	0.226	2.763	0.225	2.955
Ba	1.256	3.495	1.344	4.564	1.32	5.934	1.373	6.515	1.383	7.958
W	2.18	2.247	2.212	4.426	2.13	4.47	2.28	5.106	2.178	5.527
CC	4.77	14.456	4.08	15.738	4.07	15.004	3.78	16.149	3.91	18.931
Pb	11.501	14.592	10.643	16.254	11.313	16.88	11.949	17.608	11.846	18.224

Table 15
Computation time of different algorithms for adding a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$).

Datasets	10%		20%		30%		40%		50%	
	GMNDRS	WDNCE	GMNDRS	WDNCE	GMNDRS	WDNCE	GMNDRS	WDNCE	GMNDRS	WDNCE
S	0.004	0.145	0.006	0.167	0.005	0.17	0.002	0.233	0.005	0.224
G	0.025	0.963	0.025	0.962	0.02	0.954	0.02	0.982	0.022	1.188
Wc	0.17	2.155	0.174	2.148	0.166	2.143	0.168	2.145	0.178	2.236
ILPD	0.229	2.375	0.222	2.481	0.226	2.525	0.226	2.697	0.225	2.868
Ba	1.256	3.059	1.344	4.357	1.32	4.015	1.373	4.66	1.383	6.295
W	2.18	4.533	2.212	4.992	2.13	4.143	2.28	5.854	2.178	5.778
CC	4.77	13.022	4.08	12.985	4.07	13.006	3.78	12.994	3.91	13.125
Pb	11.501	15.482	10.643	17.028	11.313	18.029	11.949	18.054	11.846	19.702

intuitionistic fuzzy rough sets and the matrix-based GMNDRS. Generally, neighborhood relations cannot extract the samples needed by decision-makers to solve real-world problems. Therefore, we proposed a generalized multi-granulation rough set model based on neighborhood dominant relations and adjusted the strictness of the rough set model conditions by setting threshold values. In addition, we also studied two matrix dynamic mechanisms for object changes with attribute preservation in IFOIS, including adding and deleting objects. Finally, we conducted two sets of comparative experiments on eight datasets. The first set compared the times required by the matrix dynamic algorithm, general dynamic algorithm, and classical algorithm for approximate computations. The second set of experiments compared the proposed GMNDRS matrix dynamic algorithm with the other four algorithms in terms of the time needed for dynamic approximate updates. The results of these experiments confirmed the effectiveness of dynamic algorithms. They also showed that when object changes occur in IFOIS, the matrix dynamic method reduces time consumption and improves computational efficiency. However, this work has some limitations. On the one hand, the matrix-based dynamic approximation update requires more computational resources than other methods for handling data. On the other hand, although it achieves remarkable efficiency in dynamic approximation updates, it does not incorporate feature selection.

In the future, there are two important research directions in this field. First, we have only considered the matrix dynamic mechanism for object changes. We have not implemented the dynamic update mechanism for when attribute changes occur, nor the situation in which both attributes and objects change simultaneously. Because dynamic approximation updates are part of data processing, the ultimate goal is to fully utilize the approximation updates' results in the feature selection process. Therefore, the second direction is to conduct research on feature selection based on the results of the approximation updates in this method.



(a) Sobar (b) Glass (c) Wholesale customers (d) Indian Liver Patient Dataset

(e) Banknote authentication (f) Wireless (g) Customer Churn (h) Page blocks

Fig. 8. The computation time of different algorithms when adding a certain proportion of objects ($\alpha = 0.5, \beta = 0.2$)($\alpha = 0.6, \beta = 0.25$).

CRediT authorship contribution statement

Xiaoyan Zhang: Conceptualization, Investigation, Methodology, Validation, Writing – review & editing. **Jinghong Wang:** Data curation, Methodology, Software, Visualization, Writing – original draft. **Jianglong Hou:** Data curation, Methodology, Software, Validation, Writing – original draft.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

We confirm that we have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing we confirm that we have followed the regulations of our institutions concerning intellectual property.

Data availability

No data was used for the research described in the article.

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