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An incremental learning mechanism for object classification based on progressive fuzzy three-way concept

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ABSTRACT

Fuzzy concept has been an important methodology for data analysis, especially in the classification research. Particularly, fuzzy concept could directly process the continuous data through contrasting the numerical data into the membership degree of object to attribute. However, the classical fuzzy concept only focuses on the positive information, that is, the information about membership degree, while ignoring non-membership degree. Meanwhile, since the limitations of individual cognition and cognitive environment, the concept learning is progressive. Inspired by these thoughts, we design an incremental learning mechanism based on progressive fuzzy three-way concept for object classification in dynamic environment. In this paper, the object and attribute learning operators are first defined to obtain fuzzy three-way concept. Then, a progressive fuzzy three-way concept and its corresponding concept space are learned considering the progressive process of concept learning. Moreover, the object classify mechanism and dynamic update mechanism based on the progressive concept space are proposed, and their effectiveness is verified by numerical experiments. Finally, an incremental learning mechanism is further designed for dynamic increased data and compared with other fuzzy classify methods. All the experimental results carried on ten datasets from UCI and KEEL illustrate the proposed learning mechanism is an excellent object classify algorithm.

1. Introduction

Concept theory, as highly complementary to rough set theory, has been widely used in the applications of rule extraction, object classification, machine learning, and so on [14,17,20,27,30]. The classical concept describes the essential characteristics of the same kind of things through intent and extent, it provides a mathematical foundation for concept learning [12,15,21,22,28,29]. In classical concept, the relation between object and attribute is either one or the other, that is, an attribute is owned or not owned by the object. Therefore, it can only deal with the discrete data such that continuous data needs to be discretized by data preprocessing, this method would lose some useful information in learning process [12,15,21]. Also, it is noted that the intent is only composed of attributes shared by the objects in extent, but ignores the common unsuccessful attributes. The incompleteness of the information is easy to cause cognitive deviations.

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As an extension of the classical concept, the fuzzy concept can directly deal with the continuous data by transforming the specific attribute values into the membership degree of object to these attributes, which not only reduces the time consumption of discretization, but also retains the original information [4,13,17,30]. In recent years, many scholars have researched the fuzzy concept in information description, recognition pattern, concept clustering, and so on [4,6,17,22]. Cross and Kandasamy first compared the one-sided threshold approach and the fuzzy closure operator approach to obtain fuzzy concept lattice, and found the extent produced by the threshold approach is a subset of the extent produced by the fuzzy closure approach [4]. Xu et al. proposed a novel granular method of machine learning by using formal concept description of information granules, which is valuable to deal with practical issues based on two-way concept in fuzzy set [22]. Compared with classical concept, the fuzzy concept has more flexibility in the choice of concepts in fuzzy formal context. The three-way concept as an another extension of classical concept can describe the conceptual information more comprehensively by further considering the information that objects don't have in common [12,15,21], which has been widely used in many areas, such as decision making, rule extraction, knowledge discovery, data mining, and so on [2,8,9,17,20,21,24,27,30]. Zhan et al. first constructed a three-way decision model in incomplete fuzzy decision systems and applied it to the modeling of incomplete multi-attribute decision-making problems, and then provided a new perspective for realistic incomplete multi-attribute decision-making problems. Their experimental results demonstrate the validity and superiority of their three-way concept decision model [24]. Hao et al. analyzed the stability and properties of three-way concept and applied it to natural language generation [8]. In addition, they further pioneered a novel problem and method for the incremental construction of three-way concept lattice for knowledge discovery in social networks [9]. Moreover, Wei et al. performed the rules acquisition for formal decision contexts from the perspective of three-way concept lattices [21]. The three-way thought provides a new idea for dealing with the problem of data mining.

In natural circumstances, the concept learning is influenced by the cognitive environment and individual cognition, and other factors, thus the concept learning process cannot be completed in one fell swoop [1,17,25]. Recently, some scholars further investigated the concept learning based on the evolutionary computing to overcome the limitation on cognitive environment, and its effectiveness is verified by various experimental results [1,5,7,18,19]. For dealing with interactive concept-based multi-objective problems, Mukhopadhyay et al. introduced a new interactive concept-based multi-objective evolutionary algorithm, which can be used in engineering problem [18]. Dragoni tried to analyze the sentiment expressed within a document by establishing an evolutionary strategy based on the polarity values of concept-domain pairs [1]. Mi et al. further constructed a novel fuzzy concept considering the limitations of cognition and environment, and the experimental results show the proposed concept is effective in cognition recognition [17]. For the large datasets, how to effectively calculate the concept space is very important. To reduce the complexity of the concept cognitive learning process, Zhang et al. designed a concept update algorithm to represent concepts through a concept tree based on attribute topology [25]. Mi et al. designed a concurrent concept cognitive learning mechanism to improve the learning efficiency [19]. In the application of concept learning, the construction of concept and the update of concept space are two important issues, which influences the performance of learning mechanism.

Object classification based on concept similarity is one of the most important application of concept learning. The label of object can be obtained according to the concept of minimum distance from it. In this paper, we focus on designing a classify mechanism based on concept space. We first define the object and attribute learning operators to obtain fuzzy three-way concept. Then, we construct a progressive fuzzy three-way concept considering the limitations of individual cognition and incompleteness of cognitive environment. Meanwhile, the similar indicator is designed to classify objects based on the progressive concept space. To make full use of the information about added objects, we also propose an incremental learning mechanism for the further concept learning. Finally, the effectiveness of proposed classify method is verified by the numerical experiments. The block diagram of steps of the proposed approach is shown in the Fig. 1.

The remainder of this paper is organized as follows. In Section 2, we review the basic notions of regular formal context, three-way concept, fuzzy formal context, fuzzy concept, and our motivation. In addition, the learning process of progressive fuzzy three-way concept is shown in Section 3. Section 4 introduces how to classify the object and how to update the concept space dynamically, and then designs an incremental learning mechanism based on progressive fuzzy three-way concept (ILMPFTC). Moreover, to illustrate the validity of the ILMPFTC, some experiments are carried on ten datasets from UCI and KEEL in Section 5. Finally, Section 6 covers some conclusions.

2. Preliminaries

In this section, we review some basic notions about fuzzy set, fuzzy concept, and three-way concept for the concept learning process, the details of which can be obtained from their corresponding references [15,26].

2.1. Regular formal context

The $\left(X, A, \widetilde{R}\right)$ is a regular formal context, where $X = \{x_1, x_2, \ldots, x_n\}$ is the object set and $A = \{a_1, a_2, \ldots, a_m\}$ is the attribute set. $\widetilde{R} \subseteq X \times A$ is the binary relation between X and A. $\widetilde{R}(x, a) \in \{0, 1\}$, and $\widetilde{R}(x, a) = 1$ denotes $(x, a) \in \widetilde{R}$ that reflects object x has the attribute a or the a is owned by x.

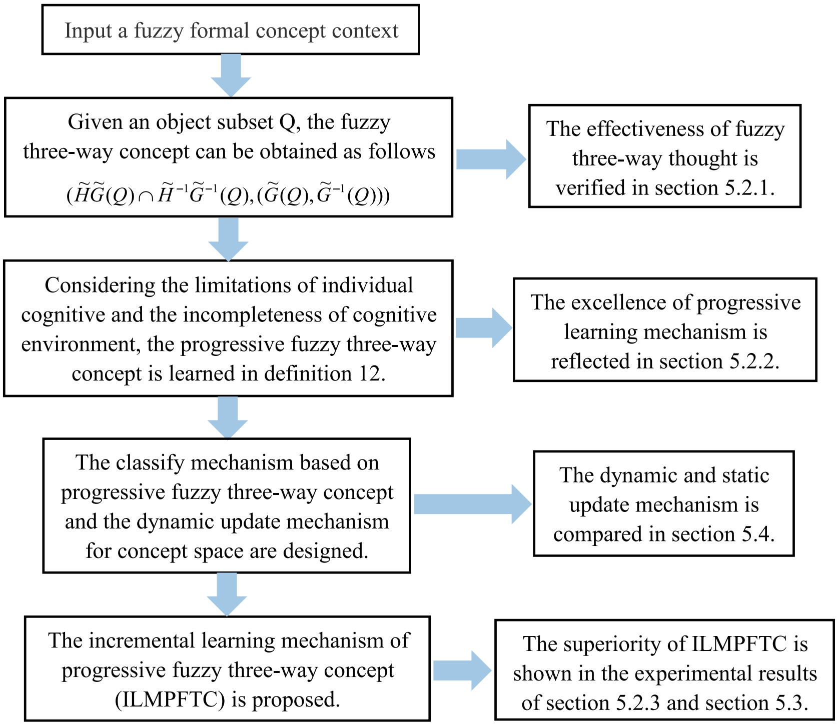

Fig. 1. Block diagram of steps of the proposed approach.

Definition 1. Let $\left(X,A,\widetilde{R}\right)$ be a regular formal context, where $\widetilde{R}$ is the binary relation. $P(X)$ and $P(A)$ represent the power sets of X and A. $\forall Q\in P(X)$ and $B\in P(A)$, the object learning operator $\diamondsuit : P(X)\longrightarrow P(A)$ and attribute learning operator $* : P(A)\longrightarrow P(X)$ are described as follows [17]:

$$Q^{\diamondsuit} = \left\{a\in A|\forall x\in Q,(x,a)\in\widetilde{R}\right\},\quad B^{*} = \left\{x\in X|\forall a\in B,(x,a)\in\widetilde{R}\right\}. \tag{1}$$

Among this definition, the operator $\diamondsuit$ and $*$ reflect the common information between object and attribute, which is called the positive learning operator.

Definition 2. Let $\left(X,A,\widetilde{R}\right)$ be a regular formal context, where $\widetilde{R}$ is the binary relation. $\forall Q\in P(X), B\in P(A)$, the (Q,B) is called concept when [17]

$$Q^{\diamondsuit} = B, B^{*} = Q. \tag{2}$$

The Q and B are called the intent and extent of this concept, it reflects the attributes shared by objects and all objects that have these attributes. In real life, some objects have the same attributes, but the attribute that they don't have is obviously different. Thus, it still has limitation to process the data only from the common information they share. The proposition of three-way concept settles this limitation well.

2.2. Three-way concept

In this subsection, we also need to define negative operators $\diamondsuit$- and $*$- to obtain the information that object and attribute don't have.

Definition 3. Let $\left(X,A,\widetilde{R}\right)$ be a regular formal context, $\forall Q\subseteq X$ and $B\subseteq A, \widetilde{R}^{c} = X\times A-\widetilde{R}$. The negative operator $\diamondsuit- : P(X)\longrightarrow P(A)$ and $*- : P(A)\longrightarrow P(X)$ are described as follows [15]:

$$Q^{\Diamond -} = \left\{a \in A | \forall x \in Q, (x, a) \in \widetilde{R}^c\right\},$$
$$B^{*-} = \left\{x \in X | \forall a \in B, (x, a) \in \widetilde{R}^c\right\}. \quad (3)$$

The negative operator could induce the information that object and attribute don't have in common. The object can be described in more detail from both positive and negative perspectives.

Definition 4. Let $\left(X, A, \widetilde{R}\right)$ be a regular formal context, where $BP(A) = P(A) \times P(A)$ is the collection of order pairs about attributes. Define three-way operator $\lhd : P(X) \to BP(A), \rhd : BP(A) \to P(X)$. $\forall Q \subseteq X$ and $B \subseteq A, Q^{\lhd} = \left(Q^{\Diamond}, Q^{\Diamond -}\right)$ and $B^{\rhd} = B^* \cap B^{*-}$. $\forall$ $Q \subseteq X$ and $B_1, B_2 \in L^A$, the $(Q, (B_1, B_2))$ is called object-oriented three-way concept when [15]

$$Q^{\lhd} = (B_1, B_2), (B_1, B_2)^{\rhd} = Q. \quad (4)$$

The Q and (B_1, B_2) are called the extent and intent of the three-way concept. Since it contains both positive and negative information, this concept can describe object more detail.

2.3. Fuzzy formal context

The three-way concept can directly process the discrete data, and the continuous data only could be learned after discretization, this is a time-consuming process and could lost some useful information. Therefore, the fuzzy set is adopted to construct fuzzy formal context.

Definition 5. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a non-empty object set, the fuzzy set $\widetilde{F}$ on X is described as follows [26]:

$$\widetilde{F} = \left\{< x, \mu_{\widetilde{F}}(x) > | x \in X\right\}, \quad (5)$$

where $\mu_{\widetilde{F}}(x) \in [0, 1]$, which denotes the membership of object x with respect to $\widetilde{F}$, and $\mu^{c}_{\widetilde{F}}(x) = 1 - \mu_{\widetilde{F}}(x)$ is the non-membership degree. The L^X denotes the set of all fuzzy sets on X.

In formal concept analysis, the concept is defined as a unit composed of intent and extent. The intent reflects the essential attributes of this concept, and the extent includes all objects that have essential attributes. The crisp relation between object and attribute is limited in numerical data. Thus, the fuzzy formal context is proposed to describe the relation between object and attribute.

Definition 6. The triplet $\left(X, A, \widetilde{R}\right)$ is a fuzzy formal context, where $X = \{x_1, x_2, \ldots, x_n\}$ is the object set and $A = \{a_1, a_2, \ldots, a_m\}$ is the attribute set. $\widetilde{R}$ is the fuzzy relation between X and A, each (x, a) has a membership degree $\widetilde{R}(x, a) \in [0, 1]$ to $\widetilde{R}$. $\widetilde{R}(x, a)$ can be explained as the membership degree of object x to attribute a or the degree to which attribute a is owned by object x, thus the $\widetilde{R}$ could be considered as a fuzzy relation between owning and being owned defined on objects and attributes. $\left(X, A, \widetilde{R}, D, \widetilde{E}\right)$ is called fuzzy formal decision context when the D is decision attribute and $\widetilde{E} : A \times D \longrightarrow \{0, 1\}$. If $\widetilde{E}(x, a) = 1$, the object has the attribute a, otherwise, x does not have a.

2.4. Fuzzy concepts

Definition 7. Let $\left(X, A, \widetilde{R}\right)$ be a fuzzy formal context, where the L^A denotes the all fuzzy sets on A. In this paper, the fuzzy set $\widetilde{B} \in L^A$ could be explained as the owned relation of attributes by objects, each $\widetilde{B}(a_j)$ is the owned degree of attribute a_j by objects. $\forall Q \in P(X)$ and $\widetilde{B} \in L^A$, the object learning operator $\widetilde{G} : P(X) \longrightarrow L^A$ and attribute learning operator $\widetilde{H} : L^A \longrightarrow P(X)$ are described as follows [17]:

$$\widetilde{G}(Q)(a_j) = \bigwedge_{x \in Q}\left(\widetilde{R}(x, a_j)\right), a_j \in A,$$
$$\widetilde{H}\left(\widetilde{B}\right) = \left\{x \in X | \widetilde{R}(x, a_j) \geqslant \widetilde{B}(a_j), j = 1, 2, \ldots, m\right\}. \quad (6)$$

The $\left(Q, \widetilde{B}\right)$ is called fuzzy concept if $\widetilde{H}\left(\widetilde{B}\right) = Q, \widetilde{G}(Q)(a_j) = \widetilde{B}$ for $j = 1, 2, \ldots, m$, and the Q and $\widetilde{B}$ are the extent and intent of this fuzzy concept, respectively. Similar to positive learning operator in three-way concept, the operators $\widetilde{G}$ and $\widetilde{H}$ describe the relation between object and attribute through membership degree that is the positive information, thus $\widetilde{G}$ and $\widetilde{H}$ are called positive object and attribute learning operators, respectively.

2.5. Motivation

The classical concept describes objects from these attributes they share, which is only suitable for processing discrete data. When we deal with numerical data, discretization and binarization of data are essential, which would lose some important information that influences further research. The proposition of fuzzy concept settles this limitation well. However, the fuzzy concept only describes the objects from the positive aspect of the commonality of the objects in the set X, ignoring the negative information of the attributes, which impacts the accuracy of the information description. The fuzzy three-way concept settles the above issues well. In this paper, we first define the fuzzy three-way concept, and then investigate the concept learning process based on it. The Example 1 is first given to illustrate the superiority of fuzzy three-way concept compared with fuzzy concept.

Example 1. Table 1 is a fuzzy formal decision context with 28 objects and 2 conditional attributes, where $X_1 = \{x_1, x_2, \ldots, x_{11}\}, X_2 = \{x_{12}, x_{13}, \ldots, x_{19}\}$, and $X_3 = \{x_{20}, x_{21}, \ldots, x_{28}\}$. They are described by b_1 and b_2. The detail corresponding distribution of objects is shown in Fig. 2.

Given the fuzzy formal decision context, suppose the intent and extent of fuzzy concept induced by X_1 are $\widetilde{B_1}$ and Q_1. According to Definition 7, we have $\widetilde{B_1}(b_1) = \bigwedge_{x \in X_1}\left(\widetilde{R}(x, b_1)\right) = 0.17$ and $\widetilde{B_1}(b_2) = \bigwedge_{x \in X_1}\left(\widetilde{R}(x, b_2)\right) = 0.08$, then the fuzzy set $\widetilde{B_1} = 0.17/b_1 + 0.08/b_2$. Also, the extent $Q_1 = \widetilde{H}\left(\widetilde{B_1}\right) = \left\{x \in X_1 | \widetilde{R}(x, b_j) \geqslant \widetilde{B}_1(b_j), j = 1, 2\right\} = (X_1 \cup \{x_{15}, x_{16}, x_{17}, x_{20}, \ldots, x_{28}\})$. The fuzzy concept $\left(Q_1, \widetilde{B}_1\right)$ is learned from the learning operator in Definition 7. Similarly, we could obtain the other two fuzzy concepts $(X_2 \cup \{x_1, x_2, x_3, x_5, x_6, x_8, x_{20}, x_{22}, \ldots, x_{26}\}, (0.02, 0.39))$ and $(X_3, (0.59, 0.32))$ induced by X_2 and X_3. The relative distribution about the extent of fuzzy concepts is shown in Fig. 3. In this figure, the shapes with red, blue and green denote the three different classes data, and the objects surrounded by a frame are in the extent of fuzzy concept. According to this figure, we find the color of the objects in the same box is different, except the extent of $(X_3, (0.59, 0.32))$. In the rose and black frames, there are respectively three and two different colors of objects, that is, there are different classes of objects in the same concept and may cause error in learning process. This example shows the fuzzy concept only induced by the positive information is not accurate in concept learning. Therefore, it is necessary to study the concept learning based on positive and negative information.

3. The learning process of progressive fuzzy three-way concept

The fuzzy concept can deal with numerical data, and the intent of it characterises the lower membership degree of object in a certain set to attribute. The membership degree only describes the positive information, which is limited in concept learning process, the Example 1 illustrates this. Therefore, we further design the negative learning operator to obtain the fuzzy three-way concept inspired by three-way concept. This section mainly investigates the learning process of fuzzy three-way concept and the construction of its corresponding progressive concept space.

3.1. Fuzzy three-way concept

To obtain the fuzzy three-way concept, we further need to design the negative learning operator based on fuzzy concept. The positive operator describes objects from membership degree, the negative depicts information from the opposite viewpoint, that is, the non-membership degree.

Table 1
A fuzzy formal decision context.

Object	b_1	b_2	Class	Object	b_1	b_2	Class
1	0.32	0.63	1	15	0.26	0.39	2
2	0.36	0.52	1	16	0.28	0.41	2
3	0.48	0.83	1	17	0.28	0.47	2
4	0.38	0.08	1	18	0.04	0.47	2
5	0.26	0.80	1	19	0.02	0.60	2
6	0.55	0.86	1	20	0.81	0.59	3
7	0.68	0.11	1	21	0.91	0.32	3
8	0.46	0.76	1	22	0.96	0.66	3
9	0.52	0.26	1	23	0.92	0.73	3
10	0.45	0.23	1	24	0.61	0.87	3
11	0.17	0.26	1	25	0.59	0.88	3
12	0.06	0.65	2	26	0.69	0.52	3
13	0.11	0.56	2	27	0.75	0.34	3
14	0.07	0.40	2	28	0.72	0.45	3

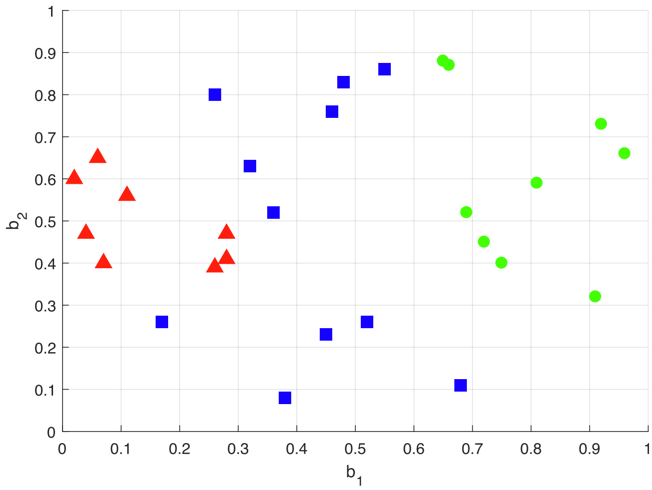

Fig. 2. The distribution of objects depicted in Table 1.

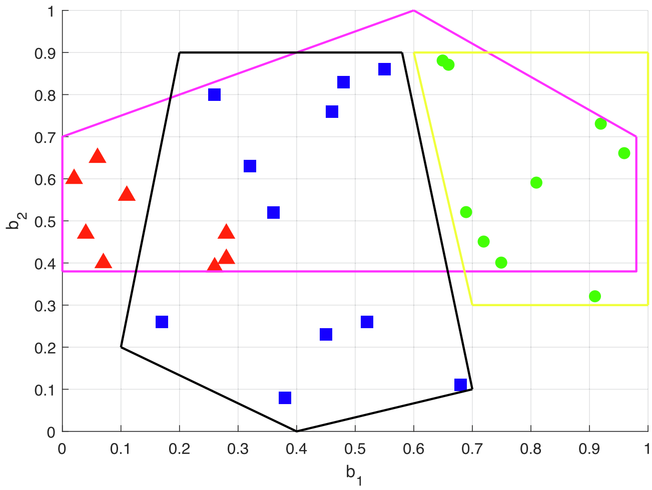

Fig. 3. The distribution of objects in the extent of fuzzy concepts.

Definition 8. Let $\left(X, A, \widetilde{R}\right)$ be a fuzzy formal context. $\widetilde{R}^c$ is a fuzzy set that reflects the non-membership degree of (x, a) to $\widetilde{R}$. $\forall Q \subseteq X$ and $\widetilde{B} \in L^A$, the negative object learning operator $\widetilde{G}^- : P(X) \longrightarrow L^A$ and negative attribute learning operator $\widetilde{H}^- : L^A \longrightarrow P(X)$ are described as follows:

$$\begin{aligned} &\widetilde{G}^-(Q)(a_j) = \bigwedge_{x \in Q}\left(\widetilde{R}^c(x, a_j)\right), a_j \in A, \\ &\widetilde{H}^-\left(\widetilde{B}\right) = \left\{x \in X | \widetilde{R}^c(x, a_j) \geqslant \widetilde{B}(a_j), j = 1, 2, \ldots, m\right\}. \end{aligned} \tag{7}$$

The $\widetilde{G}^-(Q)(a_j)$ reflects the minimum non-membership degree of (x, a) to $\widetilde{R}$ on attribute a_j, and the aim of operator $\widetilde{H}^-$ is to find the objects in X satisfying $\widetilde{R}^-(x, a_j) \geqslant \widetilde{B}$, $j = 1, 2, \ldots, m$. The positive and negative operators all have the following properties:

Property 1. *Let* $Q, Q_1, Q_2 \subseteq X, \widetilde{B}, \widetilde{B}_1, \widetilde{B}_2 \in L^A$, *we have*

(1) $\widetilde{G}(Q_2) \subseteq \widetilde{G}(Q_1)$ if $Q_1 \subseteq Q_2, \widetilde{H}\left(\widetilde{B_2}\right) \subseteq \widetilde{H}\left(\widetilde{B_1}\right)$ if $\widetilde{B_1} \subseteq \widetilde{B_2}$;

(2) $Q \subseteq \widetilde{H}\widetilde{G}(Q), \widetilde{B} = \widetilde{G}\widetilde{H}\left(\widetilde{B}\right)$;

(3) $\widetilde{G}(Q) = \widetilde{G}\widetilde{H}\widetilde{G}(Q), \widetilde{H}\left(\widetilde{B}\right) = \widetilde{H}\widetilde{G}\widetilde{H}\left(\widetilde{B}\right)$;
(4) $Q \subseteq \widetilde{H}\left(\widetilde{B}\right) \Longleftrightarrow \widetilde{B} \subseteq \widetilde{G}(Q)$;
(5) $\widetilde{G}(Q_1 \cup Q_2) = \widetilde{G}(Q_1) \cap \widetilde{G}(Q_2), \widetilde{H}\left(\widetilde{B}_1 \cup \widetilde{B}_2\right) = \widetilde{H}\left(\widetilde{B}_1\right) \cap \widetilde{H}\left(\widetilde{B}_2\right)$;
(6) $\widetilde{G}(Q_1) \cup \widetilde{G}(Q_2) \subseteq \widetilde{G}(Q_1 \cap Q_2), \widetilde{H}\left(\widetilde{B}_1\right) \cup \widetilde{H}\left(\widetilde{B}_2\right) \subseteq \widetilde{H}\left(\widetilde{B}_1 \cap \widetilde{B}_2\right)$.

Proof. (1) According to the definition of object learning operator, we have $\widetilde{G}(Q_1)(a_j) = \bigwedge_{x\in Q_1}\widetilde{R}(x, a_j)$ and $\widetilde{G}(Q_2)(a_j) = \bigwedge_{x\in Q_2}\widetilde{R}(x, a_j)$. Because $Q_1 \subseteq Q_2, \wedge_{x\in Q_2}\widetilde{R}(x, a_j) \leqslant \bigwedge_{x\in Q_1}\widetilde{R}(x, a_j)$, then $\widetilde{G}(Q_2) \subseteq \widetilde{G}(Q_1)$ holds. In addition, for $\widetilde{B}_1 \subseteq \widetilde{B}_2, \widetilde{H}\left(\widetilde{B}_1\right) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B}_1(a_j), j = 1, 2, \ldots, m\right\}$ and $\widetilde{H}\left(\widetilde{B}_2\right) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B}_2(a_j), j = 1, 2, \ldots, m\right\}$. The $\widetilde{H}\left(\widetilde{B}_2\right) \subseteq \widetilde{H}\left(\widetilde{B}_1\right)$ holds since $\widetilde{B}_1(a_j) \leqslant \widetilde{B}_2(a_j)$ for $j = 1, 2, \ldots, m$.

(2) Given the subset Q of X, it is easy to obtain $\widetilde{H}\widetilde{G}(Q) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{G}(Q)(a_j), j = 1, 2, \ldots, m\right\}$. Also, the object induced by $\widetilde{H}\left(\widetilde{B}\right)$, whose membership degree of attribute is greater than that of $\widetilde{B}$ according to $\widetilde{H}\left(\widetilde{B}\right) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B}(a_j), j = 1, 2, \ldots, m\right\}$, thus $\widetilde{G}\widetilde{H}\left(\widetilde{B}\right)(a_j) = \wedge_{x\in\widetilde{H}\left(\widetilde{B}\right)}\widetilde{R}(x, a_j) = \widetilde{B}(a_j)$ for all $a_j \in A$. $\widetilde{B} = \widetilde{G}\widetilde{H}\left(\widetilde{B}\right)$ is obtained.

(3) According to (2), we know $\widetilde{B} = \widetilde{G}\widetilde{H}\left(\widetilde{B}\right)$ for fuzzy set $\widetilde{B}$. Since $\widetilde{G}(Q)$ and $\widetilde{B}$ are the fuzzy sets on A, it is easy to obtain $\widetilde{G}(Q) = \widetilde{G}\widetilde{H}\widetilde{G}(Q)$ and $\widetilde{H}\left(\widetilde{B}\right) = \widetilde{H}\widetilde{G}\widetilde{H}\left(\widetilde{B}\right)$.

(4) Since $\widetilde{H}\left(\widetilde{B}\right) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B}(a_j), j = 1, 2, \ldots, m\right\}$, and $Q \subseteq \widetilde{H}\left(\widetilde{B}\right)$, thus $\widetilde{R}(x, a_j) \geqslant \widetilde{B}(a_j)$ for all $a_j \in A$. Further, we obtain $\widetilde{G}(Q)(a_j) = \bigwedge_{x\in Q}\widetilde{R}(x, a_j) \geqslant \widetilde{B}(a_j)$ for all $a_j \in A$, therefore, $\widetilde{B} \subseteq \widetilde{G}(Q)$. On the contrary, if $\widetilde{B} \subseteq \widetilde{G}(Q), \widetilde{B}(a_j) \leqslant \widetilde{H}(Q)(a_j) = \bigwedge_{x\in Q}\widetilde{R}(x, a_j)$ for all $a_j \in A$. Meanwhile, $\widetilde{H}\left(\widetilde{B}\right) = \left\{x \in X|\widehat{R}(x, a_j) \geqslant \widetilde{B}(a_j), j = 1, 2, \ldots, m\right\}$ and $\widetilde{B}(a_j) \leqslant \bigwedge_{x\in Q}\widetilde{R}(x, a_j) \leqslant \bigwedge_{x\in X}\widetilde{R}(x, a_j)$ for $j = 1, 2, \ldots, m$, thus $Q \subseteq \widetilde{H}\left(\widetilde{B}\right)$ holds.

(5) $\widetilde{G}(Q_1 \cup Q_2)(a_j) = \wedge_{x\in Q_1\cup Q_2}\widetilde{R}(x, a_j) = \left(\wedge_{x\in Q_1}\widetilde{R}(x, a_j)\right) \wedge \left(\wedge_{x\in Q_2}\widetilde{R}(x, a_j)\right) = \widetilde{G}(Q_1)(a_j) \wedge \widetilde{G}(Q_2)(a_j); \widetilde{H}\left(\widetilde{B_1} \cup \widetilde{B_2}\right) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \left(\widetilde{B_1} \cup \widetilde{B_2}\right)(a_j), j = 1, 2, \ldots, m\right\} = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B_1}(a_j),\right.$
$\left.j = 1, 2, \ldots, m\right\} \cap \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B_2}(a_j), j = 1, 2, \ldots, m\right\} = \widetilde{H}\left(\widetilde{B_1}\right) \cap \widetilde{H}\left(\widetilde{B_2}\right)$. Therefore, the (5) in Property 1 holds.

(6) According to Definition 7, we know $\left(\widetilde{G}(Q_1) \cup \widetilde{G}(Q_2)\right)(a_j) = \left(\wedge_{x\in Q_1}\widetilde{R}(x, a_j)\right) \vee \left(\wedge_{x\in Q_2}\widetilde{R}(x, a_j)\right) \leqslant \wedge_{x\in Q_i}\widetilde{R}(x, a_j)(i = 1, 2) \leqslant \wedge_{x\in Q_1\cap Q_2}\widetilde{R}(x, a_j) = \widetilde{G}(Q_1 \cap Q_2)(a_j)$. Also, $\widetilde{H}\left(\widetilde{B_1}\right) \cup \widetilde{H}\left(\widetilde{B_2}\right) = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B_1}(a_j), j = 1, 2, \ldots, m\right\} \cup \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B_2}(a_j), j = 1, 2, \ldots, m\right\} = \left\{x \in X|\widetilde{R}(x, a_j) \geqslant \widetilde{B_1}(a_j) \wedge \widetilde{B_2}(a_j), j = 1, 2, \ldots, m\right\} = \widetilde{H}\left(\widetilde{B_1} \cap \widetilde{B_2}\right)$. Thus, we could obtain $\widetilde{G}(Q_1) \cup \widetilde{G}(Q_2) \subseteq \widetilde{G}(Q_1 \cap Q_2)$ and $\widetilde{H}\left(\widetilde{B}_1\right) \cup \widetilde{H}\left(\widetilde{B}_2\right) \subseteq \widetilde{H}\left(\widetilde{B}_1 \cap \widetilde{B}_2\right)$ based on the above analysis.

Property 2. *Let* $Q, Q_1, Q_2 \subseteq X, \widetilde{B}, \widetilde{B}_1, \widetilde{B}_2 \in L^A$, *we have*

(1) $\widetilde{G}^-(Q_2) \subseteq \widetilde{G}^-(Q_1)$ if $Q_1 \subseteq Q_2, \widetilde{H}^-(B_2) \subseteq \widetilde{H}^-(B_1)$ if $B_1 \subseteq B_2$;
(2) $Q \subseteq \widetilde{H}^-\widetilde{G}^-(Q), \widetilde{B} = \widetilde{G}^-\widetilde{H}^-\left(\widetilde{B}\right)$;
(3) $\widetilde{G}^-(Q) = \widetilde{G}^-\widetilde{H}^-\widetilde{G}^-(Q), \widetilde{H}^-\left(\widetilde{B}\right) = \widetilde{H}^-\widetilde{G}^-\widetilde{H}^-\left(\widetilde{B}\right)$;
(4) $Q \subseteq \widetilde{H}^-\left(\widetilde{B}\right) \Longleftrightarrow \widetilde{B} \subseteq \widetilde{G}^-(Q)$;
(5) $\widetilde{G}^-(Q_1 \cup Q_2) = \widetilde{G}^-(Q_1) \cap \widetilde{G}^-(Q_2), \widetilde{H}^-\left(\widetilde{B}_1 \cup \widetilde{B}_2\right) = \widetilde{H}^-\left(\widetilde{B}_1\right) \cap \widetilde{H}^-\left(\widetilde{B}_2\right)$;
(6) $\widetilde{G}^-(Q_1) \cup \widetilde{G}^-(Q_2) \subseteq \widetilde{G}^-(Q_1 \cap Q_2), \widetilde{H}^-\left(\widetilde{B}_1\right) \cup \widetilde{H}^-\left(\widetilde{B}_2\right) \subseteq \widetilde{H}^-\left(\widetilde{B}_1 \cap \widetilde{B}_2\right)$.

Proof. Similar to Property 1, it is easy to find that the above properties about negative object and attribute learning operators hold.

Definition 9. Let $\left(X, A, \widetilde{R}\right)$ be a fuzzy formal context. The object learning operator $\triangleleft : P(X) \longrightarrow BP\left(L^A\right)$ and attribute learning operator $\triangleright : BP\left(L^A\right) \longrightarrow P(X)$. $\forall Q \subseteq X, \widetilde{B}_1, \widetilde{B}_2 \in L^A, Q^{\triangleleft} = \left(\widetilde{G}(Q), \widetilde{G}^-(Q)\right)$ and $\left(\widetilde{B}_1, \widetilde{B}_2\right)^{\triangleright} = \widetilde{H}\widetilde{B}_1 \cap \widetilde{H}\widetilde{B}_2$. We call $\left(Q, \left(\widetilde{B}_1, \widetilde{B}_2\right)\right)$ is a fuzzy three-way concept when

$$Q^{\triangleleft} = \left(\widetilde{B}_1, \widetilde{B}_2\right), \left(\widetilde{B}_1, \widetilde{B}_1\right)^{\triangleright} = Q. \tag{8}$$

The fuzzy three-way concept could depict the relation between object and attribute more detail from the membership degree and non-membership degree. The $\left(Q_1, \left(\widetilde{B}_1, \widetilde{B}_2\right)\right)$ is the sub-concept of $\left(Q_2, \left(\widetilde{B}_3, \widetilde{B}_4\right)\right)$, denoted as $\left(Q_1, \left(\widetilde{B}_1, \widetilde{B}_2\right)\right) \leqslant \left(Q_2, \left(\widetilde{B}_3, \widetilde{B}_4\right)\right)$ when $Q_1 \subseteq Q_2\left(\left(\widetilde{B}_1, \widetilde{B}_2\right) \geqslant \left(\widetilde{B}_3, \widetilde{B}_4\right)\right)$.

In a fuzzy formal context, given an object set, we could obtain a fuzzy three-way concept according to the properties of object and attribute operators.

Property 3. *Let $\left(X, A, \widetilde{R}\right)$ be a fuzzy formal context. $\forall Q \subseteq X$, $\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q), \left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)\right)$ is a fuzzy three-way concept.*

Proof. We only need to demonstrate (1) $\left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)^{\triangleright} = \widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)$ and (2) $\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)\right)^{\triangleleft} = \left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)$.

(1) According to definition of operator $\triangleright$, it is easy to obtain $\left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)^{\triangleright} = \widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)$.

(2) Because $\widetilde{G}(Q)(a_j) = \bigwedge_{x\in Q}\widetilde{R}(x, a_j)$, for $a_j \in A$ and $\widetilde{H}\widetilde{G}(Q) = \left\{x \in X | R(x, a_j) \geqslant \bigwedge_{x\in Q} R(x, a_j), j = 1, 2, \ldots, m\right\}$, $\widetilde{H}^{-}\widetilde{G}^{-}(Q) = \left\{x \in X | R^c(x, a_j) \geqslant \bigwedge_{x\in Q}\widetilde{R}^{-}(x, a_j), j = 1, 2, \ldots, m\right\} = \left\{x \in X | 1 - R^c(x, a_j) \leqslant 1 - \bigwedge_{x\in Q} R^c(x, a_j), j = 1, 2, \ldots, m\right\} = \left\{x \in X | R(x, a_j) \leqslant \bigvee_{x\in Q} R(x, a_j), j = 1, 2, \ldots, m\right\}$.

Therefore, $\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q) = \left\{x \in X | \bigwedge_{x\in Q} R(x, a_j) \leqslant R(x, a_j) \leqslant \bigvee \quad x \in Q R(x, a_j), j = 1, 2, \ldots, m\right\}$, $\widetilde{G}\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)\right)(a_j) = \bigwedge_{x\in Q} R(x, a_j) = \widetilde{G}(Q)(a_j)$ for $j = 1, 2, \ldots, m$.

In the other hand, $\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q) = \left\{x \in X | \bigwedge_{x\in Q} R(x, a_j) \leqslant R(x, a \quad j) \leqslant \bigvee_{x\in Q} R(x, a_j), j = 1, 2, \ldots, m\right\} = \{x \in X | 1 - \bigwedge_{x\in Q} R(x, a_j) \geqslant 1 - R(x, a_j) \geqslant 1 - \bigvee_{x\in Q} R(x, a_j), j = 1, 2, \ldots, m\} = \left\{x \in X | \bigvee_{x\in Q} R^c(x, a_j) \geqslant 1 - R(x, a_j) \geqslant \bigwedge_{x\in Q} R^c(x, a_j), j = 1, 2, \ldots, m\right\}$, thus $\widetilde{G}^{-}\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)\right)(a_j) = \bigwedge_{x\in Q} R^c(x, a_j) = \widetilde{G}^{-}(Q)(a_j)$, for any $a_j \in A$ holds.

In conclusion, there are $\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)\right)^{\triangleleft} = \left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)$ and $\left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)^{\triangleright} = \widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q)$, thus $\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q), \left(\widetilde{G}(Q), \widetilde{G}^{-}(Q)\right)\right)$ is a fuzzy three-way concept.

Example 2 (*Continue to Example 1*). Given an object set, we can obtain the corresponding fuzzy three-way concept according to Property 3. Given X_1, we could obtain a fuzzy three-way concept, whose intent $\left(\widetilde{G}(X_1), \widetilde{G}^{-}(X_1)\right)) = ((0.17, 0.08), (0.32, 0.14))$ and extent $\left(\widetilde{H}\widetilde{G}(Q) \cap \widetilde{H}^{-}\widetilde{G}^{-}(Q) = X_1\right.$. Similarly, the $(X_2, ((0.02, 0.39), (0.72, 0.35)))$ and $(X_3, ((0.59, 0.32), (0.04, 0.12)))$ induced by X_2 and X_3 can be obtained.

The distribution of extent induced by these three-way concepts is shown in Fig. 4. Also, we find that the objects in extent are all the same class, which is more accurate than that of fuzzy concept. The fuzzy three-way concept describes object from membership degree and non-membership degree, it limits the objects to a smaller range by adding restriction(non-membership degree). This example illustrates the comprehensiveness of fuzzy three-way concept in concept learning process.

3.2. Construction of fuzzy three-way concept space

The objects between the same groups influence each other, especially those who are extremely similar. In fuzzy formal context, the similarity of objects is usually described by the distance between their attributes, and the greater the distance, the smaller the similarity. In this paper, we adopt the Euclidean distance to describe objects' similarity.

Definition 10. Let $\left(X, A, \widetilde{R}\right)$ be a fuzzy formal context. $\forall x_i, x_j \in X$, then give their corresponding membership degrees $\widetilde{R}_{i,s}$ and $\widetilde{R}_{j,s}$ of (x, a_i) and (x, a_j), non-membership degrees $\widetilde{R}^c_{i,s}$ and $\widetilde{R}^c_{j,s}$ for $s = 1, 2, \ldots, m$. Then, their difference is described by the following Euclidean distance function [16]

$$d(x_i, x_j) = \sqrt{\sum_{s=1}^{m}\left(\|\widetilde{R}_{i,s} - \widetilde{R}_{j,s}\|^2 + |\widetilde{R}^c_{i,s} - \widetilde{R}^c_{j,s}\|^2\right)}. \tag{9}$$

Since $\widetilde{R}^c = 1 - \widetilde{R}$, we have $\|\widetilde{R}_{i,s} - \widetilde{R}_{j,s}\|^2 + |\widetilde{R}^c_{i,s} - \widetilde{R}^c_{j,s}\|^2 = 2\|\widetilde{R}_{i,s} - \widetilde{R}_{j,s}\|^2$, that is, the distance between different objects is only determined by $\|\widetilde{R}_{i,s} - \widetilde{R}_{j,s}\|^2$ for $s = 1, 2, \ldots, m$. For convenience, we adopt the following distance function to select the similar objects.

$$d(x_i, x_j) = \sqrt{\sum_{s=1}^{m} \|\widetilde{R}_{i,s} - \widetilde{R}_{j,s}\|^2}. \tag{10}$$

Usually, the objects are considered to be the same when their distance is smaller than a certain value. In this paper, we set the certain value is δ, and then we could first obtain the similar class of any object.

Definition 11. Let $\left(X, A, \widetilde{R}, D, \widetilde{E}\right)$ be a fuzzy formal decision context, where $X/D = \{X_1, X_2, \ldots, X_l\}$ and $\widetilde{E} \subseteq X \times A$ is the binary relation between X and D. For $x \in X_i (i = 1, 2, \ldots, l)$ and $a_j \in A (j = 1, 2, \ldots, m)$, the membership degree of (x, a_j) to $\widetilde{R}$ is $\widetilde{R}(x, a_j)$, non-membership degree is $\widetilde{R}^c(x, a_j)$. Then, its similar classes Q_x is described as follows:

$$Q_x = \{y \in X_i | d(x, y) \leqslant \delta\}. \tag{11}$$

Based on the similar class set Q_x, we could obtain a fuzzy three-way concept according to Definition 9. Meanwhile, it is noted that the value of δ will influence the size of similar classes, the minimum membership degree and non-membership degree will different in different object sets, and then the intent will changes with extent, so the fuzzy three-way concept will be influenced by the value of δ. Furthermore, the object classify mechanism proposed in this paper is based on fuzzy three-way concept, thus the δ will further influence the object classify performance of proposed algorithm. The fuzzy three-way concept space is constructed as follows.

Definition 12. Let $\left(X, A, \widetilde{R}, D, \widetilde{E}\right)$ be a fuzzy formal decision context, where $X = \{X_1, X_2, \ldots, X_l\}$. Given X_i, the fuzzy three-way concept space $\mathscr{C}_i$ about X_i is defined as follows:

$$\mathscr{C}_i = \left\{ \left(\widetilde{H}\widetilde{G}(Q_x) \cap \widetilde{H}^- \widetilde{G}^-(Q_x), \left(\widetilde{G}(Q_x), \widetilde{G}^-(Q_x) \right) \right) | x \in X_i \right\}. \tag{12}$$

The fuzzy three-way concept space is $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_l\}$, and each $\mathscr{C}_i$ is called the subspace of $\mathscr{C}$. In prior knowledge, each object could be learned accurately. The process of constructing fuzzy three-way concept space is given in Algorithm 1.

Algorithm 1. The construction of fuzzy three-way concept space.

Input: A fuzzy formal decision context $(X, A, \widetilde{R}, D, \widetilde{E})$ and parameter δ.
Output: The fuzzy three-way concept space C.

for $X_i \in X$ **do**
  set $C_i \leftarrow \emptyset$;
  for $x \in X_i$ **do**
    compute the similar class Q_x of $x \in X_i$ according to formula (11);
    obtain the fuzzy three-way concept $(\widetilde{H}\widetilde{G}(Q_x) \cap \widetilde{H}^-\widetilde{G}^-(Q_x), (\widetilde{G}(Q_x), \widetilde{G}^-(Q_x)))$;
    $C_i \leftarrow (\widetilde{H}\widetilde{G}(Q_x) \cap \widetilde{H}^-\widetilde{G}^-(Q_x), (\widetilde{G}(Q_x), \widetilde{G}^-(Q_x)))$;
  end
end
return: C_i.

Example 3. The Table 2 is an example of fuzzy formal decision context based on Table 1, there are ten objects are divided into three classes based on two attributes.

Given $\delta = 0.2$, for the objects in class 1, we can obtain their similar classes $Q_{x_1} = \{x_1, x_2, x_4\}, Q_{x_2} = \{x_1, x_2, x_3, x_4\}, Q_{x_3} = \{x_2, x_3\}, Q_{x_4} = \{x_1, x_2, x_4\}$. Then the fuzzy three-way concepts can be obtained according to Definition 12: $(\{x_1, x_2, x_4\}, ((0.04, 0.47), (0.89, 0.35))), (\{x_1, x_2, x_3, x_4\}, ((0.11, 0.47), (0.72, 0.35))), (\{x_2, x_3\}, ((0.04, 0.47), (0.72, 0.44))), (\{x_1, x_2, x_4\}, ((0.04, 0.47), (0.89, 0.35)))$.

For the objects in class 2 and 3, their similar classes and the fuzzy three-way concept sub-spaces induced by them in class 2 and 3 are.

$Q_{x_5} = \{x_5\}$, $Q_{x_6} = \{x_6\}$, $Q_{x_7} = \{x_7\}$;
$(\{x_5\}, ((0.32, 0.63), (0.68, 0.37)))$, $(\{x_6\}, ((0.55, 0.86), (0.45, 0.14)))$, $(\{x_7\}, ((0.68, 0.11), (0.32, 0.89)))$.
$Q_{x_8} = \{x_8, x_{10}\}$, $Q_{x_9} = \{x_9, x_{10}\}$, $Q_{x_{10}} = \{x_8, x_9, x_{10}\}$;
$(\{x_8, x_{10}\}, ((0.75, 0.40), (0.19, 0.41))), (\{x_9, x_{10}\}, ((0.75, 0.32), (0.09, 0.60))), (\{x_8, x_9, x_{10}\}, ((0.75, 0.32), (0.09, 0.41)))$.

3.3. Construction of progressive fuzzy three-way concept space

In fuzzy three-way concept space, concepts influence each other, and there is even a lot of repetitive information between them. Meanwhile, we should note that concept cognition is generally regarded as progressive because of the limitations of individual cognitive and the incompleteness of cognitive environment. Thus, in this subsection, we mainly construct a progressive fuzzy three-way concept based on the original fuzzy three-way concept space.

Definition 13. For these fuzzy three-way concepts $\left(Q_1,\left(\widetilde{G}(Q_1),\widetilde{G}^-(Q_1)\right)\right),\left(Q_2,\left(\widetilde{G}(Q_2),\widetilde{G}^-(Q_2)\right)\right),\ldots,\left(Q_u,\left(\widetilde{G}(Q_u),\widetilde{G}^-(Q_u)\right)\right)\in\mathscr{C}_i$, if there exists $Q_1\subseteq Q_2\subseteq\cdots\subseteq Q_u$, then the $\left(Q_u,\left(\widetilde{G}(Q_u),\widetilde{G}^-(Q_u)\right)\right)$ is called supremum concept. The progressive fuzzy three-way concept is defined as follows:

$$\begin{aligned}&\mathscr{Q}_{i,j}=Q_1\cup Q_2\cup Q_u;\\&\left(\mathscr{G}_{i,j},\mathscr{G}^-_{i,j}\right)=\tfrac{1}{2^{u-1}}\left(\left(\widetilde{G}(Q_1),\widetilde{G}^-(Q_1)\right)+\left(\widetilde{G}(Q_2),\widetilde{G}^-(Q_2)\right)+2\left(\widetilde{G}(Q_3),\widetilde{G}^-(Q_3)\right)+\ldots+2^{u-2}\left(\widetilde{G}(Q_u),\widetilde{G}^-(Q_u)\right)\right).\end{aligned}\tag{13}$$

The $\left(\mathscr{Q}_{i,j},\left(\mathscr{G}_{i,j},\mathscr{G}^-_{i,j}\right)\right)$ is a progressive fuzzy three-way concept, and the corresponding concept subspace is $\mathscr{P}_i=\left\{\left(\mathscr{Q}_{i,j},\left(\mathscr{G}_{i,j},\mathscr{G}^-_{i,j}\right)\right)|j=1,2,\ldots,s_i\right\}$, where s_i is the number of concept in subspace. In the learning process of progressive concept, the intent of different sub-concept is given different weight according to its corresponding extent's size. The influence of sub-concept on the generation of new concept is heightened with the increase of extent. Meanwhile, the sum of weights in all concepts is 1, that is, the sum of the total effects is 1. The detail processes of selecting supremum concept and obtaining progressive fuzzy three-way concept space are shown in Algorithm 2.

Algorithm 2. The construction of progressive fuzzy three-way concept space.

Input: Initial fuzzy three-way concept space $C=\{C_1,\ldots,C_l\}$.
Output: The progressive fuzzy three-way concept space $\mathcal{P}=\{\mathcal{P}_1,\ldots,\mathcal{P}_l\}$.

for *each* $C_i\in C$ **do**
  Set supremum concept space $S=\{(Q_1,(\widetilde{G}(Q_1),\widetilde{G}^-(Q_1)))\}$;
  $C_i=C_i-S$;
  for $(Q_i,(\widetilde{G}(Q_i),\widetilde{G}^-(Q_i)))\in C_i$ **do**
    for $(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))\in C_i-(Q_i,(\widetilde{G}(Q_i),\widetilde{G}^-(Q_i)))$ **do**
      if $(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))$ *is the sub-concept of concept in* S **then**
        $S=S,S_j=S_j\cup\{(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))\}$;
        $C_i=C_i-(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))$; `/* Find the supremum concept and its sub-concept */`
      else
        $S=S\cup\{(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))\},S_j=\{(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))\}$;
      end
      if *there exists* $(Q_i,(\widetilde{G}(Q_i),\widetilde{G}^-(Q_i)))\in S$ *is the sub-concept of* $(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))$ **then**
        $S=S-(Q_i,(\widetilde{G}(Q_i),\widetilde{G}^-(Q_i)))$ and $S=S\cup\{(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))\}$;
        $S_j=S_i\cup\{(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))\}$, and delete S_i;
        $C_i=C_i-(Q_j,(\widetilde{G}(Q_j),\widetilde{G}^-(Q_j)))$;
      end
    end
  end
  for $(Q_i,(\widetilde{G}(Q_i),\widetilde{G}^-(Q_i)))\in S$ *and its corresponding sub-concept set* S_i **do**
    Compute the progressive fuzzy three way concept $(\mathcal{Q}_{i,j},(\mathcal{G}_{i,j},\mathcal{G}^-_{i,j}))$ according to Definition 13;
    $\mathcal{P}_i\leftarrow(\mathcal{Q}_{i,j},(\mathcal{G}_{i,j},\mathcal{G}^-_{i,j}))$; `/* Obtain the progressive fuzzy three-way concept subspace */`
  end
end
return: $\mathcal{P}=\{\mathcal{P}_1,\ldots,\mathcal{P}_l\}$.

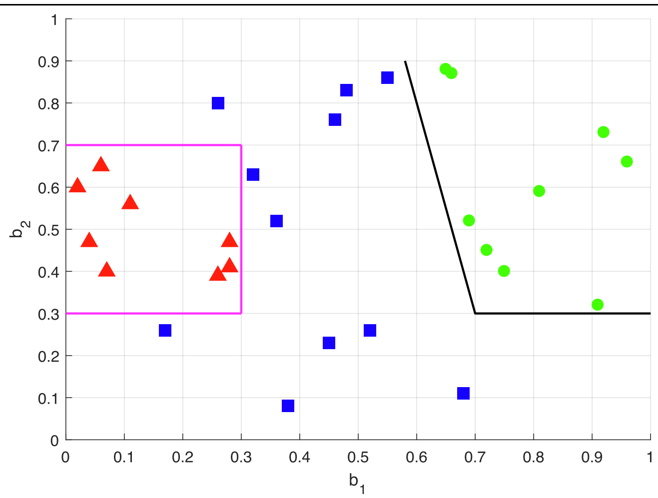

Fig. 4. The distribution of objects in the extent of fuzzy three-way concepts.

Table 2
A fuzzy formal decision context.

Object	b_1	b_2	Class
1	0.06	0.65	1
2	0.11	0.56	1
3	0.28	0.47	1
4	0.04	0.47	1
5	0.32	0.63	2
6	0.55	0.86	2
7	0.68	0.11	2
8	0.81	0.59	3
9	0.91	0.32	3
10	0.75	0.4	3

Example 4 (*Continue to Example 3*). According to Definition 13, we can further learn the progressive fuzzy three-way concept based on original fuzzy three-way concept in Example 3 as follows:

$$\mathscr{P}_{1,1} = (\{x_1, x_2, x_3, x_4\}, ((0.075, 0.47), (0.784, 0.395))),$$
$$\mathscr{P}_{2,1} = (\{x_5\}, ((0.32, 0.63), (0.68, 0.37))),$$
$$\mathscr{P}_{2,2} = (\{x_6\}, ((0.55, 0.86), (0.45, 0.14))),$$
$$\mathscr{P}_{2,3} = (\{x_7\}, ((0.68, 0.11), (0.32, 0.89))),$$
$$\mathscr{P}_{3,1} = (\{x_1, x_2, x_3\}, ((0.75, 0.34), (0.115, 0.505))).$$

In the new concept space, the first and third subspaces have only one progressive fuzzy three-way concept, while the second subspace has three progressive fuzzy three-way concepts because of the difference between extent of original concept. Compared with original concept, the progressive concept retains the original information based on the cognitive rules, and reduces the redundant concept that could improve the learning efficiency of concept recognition.

4. The incremental learning mechanism of progressive fuzzy three-way concept

When the new object is added, the progressive fuzzy three-way concept can not be updated without the class label. Thus, how to identify the class label of added object based on the concept space is a question worth exploring. In this section, we first design similar indicator that can be used for classification, and then propose an incremental learning mechanism based on progressive fuzzy three-way concept(ILMPFTC) for dynamically increasing data.

4.1. Classification mechanism based on progressive fuzzy three-way concept

In the progressive fuzzy three-way concept space, the objects in each concept could be learned accurately by the extent of concept and they all have the same class label. When adding an object x_a whose membership degree to $\widetilde{R}$ is $\widetilde{B}$, the most intuitive way to judge its category is to compare the difference between $\widetilde{B}$ and the intent of existing progressive concept.

Definition 14. Given the initial progressive fuzzy three-way concept space $\mathscr{P} = \{\mathscr{P}_1, \ldots, \mathscr{P}_l\}$, the added object x_a whose membership degree and non-membership degree to $\widetilde{R}$ are $\widetilde{B}$ and $\widetilde{B}^c$. Then, the difference between x_a and the existing $j-th$ progressive concept $\left(\mathscr{Q}_{i,j}, \left(\mathscr{G}_{i,j}, \mathscr{G}^-_{i,j}\right)\right)$ in $\mathscr{P}_i$ is described as follows:

$$DEC(x_a, \mathscr{P}_{i,j}) = \sqrt{\|\widetilde{B} - \mathscr{G}_{i,j}\|^2 + \|\widetilde{B}^c - \mathscr{G}^-_{i,j}\|^2}. \tag{14}$$

The smaller the value of $DEC(x_a, \mathscr{P}_{i,j})$, the stronger the similarity. The added object x_a can be classified according to the principle of minimum distance. This classification mechanism is illustrated by the following Example 5.

Example 5 (*Continue to Example 4*). Suppose the object $x_a = x_{28}$ in Table 1 is the added object, whose membership degree to $\widetilde{R}$ is $\widetilde{B} = (0.72, 0.45)$, non-membership degree $\widetilde{B}^c = (0.28, 0.55)$, and its class label is 3. Now, we identify its class label according to the distance between x_a and the existing progressive concepts. The distances between x_{28} and the progressive fuzzy three-way concepts are $DEC(x_a, \mathscr{P}_{1,1}) = 1.17; DEC(x_a, \mathscr{P}_{2,1}) = 0.88, DEC(x_a, \mathscr{P}_{2,2}) = 0.89, DEC(x_a, \mathscr{P}_{2,3}) = 0.68; DEC(x_a, \mathscr{P}_{3,1}) = 0.29$.

Because the progressive concept $\mathscr{P}_{3,1}$ is in the $3-th$ class, thus the x_a could be classified into the third category, which is consistent with its real label.

Algorithm 3. Classify mechanism based on the progressive fuzzy three-way concept.

Input: The progressive fuzzy three-way concept space $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_l\}$ and the added object x_a.
Output: The classification label of x_a.

for $\mathcal{P}_i \in \mathcal{P}$ **do**
  for *each* $\mathcal{P}_{i,j} \in \mathcal{P}_i$ **do**
    compute the distance $DEC(x_a, \mathcal{P}_{i,j})$;
  end
  obtain the minimum distance $s_i = min(DEC(x_a, \mathcal{P}_{i,j}))$;
end
the class label of x_a is $argmin_{i \in \{1,2,\ldots,l\}} s_i$;
return: the class label of x_a.

This classification mechanism based on the progressive fuzzy three-way concept is shown in Algorithm 3. Also, the label of x_a still can be identified by the same learning mechanism based on the progressive fuzzy concept (LMPFC), which will be compared with ILMPFTC in the experimental section.

4.2. Dynamic update mechanism of progressive fuzzy three-way concept space

When an object x_a is added, we first adjust its class label according to Algorithm 4. Suppose its label is the i, then $X' = X_i \cup \{x_a\}$, it is very time-consuming to recalculate all concepts for obtaining the progressive concept space. To reduce the learning time, we design a dynamic update mechanism to learn the concept. In this process, we only need to compute the difference between x_a and $x_c \in X_i$ to update the similar class instead of recalculating the difference between any two objects in X_i'. The dynamic update process of progressive concept space is shown in Algorithm 4.

Algorithm 4. Dynamic update mechanism of progressive fuzzy three-way concept space.

Input: The original similar classes $Q_x, x \in X_i (i = 1, 2, \ldots, l)$, fuzzy three-way concept space C;
the new added object x_a and parameter δ.
Output: The updated progressive concept space $\mathcal{P}' = \{\mathcal{P}_1', \ldots, \mathcal{P}_l'\}$.

1 input x_a, $\widetilde{B}$ and $\widetilde{B}^c$;
2 identify its class label according to Algorithm 3 and update the X_i';
3 set $\mathcal{P}' = \mathcal{P}, C' = \emptyset$;
4 **for** *each* $x_j \in X_i'$ **do**
5   compute the difference of x_a and x_j;
6   **if** $d(x_a, x_j) \leq \delta$ **then**
7     $Q_{x_j}' = Q_{x_j} \cup \{x_a\}$, $Q_{x_a} = Q_{x_a} \cup \{x_j\}$; `/* Dynamic update the original fuzzy three-way concept */`
8     $\widetilde{G}(Q_{x_j}') = min(\widetilde{G}(Q_x), \widetilde{B})$; $\widetilde{G}^-(Q_{x_j}') = min(\widetilde{G}^-(Q_x), \widetilde{B}^c)$;
9     $\widetilde{H}\widetilde{G}(Q_{x_a}') \cap \widetilde{H}^-\widetilde{G}^-(Q_{x_a}') = \widetilde{H}\widetilde{G}(Q_{x_a}) \cup \{x_s \in X_i - \widetilde{H}\widetilde{G}(Q_{x_a}) | R(x_s, a_j) \geq \widetilde{G}(Q_{x_c}')(a_j)$ and $R^c(x_s, a_j) \geq \widetilde{G}^-(Q_{x_j}')(a_j), \forall a_j \in A\}$;
10     $C_i' \leftarrow (\widetilde{H}\widetilde{G}(Q_{x_a}') \cap \widetilde{H}^-\widetilde{G}^-(Q_{x_a}'), (\widetilde{G}(Q_{x_j}'), \widetilde{G}^-(Q_{x_j}')))$;
11   **end**
12 **end**
13 $C_i' \leftarrow (\widetilde{H}\widetilde{G}(Q_{x_a}) \cap \widetilde{H}^-\widetilde{G}^-(Q_{x_a}), (\widetilde{G}(Q_{x_a}), \widetilde{G}^-(Q_{x_a})))$; `/* Obtain the progressive fuzzy three-way concept */`
14 the updated fuzzy three-way concept space C_i' is obtained;
15 the updated progressive fuzzy three-way concept subspace $\mathcal{P}_i'$ is obtained according to Definition 13;
16 **return**: $\mathcal{P}' = \{\mathcal{P}_1', \ldots, \mathcal{P}_l'\}$.

The Algorithm 4 introduces the dynamic update process of progressive fuzzy three-way concept space. The difference between dynamic update algorithm and static update algorithm lies in how to update the fuzzy three-way concept space, that is, the lines 4–12 of the pseudo-code, thus we only need to compare the difference between dynamic update and static update algorithm in obtaining new fuzzy three-way concept space. Suppose the added object is classed into $i-th$ class, that is, $X_i' = X_i \cup \{x_a\}$. When update the $j-th$ fuzzy three-way concept, we first need to compute the difference between x_a and x_j, the complexity is m. The intent of new fuzzy three-way concept can be obtained through comparing the original concept

$\left(\widetilde{G}\left(Q_{x_j}\right), \widetilde{G}^{-}\left(Q_{x_j}\right)\right)$ with $\left(\widetilde{B}, \widetilde{B}^{c}\right)$, if $x_a \in Q_{x_j}$, the time complexity of update concept is $2 \times m$. Similarly, the extent of new concept can be obtained based on the extent $\widetilde{H}\widetilde{G}\left(Q_{x_j}\right) \cap \widetilde{H}^{-}\widetilde{G}^{-}\left(Q_{x_j}\right)$ of original concept, thus the complexity of obtaining extent is $O\left(m \times \left(|X_i| - |\widetilde{H}\widetilde{G}\left(Q_{x_j}\right) \cap \widetilde{H}^{-}\widetilde{G}^{-}\left(Q_{x_j}\right)|\right)\right)$. Since there are $|X_i|$ concepts in the original concept space, thus the total complexity of pseudo-code 7–12 lines is $O\left(m \times \sum_{j=1}^{|X_i|}\left(1+2+2\left(|X_i| - |\widetilde{H}\widetilde{G}\left(Q_{x_j}\right) \cap \widetilde{H}^{-}\widetilde{G}^{-}\left(Q_{x_j}\right)|\right)\right)\right)$. Therefore, the time complexity of dynamic update fuzzy three-way concept is $O\left(m \times \sum_{j=1}^{|X_i|}\left(1+2+2\left(|X_i| - |\widetilde{H}\widetilde{G}\left(Q_{x_j}\right) \cap \widetilde{H}^{-}\widetilde{G}^{-}\left(Q_{x_j}\right)|\right)\right)\right)$. For the classic static update algorithm, it needs to recalculate the all similar classes of objects in X_i' and then regain the new fuzzy three-way concept according to Definition 14, thus the whole time complexity of static update concept space is $O\left(m \times \sum_{j=1}^{|X_i'|}|\left(|X_i'| + 2|Q_{x_j}| + 2|X_i'|\right)\right)$. Because $1 \leqslant |X_i|, 2 \leqslant |Q_{x_j}|$ and $|X_i| - |\widetilde{H}\widetilde{G}\left(Q_{x_j}\right) \cap \widetilde{H}^{-}\widetilde{G}^{-}\left(Q_{x_j}\right)| \leqslant |X_i|$, the time complexity of dynamic update algorithm is less than that of static update algorithm.

4.3. Incremental learning mechanism of progressive fuzzy three-way concept

The development of information science makes it possible to update data in real time. How to efficiently achieve the cognitive learning is important in dynamic environment. The dynamic update mechanism designed in Section 4.2 achieves concept learning based on the relationship between the added objects and original concept space, this is an incremental learning mechanism. The incremental learning mechanism based on progressive fuzzy three-way concept could be obtained for object classification by combing the classification mechanism and dynamic update mechanism. For the dynamic increased data without label, we first need to adjust its label, and then update the concept space for the further learning. After these two steps, the added objects could be used to classify new objects in the future. This incremental concept learning mechanism takes advantage of new information about added objects to make the further concept learning, which improves the learning efficiency and classify performance compared with learning mechanism of progressive three-way concept (LMPFTC). The Algorithm 5 introduces the whole process of ILMPFTC.

Algorithm 5. The incremental learning mechanism based on progressive fuzzy three-way concept.

```
Input: The original similar classes Q_x, x ∈ X_i (i = 1, 2, ..., l) and the progressive fuzzy three-way concept space
       P = {P_1, ..., P_l}; the new added data block Add = {Add_1, Add_2, ..., Add_t} and the parameter δ.
Output: The class label of added object and the updated progressive concept space P' = {P'_1, ..., P'_l}.
set Add = ∅;
for Add_i ∈ Add do
    for x_j ∈ Add_i do
        obtain it class label l_{i,j} according to Algorithm 3 and set Add(i, j) = l_{i,j};
        update the class set X';
    end
    update the progressive fuzzy three-way concept space according to Algorithm 4;
end
return: the class label Add(i, j) of data block Add and updated progressive concept space P'.
```

The Fig. 5 vividly shows the incremental learning process based on progressive fuzzy three-way concept. Given a fuzzy formal decision context with three types of objects. The objects in decision context are firstly divided into three classes according to the decision attribute, then the similar classes of objects are obtained. Subsequently, the fuzzy three-way concept is learned and its corresponding space is constructed according to Definition 9. Considering the limitations of individual cognitive and the incompleteness of cognitive environment, we further learn the progressive fuzzy three-way concept and classify the added objects according to the similarity [17]. All the classed objects are added into the original fuzzy formal decision context and are used to the further concept learning process. This mechanism achieves the incremental learning for dynamic increased data, and its classify performance has been improved due to the learned of new information.

5. Experimental analysis

In this section, we verify the effectiveness of proposed algorithm through numerical experiments, which is mainly reflected in the following aspects: (1) the classification performance of ILMPFTC; (2) the convergence of incremental classify mechanism; (3) the efficiency of dynamic update mechanism.

5.1. Experimental design

To verify the accuracy performance of the proposed learning mechanism (ILMPFTC), we compare it with LMPFC [16], KNN [3], FuzzyKNN [10], IF-KNN [11], and FENN [23]. According to the analysis of Definition 11, we know the parameter δ is an important parameter that influences the classify accuracy in ILMPFC. In the experiments, the optimal δ is selected according

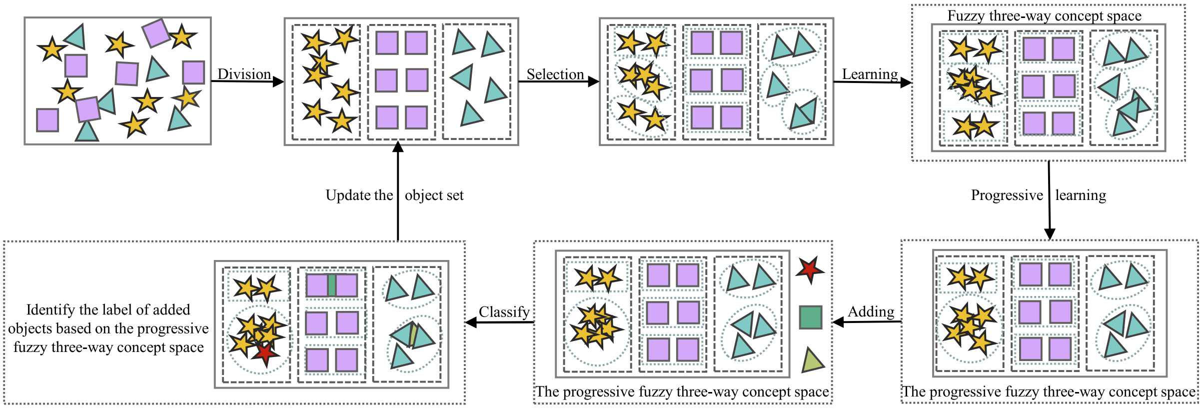

Fig. 5. The incremental learning mechanism based on progressive fuzzy three-way concept.

to the classify accuracy of added objects, the select range is $[0,1]$ with step 0.001. Similarly, the parameter δ in LMPFC can be obtained. The other four compared algorithms all have the same neighbor parameter k, where $k=3$ in this paper. Ten experimental datasets are obtained from UCI Machine Learning Repository and KEEL, their detail information is shown in Table 3. In the preprocessing stage, the dataset is fuzzified to get the membership degree belonging to the interval [0,1]. Therefore, these datasets are first fuzzified by using [6]

$$\widetilde{R}(x_i,a_j)=\frac{f(x_i,a_j)-min(f(a_j))}{max(f(a_j))-min(f(a_j))}. \tag{15}$$

where the $f(x_i,a_j)$ denotes the value of x_i in attribute a_j, the $max(f(a_j))$ and $min(f(a_j))$ denote the maximum and minimum value of all objects in attribute a_j. In the fuzzy formal decision context, the fuzzy value of $\widetilde{R}(x,a)$ reflects the membership degree of (x,a) to $\widetilde{R}$. Usually, the fuzzy set $\widetilde{R}$ could be understood as the ownership degree of object to attribute. The greater the value of $f(x,a)$, the greater the degree to which x owns attribute a, the formula (15) as a method of fuzzification could convert the original numerical into fuzzy formal decision context.

In each dataset, 70% of the data is used to train the model, the remaining data is divided into ten equal parts and added to the test set to verify the classify performance of the ILMPFTC and the effectiveness of dynamic update mechanism. All the algorithms are implemented in Matlab 2016b and carry out on a personal computer with Intel(R) Core(TM) i5-1135G7 CPU@2.40GH 2.42GH, and 16 GB memory.

5.2. The comparison of classify performance between different algorithms

In this subsection, we verify the excellence of ILMPFTC through classify performance, which includes the following three aspects: (1) to illustrate the effectiveness of fuzzy three-way thought through the comparison between LMPFTC and LMPFC; (2) to show the advantages of incremental mechanism through the comparison between ILMPFTC and LMPFTC; (3) to verify the superiority of proposed classify mechanism through the comparison between ILMPFTC and other four fuzzy classify methods.

5.2.1. The classification performance comparison between LMPFTC and LMPFC

Table 4 records the optimal δ and classify accuracy of LMPFTC and LMPFC on ten datasets. The last column represents the average accuracy and standard deviation(std) of accuracy. On all datasets, we find that the accuracy of LMPFTC is almost greater than or equal to that of LMPFC every time, except set 5. The LMPFTC is better than LMPFC in 9 datasets in accuracy and excellent on set 1, set 2, set 6, set 8, set 9 in standard deviation. Meanwhile, the data in Table 5 shows that the average classify accuracy of LMPFTC in ten datasets is higher than that of LMPFC, which verifies the effectiveness of fuzzy three-way concept. Further, to test whether there is a significant difference between LMPFTC and LMPFC from a statistical perspective, we adopt the Wilcoxon pairwise test to compare this experiment. Given the test threshold is 0.05, the test P-value is $0.006<0.05$, we could reject the null hypothesis(there is no difference between the two algorithms) and consider there is a significant difference between LMPFTC and LMPFC. The more vividly comparison is shown in Fig. 6, it can be obtained that the classify accuracy of LMPFTC is significantly higher than that of LMPFC in most datasets. Compared with fuzzy concept, the fuzzy three-way concept also uses negative information to describe the object, which adds constraint on concept learning but can make the learning concept more accurate, thus improving the classify performance. The above experimental results confirm this point.

Table 3
Dataset description.

No.	Dataset	Sample	Attribute	Class
set1	AuditData	772	17	2
set2	BreastCancer	683	9	2
set3	BreastCancerCoimbra	116	9	2
set4	Contraceptive Method Choice	1473	9	3
set5	glass	214	9	6
set6	Liver Disorders	345	6	2
set7	Nursery	12960	8	5
set8	Occupancy	20560	5	2
set9	Wireless Indoor Localization	2000	7	4
set10	Wilt	4839	5	2

Table 4
The optimal δ and classify accuracy (%) of LMPFTC and LMPFC when adding new objects.

Dataset	Method	δ	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	Average ± std
set1	LMPFTC	0.679	100.00	100.00	100.00	100.00	100.00	100.00	92.55	89.13	82.61	77.83	**94.21 ± 8.38**
	LMPFC	0.139	100.00	100.00	100.00	100.00	100.00	100.00	92.55	88.59	82.13	76.09	93.93 ± 8.88
set2	LMPFTC	0.222	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.38	98.89	99.00	**99.73 ± 0.46**
	LMPFC	1	100.00	100.00	100.00	100.00	100.00	100.00	99.29	95.00	93.33	91.50	97.91 ± 3.31
set3	LMPFTC	0.674	100.00	100.00	100.00	100.00	93.33	88.89	76.19	66.67	59.26	53.33	**83.77 ± 18.40**
	LMPFC	0.473	66.67	66.67	66.67	66.67	66.67	61.11	57.14	58.33	55.56	56.67	62.21 ± 4.90
set4	LMPFTC	0.100	61.36	55.68	51.52	53.41	52.73	48.86	48.05	47.44	45.96	44.77	**50.98 ± 5.04**
	LMPFC	0.220	31.82	42.05	40.91	40.34	41.36	42.05	40.26	38.35	36.87	34.77	38.88 ± 3.42
set5	LMPFTC	0.100	83.33	75.00	66.67	75.00	76.67	75.00	71.43	70.83	70.37	70.00	73.43 ± 4.64
	LMPFC	0.032	83.33	83.33	72.22	75.00	76.67	75.00	71.43	70.83	70.37	70.00	**74.82 ± 5.01**
set6	LMPFTC	0.735	100.00	100.00	96.67	95.00	86.00	75.00	65.71	61.25	55.56	52.00	**78.72 ± 19.10**
	LMPFC	0.575	100.00	100.00	100.00	97.50	84.00	71.67	62.86	56.25	50.00	46.00	76.83 ± 22.12
set7	LMPFTC	0.200	65.21	62.50	44.50	55.93	64.74	70.62	72.53	69.33	68.27	71.44	**64.51 ± 8.60**
	LMPFC	0.500	0.00	0.00	0.00	0.00	0.00	0.00	4.49	11.86	17.67	22.58	5.66 ± 8.58
set8	LMPFTC	0.010	99.03	99.35	99.57	99.68	99.74	99.78	99.74	99.21	97.24	96.64	**99.00 ± 1.12**
	LMPFC	0.005	96.10	89.29	92.21	94.16	95.32	96.10	92.35	93.30	93.96	92.79	93.56 ± 2.07
set9	LMPFTC	0.255	100.00	100.00	100.00	100.00	99.66	99.15	98.31	97.46	97.74	97.97	**99.03 ± 1.05**
	LMPFC	0.220	61.02	70.34	66.10	74.15	78.98	81.64	83.78	85.38	87.01	88.14	77.65 ± 9.36
set10	LMPFTC	0.005	58.62	78.62	85.52	88.79	90.62	91.84	92.91	93.71	93.33	93.45	**86.74 ± 10.95**
	LMPFC	0.050	57.24	54.83	49.66	45.00	42.48	42.30	41.87	42.24	44.75	46.48	46.69 ± 5.51

Table 5
The average accuracy and their Wilcoxon test result of LMPFTC and LMPFC.

Mechanism	set1	set2	set3	set4	set5	set6	set7	set8	set9	set10	Average	P-value
LMPFTC	**94.21**	**99.73**	**83.77**	**50.98**	73.43	**78.72**	**64.51**	**99.00**	**99.03**	**86.74**	**83.01**	–
LMPFC	93.93	97.91	62.21	38.88	74.82	76.83	5.66	93.56	77.65	46.69	66.81	0.006

5.2.2. The classification performance comparison between ILMPFTC and LMPFTC

For the object classification problem of dynamic increased data, it is limited to use only the original information for classification. The incremental learning mechanism further utilizes the information about added objects based on the original data, it can update the data table in time and avoid the lack of useful information. The Table 6 records the classify accuracy of added objects under ILMPFTC and LMPFTC. From this table, we find the ILMPFTC achieves the highest average accuracy in 9 datasets, and the average accuracy of ten datasets is also higher than that of LMPFTC, which shows the incremental mechanism could improve the object classify performance compared with the LMPFTC.

5.2.3. The classification performance comparison between ILMPFTC and other fuzzy classify algorithms

In this subsection, we verify the advantages of LMPFTC compared with the other four fuzzy classify algorithms based on the KNN classifier. Table 7 records the classify accuracy with the increase of objects, the average accuracy, and corresponding

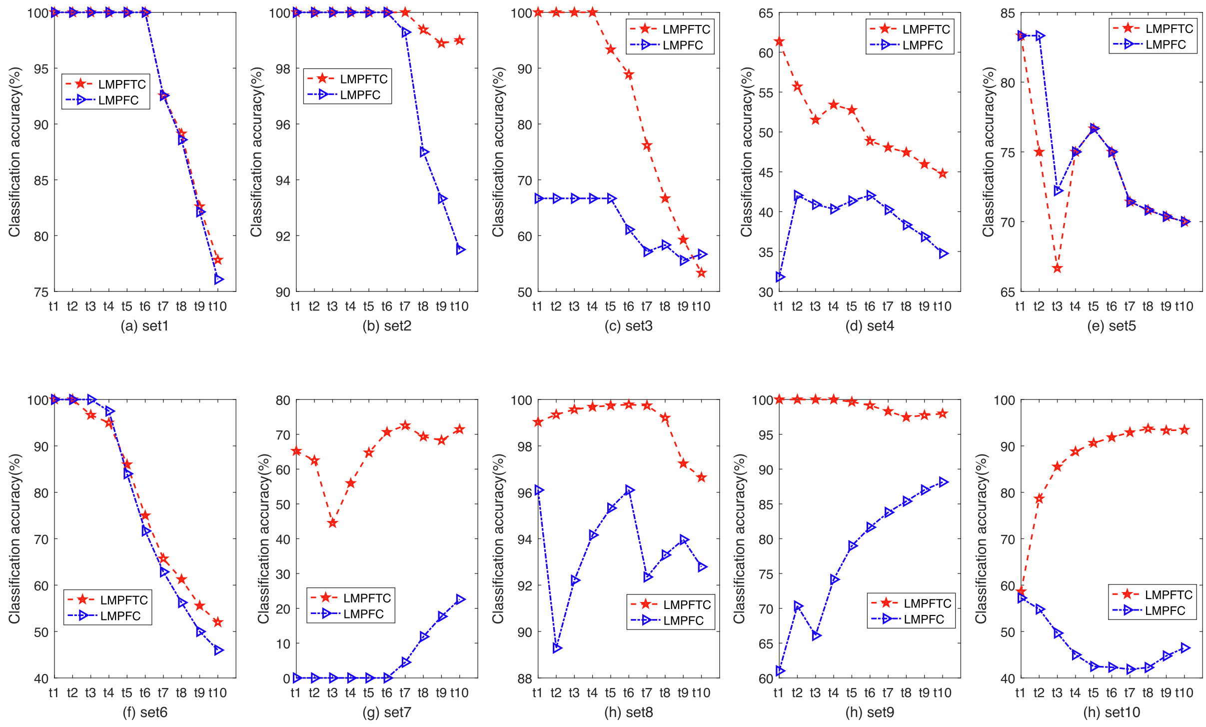

Fig. 6. The comparison of classify performance between LMPFTC and LMPFC.

Table 6
The optimal δ and claffify accuracy (%) of ILMPFTC and LMPFTC when adding new objects.

Dataset	Method	δ	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	Average ± std
set1	ILMPFTC	0.740	100.00	100.00	100.00	100.00	100.00	100.00	91.93	88.04	86.47	87.39	**95.38 ± 5.80**
	LMPFTC	0.679	100.00	100.00	100.00	100.00	100.00	100.00	92.55	89.13	82.61	77.83	94.21 ± 7.95
set2	ILMPFTC	0.240	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.38	98.89	99.00	**99.73 ± 0.43**
	LMPFTC	0.222	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.38	98.89	99.00	**99.73 ± 0.43**
set3	ILMPFTC	0.690	100.00	100.00	100.00	100.00	100.00	88.89	90.48	79.17	70.37	66.67	**89.56 ± 12.44**
	LMPFTC	0.674	100.00	100.00	100.00	100.00	93.33	88.89	76.19	66.67	59.26	53.33	83.77 ± 17.46
set4	ILMPFTC	0.150	52.27	55.68	54.55	55.68	55.00	51.89	50.97	51.14	48.48	47.05	**52.27 ± 2.84**
	LMPFTC	0.100	61.36	55.68	51.52	53.41	52.73	48.86	48.05	47.44	45.96	44.77	50.98 ± 4.78
set5	ILMPFTC	0.099	83.33	75.00	61.11	70.83	73.33	72.22	69.05	68.75	68.52	68.33	71.05 ± 5.43
	LMPFTC	0.100	83.33	75.00	66.67	75.00	76.67	75.00	71.43	70.83	70.37	70.00	**73.43 ± 4.40**
set6	ILMPFTC	0.730	100.00	100.00	96.67	95.00	94.00	83.33	72.86	67.50	61.11	56.00	**82.65 ± 16.06**
	LMPFTC	0.735	100.00	100.00	96.67	95.00	86.00	75.00	65.71	61.25	55.56	52.00	78.72 ± 18.12
set7	ILMPFTC	0.240	65.21	64.05	68.90	73.52	78.81	82.35	82.58	76.19	70.13	68.97	**73.07 ± 6.37**
	LMPFTC	0.200	65.21	62.50	44.50	55.93	64.74	70.62	72.53	69.33	68.27	71.44	64.51 ± 8.16
set8	ILMPFTC	0.016	99.35	99.51	99.68	99.76	99.81	99.84	99.86	98.97	96.99	96.35	**99.01 ± 1.21**
	LMPFTC	0.010	99.03	99.35	99.57	99.68	99.74	99.78	99.74	99.21	97.24	96.64	99.00 ± 1.06
set9	ILMPFTC	0.108	100.00	100.00	99.44	99.15	98.98	98.87	98.55	98.73	98.87	98.47	**99.11 ± 0.52**
	LMPFTC	0.255	100.00	100.00	100.00	100.00	99.66	99.15	98.31	97.46	97.74	97.97	99.03 ± 1.00
set10	ILMPFTC	0.014	63.45	80.00	85.52	88.62	90.21	91.15	92.32	93.10	92.72	92.83	**86.99 ± 8.76**
	LMPFTC	0.005	58.62	78.62	85.52	88.79	90.62	91.84	92.91	93.71	93.33	93.45	86.74 ± 10.38

stand deviation. From this table, we find that the ILMPFTC gets the maximum value of 7 times in 10 datasets, while the other four algorithms have achieved the maximum value for 3, 1, 1 and 0 times, respectively. Meanwhile, we obtain the average rank of classification accuracy on ILMPFTC is 1.9((1 + 1 + 1 + 2+5 + 1 + 1 + 4+1 + 1)/10), and that of others are respectively 3.4, 2.8, 3.4, 2.5, which illustrates the superior of ILMPFTC in object classification. At the same time, the stand deviation of ILMPFTC in set1, set2, set4, set5, set7, set9, set10 is smaller than that of KNN, FuzzyKNN, IF-KNN, and FENN, indicating that ILMPFTC has relatively better robustness than other algorithms.

Table 7
The parameter and classify accuracy (%) of five algorithms when adding objects.

Dataset	Method	δ	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	Average ± std	Rank
set1	ILMPFTC	0.740	100.00	100.00	100.00	100.00	100.00	100.00	91.93	88.04	86.47	87.39	**95.38 ± 5.80**	1
	KNN	3	100.00	100.00	100.00	100.00	100.00	100.00	92.55	88.59	82.13	77.39	94.07 ± 8.15	2
	Fuzzy KNN	3	100.00	100.00	100.00	100.00	100.00	100.00	92.55	88.59	82.13	77.39	94.07 ± 8.15	2
	IF-KNN	3	100.00	100.00	100.00	100.00	100.00	100.00	92.55	88.59	82.13	77.39	94.07 ± 8.15	2
	FENN	3	100.00	100.00	100.00	100.00	100.00	100.00	92.55	88.59	82.13	77.39	94.07 ± 8.15	2
set2	ILMPFTC	0.240	100.00	100.00	100.00	100.00	100.00	100.00	100.00	99.38	98.89	99.00	**99.73 ± 0.43**	1
	KNN	3	100.00	100.00	100.00	100.00	99.02	99.18	99.30	98.16	98.36	98.04	99.21 ± 0.76	5
	Fuzzy KNN	3	100.00	100.00	100.00	100.00	100.00	100.00	100.00	98.77	98.91	99.02	99.67 ± 0.51	2
	IF-KNN	3	100.00	100.00	100.00	100.00	99.02	99.18	99.30	98.16	98.36	98.53	99.25 ± 0.69	4
	FENN	3	100.00	100.00	100.00	100.00	99.02	99.18	99.30	98.77	98.91	99.02	99.42 ± 0.49	3
set3	ILMPFTC	0.690	100.00	100.00	100.00	100.00	100.00	88.89	90.48	79.17	70.37	66.67	**89.56 ± 12.44**	1
	KNN	3	33.33	66.67	60.00	69.23	70.59	70.00	65.22	55.56	56.67	55.88	60.31 ± 10.62	5
	Fuzzy KNN	3	66.67	83.33	70.00	76.92	70.59	70.00	65.22	55.56	53.33	52.94	66.46 ± 9.55	3
	IF-KNN	3	66.67	83.33	70.00	76.92	70.59	70.00	65.22	55.56	53.33	52.94	66.46 ± 9.55	3
	FENN	3	66.67	83.33	90.00	76.92	70.59	70.00	60.87	51.85	50.00	47.06	66.73 ± 13.67	2
set4	ILMPFTC	0.150	52.27	55.68	54.55	55.68	55.00	51.89	50.97	51.14	48.48	47.05	52.27 ± 2.84	2
	KNN	3	56.82	57.09	57.58	60.36	56.36	50.38	48.38	46.88	45.96	44.55	**52.43 ± 5.49**	1
	Fuzzy KNN	3	56.82	55.68	50.76	54.55	50.45	46.97	47.73	47.44	46.97	45.45	50.28 ± 3.87	3
	IF-KNN	3	59.09	53.41	48.48	52.84	49.09	45.08	45.78	45.45	45.20	43.86	48.83 ± 4.64	5
	FENN	3	63.64	53.41	49.24	50.57	46.82	43.18	44.48	46.31	46.97	47.05	49.17 ± 5.57	4
set5	ILMPFTC	0.099	83.33	75.00	61.11	70.83	73.33	72.22	69.05	68.75	68.52	68.33	71.05 ± 5.43	5
	KNN	3	83.33	83.33	72.22	75.00	76.67	75.00	71.43	64.58	64.81	63.33	**72.97 ± 6.85**	2
	Fuzzy KNN	3	83.33	83.33	72.22	75.00	76.67	72.22	69.05	64.58	64.81	65.00	72.62 ± 6.70	3
	IF-KNN	3	83.33	83.33	72.22	75.00	76.67	72.22	69.05	60.42	61.11	60.00	71.34 ± 8.30	4
	FENN	3	83.33	91.67	83.33	83.33	80.00	75.00	69.05	60.42	61.11	60.00	74.72 ± 10.85	1
set6	ILMPFTC	0.730	100.00	100.00	96.67	95.00	94.00	83.33	72.86	67.50	61.11	56.00	**82.65 ± 16.06**	1
	KNN	3	50.00	50.00	43.33	45.00	39.22	49.18	54.93	59.26	61.54	59.80	51.23 ± 7.15	3
	Fuzzy KNN	3	40.00	45.00	36.67	42.50	39.22	49.18	56.34	60.49	62.64	60.78	49.28 ± 9.47	4
	IF-KNN	3	40.00	45.00	36.67	42.50	37.25	45.90	53.52	58.02	60.44	58.82	47.81 ± 8.68	5
	FENN	3	70.00	45.00	36.67	40.00	41.18	50.82	57.75	61.73	64.84	66.67	53.46 ± 11.65	2
set7	ILMPFTC	0.240	65.21	64.05	68.90	73.52	78.81	82.35	82.58	76.19	70.13	68.97	**73.07 ± 6.37**	1
	KNN	3	62.11	60.88	43.52	55.21	64.16	70.14	72.12	69.98	70.25	73.23	64.16 ± 8.79	4
	Fuzzy KNN	3	64.18	61.90	44.21	55.73	64.57	70.48	71.53	69.47	69.79	72.82	64.47 ± 8.38	3
	IF-KNN	3	70.88	65.25	46.44	57.40	65.91	71.60	72.34	73.26	73.17	75.86	67.21 ± 8.60	2
	FENN	3	64.18	58.82	42.15	54.18	63.34	69.46	70.95	72.04	72.08	74.88	64.21 ± 9.63	5
set8	ILMPFTC	0.016	99.35	99.51	99.68	99.76	99.81	99.84	99.86	98.97	96.99	96.35	99.01 ± 1.21	4
	KNN	3	98.38	99.03	99.35	99.51	99.61	99.68	99.65	98.80	98.79	97.91	**99.07 ± 0.57**	1
	Fuzzy KNN	3	98.38	99.03	99.35	99.51	99.61	99.68	99.65	98.80	98.79	97.91	**99.07 ± 0.57**	1
	IF-KNN	3	98.38	99.03	99.35	99.51	99.61	99.68	99.65	98.80	98.79	97.91	**99.07 ± 0.57**	1
	FENN	3	98.38	99.03	99.35	99.51	99.61	99.68	99.65	98.50	98.54	97.71	99.00 ± 0.64	5
set9	ILMPFTC	0.108	100.00	100.00	99.44	99.15	98.98	98.87	98.55	98.73	98.87	98.47	**99.11 ± 0.52**	1
	KNN	3	100.00	100.00	100.00	99.58	97.99	98.32	97.84	98.11	98.32	98.15	98.83 ± 0.89	3
	Fuzzy KNN	3	100.00	100.00	99.44	99.16	97.32	97.76	97.36	97.69	97.95	97.82	98.45 ± 1.02	5
	IF-KNN	3	100.00	100.00	99.44	99.16	97.32	97.76	97.36	97.69	97.95	97.99	98.47 ± 1.01	4
	FENN	3	100.00	100.00	99.44	99.16	97.32	97.76	97.60	97.90	98.13	98.32	98.56 ± 0.95	2
set10	ILMPFTC	0.014	63.45	80.00	85.52	88.62	90.21	91.15	92.32	93.10	92.72	92.83	**86.99 ± 8.76**	1
	KNN	3	53.10	75.52	83.22	87.41	89.79	89.38	92.61	93.53	93.56	93.94	85.21 ± 12.01	5
	Fuzzy KNN	3	52.41	75.86	83.68	87.76	90.21	91.72	92.91	93.79	94.02	94.42	85.68 ± 12.38	2
	IF-KNN	3	51.72	75.52	83.22	87.41	89.93	91.49	92.71	93.62	93.87	94.28	85.38 ± 12.53	4
	FENN	3	51.72	75.86	83.45	87.59	90.07	91.61	92.81	93.71	93.87	94.28	85.50 ± 12.52	3

The results of the Wilcoxon pairwise test are shown in Table 8, it can be obtained that there exists a significant difference between KNN, FuzzyKNN, IF-KNN, and ILMPFTC when the test threshold is 0.05, and the ILMPFTC is different from FENN when the test threshold is 0.1. Also, the average accuracy of ten datasets on ILMPFTC is 84.88, which is higher than that of other four algorithms. Therefore, we could consider the ILMPFTC is excellent in object classification compared with algorithms based on KNN classifier.

5.3. The comparison of overall classify performance when adding objects

The Fig. 7 shows the comparison of classify accuracy between ILMPFC and other four fuzzy classify algorithm. It is noted that the five classify methods all have the downward trend as a whole with the increase of new objects to be identified. To fully describe the classify performance of t time, it needs to combine the Acc_t(the classify accuracy at t time) and the change

Table 8
The average accuracy of different classify algorithm and their corresponding Wilcoxon test results.

Mechanism	set1	set2	set3	set4	set5	set6	set7	set8	set9	set10	Average	P–value
ILMPFTC	**95.38**	**99.73**	**89.56**	52.27	71.05	82.65	73.07	99.01	**99.11**	**86.99**	**84.88**	
KNN	94.07	99.21	60.31	52.43	72.97	51.23	64.16	**99.07**	98.83	85.21	77.75	0.084
Fuzzy KNN	94.07	99.67	66.46	50.28	72.62	49.28	64.47	**99.07**	98.45	85.68	78.00	0.043
IF-KNN	94.07	99.25	66.46	48.83	71.34	47.81	**67.21**	**99.07**	98.47	85.38	77.79	0.010
FENN	94.07	99.42	66.73	49.17	74.72	53.46	64.21	99.00	98.56	85.50	78.48	0.037

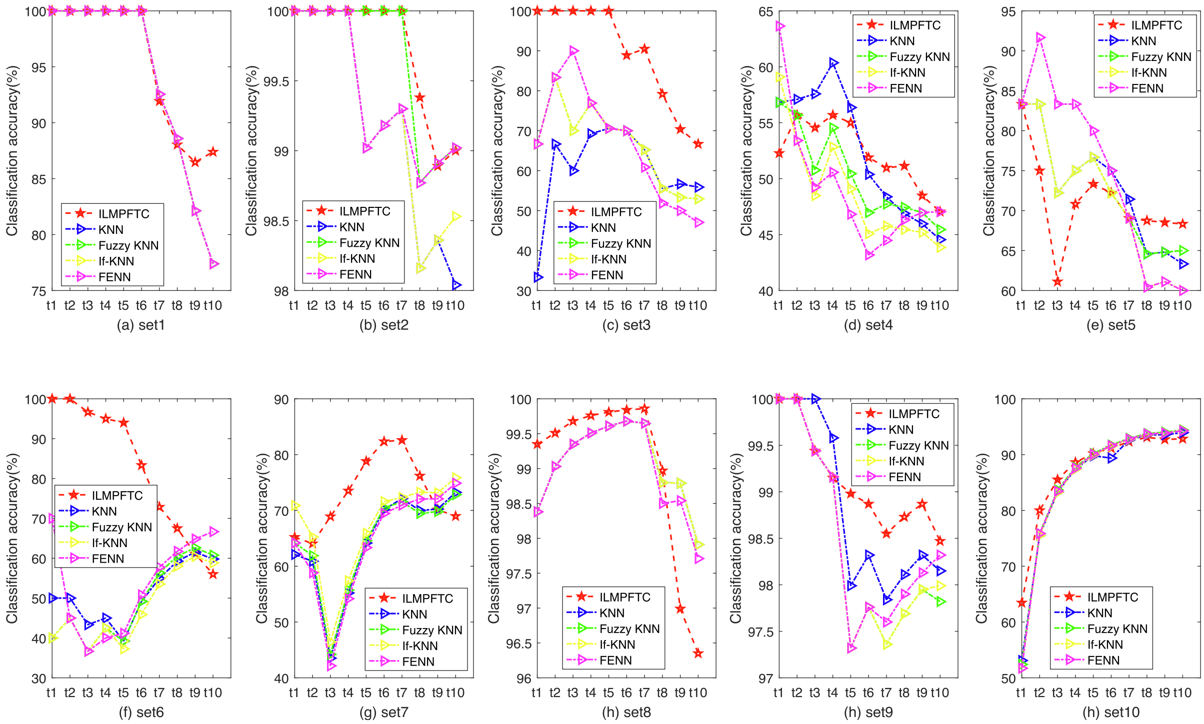

Fig. 7. The object classify performance of different algorithms when adding objects.

of accuracy from $t-1$ time to t time. Therefore, we design an indicator to measure the accuracy performance of different algorithms as follows:

$$IAP = \sum_{t=1}^{9}\left(w_1 \times \frac{Acc_{t+1} - Acc_t}{Acc_t} \times 100\% + w_2 \times Acc_{t+1}\right). \tag{16}$$

where Acc_t and Acc_{t+1} denote the classify accuracy at $t-th$ and $t+1-th$ time, and $\frac{Acc_{t+1}-Acc_t}{Acc_t} \times 100$ reflects the increase in accuracy from $t-th$ to $t+1-th$. The w_1 and w_2 denote the weight of $\frac{Acc_{t+1}-Acc_t}{Acc_t} \times 100\%$ and Acc_{t+1}. At any time, the higher the classify accuracy of classification, the better of classify mechanism. Therefore, the larger the index value of IAP is, the better the overall classify performance of the algorithm is with the increase of objects. In this paper, the weights of w_1 and w_2 are set to 0.4 and 0.6, respectively.

Table 9 records the values of indicator *IAP* under the different algorithms. From this table, we know the ILMPFTC obtains the maximum value of IAP 7 times in 10 datasets, the other four fuzzy algorithms get the maximum value 0, 1, 2, 2 times,

Table 9
The values of IAP for different algorithms when adding new objects.

Mechanism	set1	set2	set3	set4	set5	set6	set7	set8	set9	set10	Average
ILMPFTC	**56.34**	**59.77**	**51.34**	**30.92**	41.10	**45.98**	44.68	**59.25**	**59.34**	55.60	**50.43**
KNN	54.94	59.38	41.79	30.14	41.95	31.87	39.81	59.47	59.14	56.14	47.46
Fuzzy KNN	54.94	59.74	39.12	28.80	41.82	32.41	39.69	59.47	58.87	**56.62**	47.15
IF-KNN	54.94	59.44	39.12	27.38	40.64	31.31	**40.80**	**59.47**	58.89	56.53	46.85
FENN	54.94	59.57	38.80	27.31	**42.91**	31.40	39.68	59.41	58.97	**56.62**	46.96

respectively. Meanwhile, the average *IAP* of ten datasets is the highest among five classify algorithms, thus we could obtain the classify performance of ILMPFTC is better than other algorithms as the number of object increases.

5.4. The efficiency of dynamic update mechanism

A key issue of the incremental learning mechanism is to further use the information about added object for next learning, therefore, how to efficiently update the original concept space after judging the category of the added objects is very impor-

Table 10
The consume time(s) of updating evolutionary concepts space on optimal δ.

Dataset	Algorithm	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
set1	Dynamic	1.15	1.17	1.23	1.54	1.69	1.78	1.85	1.96	2.04	2.09
	Static	1.23	1.24	1.30	1.64	1.70	1.81	1.98	2.03	2.11	2.20
set2	Dynamic	1.04	1.07	1.13	1.18	1.20	1.25	1.41	1.42	1.45	1.49
	Static	1.13	1.14	1.18	1.19	1.23	1.31	1.47	1.50	1.52	1.56
set3	Dynamic	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.06	0.06	0.06
	Static	0.04	0.05	0.05	0.05	0.06	0.06	0.06	0.07	0.07	0.08
set4	Dynamic	3.12	3.17	3.18	3.20	3.27	3.26	3.20	2.99	3.08	3.19
	Static	3.10	3.41	3.77	3.87	3.89	4.20	4.54	4.52	4.70	4.90
set5	Dynamic	0.07	0.08	0.08	0.09	0.10	0.10	0.11	0.11	0.12	0.12
	Static	0.08	0.09	0.09	0.10	0.11	0.11	0.13	0.13	0.14	0.14
set6	Dynamic	0.30	0.31	0.35	0.35	0.37	0.39	0.40	0.41	0.43	0.45
	Static	0.30	0.31	0.35	0.36	0.39	0.41	0.42	0.43	0.45	0.49
set7	Dynamic	4.37	4.72	5.41	5.66	6.26	6.79	7.04	7.32	7.67	8.21
	Static	4.72	4.98	5.55	5.65	6.29	6.87	7.38	7.83	8.25	8.81
set8	Dynamic	84.15	99.02	109.75	113.51	128.56	133.60	134.50	140.67	153.24	160.20
	Static	91.44	109.94	126.77	150.44	188.28	227.17	262.26	269.38	271.84	278.17
set9	Dynamic	218.80	288.74	364.44	454.82	514.33	538.72	619.72	646.72	702.72	738.72
	Static	236.47	308.13	393.26	490.10	542.48	579.22	658.22	696.22	763.22	836.89
set10	Dynamic	593.66	692.62	816.93	1002.48	1180.78	1284.78	1407.78	1434.78	1563.78	1663.58
	Static	646.66	843.34	1084.78	1394.46	1624.46	1628.46	1811.67	1845.92	2084.92	2284.92

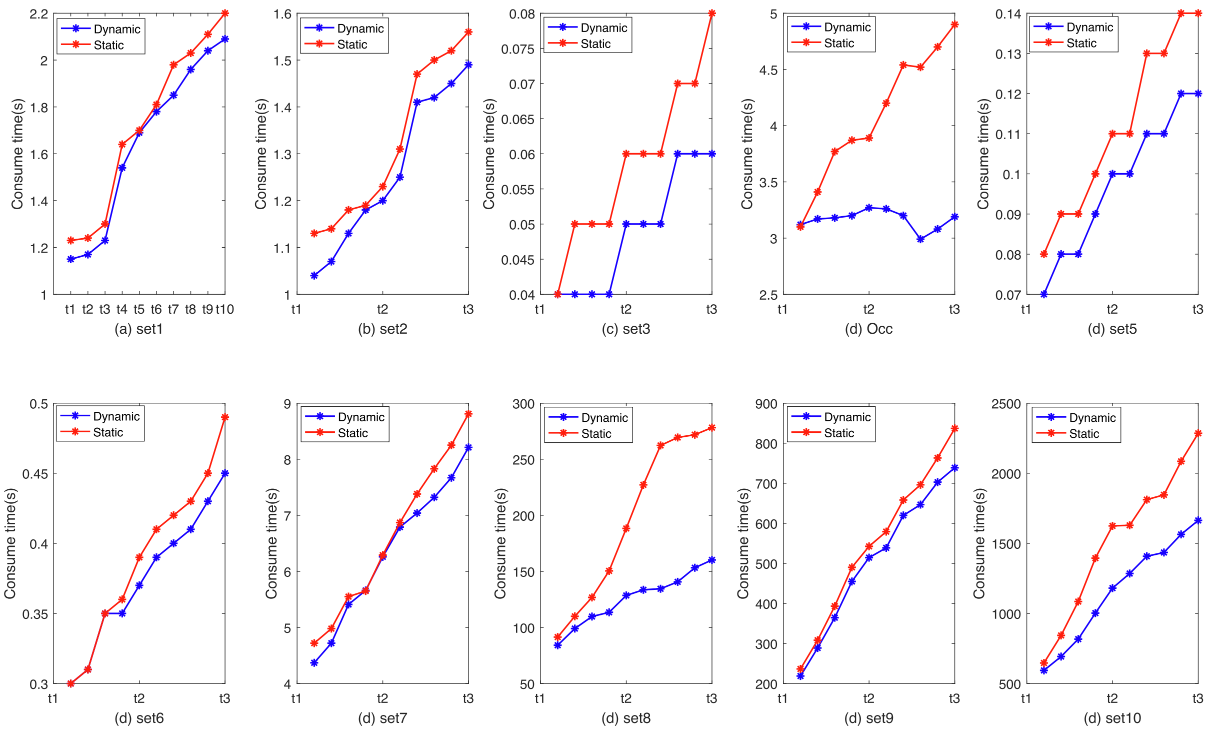

Fig. 8. The consume time of updating concepts on dynamic and static mechanisms.

tant. The update mechanism in ILMPFTC is designed to be dynamic for saving time, and its effectiveness is verified by the comparison between dynamic algorithm and static algorithm.

The Table 10 and Fig. 8 record the elapsed time of the dynamic algorithm and static algorithm for concept update with object increases. The computational time increases with the object increases in each dataset. When the data scale is small, the difference of time between dynamic and static algorithms is also small. The time gap increases as the data size increases, especially in large datasets. These results fully illustrate the advantages of the dynamic mechanism in updating concept space.

According to the above analysis, the incremental learning mechanism is an efficient for object classification:

1) The classify performance of learning mechanism based on fuzzy three-way concept is superior to the learning mechanism based on fuzzy concept, which is verified in Section 5.2.1.

2) The incremental learning mechanism can further improve the classify accuracy by making further use of the information of added objects and realizing the update of concept space in time.

3) In the aspect of object classification, the ILMPFTC is better than the other four fuzzy classify algorithms based on KNN classifier, and which is further verified by the statistical test. Meanwhile, the classify performance of ILMPFTC is the best among five classify algorithms with the increases of object.

4) Compared with static algorithm, the dynamic update mechanism could significantly improve the calculation efficiency of concept updating.

According to the above analysis, the ILMPFTC based on progressive fuzzy three-way concept is an excellent algorithm in object classification.

6. Conclusion

With the development of information technology, the data updates rapidly and the data without label makes it difficult for us to learn. How to identify the class label of added objects is important in data mining. Recently, the learning algorithm based on the concept thought has been widely used in object classification. In regular formal context, the concept describes object through the intent and extent, but it is limited in dealing with numerical data. Meanwhile, the absence of negative information in classical concept could also reduce the learn accuracy. The proposed ILMPFTC algorithm based on progressive fuzzy concept settles the above issues and can achieve a better classify performance. As the updated concept is further used for learning, the concept space contains new information about new objects, so the incremental mechanism can improve the classification accuracy. Meanwhile, the dynamic update mechanism in ILMPFTC is introduced to obtain the updated concept space after adding new objects. All the above concludes are verified in experiments on ten public datasets. In real life, as long as you can describe the object from concept thought, the ILMPFTC can be used to settle its corresponding classify problem, such as pattern recognition and face recognition. At the same time, the idea of incremental learning can also be applied to dynamic increased data, such as time series data. Moreover, the concept learning itself is a complex problem, so how to describe a concept and how to efficiently construct the concept space is a point worthy our further study.

CRediT authorship contribution statement

Kehua Yuan: Data curation, Methodology, Software, Visualization, Writing - original draft, Writing - review & editing. **Weihua Xu:** Conceptualization, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation. **Wentao Li:** Investigation. **Weiping Ding:** Investigation, Supervision.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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