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Updating approximations with dynamic objects based on local multigranulation rough sets in ordered information systems

Wentao Li[1] · Weihua Xu[1] · Xiaoyan Zhang[1] · Jia Zhang[2]

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Abstract

The main task of local rough set model is to avoid the interference of complicated calculation and invalid information in the formation of approximation space. In this paper, we first present a local rough set model based on dominance relation to make the local rough set theory applicable to the ordered information system, then two kinds of local multigranulation rough set models in the ordered information system are constructed by extending the single granulation environment to a multigranulation case. Moreover, the updating processes of dynamic objects based on global (classical) and local multigranulation rough sets in the ordered information system are analyzed and compared carefully. It is addressed about how the rough approximation spaces of global multigranulation rough set and local multigranulation rough set change when the object set increase or decrease in an ordered information system. The relevant algorithms for updating approximations with dynamic objects on global and local multigranulation rough sets are provided in ordered information systems. To illustrate the superiority and the effectiveness of the proposed dynamic updating approaches in the ordered information system, experimental evaluation is performed using six datasets coming from the University of California-Irvine repository.

Keywords Dynamic updating · Local rough set · Multigranulation rough set · Granular computing · Ordered information system

✉ Weihua Xu
chxuwh@gmail.com

Wentao Li
drliwentao@gmail.com

Xiaoyan Zhang
zxy19790915@163.com

Jia Zhang
Krystalgaga@163.com

[1] College of Artificial Intelligence, Southwest University, Chongqing 400715, People's Republic of China

[2] School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, People's Republic of China

1 Introduction

Pawlak rough set model (Pawlak 1982), as a mathematical tool to describe imprecise and incomplete information, can effectively discover hidden knowledge and reveal potential decision rules in an information system (Hu and Cercone 1995; Jeon et al. 2016; Sun and Ma 2015; Xu et al. 2013). This relatively new soft computing methodology has received great attention in recent years, and its usefulness has been confirmed through successful applications in many areas, such as pattern recognition, data mining, image processing, decision analysis, and medical diagnosis (Inbarani 2015; Li et al. 2016; Sang et al. 2018; Xu and Guo 2016; Yu et al. 2018; Zhou et al. 2020). Pawlak rough set theory is built on the basis of the classification mechanism, it is classified as the equivalence relation in a specific universe, and the equivalence relation constitutes a partition of the universe. However, in many real-life circumstances, the information system is no longer classical, the binary relations in the information systems are never equivalence relations (Greco et al. 2007; Huang et al. 2013; Mandal and Ranadive 2019), but preference relations, such as dominance relation. This kind of information system is called an ordered information system.

For an ordered information system, it is vital to propose an extension called the dominance-based rough set approach (DRSA) to take into account the ordering properties of criteria (Greco et al. 2002). The innovation is mainly based on substitution of the indiscernibility relation (equivalence relation) in the ordered information system by a dominance relation. Since Greco et al. initially studied DRSA in the year of 1998, many scholars have investigated a variaty of rough set models based on dominance relation to solve different problems (Greco et al. 2001; Li et al. 2020; Shao and Zhang 2005; Susmaga 2014; Zhang et al. 2013). Among these achievements, Azar et al. improved a dominance rough set-based classification system (Azar et al. 2017); Chen et al. developed a parallel attribute reduction method in dominance-based neighborhood rough set model (Chen et al. 2016); Kusunoki and Inuiguchi proposed a unified approach to attribute reduction in DRSA (Kusunoki and Inuiguchi 2010); Li and Xu first introduced the probabilistic rough set model based on dominance relation (Li and Xu 2015) and then further investigated the formation of multi-granulation decision-theoretic rough sets in ordered information systems (Li and Xu 2014); Sun et al. presented the dominance-based rough set theory over interval-valued information systems (Sun et al. 2014); Xu et al. constructed a multiple granulation rough set approach to ordered information systems (Xu et al. 2012); Yang et al. provided the notation of α -dominance relation and the corresponding rough set models in interval-valued information systems (Yang et al. 2015), and many other relevant generalizations.

It should be noted that Pawlak and its generalized rough sets are constructed based on one set of classes, these classes are also regarded as information granules, which are generated by a partition or a covering of the universe. In 1985, Hobbs put forward the concept of granularity (Hobbs 1985), and Zadeh first explored the concept of granular computing between 1996 and 1997 (Zadeh 1997). They all think that information granules refer to pieces, classes and groups into which complex information are divided in accordance with the characteristics and processes of the understanding and decision-making. At present, granular computing is an emerging information processing computing paradigm. It concerns the processing of complex information entities called information granules (Xu and Li 2016a). Information granules, as encountered in natural language, are implicit in their nature (Pedrycz 2013). To make full use of it and make it effectively used in the analysis and design of intelligent systems, we need to make information granules explicit. This can be achieved through a prudent formalization provided in the area of granular computing.

Pal et al. presented the relationship among granular computing, rough entropy and target extraction (Pal et al. 2005). Skowron et al. introduced the basic concepts related to granular computing, including the intersection and semantics of information granules as well as the inclusion and similarity relations of granules (Skowron and Stepaniuk 2004), and the basis of rough-neural computing (Skowron and Stepaniuk 2001). Yao first proposed the relationship between information granulation and rough approximation theory (Yao 2001). Peters et al. presented a method of measuring information granules based on rough set theory (Peters et al. 2002). Especially, Rasiowa (1991), Rasiowa and Marek (1989) investigated the approaches to approximation based on many indiscernibility relations for rough approximations. However, the approximations in these approaches are still based on a singleton granulation induced from an indiscernibility relation, which can be applied to knowledge representation in distributive systems and groups of intelligent systems. Later on, Rauszer (1992) discovered the rough logic for the multi-agent systems to make rough set model with a group of intelligent systems more explicit. In order to make rough set theory more widely used, Qian et al. extended Pawlak's single granulation to a multiple granulation case (Qian and Liang 2006). This generalization has a significant impact on the expansion of granular computing in rough set theory. After that, many researchers extended the multigranulation rough sets to different environments, inducing aplenty and significant results (Chen and Zhang 2014; Qian et al. 2010; Xu et al. 2012; Xu and Li 2016b; Zhou et al. 2021).

With the advent of information era, the data in various fields of society are constantly changing, which makes the data stored in each database continuously updated. These updates are mainly reflected on adding new data and eliminating old data. For different types of information system, the approximation space in rough set models will change accordingly with the variation of object set or attribute set, and the decision rules derived from the approximation sets will have corresponding changes. There have been a lot of studies on dynamic information system in the literature of rough sets, including the change of attribute values, object set, and attribute set. Among these studies, Li et al. developed a rough set based characteristic relation for dynamic attribute generalization and dynamic maintenance of decision rules in data mining (Li et al. 2007a, b); Chen et al. investigated an incremental approach to rough approximation updating under dynamic maintenance environments (Chen et al. 2013, 2015), then further explored a rough set based dynamic maintenance method for approximation in classical and incomplete ordered decision information system while attribute values coarsening or refining (Chen et al. 2010, 2012); Liu et al. studied incremental updating approximations in dynamic incomplete information systems with the variation of attribute sets (Liu et al. 2009, 2011, 2014, 2015); Li et al. (2013) and Luo et al. (2013) proposed dynamic maintenance approaches of approximations in the ordered information system under the variation of the object set; Zhang et al. presented different formation methods of rough set models for dynamic data mining (Zhang et al. 2012, 2014), and others (Cheng 2011; Guo et al. 2020; Hu et al. 2017a, b; Huang et al. 2017; Lang et al. 2017; Li and Li 2015; Li et al. 2018, 2019; Liu et al. 2014; Wang et al. 2013; Yang et al. 2014, 2017). From the dynamic updating results reported by these traditional rough approximations, it is not hard to discover that all of these methods have a common limitation, namely a lot of time for repeated computing that should be saved has also been implemented, and this leads to the updating efficiency of the algorithm being not explicit in the processes.

As a relatively new research idea in the field of rough set theory, the local rough set approach is presented by Qian et al. (2018). This novel idea does not need to refer to all the objects in the universe to approximate the target concept. Instead, it only needs to consider the objects in the target concept, providing a convenient way to search the required information directly and effectively reducing the filtering time. Let us first give an example of local idea, independent of rough sets, to show that local approach is more efficient than global method in information retrieval. Southwest University is located in China. Now assume that we need to find a scientific researcher in China, who studies Granular Computing (or other fields), named ABC. A natural search method is to screen all scientific researchers in China one by one until ABC is finally found. This natural retrieval method reflects the global idea. Although ABC can be found in the end, it will undoubtedly take a lot of time. However, if we know some prior knowledge of ABC, such as the affiliation of ABC: Southwest University, then we only need to find it from the scientific researchers within the local area, namely Southwest University, rather than from the global scientific researchers in China. It is more efficient to search knowledge from local information than from global information. Because of this visible characteristic of local rough set, several generalized local rough set models regarding to different information systems have been researched emphatically by scholars (Qian et al. 2017; Zhang et al. 2019).

Inspired by the above discussed studies, in this paper, we want to construct an expression of local rough sets in ordered information systems from the multigranulation viewpoint to overcome the mentioned limitation about the time consumption of repeated computing, and investigate the corresponding variation rules for updating approximations with dynamic objects. This is the motivation behind the research presented here. The main contents and innovation of this paper are shown as: (1) The local rough set theory is extended to the multiple granulation ordered information system, and then two kinds of local multigranulation rough set models in ordered information systems are presented. (2) The theories of updating approximations with dynamic objects based on multigranulation rough set are presented in ordered information systems, including classical (global) and local multigranulation rough set in ordered information systems. (3) The related algorithms for updating approximations with dynamic objects on classical and local multigranulation rough sets are carefully discussed in ordered information systems, and the experimental evaluation is performed using six public avaible datasets. Moreover, The superiority of dynamic object updating in the local multigranulation rough set models in ordered information systems is verified by the analysis of experimental results.

The paper is organized as follows. Related concepts are reivewed briefly in Sect. 2. In Sect. 3, we present the notion of local rough set model in ordered information system, and investigate the two kinds of local multigranulation rough approximations in ordered information systems, which are optimistic and pessimistic local multigranulation rough set models in ordered information systems. In Sect. 4, we mainly discuss and make a comparison on the update methods of local and classical multigranulation rough set models in an ordered information system with both dynamic and static background. In Sect. 5, we first provide the corresponding updating algorithms for deriving classical and local multigranulation rough approximations in ordered information systems, and then we do the experimantal testing by six datasets from the UCI datasets in Sect. 6, to make the comparisons on computing time of static updating and dynamic updating with object variation for the novel models. Finally, Sect. 7 covers some conclusions.

2 Basic notions

In this section, some basic concepts on rough set theory in an ordered information system are reviewed briefly, detailed descriptions could be referred to the relevant references.

Definition 2.1 An information system is a triple $I = (U, AT, F)$, where $U = \{x_1, x_2, \dots, x_n\}$ is a non-empty and finite set of objects; $AT = \{a_1, a_2, \dots, a_m\}$ is a non-empty and finite set of attributes; $F = \{f_j | U \rightarrow V_j, j \leq m\}$, where f_j is the value of a_j on $x \in U$, and V_j is the domain of $a_j \in AT$.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion (Greco et al. 2002, 2007). An information system is called an ordered information system if all condition attributes are criteria. As the decreasing preference can be converted to increasing preference, in this paper we only consider the increasing preference without any loss of generality. In an ordered information system, $\geq_{a_j}$ is defined to denote the preference-ordered relation based on the condition attribute a, then $x \geq_{a_j} y$ means that x is at least as good as y with respect to criterion a. $\forall a_j \in AT$, if $x \geq_{a_j} y$, then x dominates y in AT. We use $I^{\geq} = (U, AT, F)$ to denote the ordered information system.

Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$. We call $R_A^{\geq} = \{(x, y) \in U \times U | f_a(y) \geq f_a(x), \forall a \in A\}$ as a dominance relation. The set of dominance classes induced by a dominance relation $R_A^{\geq}$ is called $U/R_A^{\geq} = \{[x]_{R_A^{\geq}} | x \in U\}$, where $[x]_{R_A^{\geq}} = \{y \in U | (x, y) \in R_A^{\geq}\}$ is the dominance class containing x.

Definition 2.2 (*Xu et al.* 2006) Let $I^{\geq} = (U, AT, F)$ be an ordered information system. $R^{\geq}$ is a dominance relation. For any $X \subseteq U$, the lower and upper approximations of X with respect to A in the ordered information system are defined as

$$\underline{R_A^{\geq}}(X) = \{x \in U | [x]_{R_A}^{\geq} \subseteq X\},$$

$$\overline{R_A^{\geq}}(X) = \{x \in U | [x]_{R_A}^{\geq} \cap X \neq \emptyset\}.$$

If $\underline{R_A^{\geq}}(X) = \overline{R_A^{\geq}}(X)$, X is called definable set in the ordered information system; and if $\underline{R_A^{\geq}}(X) \neq \overline{R_A^{\geq}}(X)$, then X is called a rough set in the ordered information system. Moreover, the lower and upper approximations satisfy $\underline{R^{\geq}}(X) \subseteq X \subseteq \overline{R^{\geq}}(X)$. Three disjoint decision regions of X are shown as

$$pos(X) = \underline{R_A^{\geq}}(X),$$

$$neg(X) = \sim \overline{R_A^{\geq}}(X),$$

$$bnd(X) = \overline{R_A^{\geq}}(X) - \underline{R_A^{\geq}}(X),$$

where $pos(X)$, $neg(X)$ and $bnd(X)$ are called the positive region, negative region and boundary region, respectively.

The multigranulation rough set model was determined by Qian and Liang (2006), Qian et al. (2010) in the year of 2006, and it was extended to the ordered information system by Xu et al. (2012). Let us introduce two kinds of rough approximations regard to multigranulation rough sets in ordered information systems as follows.

Definition 2.3 (*Xu et al.* 2012) Let $I^{\geq} = (U, AT, F)$ be an ordered information system and $R_i (i = 1, 2, \dots, m)$ be dominance relations. $\forall X \subseteq U$, $[x]_{R_i}^{\geq} = \{y | (x, y) \in R_i^{\geq}\}$ is called the i-th dominance class contains x with respect to the i-th dominance relation R_i. The optimistic multigranulation lower and upper approximations of the target set X are defined as

$$\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{ x \in U \mid \bigvee_{i=1}^{m} ([x]_{R_i}^{\geq} \subseteq X) \right\},$$

$$\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{ x \in U \mid \bigwedge_{i=1}^{m} ([x]_{R_i}^{\geq} \cap X \neq \emptyset) \right\},$$

where the logical operations }} $\bigvee''$ and }} $\bigwedge''$ represent for "*or*" and "*and*", respectively. If $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \neq \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, then we call X as the optimistic multigranulation rough set in the ordered information system. In a similar way, we could obtain the pessimistic multigranulation lower and upper approximations in the ordered information system.

Definition 2.4 (*Xu et al.* 2012) Let $I^{\geq} = (U, AT, F)$ be an ordered information system and R_i $(i = 1, 2, \ldots, m)$ be dominance relations. $\forall X \subseteq U$, $[x]_{R_i}^{\geq} = \{y | (x, y) \in R_i^{\geq}\}$ is called the i-th dominance class contains x with respect to the i-th dominance relation R_i. The pessimistic multigranulation lower and upper approximations of the target set X are defined as

$$\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{ x \in U \mid \bigwedge_{i=1}^{m} ([x]_{R_i}^{\geq} \subseteq X) \right\},$$

$$\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{ x \in U \mid \bigvee_{i=1}^{m} ([x]_{R_i}^{\geq} \cap X \neq \emptyset) \right\},$$

where }} $\bigvee''$ and }} $\bigwedge''$ represent for "*or*" and "*and*", respectively. If $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \neq \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, then X is called the pessimistic multigranulation rough set in the ordered information system.

To meet the requirements of the following sections, we need to review some basic properties of the lower and upper approximations of multigranulation rough sets in ordered information systems.

Proposition 2.1 (Xu et al. 2012) *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, and R_i $(i = 1, 2, \ldots, m)$ be dominance relations, $X, Y \in F(U)$. The lower and upper approximations with respect to dominance relations R_i $(i = 1, 2, \ldots, m)$ meet the following properties.*

(1) $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \subseteq X \subseteq \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \subseteq X \subseteq \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$;

(2) $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(\sim X) =\sim \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(\sim X) =\sim \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$;

(3) $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(\sim X) =\sim \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(\sim X) =\sim \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$;

(4) $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(U) = \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(U) = U$, $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(\emptyset) = \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(\emptyset) = \emptyset$;

(5) $X \subseteq Y \Longrightarrow \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \subseteq \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y)$, $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \subseteq \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y)$;

(6) $X \subseteq Y \Longrightarrow \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \subseteq \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y)$, $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \subseteq \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y)$;

(7) $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcap Y) \subseteq \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcap \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y)$,

$\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcap Y) = \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcap \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y)$;

$$
\begin{aligned}
(8)\quad & \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcup Y) \supseteq \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcup \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y),\\
& \quad \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcup Y) = \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcup \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y);\\
(9)\quad & \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcup Y) \supseteq \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcup \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y),\\
& \quad \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcup Y) \supseteq \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcup \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y);\\
(10)\quad & \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcap Y) \subseteq \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcap \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y),\\
& \quad \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X \bigcap Y) \subseteq \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \bigcap \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(Y).
\end{aligned}
$$

Proof Detailed proof process could be referred to reference (Xu et al. 2012). □

In the next section, we will investigate the formation of local multigranulation rough sets in ordered information systems to further generalize the theory of local rough set model.

3 Local multigranulation rough set models in ordered information systems

Decision-makers have got great convenience in the era of big data because of the development of information technology. However, this convenience brings a lot of useless information. In an information system, if we take all the global information into account, it will have a large number of negative effects, such as: low efficiency, memory consumption, time wasting, and many others, the same to ordered information systems. In most cases, the useful information that decision-makers need to consider accounts for a very small proportion of the global information. Therefore, it is an innovation worth trying if decision-makers could retrieve the useful information from the target local information. The purpose of constructing local rough set model in ordered information systems is to attempt to solve the mentioned issue. Let us focus on the following definition of local rough approximations in ordered information systems.

Definition 3.1 Let $I^{\geq} = (U, AT, F)$ be an ordered information system. $R^{\geq}$ is a dominance relation, and $X \subseteq U$. The local lower and upper approximations of the set X with respect to R are defined as follows.

$$\underline{R_L^{\geq}}(X) = \{x \mid [x]_R^{\geq} \subseteq X, x \in X\},$$

$$\overline{R_L^{\geq}}(X) = \cup \{[x]_R^{\geq} \mid [x]_R^{\geq} \cap X \neq \emptyset, x \in X\}.$$

If $\underline{R_L^{\geq}}(X) = \overline{R_L^{\geq}}(X)$, X is called local definable set in the ordered information system; and if $\underline{R_L^{\geq}}(X) \neq \overline{R_L^{\geq}}(X)$, then X is called a local rough set in an ordered information system, denoted as the pair $\langle \underline{R_L^{\geq}}(X), \overline{R_L^{\geq}}(X) \rangle$. It could be seen from the definition that $\underline{R_L^{\geq}}(X)$ belongs to X, and X belongs to $\overline{R_L^{\geq}}(X)$, namely $\underline{R_L^{\geq}}(X) \subseteq X \subseteq \overline{R_L^{\geq}}(X)$. We define the positive region, negative region and boundary region of X for the local rough set in an ordered information system.

$$
\begin{aligned}
pos(X) &= \underline{R_L^{\geq}}(X) = \{x \mid [x]_R^{\geq} \subseteq X, x \in X\},\\
neg(X) &= \sim \overline{R_L^{\geq}}(X) = U - \cup \{[x]_R^{\geq} \mid [x]_R^{\geq} \cap X \neq \emptyset, x \in X\},\\
bnd(X) &= \overline{R_L^{\geq}}(X) - \underline{R_L^{\geq}}(X),
\end{aligned}
$$

Compared with the global rough set defined in Definition 2.2, it is not difficult to derive the following relationship among global and local approximations and the approximated set,

$$\underline{R_L^{\geq}}(X) \subseteq \underline{R^{\geq}}(X) \subseteq X \subseteq \overline{R_L^{\geq}}(X) \subseteq \overline{R^{\geq}}(X).$$

Then the following properties hold.

Proposition 3.1 *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R^{\geq}$ is a dominance relation induced by AT, $X, Y \subseteq U$. The following items hold.*

(1) $\underline{R_L^{\geq}}(X) \subseteq X \subseteq \overline{R_L^{\geq}}(X)$;
(2) $\underline{R_L^{\geq}}(U) = \overline{R_L^{\geq}}(U) = U$;
(3) $\overline{R_L^{\geq}}(\emptyset) = \underline{R_L^{\geq}}(\emptyset) = \emptyset$;
(4) $X \subseteq Y \Longrightarrow \underline{R_L^{\geq}}(X) \subseteq \underline{R_L^{\geq}}(Y)$;
(5) $X \subseteq Y \Longrightarrow \overline{R_L^{\geq}}(X) \subseteq \overline{R_L^{\geq}}(Y)$;
(6) $\underline{R_L^{\geq}}(X \bigcap Y) = \underline{R_L^{\geq}}(X) \bigcap \underline{R_L^{\geq}}(Y)$;
(7) $\overline{R_L^{\geq}}(X \bigcup Y) = \overline{R_L^{\geq}}(X) \bigcup \overline{R_L^{\geq}}(Y)$;
(8) $\underline{R_L^{\geq}}(X \bigcup Y) \supseteq \underline{R_L^{\geq}}(X) \bigcup \underline{R_L^{\geq}}(Y)$;
(9) $\overline{R_L^{\geq}}(X \bigcap Y) \subseteq \overline{R_L^{\geq}}(X) \bigcap \overline{R_L^{\geq}}(Y)$.

Proof The above items could be derived directly by Definition 3.1. □

In an ordered information system, the construction of local multigranulation rough sets is different from local rough set because the former is constructed based on a family of approximation spaces.

Definition 3.2 Let $I^{\geq} = (U, AT, F)$ be an ordered information system and $R_i^{\geq}$ $(i = 1, 2, \ldots, m)$ be dominance relations, $\forall X \subseteq U$. The optimistic local multigranulation lower and upper approximations of the set X with respect to $R_i^{\geq}$ are defined as

$$
\begin{aligned}
\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) &= \{x \mid \bigvee_{i=1}^{m} ([x]_{R_i}^{\geq} \subseteq X), x \in X\},\\
\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) &= \bigcap_{i=1}^{m} \overline{R_i^{\geq}}_L,
\end{aligned}
$$

where $\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)$ represents the intersection of the upper approximation under each granularity. And $\overline{R_i^{\geq}}_L = \cup \{ [x]_{R_i}^{\geq} \mid [x]_{R_i}^{\geq} \cap X \neq \emptyset, x \in X\}$.

We can define the positive region, negative region and boundary region of X for the optimistic local multigranulation in an ordered information system.

$$pos(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) = \{x | \bigvee_{i=1}^{m}([x]_{R_i}^{\geq} \subseteq X), x \in X\},$$

$$neg(X) = \sim \overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) = U - \bigcap_{i=1}^{m} \overline{R_i^{\geq}}_L,$$

$$bnd(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X),$$

Definition 3.3 Let $I^{\geq} = (U, AT, F)$ be an ordered information system and $R_i^{\geq}$ $(i = 1, 2, \dots, m)$ be dominance relations, $\forall X \subseteq U$. The pessimistic local multigranulation lower and upper approximations of the set X with respect to R_i are defined as

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) = \{x | \bigwedge_{i=1}^{m}([x]_{R_i}^{\geq} \subseteq X), x \in X\},$$

$$\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) = \bigcup_{i=1}^{m} \overline{R_i^{\geq}}_L,$$

where $\overline{R_i^{\geq}}_L = \cup \{ [x]_{R_i}^{\geq} \mid [x]_{R_i}^{\geq} \cap X \neq \emptyset, x \in X\}$, which represents local upper approximation under each granule. And pessimistic local multigranulation upper approximation is the integration of local upper approximation of each granules.

We can define the positive region, nagative region and boundary region of X for the pessimistic local multigranulation in an ordered information system.

$$pos(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) = \{x | \bigwedge_{i=1}^{m}([x]_{R_i}^{\geq} \subseteq X), x \in X\},$$

$$neg(X) = \sim \overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) = U - \bigcup_{i=1}^{m} \overline{R_i^{\geq}}_L,$$

$$bnd(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X),$$

Remark According to the definition of upper approximation, local optimistic and pessimistic approximation should be defined as $\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) = \bigcup\{[x]_{R_i}^{\geq} | \bigwedge_{i=1}^{m}([x]_{R_i}^{\geq} \cap X) \neq \emptyset, x \in X\}$ and $\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) = \bigcup\{[x]_{R_i}^{\geq} | \bigvee_{i=1}^{m}([x]_{R_i}^{\geq} \cap X) \neq \emptyset, x \in X\}$. Due to $x \in X$, the above optimistic and pessimistic approximation are the same result, that is, $\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)$. The optimistic upper approximation should be the minimum lower bound containing the target set, and the pessimistic upper approximation should include all the objects related to the target set as much as possible, so it is the maximum upper bound containing the target set. Therefore, the optimistic and pessimistic upper approximations are respectively defined as $\overline{\sum_{i=1}^{m} R_{i}^{\geq}}_{L}^{O}(X) = \bigcap_{i=1}^{m} \overline{R_{i}^{\geq}}_{L}$ and $\overline{\sum_{i=1}^{m} R_{i}^{\geq}}_{L}^{P}(X) = \bigcup_{i=1}^{m} \overline{R_{i}^{\geq}}_{L}$. At this time, it satisfies the above requirements.

Example 3.1 Supposing Table 1 is an ordered information system about a case study of the risk investment. There are ten choices and six risk factors, which are purchasing power risk (a_1), financial risk (a_2), interest rate risk (a_3), market risk (a_4), liquidity risk (a_5), event risk (a_6). The higher the value of the attributes, the lower the risk rate. Assume that target is $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$ and consider two granulations R_1 and R_2, where R_1 is induced by the attribute set $\{a_2, a_4, a_5, a_6\}$ and R_2 is induced by the attribute set $\{a_1, a_3, a_4, a_6\}$. It means that there are two ways to choose. The one is to consider financial risk, market risk, liquidity risk and event risk. The other is to focus on power risk, interest rate risk, market risk, event risk. Therefore, we should combine two kinds of granularity to choose safer investment scope by lower and upper approximations.

We calculate the dominance classes of elements in a target set.

According to the granulation R_1, we obtain the following dominance classes:
$[x_2]_{R_1}^{\geq} = \{x_1, x_2, x_{10}\}, [x_4]_{R_1}^{\geq} = \{x_4, x_6, x_{10}\}, [x_5]_{R_1}^{\geq} = \{x_3, x_5\}$,
$[x_7]_{R_1}^{\geq} = \{x_7, x_8, x_9\}, [x_8]_{R_1}^{\geq} = \{x_8\}, [x_9]_{R_1}^{\geq} = \{x_8, x_9\}$.

According to the granulation R_2, we obtain the following dominance classes:
$[x_2]_{R_2}^{\geq} = \{x_2, x_3, x_4, x_6\}, [x_4]_{R_2}^{\geq} = \{x_4, x_6\}, [x_5]_{R_2}^{\geq} = \{x_3, x_5\}$,
$[x_7]_{R_2}^{\geq} = \{x_7\}, [x_8]_{R_2}^{\geq} = \{x_7, x_8\}, [x_9]_{R_2}^{\geq} = \{x_3, x_5, x_7, x_8, x_9\}$.

Thus, based on the definition of optimistic and pessimistic local multigranulation lower and upper approximations, we obtain that

$\underline{\sum_{i=1}^{m} R_{i}^{\geq}}_{L}^{O}(X) = \{x_7, x_8, x_9\}$,
$\overline{\sum_{i=1}^{m} R_{i}^{\geq}}_{L}^{O}(X) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$;
$\underline{\sum_{i=1}^{m} R_{i}^{\geq}}_{L}^{P}(X) = \{x_7, x_8\}$,
$\overline{\sum_{i=1}^{m} R_{i}^{\geq}}_{L}^{P}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$.

From the definition of local multigranulation rough sets proposed in this paper and global multigranulation rough sets presented in reference (Xu et al. 2012), there is no doubt that the time for calculating the upper and lower approximation of local multigranulation

Table 1 An ordered information system

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	1	2	1	2	2	3
x_2	2	2	3	2	2	3
x_3	2	2	3	3	1	4
x_4	4	1	4	2	4	3
x_5	1	2	1	3	1	4
x_6	4	1	4	2	4	3
x_7	4	1	4	3	3	2
x_8	3	4	2	3	3	2
x_9	1	2	1	3	3	2
x_{10}	3	4	2	2	4	3

rough sets in ordered information system is smaller than that of global multigranulation rough sets in ordered information system. To be sure, the closer the approximated concept is to the whole universe, the less significant the difference in the time they take.

4 Multigranulation-based updating approximations with dynamic objects in ordered information systems

With the rapid development of science and technology, information contained in each information system needs to be constantly updated, including deleting old information and adding new information. With the information updating in information systems, the information granularity in the information system will change accordingly, which will lead to the variation of knowledge structure. As to rough set theory in ordered information systems, there are three possible variations for the dynamic updates, namely variation of object set, variation of attribute set, and variation of attribute values. The main objective of this paper is to study the variation of object set, including the increase and decrease of objects. Whenever an information system is updated (adding new objects or deleting old ones), it generates a new information table.

After generating a new information table, if every update is batch processed from scratch, it will lead to huge time consumption and increase the complexity of calculation. In fact, there is a great relationship between the updated ordered information system and the original one. The dynamic update method is proposed to avoid unnecessary time wasting. Updating with dynamic objects includes two cases, the one is to delete old objects and the other is to add new objects. Since the target set is a subset of the universe, the elements in the target set may be deleted when the objects of the universe is deleted. Therefore, it is divided into two parts when removing objects from the information system: (1) the deleted objects are not elements in the target set; (2) the deleted objects are elements in the target set. For the above two issues, the situation of dominance classes of the target set after dynamic updating is different. In particular, there is only one case when adding new objects, because the target set does not change. It just increases the cardinal number of the universe. In the following, we will study the dynamic processing of information variation and compare the global dynamic updating with the local dynamic updating.

4.1 Dynamic updating approximations on global multigranulation rough set in ordered information systems

We discuss the global updating with dynamic objects from two aspects: deleting objects and adding objects. Let $I^{\geq} = (U, AT, F)$ be the initial order information system, and $[x]^{\geq}_{R_i}$ is the original dominance classes of the object under the relation $R_i^{\geq}$. The target set is represented by X, and the lower and upper approximations of the optimistic multigranulation rough set are defined as $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$. Moreover, the lower and upper approximations of the pessimistic multigranulation rough set are $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$ and $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$. After dynamically deleting objects, dominance classes will change, and then the lower and upper approximations of multigranulation rough sets will make the corresponding changes. Denote the changed new ordered information system as $I^{\geq\prime} = (U', AT, F)$, and the new dominance classes are denoted as $[x]^{\geq\prime}_{R_i}$. Moreover, $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$, $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$, $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$ and $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$ are respectively optimistic and pessimistic approximations after deleting objects. Because the target concept is a subset of the universe and selected objects are deleted randomly, the objects in the target concept may also be deleted. Therefore, this paper mainly discusses from two parts: the removed object belongs to the target concept and the removed object does not belong to the target concept.

Proposition 4.1 *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}$ $(i = 1, 2, \dots, m)$ is a dominance relation, and $X \subseteq U$. If the object x_t not in the target concept X is deleted, which means $x_t \notin X$, we obtain that*

(1) *If $x_s \notin \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}(x_s \neq x_t)$ and exists a granule R_i such that $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$, then $x_s \in \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}'$.*
 If $x_s \notin \overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)}(x_s \neq x_t)$ and $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$ for every granule $R_i (i = 1, 2, \dots, m)$, then $x_s \in \overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)}'$.

(2) $\overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}' = \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)} - \{x_t\}$, $\overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)}' = \overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)} - \{x_t\}$.

Proof

(1) According to $x_s \notin \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}(x_s \neq x_t)$ and the definition of $\overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}$, we have $[x_s]_{R_i}^{\geq} \nsubseteq X$ for every R_i $(i = 1, 2 \dots, m)$. From known condition that it exists a R_i such that $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$, it means after removing the object x_t we have $[x_s]_{R_i}^{\geq\prime} \subseteq X$ for one granule R_i. Finally, we get $x_s \in \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}'$.
 Similarly, due to $x_s \notin \overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)}(x_s \neq x_t)$ and the definition of pessimistic lower approximation $\overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)}$, we know $[x_s]_{R_i}^{\geq} \subseteq X$ that does not suitable for every granule. When this condition $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$ holds at all granularity $R_i (i = 1, 2, \dots, m)$, we have $[x_s]_{R_i}^{\geq\prime} \subseteq X$ for every granule after removing the object x_t. Namely, $\bigwedge_{i=1}^m ([x_s]_{R_i}^{\geq\prime} \subseteq X)$ holds. Thus $x_s \in \overline{PM_{\sum_{i=1}^m R_i^{\geq}}(X)}'$.

(2) If $x_s \in \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}$ $(x_s \neq x_t)$, then $[x_s]_{R_i}^{\geq} \cap X \neq \emptyset$ for any granule $R_i (i = 1, 2, \dots, m)$. That is to say, its dominance class of each granule has the elements in X, but $x_t \notin X$. So after deleting x_t, the part of the dominance classes that intersect with the X does not change, namely, $[x_s]_{R_i}^{\geq\prime} \cap X \neq \emptyset$ .Thus, $x_s \in \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}'$.
 If $x_s \notin \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}$ $(x_s \neq x_t)$, then $\exists R_i$ such that $[x_s]_{R_i}^{\geq} \cap X = \emptyset$. As a result of $x_t \notin X$, there is still no same element with X after deleting the object x_t. That means $[x_s]_{R_i}^{\geq\prime} \cap X = \emptyset$. Therefore, $x_s \notin \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}'$.
 Moreover, the optimistic upper approximation does not change except x_t. Since x_t is deleted, there will be no more the object x_t in the new lower and upper approximations, namely, $\overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)}' = \overline{OM_{\sum_{i=1}^m R_i^{\geq}}(X)} - \{x_t\}$.
 The proof process of pessimistic upper approximation is similar to optimistic upper approximation.

□

Example 4.1 (Continuation of Example 3.1) Consider the target concept $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$, the optimistic and pessimistic approximations of the target set can be obtained. In this example, we discuss the updated lower and upper approximations after deleting elements that do not belong to the target set X, such as x_3, x_6.

(1) If x_3 is deleted, then the changed dominance classes are listed in the following $[x_1]_{R_1}^{\geq\prime} = [x_2]_{R_1}^{\geq\prime} = \{x_1, x_2, x_{10}\}, [x_5]_{R_1}^{\geq\prime} = \{x_5\}, [x_4]_{R_1}^{\geq\prime} = [x_6]_{R_1}^{\geq\prime} = \{x_4, x_6, x_{10}\}$, $[x_7]_{R_1}^{\geq\prime} = \{x_7, x_8, x_9\}, [x_8]_{R_1}^{\geq\prime} = \{x_8\}, [x_9]_{R_1}^{\geq\prime} = \{x_8, x_9\}, [x_{10}]_{R_1}^{\geq\prime} = \{x_{10}\}$. $[x_1]_{R_2}^{\geq\prime} = \{x_1, x_2, x_4, x_5, x_6, x_{10}\}, [x_2]_{R_2}^{\geq\prime} = \{x_2, x_4, x_6\}, [x_4]_{R_2}^{\geq\prime} = [x_6]_{R_2}^{\geq} = \{x_4, x_6\}$, $[x_5]_{R_2}^{\geq\prime} = \{x_5\}$, $[x_7]_{R_2}^{\geq\prime} = \{x_7\}$, $[x_8]_{R_2}^{\geq\prime} = \{x_7, x_8\}$, $[x_9]_{R_2}^{\geq\prime} = \{x_5, x_7, x_8, x_9\}$, $[x_{10}]_{R_2}^{\geq\prime} = \{x_4, x_6, x_{10}\}$.

Thus, due to the definition of optimistic and pessimistic approximations, we have

$$\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_5, x_7, x_8, x_9\}$$
$$\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9\}$$
$$\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_5, x_7, x_8, x_9\}$$
$$\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$$

Finally, we find $[x_5]_{R_1}^{\geq} - \{x_3\} \in X$ and $[x_5]_{R_2}^{\geq} - \{x_3\} \in X$, so $x_5 \in \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$ and $x_5 \in \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$.

Meanwhile, $[x_9]_{R_1}^{\geq} - \{x_3\} \in X$ and $[x_9]_{R_2}^{\geq} - \{x_3\} \in X$, so $x_9 \in \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$.

And $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_3\}$, $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_3\}$.

(2) If x_6 is deleted, then the changed dominance classes are listed in the following

$[x_1]_{R_1}^{\geq\prime} = [x_2]_{R_1}^{\geq\prime} = \{x_1, x_2, x_{10}\}$, $[x_3]_{R_1}^{\geq\prime} = [x_5]_{R_1}^{\geq\prime} = \{x_3, x_5\}$, $[x_4]_{R_1}^{\geq\prime} = \{x_4, x_{10}\}$,

$[x_7]_{R_1}^{\geq\prime} = \{x_7, x_8, x_9\}$, $[x_8]_{R_1}^{\geq\prime} = \{x_8\}$, $[x_9]_{R_1}^{\geq\prime} = \{x_8, x_9\}$, $[x_{10}]_{R_1}^{\geq\prime} = \{x_{10}\}$.

$[x_1]_{R_2}^{\geq\prime} = \{x_1, x_2, x_3, x_4, x_5, x_{10}\}$, $[x_2]_{R_2}^{\geq\prime} = \{x_2, x_3, x_4\}$, $[x_3]_{R_2}^{\geq\prime} = \{x_3\}$, $[x_4]_{R_2}^{\geq\prime} = \{x_4\}$,

$[x_5]_{R_2}^{\geq\prime} = \{x_3, x_5\}$, $[x_7]_{R_2}^{\geq\prime} = \{x_7\}$, $[x_8]_{R_2}^{\geq\prime} = \{x_7, x_8\}$, $[x_9]_{R_2}^{\geq\prime} = \{x_3, x_5, x_7, x_8, x_9\}$, $[x_{10}]_{R_2}^{\geq\prime} = \{x_4, x_{10}\}$.

Thus, due to the definition of optimistic and pessimistic approximations, we have

$$\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_4, x_7, x_8, x_9\},$$
$$\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_1, x_2, x_4, x_5, x_7, x_8, x_9\};$$
$$\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_7, x_8\},$$
$$\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

Finally, we find $[x_4]_{R_2}^{\geq} - \{x_6\} \in X$, so $x_4 \in \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$.

Meanwhile $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_6\}$, $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_6\}$.

Proposition 4.2 *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}$ $(i = 1, 2, \dots, m)$ is a dominance relation in U, and $X \subseteq U$. If the object x_t in the target concept X is deleted, which means $(x_t \in X)$, we obtain that*

(1) $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t\}$, $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t\}$.

(2) *If exists a granule* R_i *such that* $([x_s]_{R_i}^{\geq} - \{x_t\}) \cap X = \emptyset$ $(x_s \neq x_t)$, *then* $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t, x_s\}$.
If it satisfies that $([x_s]_{R_i}^{\geq} - \{x_t\}) \cap X = \emptyset$ $(x_s \neq x_t)$ *for all* $R_i (i = 1, 2, \ldots, m)$, *then* $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t, x_s\}$.

Proof

(1) We know that $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{x \in U \mid \bigvee_{i=1}^{m}([x]_{R_i}^{\geq} \subseteq X)\right\}$. If $\forall x_s \in U, x_s \neq x_t$, we just consider that $x_s \in \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$ because another situation is easy to get to the conclusion. Due to $x_s \in \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, so it exists R_i such that $[x_s]_{R_i}^{\geq} \subseteq X$. After removing x_t, the elements of $[x_s]_{R_i}^{\geq}$ based on the R_i still belong to X except x_t, namely, $[x_s]_{R_i}^{\geq\,\prime} \subseteq X$. Therefore, $x_s \in \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$. In addition to x_t, it has not changed on $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$. Finally, we have $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t\}$. In a similar manner, we have $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t\}$.

(2) It's easy to prove that $x_s \notin \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$ and $x_s \notin \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, so we only consider in this case when $x_s \in \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X), x_s \in \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$. If exists R_i such that $([x_s]_{R_i^{\geq}} - \{x_t\}) \cap X = \emptyset$ for any $x_s \in U, x_s \neq x_t$. According to the definition of the optimistic upper approximation, that is, $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{x \in U \mid \bigwedge_{i=1}^{m}([x]_{R_i}^{\geq} \cap X \neq \emptyset)\right\}$, we have $x_s \notin \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$ after removing the object x_t. The same method is also used to prove the pessimistic upper approximation. If $([x_s]_{R_i^{\geq}} - \{x_t\}) \cap X = \emptyset$ for any R_i, combing the definition of the pessimistic upper approximation ($\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{x \in U \mid \bigvee_{i=1}^{m}([x]_{R_i}^{\geq} \cap X \neq \emptyset)\right\}$), it is obtained that $x_s \notin \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$. Due to removing x_t, so $x_t \notin \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)', x_t \notin \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)'$. Therefore, we have $\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t, x_s\}$, $\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) - \{x_t, x_s\}$. □

Example 4.2 (Continuation of Example 3.1) Consider the target set $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$, the optimistic and pessimistic approximations of the target set can be obtained. Furthermore, we discuss the lower and upper approximations after deleting objects that belong to the target set X.

If x_4 is deleted, then the changed dominance classes are listed in the following

$$[x_1]_{R_1}^{\geq\,\prime} = [x_2]_{R_1}^{\geq\,\prime} = \{x_1, x_2, x_{10}\},\ [x_3]_{R_1}^{\geq\,\prime} = [x_5]_{R_1}^{\geq\,\prime} = \{x_3, x_5\},\ [x_6]_{R_1}^{\geq\,\prime} = \{x_6, x_{10}\},$$
$$[x_7]_{R_1}^{\geq\,\prime} = \{x_7, x_8, x_9\},\ [x_8]_{R_1}^{\geq\,\prime} = \{x_8\},\ [x_9]_{R_1}^{\geq\,\prime} = \{x_8, x_9\},\ [x_{10}]_{R_1}^{\geq\,\prime} = \{x_{10}\}.$$
$$[x_1]_{R_2}^{\geq\,\prime} = \{x_1, x_2, x_3, x_5, x_6, x_{10}\},\ [x_2]_{R_2}^{\geq\,\prime} = \{x_2, x_3, x_6\},\ [x_3]_{R_2}^{\geq\,\prime} = \{x_3\},\ [x_5]_{R_2}^{\geq\,\prime} = \{x_3, x_5\},$$
$$[x_6]_{R_2}^{\geq\,\prime} = \{x_6\},\ [x_7]_{R_2}^{\geq\,\prime} = \{x_7\},\ [x_8]_{R_2}^{\geq\,\prime} = \{x_7, x_8\},\ [x_9]_{R_2}^{\geq\,\prime} = \{x_3, x_5, x_7, x_8, x_9\},\ [x_{10}]_{R_2}^{\geq\,\prime} = \{x_6, x_{10}\}.$$

Thus, due to the definition of optimistic and pessimistic approximations, we have

$$
\begin{aligned}
\underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' &= \{x_7, x_8, x_9\},\\
\overline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' &= \{x_1, x_2, x_5, x_7, x_8, x_9\};\\
\underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' &= \{x_7, x_8\},\\
\overline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' &= \{x_1, x_2, x_3, x_5, x_7, x_8, x_9\}.
\end{aligned}
$$

Finally, we find $\underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_4\}$, $\underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_4\}$.

According to R_1 and R_2, it is easy to see that $([x_6]_{R_i}^{\geq} - \{x_4\}) \cap X = \emptyset$ and $([x_{10}]_{R_i}^{\geq} - \{x_4\}) \cap X = \emptyset$. Therefore, $\overline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \overline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_4, x_6, x_{10}\}$ and $\overline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \overline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_4, x_6, x_{10}\}$.

For an ordered information system, the dynamic updating should include adding new information, not only deleting invalid information. Because the target concept is given in advance, the added objects will only increase the cardinality of the universe, and will not change the target concept. However, the new objects will affect the overall classification, so the lower and upper approximations will be changed. $\underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)'$, $\overline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)'$, $\underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)'$, $\overline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)'$ are respectively represent for the new optimistic and pessimistic approximations. The changed dominance classes are represented by $[x_s]_{R_i}^{\geq\prime}$.

Proposition 4.3 *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}$ $(i = 1, 2, \dots, m)$ is a dominance relation, and $X \subseteq U$. If a new object $x_t (x_t \notin U, x_t \in U')$ is added, then the lower and upper approximations of optimistic and pessimistic multigranulation are shown as follows.*

(1) $\forall x_s \in \underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)$ *if* $x_t \in [x_s]_{R_i}^{\geq\prime}$ *for any* $R_i (i = 1, 2, \dots, m)$, *then* $\underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_s\}\,(x_s \neq x_t)$.
   $\forall x_s \in \underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)$ *if exists a granule* R_i *such that* $x_t \in [x_s]_{R_i}^{\geq\prime}$, *then* $\underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_s\}\,(x_s \neq x_t)$.

(2) *If it is satisfied that* $[x_t]_{R_i}^{\geq\prime} \cap X \neq \emptyset$ *for every* $R_i (i = 1, 2, \dots, m)$, *then* $\overline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \overline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) \cup \{x_t\}$.
   If exists a granule R_i *such that* $[x_t]_{R_i}^{\geq\prime} \cap X = \emptyset$, *then* $\overline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \overline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) \cup \{x_t\}$.

Proof

(1) Due to $x_s \in \underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)$, so it is obtained that $[x_s]_{R_i}^{\geq} \subseteq X$ at least a granule R_i. Because of $x_t \notin X$ and $x_t \in [x_s]_{R_i}^{\geq\prime}$ for every R_i, then we have $[x_s]_{R_i}^{\geq\prime} \nsubseteq X$ for all granules. Moreover, $x_s \notin \underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)'$. Thus, $\underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \underline{OM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_s\}\,(x_s \neq x_t)$. Similarly, the pessimistic lower approximation can be proved, namely, $\underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X)' = \underline{PM}_{\sum_{i=1}^{m} R_i^{\geq}}(X) - \{x_s\}$.

(2) It is easy to obtain the related results.

□

Example 4.3 (Continuation of Example 3.1) Consider the target set $X=\{x_2,x_4,x_5,x_7,x_8,x_9\}$, then the optimistic and pessimistic approximations of the target set can be obtained. Furthermore, we discuss the lower and upper approximations after adding objects x_{11} and x_{12}. As shown in the following Table 2, it is a new information table after objects are added.

After adding new objects x_{11},x_{12}, the dominance classes are listed as follows.

$$[x_1]_{R_1}^{\geq\,\prime}=[x_2]_{R_1}^{\geq\,\prime}=\{x_1,x_2,x_{10},x_{12}\},\ [x_3]_{R_1}^{\geq\,\prime}=[x_5]_{R_1}^{\geq\,\prime}=\{x_3,x_5\},\ [x_4]_{R_1}^{\geq\,\prime}=[x_6]_{R_1}^{\geq\,\prime}=\{x_4,x_6,x_{10},x_{12}\},$$
$$[x_7]_{R_1}^{\geq\,\prime}=\{x_7,x_8,x_9,x_{11}\},\ [x_8]_{R_1}^{\geq\,\prime}=\{x_8\},\ [x_9]_{R_1}^{\geq\,\prime}=\{x_8,x_9,x_{11}\},\ [x_{10}]_{R_1}^{\geq\,\prime}=\{x_{10},x_{12}\},$$
$$[x_{11}]_{R_1}^{\geq\,\prime}=\{x_8,x_{11}\},\ [x_{12}]_{R_1}^{\geq\,\prime}=\{x_{10},x_{12}\}.[x_1]_{R_2}^{\geq\,\prime}=\{x_1,x_2,x_3,x_4,x_5,x_6,x_{10},x_{12}\},$$
$$[x_2]_{R_2}^{\geq\,\prime}=\{x_2,x_3,x_4,x_6\},\ [x_3]_{R_2}^{\geq\,\prime}=\{x_3\},\ [x_4]_{R_2}^{\geq\,\prime}=[x_6]_{R_2}^{\geq\,\prime}=\{x_4,x_6\},$$
$$[x_5]_{R_2}^{\geq\,\prime}=\{x_3,x_5\},\ [x_7]_{R_2}^{\geq\,\prime}=\{x_7\},\ [x_8]_{R_2}^{\geq\,\prime}=\{x_7,x_8\},\ [x_9]_{R_2}^{\geq\,\prime}=\{x_3,x_5,x_7,x_8,x_9,x_{11}\},$$
$$[x_{10}]_{R_2}^{\geq\,\prime}=[x_{12}]_{R_2}^{\geq\,\prime}=\{x_4,x_6,x_{10},x_{12}\},\ [x_{11}]_{R_2}^{\geq\,\prime}=\{x_3,x_7,x_8,x_{11}\}.$$

Before adding the objects x_{11},x_{12}, $\{x_7,x_8,x_9\}=\underline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X)$, $\{x_7,x_8\}=\underline{PM_{\sum_{i=1}^m R_i^{\geq}}}(X)$, and $x_{11}\in[x_7]_{R_1}^{\geq\,\prime}$, so we have $x_7\notin\underline{PM_{\sum_{i=1}^m R_i^{\geq}}}(X)'$. Meanwhile, $x_{11}\in[x_9]_{R_1}^{\geq\,\prime}$ and $x_{11}\in[x_9]_{R_2}^{\geq\,\prime}$, then $x_9\notin\underline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X)'$. Thus, $\underline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X)'=\underline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X)-\{x_9\}=\{x_7,x_8\}$ and $\underline{PM_{\sum_{i=1}^m R_i^{\geq}}}(X)'=\underline{PM_{\sum_{i=1}^m R_i^{\geq}}}(X)-\{x_7\}=\{x_8\}$.

At the same time, $[x_{11}]_{R_1}^{\geq\,\prime}\cap X\neq\emptyset$, $[x_{11}]_{R_2}^{\geq\,\prime}\cap X\neq\emptyset$, so $x_{11}\in\overline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X)'$. Due to $[x_{12}]_{R_2}^{\geq\,\prime}\cap X\neq\emptyset$, we have $x_{12}\in\overline{PM_{\sum_{i=1}^m R_i^{\geq}}}(X)'$.

According to the lower and upper approximations definitions of optimistic and pessimistic, we have

Table 2 New ordered information system

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	1	2	1	2	2	3
x_2	2	2	3	2	2	3
x_3	2	2	3	3	1	4
x_4	4	1	4	2	4	3
x_5	1	2	1	3	1	4
x_6	4	1	4	2	4	3
x_7	4	1	4	3	3	2
x_8	3	4	2	3	3	2
x_9	1	2	1	3	3	2
x_{10}	3	4	2	2	4	3
x_{11}	2	3	1	3	3	2
x_{12}	3	4	2	2	4	3

$$\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_7, x_8\},$$
$$\overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}\};$$
$$\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_8\},$$
$$\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)' = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}.$$

From the above discuss, it is easy to verify the correctness of the Proposition 4.3.

4.2 Dynamic updating approximations on local multigranulation rough set in ordered information systems

Since the local update with dynamic objects is mainly aimed at the object in the target set, it is necessary to classify and discuss whether the deleted object is in the target set. Meanwhile, because the added object will not appear in the target set, only one case should be considered when adding objects. Next, based on the global updating representation method, we only change the local multigranulation lower and upper approximations of optimistic and pessimistic, which are denoted as $\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)'$, $\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)'$, $\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^P(X)'$, $\overline{\sum_{i=1}^{m} R_i^{\geq}}_L^P(X)'$.

Proposition 4.4 *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}$ $(i = 1, 2, \dots, m)$ is a dominance relation, and $X \subseteq U$. After deleting x_t, we obtain the following properties of local multigranulation optimistic and pessimistic approximations.*

(1) *If $x_t \notin X$, then $\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)' = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X) - \{x_t\}$, $\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^P(X)' = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^P(X) - \{x_t\}$.*

(2) *If $x_t \in X$, $\forall x_s \in X$ and $x_s \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)$, if there exists R_i such that $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$, then $x_s \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)'$.*
If $x_t \in X$, $\forall x_s \in X$ and $x_s \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^P(X)$, if $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$ for any $R_i (i = 1, 2, \dots, m)$, then $x_s \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^P(X)'$.

Proof

(1) It is easy to obtain the related results.
(2) $\forall x_s \in X$ and $x_s \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)$, we have $[x_s]_{R_i}^{\geq} \nsubseteq X$ for every $R_i (i = 1, 2, \dots, m)$. If there exists R_i such that $([x_s]_{R_i}^{\geq} - \{x_t\}) \subseteq X$, from Definition 3.2, we can get that $[x_s]_{R_i}^{\geq\prime} \subseteq X$. That means $x_s \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^O(X)'$.

The proof of pessimistic lower approximation is similar to the above optimistic lower approximation. □

Example 4.4 (Continuation of Example 3.1) Consider the target set $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$, the optimistic and pessimistic approximations of the target set can be obtained. Furthermore, we discuss the lower and upper approximations after deleting objects x_3 and x_4.

After removing $x_3 \notin X$, the dominance classes are as follows

$[x_2]^{\geq}_{R_1}{}' = \{x_1, x_2, x_{10}\}$, $[x_4]^{\geq}_{R_1}{}' = \{x_4, x_6, x_{10}\}$, $[x_5]^{\geq}_{R_1}{}' = \{x_5\}$, $[x_7]^{\geq}_{R_1}{}' = \{x_7, x_8, x_9\}$, $[x_8]^{\geq}_{R_1}{}' = \{x_8\}$, $[x_9]^{\geq}_{R_1}{}' = \{x_8, x_9\}$.

$[x_2]^{\geq}_{R_2}{}' = \{x_2, x_4, x_6\}$, $[x_4]^{\geq}_{R_2}{}' = \{x_4, x_6\}$, $[x_5]^{\geq}_{R_2}{}' = \{x_5\}$, $[x_7]^{\geq}_{R_2}{}' = \{x_7\}$, $[x_8]^{\geq}_{R_2}{}' = \{x_7, x_8\}$, $[x_9]^{\geq}_{R_2}{}' = \{x_5, x_7, x_8, x_9\}$.

We find $([x_5]^{\geq}_{R_1} - \{x_3\}) \subseteq X$, $([x_5]^{\geq}_{R_2} - \{x_3\}) \subseteq X$, so it is that $x_5 \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)'$ and $x_5 \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)'$. Meanwhile, $x_9 \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)$, but $([x_9]^{\geq}_{R_2} - \{x_3\}) \subseteq X$, therefore $x_9 \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)'$.

When removing $x_4 \in X$, we have

$[x_2]^{\geq}_{R_1}{}' = \{x_1, x_2, x_{10}\}$, $[x_5]^{\geq}_{R_1}{}' = \{x_3, x_5\}$, $[x_7]^{\geq}_{R_1}{}' = \{x_7, x_8, x_9\}$, $[x_8]^{\geq}_{R_1}{}' = \{x_8\}$, $[x_9]^{\geq}_{R_1}{}' = \{x_8, x_9\}$.

$[x_2]^{\geq}_{R_2}{}' = \{x_2, x_3, x_6\}$, $[x_5]^{\geq}_{R_2}{}' = \{x_3, x_5\}$, $[x_7]^{\geq}_{R_2}{}' = \{x_7\}$, $[x_8]^{\geq}_{R_2}{}' = \{x_7, x_8\}$, $[x_9]^{\geq}_{R_2}{}' = \{x_3, x_5, x_7, x_8, x_9\}$.

We can easily get that $\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)' = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X) - \{x_4\}$, $\underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)' = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X) - \{x_4\}$.

Proposition 4.5 *Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}$ $(i = 1, 2, \ldots, m)$ is a dominance relation, and $X \subseteq U$. If a new object $x_t (x_t \notin U, x_t \in U')$ is added, then we obtain*

(1) $\forall x_s \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)$, *if $x_t \in [x_s]^{\geq}_{R_i}{}'$ for every $R_i (i = 1, 2, \ldots, m)$, then $x_s \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)'$*
.

(2) $\forall x_s \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)$, *if exists a granule R_i such that $x_t \in [x_s]^{\geq}_{R_i}{}'$, then $x_s \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)'$*
.

Proof

(1) Because $x_s \in \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)$, then we have $x_s \in X$. However, $x_t \notin X$, so $x_s \neq x_t$. Due to $x_t \in [x_s]^{\geq}_{R_i}{}'$, $[x_s]^{\geq}_{R_i}{}' \nsubseteq X$ for each $R_i (i = 1, 2, \ldots, m)$. Then $x_s \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)'$.
(2) It is easy to obtain the related results.

□

Example 4.5 (Continuation of Example 4.3) Consider the target set $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$, adding objects x_{11} and x_{12}, we discuss local multigranulation optimistic and pessimistic approximations as follows.

After adding new objects x_{11}, x_{12}, the local dominance classes are
$[x_2]^{\geq}_{R_1}{}' = \{x_1, x_2, x_{10}, x_{12}\}$, $[x_4]^{\geq}_{R_1}{}' = \{x_4, x_6, x_{10}, x_{12}\}$, $[x_5]^{\geq}_{R_1}{}' = \{x_3, x_5\}$,
$[x_7]^{\geq}_{R_1}{}' = \{x_7, x_8, x_9, x_{11}\}$, $[x_8]^{\geq}_{R_1}{}' = \{x_8\}$, $[x_9]^{\geq}_{R_1}{}' = \{x_8, x_9, x_{11}\}$.
$[x_2]^{\geq}_{R_2}{}' = \{x_2, x_3, x_4, x_6\}$, $[x_4]^{\geq}_{R_2}{}' = [x_6]^{\geq}_{R_2}{}' = \{x_4, x_6\}$, $[x_5]^{\geq}_{R_2}{}' = \{x_3, x_5\}$,
$[x_7]^{\geq}_{R_2}{}' = \{x_7\}$, $[x_8]^{\geq}_{R_2}{}' = \{x_7, x_8\}$, $[x_9]^{\geq}_{R_2}{}' = \{x_3, x_5, x_7, x_8, x_9, x_{11}\}$.

Due to $\{x_7, x_8, x_9\} = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)$, $x_{11} \in [x_9]^{\geq}_{R_1}{}'$ and $x_{11} \in [x_9]^{\geq}_{R_2}{}'$, then $x_9 \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{O}(X)'$. Meanwhile, $\{x_7, x_8\} = \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)$ and $x_{11} \in [x_7]^{\geq}_{R_1}{}'$, so we have $x_7 \notin \underline{\sum_{i=1}^{m} R_i^{\geq}}_L^{P}(X)'$.

Proposition 4.1 is about the decision rules for updating approximations on global multigranulation rough set in ordered information systems when the object x_t not belonged to the target concept X is deleted, Proposition 4.2 is about the decision rules for updating approximations on global multigranulation rough set in ordered information systems when the object x_t belonged to the target concept X is deleted, and Proposition 4.3 is about the decision rules for the global method when a new object is added. Propositions 4.4 and 4.5 are about the decision rules for updating approximations on local multigranulation rough set in ordered information systems when an object x_t is deleted and added, respectively.

5 The algorithm of updating approximations with dynamic objects

Before discussing the updating algorithms of dynamic objects in global and local background, we need to point out that the pessimistic multigranulation rough approximation dynamic updating algorithm in ordered information system is similar to the optimistic multigranulation rough approximation dynamic updating algorithm. In this section, we only investigate the optimistic multigranulation rough approximation dynamic updating algorithm in ordered information systems, the algorithms in pessimistic situation could be deduced similarly. In the following, we provide different algorithms based on adding objects and deleting objects, and compare static updating and dynamic updating about global and local multigranulation rough approximations in ordered information systems. Since the pessimistic multigranulation algorithm is similar to optimistic multigranulation algorithm, it could be obtained similarly, so we just discuss the related algorithms for optimistic multigranulation rough set approximations here. Table 3 is about necessary notations and symbols used in the algorithm.

Let us focus on the following algorithms for dynamic updating. Algorithm 1 is about the dynamic updating algorithm for computing the global optimistic lower and upper approximations in ordered information systems when deleting objects from the whole universe. Firstly, we give the initial value, that is, optimistic lower and upper approximations are empty sets. Then the second step computes the dominance classes of all objects. After the third step about deleting the object x_t, the fourth step is to adopt the dynamic updating method, which is based on the dominance classes in the second step, and only needs to delete the corresponding elements from the existing dominance classes. Finally, the global optimistic lower and upper approximations are calculated based on the changed dominance classes. Algorithm 2 is the dynamic updating algorithm for computing the local optimistic lower and upper approximations in ordered information systems when removing objects from the whole universe. Firstly, we give the initial value, that is, optimistic lower and upper approximations are empty sets. Then, the second step computes the dominance classes in the target set. After deleting the object x_t in the third step, the fourth step is to adopt the dynamic updating method, which is based on the dominance classes in the second step, and only needs to delete the corresponding elements from the existing dominance classes in the target set. Finally, the local optimistic lower and upper approximations are calculated based on the changed dominance classes. The time complexity for each step in Algorithms 1 and 2 is shown in Table 4.

Table 3 Symbolic notation

Name	Symbolic
An ordered information system	$I^{\geq}$
Dominance relations	R_i
A target concept	X
Domiance classes based on R_i	$[x_j]^{\geq}_{R_i}$
Original lower approximation	LA
Original upper approximation	UA
Global optimistic multigranulation lower approximation	GML
Global optimistic multigranulation upper approximation	GMU
Local optimistic multigranulation lower approximation	LML
Local optimistic multigranulation upper approximation	LMU

Algorithm 1: The dynamic updating algorithm for global optimistic multigranulation approximations when deleting objects in ordered information systems

Input : (1) $I^{\geq} = (U, AT, F)$, $R_i \subseteq AT(i = 1, 2, \ldots, m)$; (2) X, $x_t \in U$.
Output : GML, GMU.
begin
 set $GMU \leftarrow \emptyset, GML \leftarrow \emptyset$;
 for $i = 1 : m$ **do**
  for $j = 1 : |U|$ **do**
   compute $[x_j]^{\geq}_{R_i}$; /* global dominance classes */
  end
 end
 delete the object x_t;
 for $i = 1 : m$ **do**
  for $j = 1 : (|U| - 1)$ **do**
   if $x_t \in [x_j]^{\geq}_{R_i}$ **then**
    $[x_j]^{\geq'}_{R_i} = [x_j]^{\geq}_{R_i} - \{x_t\}$; /* global dominance classes after deleting x_t */
   else
    $[x_j]^{\geq'}_{R_i} = [x_j]^{\geq}_{R_i}$;
   end
  end
 end
 for $j = 1 : (|U| - 1)$ **do**
  for $i = 1 : m$ **do**
   if $[x_j]^{\geq'}_{R_i} \subseteq X$ **then**
    $GML = GML \cup \{x_j\}$; /* global optimistic approximations after removing x_t */
   end
   if $[x_j]^{\geq'}_{R_i} \cap X = \emptyset$ **then**
    break;
   end
  end
  $GMU = GMU \cup \{x_j\}$;
 end
 return : GMU, GML;
end

Before deleting objects dynamically, the global and local algorithms in the second step are calculating the corresponding dominance classes. The dominance classes of all objects based on each granularity need to be computed in the global algorithm and its complexity is $O(m \times |U|^2)$. The local algorithm only need to compute the dominance classes of objects that are included in target concepts. Therefore, its complexity is

Table 4 The time complexity of Algorithms 1 and 2

Step	Global	Local
2	$O(m \times \lvert U\rvert^2)$	$O(m \times \lvert X\rvert\lvert U\rvert)$
4	$O(m \times \lvert U-1\rvert)$	$O(m \times \lvert X\rvert)$
5	$O(m \times \lvert U-1\rvert)$	$O(m \times \lvert X\rvert)$
Others	0	0

$O(m \times |X||U|)$. Due to $|X| \ll |U|$, we have $O(m \times |X||U|) \ll O(m \times |U|^2)$. In the following steps, the complexity of the local algorithm is less than that of the global algorithm. Thus, when the capacity of dataset becomes larger and larger, the complexity of the local algorithm is much less than that of the global algorithm, and the advantages are more obvious.

Algorithm 2: The dynamic updating algorithm for local optimistic multigranulation approximations when deleting the object in ordered information systems

```
   Input  : (1) I^≥ = (U, AT, F), R_i ⊆ AT (i = 1, 2, ..., m); (2) X, x_t ∈ U.
   Output : LML, LMU.
1  begin
2      set LMU ← ∅, LML ← ∅;
3      for i = 1 : m do
4          for j = 1 : |U| do
5              if x_j ∈ X then
6                  compute [x_j]^≥_{R_i};                      /* local dominance classes */
7              end
8          end
9      end
10     delete the object x_t;
11     for i = 1 : m do
12         for j = 1 : (|U| − 1) do
13             if x_j ∈ X then
14                 if x_t ∉ X then
15                     [x_j]^≥'_{R_i} = [x_j]^≥_{R_i} − {x_t};  /* local dominance classes after deleting x_t */
16                 else
17                     delete [x_t]^≥_{R_i};
18                 end
19             end
20         end
21     end
22     for i = 1 : m do
23         for j = 1 : (|U| − 1) do
24             if x_j ∈ X then
25                 if [x_j]^≥'_{R_i} ⊆ X then
26                     LML = LML ∪ {x_j};       /* local optimistic approximations after removing x_t */
27                 end
28                 LMU_i = LMU_i ∪ [x_j]^≥'_{R_i};
29             end
30         end
31         LMU = ⋂_{i=1}^{m} LMU_i;
32     end
       return : LMU, LML;
33 end
```

Algorithms 3 and 4 are respectively dynamic updating algorithms for computing the global and local optimistic lower and upper approximations with adding the object x_t in ordered information systems. The method of adding new objects is same to the former two algorithms. For the incremental algorithm, take adding a single object as an example, the new objects are inserted in a dynamic way according to the steps, and the approximation values after updating are calculated. The explanation for each step is based on the steps previously deleted. Finally, based on the definition of optimistic approximation, global updating and local updating algorithms are presented.

Algorithm 3: The dynamic updating algorithm for global optimistic multigranulation approximations when adding the object in ordered information systems

Input : (1) Original $[x_j]^{\geqslant}_{R_i}$, LA, UA; (2) X, $x_t \notin U$.
Output : GML, GMU.

```
begin
    set GMU ← ∅, GML ← ∅;
    add the object x_t;
    for j = 1 : |LA| do
        for i = 1 : m do
            if all (x_t(i) ≥ x_j(i)) then
                GML = LA − {x_j};
                                        /* global lower approximation after adding x_t */
            else
                GML = LA ∪ {x_j};
                break ;
            end
        end
    end
    for j = 1 : |UA| do
        for i = 1 : m do
            if all (x_t(i) ≥ x_j(i)) then
                GMU = UA ∪ {x_t};
                                        /* global upper approximation after adding x_t */
            else
                GMU = UA − {x_t};
                break ;
            end
        end
    end
    return : GMU, GML;
end
```

Algorithm 4: The dynamic updating algorithm for local optimistic multigranulation approximations when adding the object in ordered information systems

```
Input   : (1) I^≥ = (U, AT, F), R_i ⊆ AT (i = 1, 2, ..., m); (2) X, x_t ∉ U.
Output  : LML, LMU.
begin
    set LMU ← ∅, LML ← ∅;
    for i = 1 : m do
        for j = 1 : |U| do
            if x_j ∈ X then
                compute [x_j]^≥_{R_i};          /* initial local dominance classes */
            end
        end
    end
    add the object x_t;
    for i = 1 : m do
        for j = 1 : (|U| + 1) do
            if x_j ∈ X then
                if all (x_t(i) ≥ x_j(i)) then
                    [x_j]^≥'_{R_i} = [x_j]^≥_{R_i} − {x_t};   /* local dominance classes after adding x_t */
                end
            end
        end
    end
    for i = 1 : m do
        for j = 1 : (|U| + 1) do
            if x_j ∈ X then
                if [x_j]^≥'_{R_i} ⊆ X then
                    LML = LML ∪ {x_j};   /* local optimistic approximations after adding x_t */
                end
                LMU_i = LMU_i ∪ [x_j]^≥'_{R_i};
            end
        end
        LMU = ⋂_{i=1}^{m} LMU_i;
    end
    return  : LMU, LML;
end
```

To facilitate comparison, we also need to analyze the time complexity of static methods after adding objects into the universe. After adding the object x_t, the time complexity of static global method is $O(m \times |U+1|^2)$, while that of static local method is $O(m \times |X||U+1|)$. Due to $|X| \ll |U|$, the complexity of local method is much less than that of global method for static updating. Compared with static method, dynamic updating reduces many ineffective and onerous computational processes. After adding new object x_t, there is no need to compute dominance classes. The target information is dynamically depicted by the approximation before updating. Therefore, the time complexity of dynamically increasing objects is the sum of the lower approximation complexity and upper approximation complexity, which is $O(m \times |LA|) + O(m \times |UA|) = O(m \times |UA|)$. As for the dynamically increasing local algorithm, it only updates the dominance classes of the target object, so its local time complexity is $O(m \times |X|)$. The above results have been shown in Table 5. Usually, the upper approximation contains the concept of the target $|UA| > |X|$. Therefore, the cardinality of the upper approximation is larger than that of the target set. Therefore, when adding objects dynamically, the local time complexity is less than the global time complexity.

6 Experimental studies

In order to further illustrate the advantages of local multigranulation algorithms under the background of dynamic updating to deal with changing information in an ordered information system, some experiments are carried out using six datasets from UCI (http://archive.ics.uci.edu/ml/datasets.html). The time consumption of global algorithm and local algorithm using dynamic updating method is compared. At the same time, the local algorithm under dynamic updating and local algorithm under static updating are also compared. Detailed information for the tested datasets is shown in Table 6. These experiments are implemented by using Matlab R2014b and performed on a personal computer with an Intel Core i7-6500, 2.50 GHz CPU, 4.0 GB of memory, and 64-bit Windows 10.

The experimental part includes two parts: the deletion of existing objects and the addition of new objects. Each dataset is added and deleted according to the proportion of its own data. The possible differences between global method and local method depend on the size of the target concept. No matter how large the target concept is, the target concept will be a subset of the whole universe. To be sure, the closer the approximated concept is to the whole universe, the less significant the difference in the time they take. We just select and show one of the target concept to verify and illustrate our approach, so we use the computer program to randomly generate a target set to make sure it is randomly selected. The multigranulation means two or more than two granulations. In the study of rough set theory, a subset of attribute set can generate a granulation for the multigranulation, which also forms a binary relation (in this case, dominance relation). Without loss of generality, we considered two granulations in the experiments for each dataset.

In terms of deleting objects, we randomly delete objects in the original dataset according to percentages. Since the deleted objects are randomly selected, it is possible to delete the elements in the target set, thus affecting the time results of the local multigranulation dynamic updating algorithm. That is to say, it takes less time to compute approximation after deleting fewer objects than after deleting more objects. However, the comparison between the global algorithm and the local algorithm will not be affected. Because the time difference is small, this does not affect the final overall trend when comparing the time results of global and local algorithms. According to Algorithms 1 and 2, the global and local dynamic updating algorithms are given to obtain the approximation time for the constantly changing information. Similarly, after deleting objects, the time consumption of local multigranulation rough sets with static updating and dynamic updating is compared. On the other hand, 50% of the original dataset elements are used as the initial data before adding objects, and then the remaining datasets are proportionally added to the initial data. Therefore, according to Algorithms 3 and 4, the computational time of global dynamic increment algorithm and local dynamic increment algorithm is compared. Furthermore, the approximation time of local multigranulation rough sets is calculated by dynamic and static updating methods respectively after adding objects, and then further compared.

In order to ensure that the experiment could be carried out effectively, data are pre-processed in the experiment. The global and local comparisons of the optimistic approximation are given below. The pessimistic approximation is similar to the optimistic approximation and is not given repeatedly. Similarly, only static and dynamic comparison results of optimistic approximation are given, and the comparison results of pessimistic approximation are omitted. Before we analyze the data shown from Tables 7, 8, 9, and 10, we need to point out that the unit of the time consumption is seconds.

Table 7 concludes the time results of six datasets about optimistic approximations, which represent the time consumption required lower and upper approximations for global and local multigranulation rough sets under dynamic deletion algorithms. From Table 7, the computation time of local dynamic algorithm is much less than that of the global dynamic algorithm after objects are deleted. The ratio of deleted objects as abscissa and time consumption is used as ordinate. With the increase of the ratio of deleted objects, the time changing trend chart of dynamic updating consumption can be obtained in Fig. 1. As the number of deleted objects increases, computing the approximation time becomes shorter. However, the advantage of local dynamic algorithm time still can be seen intuitively. The local dynamic algorithm is much better than the global dynamic algorithm in ordered information systems.

Table 8 shows the time consumed by dynamic deletion and static deletion of local multigranulation rough sets to obtain approximations respectively. Although the time difference between the dynamic deletion algorithm and the static deletion algorithm is not as big as that between the global and local algorithms, we can see that the static algorithm and the dynamic algorithm change with the increase of the number of objects in the dataset. When the dataset is bigger, the advantage of the dynamic algorithm of the local multigranulation rough set is more obvious based on dominance relation. The specific change trend chart is shown in Fig. 2.

Table 9 shows the time consumption of lower and upper approximations required for global and local multigranulation rough sets based on dynamic increase algorithms. As it can be seen from Table 9, with the increase of the number of objects in the dataset, the difference between the computation time of the local dynamic algorithm and that of the global dynamic algorithm becomes much more obvious. Similarly, a rectangular coordinate system is established based on the increased proportion and the time consumed. The time change trend chart of the dynamic updating of the six datasets is shown in Fig. 3. As the number of objects increases, the approximation time of computation becomes longer. After adding new objects, the local dynamic algorithm is superior to the global dynamic algorithm in ordered information systems.

Table 10 shows the time consumed to obtain the lower and upper approximations of local multigranulation rough sets by dynamic increment and static increment algorithms, respectively. Since they are all based on the local multigranulation rough set, only the objects related to the target set need to be concerned. Therefore, dynamic local algorithm and static local algorithm can only show the gap on large data, reflecting the advantages of dynamic local algorithm. In Fig. 4, the time difference between static algorithm and dynamic algorithm is getting larger and larger with the increase of objects. In ordered information systems, it shows that the algorithm of local dynamic increase is better.

Table 5 The time complexity of Algorithms 3 and 4

Update	Global	Local
Add statically	$O(m \times \|U+1\|^2)$	$O(m \times \|X\|\|U+1\|)$
Add dynamically	$O(m \times \|UA\|)$	$O(m \times \|X\|)$

7 Conclusions

The information contained in various information systems is constantly updated and changed with the development of information technology. How to efficiently extract useful information from the dynamic information system is a very challenging work in the multi-granulation structure of complex and different information sources. To solve certain limitations of classical rough set models regard to large datasets, the local rough set was first presented by Qian et al. (2017, 2018). It is stated that local rough sets do not need to deal with the information granules of all the objects in the whole dataset, but only need to calculate the information granules of objects in the target concepts, which can significantly improve the computing performance. The main objective of this paper is to build novel local rough set models for a wider application environment by considering local rough approximations from the multigranulation viewpoint in ordered information systems. The dynamic updating algorithms of the constructed local rough approximations are provided with the object variation in ordered information systems. Moreover, we also tested the time consumption of the proposed dynamic algorithm from six different datasets and compared it with the global dynamic algorithm and the static algorithm. From the results reported by the experiment studies about adding and deleting objects from the ordered information systems, we could summarize the following two points about the superiority of dynamic object updating based on the local multigranulation rough set models in ordered information systems: (1) the computation time of the local dynamic updating algorithm is much less than that of the global dynamic updating algorithm; (2) the computation time of the local dynamic updating algorithm is much less than that of the local static updating algorithm. Among this article, we use examples to interpret and analyze the concepts we studied. The potential applications of this study could mainly embodied in deriving decision rules based on the proposed local multigranulation rough sets in ordered information system. These decision rules could be obtained from the disjoint decision regions. In other words, we can apply our results to classifiers. In terms of classifier, we could treat decision class as target set, and determine the label of objects by observing whether they fall in the positive region of the target set. Also, this dynamic update mechanism in this paper can also be applied to judge whether the object falls within the updated upper and lower approximation, so as to further dynamic classify the objects. In the future work, we will focus on our model as a classifier, and compare it with other existing classical classifiers to obtain its classification effect.

Table 6 The basic information of datasets

No.	Dataset	Abbreviation	Objects	Attributes
1	Absenteeism at work	A-aw	740	21
2	Statlog (image segmentation)	S(IS)	2310	19
3	Page blocks classification	PBC	5473	11
4	Combined cycle power plant	CCPP	9568	5
5	Crowdsourced mapping	CM	10,545	28
6	MAGIC gamma telescope	MGT	19,020	11

Table 7 Computation time between local and global optimistic multigranulation when deleting objects

Percentage (%)	A-aw		S(IS)		PBC		CCPP		CM		MGT	
	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local
5	0.4691	0.0623	2.5630	0.0469	6.0158	0.0310	9.1559	0.0313	9.8751	0.0780	19.4281	0.0316
10	0.3903	0.0623	2.5154	0.0313	5.5159	0.0310	9.0629	0.0313	8.9228	0.0310	16.4534	0.0316
15	0.3596	0.0623	2.4224	0.0313	4.8911	0.0160	8.3599	0.0313	8.0628	0.0310	16.8611	0.0160
20	0.3593	0.0000	2.2658	0.0156	4.6114	0.0160	8.0310	0.0160	7.5624	0.0310	15.5941	0.0160
25	0.3286	0.0000	2.0629	0.0156	4.1561	0.0160	6.8129	0.0160	7.0316	0.0310	14.5628	0.0316
30	0.3489	0.0000	1.7661	0.0000	4.1206	0.0160	6.6720	0.0000	6.2816	0.0310	12.8906	0.0160
35	0.2810	0.0000	1.7028	0.0000	3.8441	0.0160	6.0470	0.0160	6.2030	0.0310	11.5941	0.0160
40	0.2498	0.0000	1.4845	0.0000	3.4376	0.0160	5.3121	0.0160	5.6561	0.0310	11.2811	0.0160
45	0.2188	0.0000	1.3436	0.0000	2.9528	0.0000	4.9998	0.0160	4.7341	0.0160	9.8805	0.0160
50	0.2028	0.0000	1.1715	0.0000	2.6409	0.0000	4.5783	0.0000	4.4216	0.0000	8.6724	0.0160

Table 8 Optimistic local approximations computation time between static and dynamic updating when deleting objects

Percentage (%)	A-aw		S(IS)		PBC		CCPP		CM		MGT	
	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic
5	0.0623	0.0160	0.1409	0.0469	0.2190	0.0310	0.2810	0.0310	0.4380	0.0780	0.5936	0.0316
10	0.0623	0.0160	0.1253	0.0313	0.1720	0.0310	0.2500	0.0160	0.3910	0.0310	0.5310	0.0160
15	0.0623	0.0160	0.1094	0.0313	0.1640	0.0160	0.2156	0.0160	0.3750	0.0310	0.5040	0.0160
20	0.0310	0.0000	0.0892	0.0156	0.1560	0.0160	0.1720	0.0000	0.3130	0.0310	0.4220	0.0160
25	0.0310	0.0000	0.0781	0.0156	0.1090	0.0160	0.1560	0.0000	0.2750	0.0310	0.3716	0.0316
30	0.0310	0.0000	0.0541	0.0000	0.0840	0.0160	0.1420	0.0000	0.1880	0.0310	0.3126	0.0316
35	0.0160	0.0000	0.0496	0.0000	0.0561	0.0000	0.1130	0.0000	0.1720	0.0310	0.2853	0.0160
40	0.0160	0.0000	0.0310	0.0000	0.0561	0.0000	0.0940	0.0000	0.1410	0.0310	0.2660	0.0160
45	0.0000	0.0000	0.0310	0.0000	0.0561	0.0000	0.0940	0.0000	0.1090	0.0160	0.2310	0.0160
50	0.0000	0.0000	0.0160	0.0000	0.0470	0.0000	0.0940	0.0000	0.0940	0.0000	0.1560	0.0160

Table 9 Computation time between local and global optimistic multigranulation when adding objects

Percentage (%)	A-aw		S(IS)		PBC		CCPP		CM		MGT	
	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local
10	0.0780	0.0000	0.1560	0.0160	0.3590	0.0160	0.9220	0.0310	0.5470	0.0470	2.2190	0.0630
20	0.1560	0.0000	0.2970	0.0316	0.6560	0.0313	2.0000	0.0630	1.1250	0.0936	4.3130	0.1090
30	0.2190	0.0000	0.4530	0.0473	1.0310	0.0630	2.8130	0.0780	2.7810	0.1716	6.4390	0.2030
40	0.2660	0.0316	0.6090	0.0623	1.5160	0.0786	3.7660	0.1250	4.3440	0.2813	8.5470	0.2500
50	0.3440	0.0316	0.7970	0.0783	2.0780	0.0940	4.6880	0.1560	4.5160	0.3096	10.8440	0.2810
60	0.3910	0.0316	0.9220	0.0783	2.3750	0.1090	5.7030	0.1876	4.8310	0.3906	12.7030	0.3750
70	0.4690	0.0316	1.2190	0.0940	2.6090	0.1250	6.5310	0.2190	5.1720	0.5073	15.1560	0.4216
80	0.5470	0.0316	1.2190	0.0940	3.0780	0.1716	7.5160	0.2496	6.5470	0.5629	16.5630	0.5276
90	0.6410	0.0466	1.3590	0.0940	3.8590	0.2036	8.5630	0.2816	7.2810	0.6093	18.8750	0.5310
100	0.7340	0.0466	1.5470	0.1096	4.3090	0.2343	9.5470	0.2970	7.4380	0.7035	20.4060	0.5786

Table 10 Optimistic local approximations computation time between static and dynamic updating when adding objects

Percentage (%)	A-aw		S(IS)		PBC		CCPP		CM		MGT	
	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic	Static	Dynamic
10	0.0310	0.0000	0.0940	0.0160	0.2340	0.0160	0.3280	0.0310	0.5160	0.0470	0.6250	0.0630
20	0.0310	0.0000	0.1246	0.0316	0.2500	0.0313	0.3590	0.0630	0.5625	0.0936	0.6560	0.1090
30	0.0310	0.0000	0.1403	0.0473	0.2560	0.0630	0.3910	0.0780	0.7346	0.1716	0.7190	0.2030
40	0.0466	0.0000	0.1403	0.0623	0.2956	0.0786	0.4220	0.1250	0.9063	0.2813	0.7810	0.2500
50	0.0466	0.0156	0.1563	0.0783	0.2970	0.0940	0.4380	0.1560	1.0156	0.3096	0.8590	0.2810
60	0.0542	0.0316	0.1563	0.0783	0.3280	0.1090	0.5156	0.1876	1.1096	0.3906	0.8910	0.3750
70	0.0626	0.0316	0.1410	0.0940	0.3440	0.1250	0.5270	0.2190	1.1563	0.5073	0.9846	0.4216
80	0.0692	0.0316	0.1410	0.0940	0.3746	0.1716	0.5466	0.2496	1.2059	0.5629	1.0246	0.5276
90	0.0786	0.0466	0.1560	0.0940	0.4376	0.2036	0.5786	0.2816	1.2813	0.6093	1.1090	0.5310
100	0.0786	0.0466	0.1716	0.1096	0.4933	0.2343	0.6090	0.2970	1.3375	0.7035	1.1406	0.5786

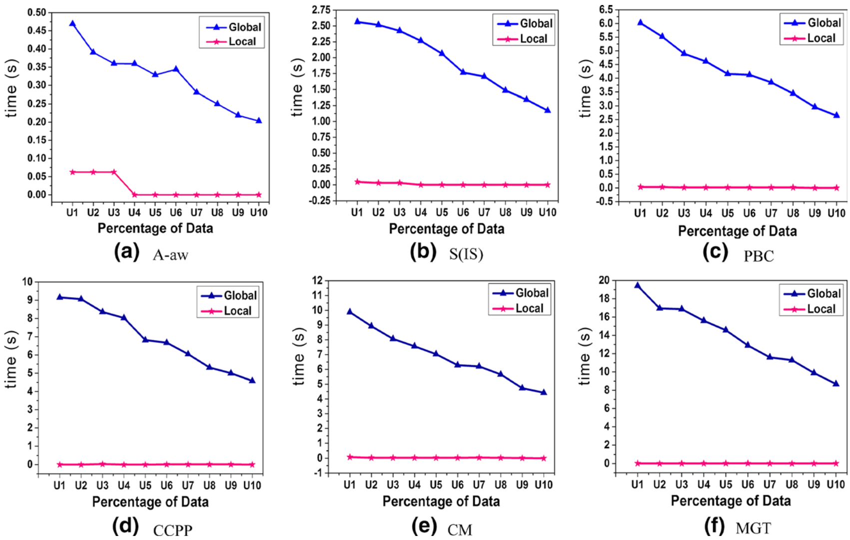

Fig. 1 The computation time comparison of dynamic updating between local algorithm and global algorithm when deleting objects

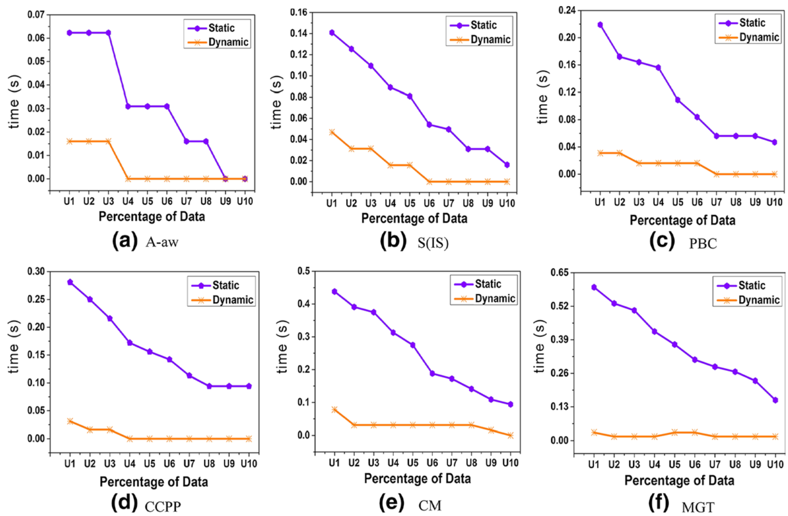

Fig. 2 The computation time comparison of local algorithm between static updating and dynamic updating when deleting objects

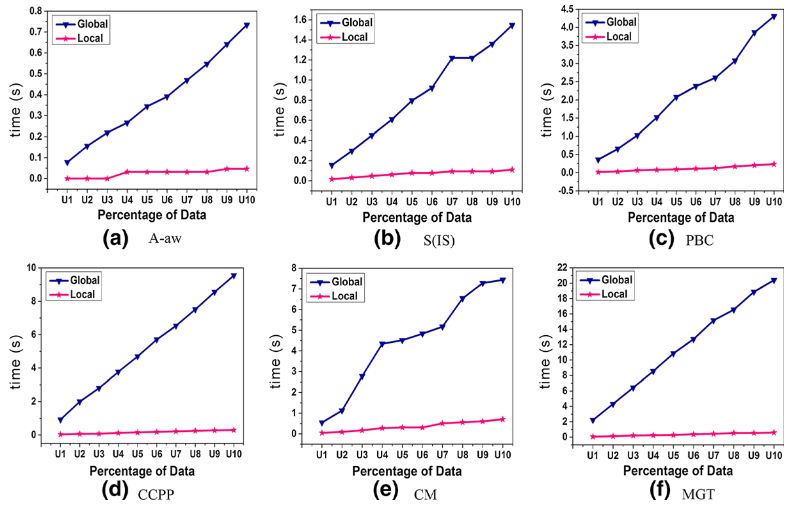

Fig. 3 The computation time comparison of dynamic updating between local algorithm and global algorithm when adding objects

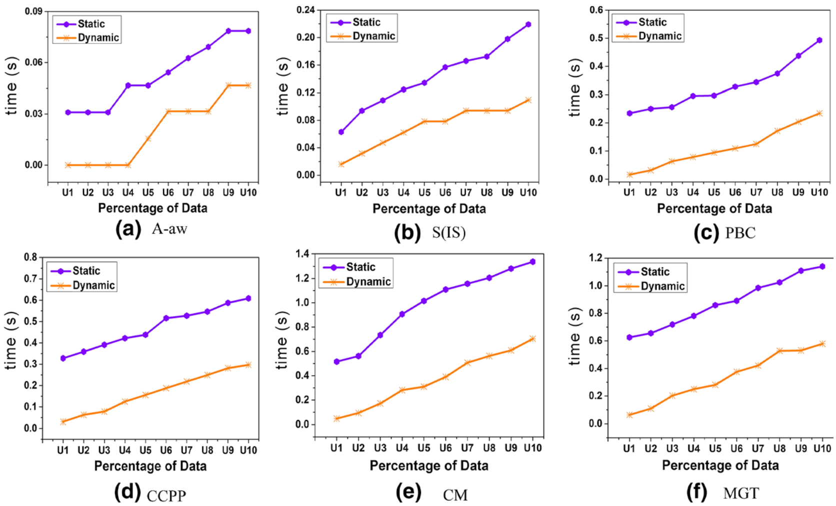

Fig. 4 The computation time comparison of local algorithm between static updating and dynamic updating when adding objects

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Declarations

Conflict of interest There is no conflict of interest.

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