Knowledge-Based Systems 227 (2021) 107223

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

# Feature selection for dynamic interval-valued ordered data based on fuzzy dominance neighborhood rough set

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ARTICLE INFO

*Article history:*
Received 28 November 2020
Received in revised form 3 June 2021
Accepted 9 June 2021
Available online 12 June 2021

*Keywords:*
Interval-valued ordered decision system
Fuzzy dominance neighborhood rough set
Conditional entropy
Incremental learning
Feature selection
Attribute reduction

ABSTRACT

Incremental learning strategy based feature selection approaches can improve the efficiency of reduction algorithm used for datasets with dynamic characteristic, which has attracted increasing research attention. Nevertheless, there is currently no work on incremental feature selection approaches for dynamic interval-valued ordered data. Interval-valued ordered data is a generalized form of single-valued ordered data, which is more widely used in practice. However, the endpoints of the interval numbers are easily polluted by noise, thereby the knowledge granules are very sensitive. Motivated by these two issues, we study incremental feature selection approaches based on a fuzzy dominance neighborhood rough set (FDNRS) for dynamic interval-valued ordered data in this work. First, we propose the FDNRS model for an interval-valued ordered decision system (IvODS) and investigate its related properties. Second, a conditional entropy with robustness is proposed based on the proposed model. This conditional entropy can measure the degree of monotonic consistency of the IvODS, so it is used as a metric and combined with a heuristic feature selection algorithm. Finally, two incremental feature selection algorithms are proposed on the basis of the above researches. Experiments are performed on nine public datasets to evaluate the robustness of the proposed metric and the performance of the incremental algorithms. Experimental results verify that the proposed metric is robust and our incremental algorithms are effective and efficient for updating reducts in dynamic IvODS.



## 1. Introduction

With the development of the information age, various complex data need to be dealt with in different fields, among which interval-valued data is one of the important representatives. Interval-valued data is widely used in the real world, it is usually used to characterize inaccurate and ambiguous information, such as fluctuations of commodity prices [1], changes of temperature [2], and the range of physiological indicators [3]. In multi-criteria decision analysis problems, interval-valued data follows a preference-ordered relation, which is called interval-valued ordered data [4]. In practical applications, interval-valued ordered data evolves over time, i.e., dynamic interval-valued ordered data [5,6], which brings challenges for efficient data mining in such data.

Feature selection is a common data dimensionality reduction method in data mining, it can identify more relevant features and reduce the dimension of data, thereby improving the classification ability of the learning models [7–11]. For dynamic data, some traditional feature selection methods have exposed the defects of low computational efficiency. To improve efficiency, feature selection algorithms with incremental technology have attracted increasing research attention [12–16]. Nevertheless, up to now, there is no incremental feature selection method for dynamic interval-valued ordered data. In order to further complete the research in this field, we study the feature selection method with incremental technology on dynamic interval-valued ordered data.

Rough set theory (RST) is a granular computing tool, which is widely used to deal with uncertain and vague information. Interval-valued data is called interval-valued information system (IvIS) in RST. In recent years, some extended rough set models for IvIS have been successively proposed, as shown in Table 1.

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https://doi.org/10.1016/j.knosys.2021.107223

**Table 1**
The review of some extended rough set models for IvIS.

| Year | Authors | Extended models | Reference |
|---|---|---|---|
| 2008 | Gong et al. | Rough set model of interval-valued fuzzy information system | [17] |
| 2008 | Sun et al. | Fuzzy rough set model of interval-valued fuzzy information system | [18] |
| 2008 | Leung et al. | Rough set approach for the discovery of classification rules in IvIS | [19] |
| 2008 | Qian et al. | Dominance-based rough set approach of ordered IvIS | [4] |
| 2009 | Yang et al. | Dominance-based rough set approach of incomplete ordered IvIS | [20] |
| 2013 | Zhang et al. | Variable-precision dominance-based rough set approach of ordered IvIS | [21] |
| 2015 | Yang et al. | $\alpha$-dominance relation based rough set model of ordered IvIS | [22] |
| 2017 | Dai et al. | Probability approach based dominance fuzzy rough set model of IvODS | [23] |
| 2018 | Dai et al. | Dominance-based fuzzy rough set model of incomplete ordered IvIS | [24] |

Although some of the dominance-based rough set approach (DRSA) models have been extended to IvODS in the above researches, these models cannot describe the preference-ordered relation between objects in IvODS both qualitatively and quantitatively. The fuzzy preference based rough sets model [25], proposed by Hu et al. can make up for this deficiency. Therefore, it is very meaningful to extend this model to IvODS. But this model is not robust, because it does not consider that the boundaries of interval numbers are easily disturbed by noise, then cause the perturbation of the endpoint values. This shortcoming makes the knowledge granule lack of fault tolerance (flexibility), thus providing decision-makers with wrong information, which may eventually lead to wrong decisions. Inspired by this, we introduce the idea of neighborhood into the fuzzy preference based rough sets model, and propose a new model to make the knowledge granule robust, i.e., the FDNRS model of IvODS.

Uncertainty metric is an important research content of RST. In recent years, RST-based uncertainty metrics for interval-valued data have attracted the attention of many scholars. Some representative works are shown in Table 2. However, these metrics do not take into account the preference-ordered relation of between objects in IvODS. For ordered data, Hu et al. proposed rank conditional entropy and fuzzy rank conditional entropy [26], and then they were applied to feature selection [27] and decision trees [28] for monotonic classification tasks. Inspired by this, we introduce a FDNRS based conditional entropy (called fuzzy dominance neighborhood conditional entropy (FDNCE)) to evaluate the consistency degree of the ordering of samples under features and decisions in IvODS. In this study, the FDNCE is used as a feature evaluation index for feature selection in IvODS.

Feature selection is also called attribute reduction in RST. Some RST-based attribute reduction methods have been extended or further improved for interval-valued data, as shown in Table 3. However, the above attribute reduction method has two insufficiencies. On the one hand, these methods do not consider interval-valued data with a preference-ordered relation. On the other hand, for interval-valued data with dynamic characteristics, these methods expose the disadvantage of high time cost. Because these attribute reduction methods must be executed repeatedly when new data arrives or old data is removed, which causes a lot of unnecessary calculations. Therefore, it is very meaningful to study an efficient attribute reduction method that can be applied to data with dynamic interval-valued ordered data.

The feature selection with incremental mechanism can efficiently extract the necessary attributes from dynamic datasets. In recent years, the research on incremental feature selection has attracted the attention of many scholars. Some recent research works are presented in Table 4. Although scholars have done a lot of works on the research of incremental feature selection methods, these existing methods are not suitable for dynamic interval-valued ordered data. This flaw inspires our study.

In this study, we propose incremental feature selection methods based on FDNRS model for dynamic interval-valued ordered datasets with time-evolving objects. The major contributions of this study are as follows.

- We propose a new rough set model FDNRS for IvODS, and give reasonable explanations of the approximate operators of this model. Moreover, the relevant properties of this model are presented and proved.
- We define a robust uncertainty metric FDNCE based on FDNRS model, which is used as an uncertainty metric to evaluate the degree of ranking consistency of objects in IvODS. This metric is proven to be non-monotonic, and then is combined with the heuristic feature selection strategy.
- Based on the above researches, we propose two incremental feature selection algorithms when a group objects are added to or deleted from an IvODS, respectively.
- Comparison experiments are performed on public datasets, and the results indicate that the robustness of the proposed metric and the effectiveness and efficiency of the proposed incremental algorithms.

The remaining of the paper is organized as follows. Section 2 introduces the related knowledge. In Section 3, the FDNRS model of IvODS is proposed, and its relevant properties are investigated. Section 4 proposes FDNCE and a FDNCE-based heuristic non-monotonic feature selection algorithm for IvODS. In Section 5, two incremental feature selection methods are introduced. The results and analysis of our experiments are reported in Section 6. Finally, Section 7 summarizes the study and outlines the further work.

## 2. Preliminaries

In this section, some basic concepts are introduced, which can be found in literatures [4,54].

### 2.1. Interval-valued ordered decision system

**Definition 2.1.** Let $S = \langle U, A \cup \{d\}, V\rangle$ be a decision system, where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects; $A$ is a nonempty finite set of conditional attributes, $d$ is a decision attribute; $V = \bigcup V_{a_k}$ $(a_k \in A \cup \{d\})$, $V_{a_k} = \{v(x_i, a_k)|\forall x_i \in U\}$, $v(x_i, a_k)$ is the value of $x_i$ under attribute $a_k$, which is also denoted by $v_{ik}$.

**Definition 2.2** (*[54]*). Let $IS = \langle U, A\cup\{d\}, V\rangle$ be an interval-valued decision system, for any $x_i \in U$, $a_k \in A$, $v(x_i, a_k)$ is an interval-valued number, i.e., $v(x_i, a_k) = [v^l_{a_k}(x_i), v^r_{a_k}(x_i)] = \{t|v^l_{a_k}(x_i) \le t \le v^r_{a_k}(x_i), v^l_{a_k}(x_i), v^r_{a_k}(x_i) \in R\}$, $v^l_{a_k}(x_i)$ and $v^r_{a_k}(x_i)$ are called the left and right boundaries of $v(x_i, a_k)$, respectively, and they can also be denoted by $v^l_{ik}$ and $v^r_{ik}$. Furthermore, for any $x_i \in U$, $v(x_i, d)$ is a single value under decision attribute $d$.

In an interval-valued decision system, for any $x_i \in U$, $a_k \in A$, $v(x_i, a_k)$ degenerates to a single value when $v^l_{a_k}(x_i) = v^r_{a_k}(x_i)$. Therefore, a single-valued decision system is a special form of the interval-valued decision system.

**Table 2**
The review of some RST-based uncertainty metrics for IvIS.

| Year | Authors | Uncertainty metrics | Reference |
|---|---|---|---|
| 2012 | Dai et al. | Possible degree based conditional entropy for interval-valued decision systems | [29] |
| 2013 | Dai et al. | Similarity relation based accuracy and roughness for IvIS | [30] |
| 2013 | Huang et al. | Information entropy for interval-valued intuitionistic fuzzy information systems | [31] |
| 2017 | Dai et al. | Accuracy, roughness, and approximation accuracy based on $\alpha$-weak similarity for incomplete IvIS | [32] |
| 2019 | Xie et al. | $\theta$-information granulation, $\theta$-information amount, $\theta$-rough entropy, and $\theta$-information entropy for IvIS | [33] |

**Table 3**
The review of some RST-based attribute reduction methods for IvIS.

| Year | Authors | Attribute reduction methods | Reference |
|---|---|---|---|
| 2014 | Zhang et al. | Confidence preserved based attribute reduction method for IvIS | [34] |
| 2016 | Dai et al. | Information entropy based attribute reduction method for IvIS | [35] |
| 2019 | Shu et al. | $\theta$-conditional entropy based attribute reduction method for incomplete IvIS | [36] |
| 2020 | Liu et al. | $\alpha$-mutual information based unsupervised attribute reduction method for IvIS | [37] |
| 2020 | Dai et al. | Kernel density estimation based attribute reduction approach for IvIS | [38] |

**Table 4**
The review of some incremental feature selection methods.

| Year | Authors | Incremental feature selection methods | Reference |
|---|---|---|---|
| 2014 | Liang et al. | Incremental feature selection based on information entropy for dynamic data with samples change | [39] |
| 2015 | Zeng et al. | Incremental feature selection based on fuzzy rough set for dynamic hybrid information systems | [40] |
| 2017 | Lang et al. | Incremental updating reducts approaches for dynamic covering information systems | [41] |
| 2018 | Das et al. | Incremental feature selection for classification using RST-based genetic algorithm | [42] |
| 2018 | Yang et al. | Fuzzy rough sets based incremental attribute reduction algorithms by active sample selection strategy | [43] |
| 2018 | Wei et al. | Discernibility matrix based incremental attribute reduction method when attribute values change | [44] |
| 2019 | Shu et al. | Two incremental feature selection methods when multiple objects are added or deleted from data | [16] |
| 2019 | Zhang et al. | Information entropy based incremental feature selection approach using the active sample selection strategy under the framework of fuzzy rough set theory | [45] |
| 2019 | Wei et al. | Accelerated incremental attribute reduction method by combining the method of compressing information table with the incremental technology | [46] |
| 2019 | Cai et al. | Two incremental methods for attribute reduction from the perspective of the coarsening and refining covering granularity | [47] |
| 2020 | Ni et al. | Fuzzy rough set based incremental feature selection approach by introducing a key instance set containing representative instances | [48] |
| 2020 | Shu et al. | Incremental attribute reduction method based on neighborhood rough set for dynamic hybrid data | [49] |
| 2020 | Yang et al. | Incremental attribute reduction approach for heterogeneous data with the ordered arrival of objects | [50] |
| 2020 | Liu et al. | Discernibility matrix based incremental feature selection method for fused information system | [51] |
| 2020 | Chen et al. | Incremental attribute reduction approach using discernible relations when multiple attributes are added simultaneously | [52] |
| 2020 | Dong et al. | Incremental update reduction method when multiple objects and attributes are added to an information table simultaneously | [53] |

**Definition 2.3** ([4]). Let $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$ be an IvODS, for any $a_k \in A$ is a criterion, $V_{a_k}$ is completely pre-ordered by the relation $\preceq_{a_k}$: $\forall x_i, x_j \in U$, $x_i \preceq_{a_k} x_j \Leftrightarrow v(x_i, a_k) \le v(x_j, a_k)$ (i.e. an increasing preference) or $x_i \preceq_{a_k} x_j \Leftrightarrow v(x_i, a_k) \ge v(x_j, a_k)$ (i.e. a decreasing preference).

In real-world applications, decision makers usually know the order of criterion values within their domain or prior knowledge. Such as, for the test score and operating profit, the higher the better. For risk assessment, the lower the better with all other things being equal. For simplicity and without any loss of generality, the following we only consider criteria with increasing preferences.

### 2.2. Dominance-based rough set approach to IvODS

**Definition 2.4** ([4]). Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, $\forall B \subseteq A$, the dominance relation $D_B^{\preceq}$ is defined as

$$D_B^{\preceq} = \{(x_i, x_j) \in U \times U | v_{a_k}^l(x_i) \le v_{a_k}^l(x_j),\ v_{a_k}^r(x_i) \le v_{a_k}^r(x_j), \forall a_k \in B\}. \tag{1}$$

From Eq. (1), we easily find that the dominance relation $D_B^{\preceq}$ is reflexive, asymmetric, and transitive. Moreover, the dominance relation on decision attribute $d$ is denoted as $D_d^{\preceq} = \{(x_i, x_j) \in U \times U | v(x_i, d) \le v(x_j, d)\}$.

**Definition 2.5** ([4]). Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, $\forall B \subseteq A$, the dominating and dominated sets of $x_i \in U$ in term of $B$ are defined as

$$D_B^{+}(x_i) = \{x_j \in U | x_i D_B^{\preceq} x_j\}; \tag{2}$$

$$D_B^{-}(x_i) = \{x_j \in U | x_j D_B^{\preceq} x_i\}, \tag{3}$$

which are call knowledge granules induced by $D_B^{\preceq}$.

**Property 2.1** ([4]). *For any $B_1, B_2 \subseteq A$ and $\forall x \in U$, the following properties hold.*

(1) *If $B_1 \subseteq B_2$, then $D_{B_2}^{+}(x) \subseteq D_{B_1}^{+}(x)$ and $D_{B_2}^{-}(x) \subseteq D_{B_1}^{-}(x)$;*
(2) *$D_{B_1}^{+}(x) \cap D_{B_2}^{+}(x) = D_{B_1 \cup B_2}^{+}(x)$ and $D_{B_1}^{-}(x) \cap D_{B_2}^{-}(x) = D_{B_1 \cup B_2}^{-}(x)$.*

In IvODS, $d$ is a decision attribute, $U/d = \{Cl_t | t \in \{1, \ldots, T\}\}$ ($T \le |U|$), where for each $Cl_t$ be an equivalence class, and $Cl_T \succ \ldots \succ Cl_t \succ \ldots \succ Cl_1$. The upward and downward unions in DRSA are expressed as $Cl_t^{\succeq} = \bigcup Cl_{t'}(t' \ge t)$ and $Cl_t^{\preceq} = \bigcup Cl_{t'}(t' \le t)$, where $t, t' \in \{1, \ldots, T\}$. If $x \in Cl_t^{\succeq}$, then the decision of $x$ cannot be worse than $Cl_t$; if $x \in Cl_t^{\preceq}$, then the decision of $x$ cannot be better than $Cl_t$. Note that $Cl_0^{\preceq} = Cl_{T+1}^{\succeq} = \emptyset$ and $Cl_T^{\preceq} = Cl_1^{\succeq} = U$.

**Definition 2.6** ([54]). Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, $\forall B \subseteq A$ and $t \in \{1, \ldots, T\}$, the lower and upper approximations of

**Table 5**
An interval-valued ordered decision system.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $d$ |
|---|---|---|---|---|---|
| $x_1$ | [0.28, 0.30] | [0.33, 0.40] | [0.54, 0.66] | [0.53, 0.65] | 1 |
| $x_2$ | [0.27, 0.29] | [0.49, 0.60] | [0.36, 0.44] | [0.41, 0.50] | 3 |
| $x_3$ | [0.40, 0.43] | [0.41, 0.50] | [0.27, 0.33] | 0 | 2 |
| $x_4$ | [0.41, 0.50] | [0.08, 0.10] | [0.20, 0.24] | [0.41, 0.50] | 3 |
| $x_5$ | [0.42, 0.44] | [0.16, 0.20] | 0 | [0.16, 0.20] | 1 |
| $x_6$ | [0.55, 0.60] | [0.82, 1.00] | [0.72, 0.88] | [0.82, 1.00] | 2 |
| $x_7$ | [0.78, 0.81] | [0.65, 0.80] | [0.36, 0.44] | [0.08, 0.10] | 1 |

the upward union $Cl_t^{\succeq}$ and downward union $Cl_t^{\preceq}$ are respectively defined as

$$\underline{D_B^{\preceq}}(Cl_t^{\succeq}) = \{x \in U | D_B^{+}(x) \subseteq Cl_t^{\succeq}\}, \tag{4}$$

$$\overline{D_B^{\preceq}}(Cl_t^{\succeq}) = \{x \in U | D_B^{-}(x) \cap Cl_t^{\succeq} \neq \emptyset\}; \tag{5}$$

$$\underline{D_B^{\preceq}}(Cl_t^{\preceq}) = \{x \in U | D_B^{-}(x) \subseteq Cl_t^{\preceq}\}, \tag{6}$$

$$\overline{D_B^{\preceq}}(Cl_t^{\preceq}) = \{x \in U | D_B^{+}(x) \cap Cl_t^{\preceq} \neq \emptyset\}. \tag{7}$$

The boundary regions of $Cl_t^{\succeq}$ and $Cl_t^{\preceq}$ are defined as

$$Bn_B(Cl_t^{\succeq}) = \overline{D_B^{\preceq}}(Cl_t^{\succeq}) - \underline{D_B^{\preceq}}(Cl_t^{\succeq}), \tag{8}$$

$$Bn_B(Cl_t^{\preceq}) = \overline{D_B^{\preceq}}(Cl_t^{\preceq}) - \underline{D_B^{\preceq}}(Cl_t^{\preceq}). \tag{9}$$

In addition, $\underline{D_B^{\preceq}}(\emptyset) = \overline{D_B^{\preceq}}(\emptyset) = \emptyset$, $\underline{D_B^{\preceq}}(U) = \overline{D_B^{\preceq}}(U) = U$, and $Bn_B(\emptyset) = Bn_B(U) = \emptyset$.

**Property 2.2** *([54]). For any $B \subseteq A$, the approximations of $Cl_t^{\succeq}$ and $Cl_t^{\preceq}$ ($t \in \{1, \ldots, T\}$) have the following properties.*

(1) $\underline{D_B^{\preceq}}(Cl_t^{\succeq}) \subseteq Cl_t^{\succeq} \subseteq \overline{D_B^{\preceq}}(Cl_t^{\succeq})$ *and* $\underline{D_B^{\preceq}}(Cl_t^{\preceq}) \subseteq Cl_t^{\preceq} \subseteq \overline{D_B^{\preceq}}(Cl_t^{\preceq})$.

(2) $\underline{D_B^{\preceq}}(Cl_t^{\succeq}) = U - \overline{D_B^{\preceq}}(Cl_{t-1}^{\preceq})$ *and* $\underline{D_B^{\preceq}}(Cl_t^{\preceq}) = U - \overline{D_B^{\preceq}}(Cl_{t+1}^{\succeq})$.

(3) $Bn_B(Cl_t^{\succeq}) = Bn_B(Cl_{t-1}^{\preceq})$.

The following, we give an example to illustrate these definitions and properties mentioned above.

**Example 1.** Table 5 shows an IvODS, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $A = \{a_1, a_2, a_3, a_4\}$, and $d$ is decision attribute.

According to Definition 2.5, the dominating and dominated sets of each object are calculated as $D_A^{+}(x_1) = \{x_1, x_6\}$, $D_A^{+}(x_2) = \{x_2, x_6\}$, $D_A^{+}(x_3) = \{x_3, x_6, x_7\}$, $D_A^{+}(x_4) = \{x_4, x_6\}$, $D_A^{+}(x_5) = \{x_5, x_6\}$, $D_A^{+}(x_6) = \{x_6\}$, $D_A^{+}(x_7) = \{x_7\}$; $D_A^{-}(x_1) = \{x_1\}$, $D_A^{-}(x_2) = \{x_2\}$, $D_A^{-}(x_3) = \{x_3\}$, $D_A^{-}(x_4) = \{x_4\}$, $D_A^{-}(x_5) = \{x_5\}$, $D_A^{-}(x_6) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $D_A^{-}(x_7) = \{x_3, x_7\}$. Then, the upward and downward unions are $Cl_1^{\succeq} = U$, $Cl_2^{\succeq} = \{x_2, x_3, x_4, x_6\}$, $Cl_3^{\succeq} = \{x_2, x_4\}$; $Cl_1^{\preceq} = \{x_1, x_5, x_7\}$, $Cl_2^{\preceq} = \{x_1, x_3, x_5, x_6, x_7\}$, $Cl_3^{\preceq} = U$. According to Definition 2.6, the approximations of the upward unions are calculated as $\underline{D_A^{\preceq}}(Cl_1^{\succeq}) = U$, $\underline{D_A^{\preceq}}(Cl_2^{\succeq}) = \{x_2, x_4, x_6\}$, $\underline{D_A^{\preceq}}(Cl_3^{\succeq}) = \emptyset$; $\overline{D_A^{\preceq}}(Cl_1^{\succeq}) = U$, $\overline{D_A^{\preceq}}(Cl_2^{\succeq}) = \{x_2, x_3, x_4, x_6, x_7\}$, $\overline{D_A^{\preceq}}(Cl_3^{\succeq}) = \{x_2, x_4, x_6\}$. The approximations of the downward unions are calculated as $\underline{D_A^{\preceq}}(Cl_1^{\preceq}) = \{x_1, x_5\}$, $\underline{D_A^{\preceq}}(Cl_2^{\preceq}) = \{x_1, x_3, x_5, x_7\}$, $\underline{D_A^{\preceq}}(Cl_3^{\preceq}) = U$; $\overline{D_A^{\preceq}}(Cl_1^{\preceq}) = \{x_1, x_3, x_5, x_7\}$, $\overline{D_A^{\preceq}}(Cl_2^{\preceq}) = U$, $\overline{D_A^{\preceq}}(Cl_3^{\preceq}) = U$. The boundary regions of the upward and downward unions are calculated as $Bn_A(Cl_1^{\succeq}) = \emptyset$, $Bn_A(Cl_2^{\succeq}) = \{x_3, x_7\}$, $Bn_A(Cl_3^{\succeq}) = \{x_2, x_4, x_6\}$; $Bn_A(Cl_1^{\preceq}) = \{x_3, x_7\}$, $Bn_A(Cl_2^{\preceq}) = \{x_2, x_4, x_6\}$, $Bn_A(Cl_3^{\preceq}) = \emptyset$. Next, we verify Property 2.2 as follows. Let $t = 2$, (1) $\underline{D_A^{\preceq}}(Cl_2^{\succeq}) \subseteq Cl_2^{\succeq} \subseteq \overline{D_A^{\preceq}}(Cl_2^{\succeq})$ and $\underline{D_A^{\preceq}}(Cl_2^{\preceq}) \subseteq Cl_2^{\preceq} \subseteq \overline{D_A^{\preceq}}(Cl_2^{\preceq})$; (2) $\underline{D_A^{\preceq}}(Cl_2^{\succeq}) = U - \overline{D_A^{\preceq}}(Cl_1^{\preceq})$ and $\underline{D_A^{\preceq}}(Cl_2^{\preceq}) = U - \overline{D_A^{\preceq}}(Cl_3^{\succeq})$; (3) $Bn_A(Cl_2^{\succeq}) = Bn_A(Cl_1^{\preceq})$.

The DRSA for interval-valued ordered data [4,54] is an important extension model of the classic DRSA [55]. It is worth noting that the extended model only qualitatively considers the preference relation (i.e., Definition 2.4) between interval numbers, which is a boolean relation. However, in practical applications, decision makers (or users) usually need to not only qualitatively consider the preference relation between samples, but also quantitatively consider the degree of preference between samples. Consequently, the extended model needs to be further improved to make up for its shortcomings. After research, we found that the fuzzy set theory can make up for this defect, because it can quantify the degree of uncertainty of the concept, which meets the requirements of practical application. As pointed out by Zadeh [56], in human reasoning and concept formation, the granules used are fuzzy rather than Boolean. Therefore, we introduce the idea of fuzzy set into DRSA based on IvODS, which is necessary and meaningful.

## 3. Fuzzy dominance neighborhood rough set to IvODS

In this section, we propose a new model to IvODS, called FDNRS model. This model qualitatively and quantitatively considers the preference-ordered relation between objects in IvODS. Not only that, the proposed model also combines the idea of neighborhood to avoid the influence of noise for knowledge. The relevant definitions and properties are introduced as follow.

### 3.1. Fuzzy dominance neighborhood relation and fuzzy knowledge granules

The fuzzy dominance degree is firstly defined to describe the preference relation between interval numbers more precisely. Then, we introduce the idea of neighborhood, and propose the fuzzy dominance neighborhood relation between objects in IvODS. Final, the fuzzy knowledge granules of IvODS induced by fuzzy dominance neighborhood relation are introduced.

The following, we review some basic knowledge used in this subsection on fuzzy set theory [57].

Let $U = \{x_1, x_2, \ldots, x_n\}$, if $\mathcal{A}$ is a map of $U$ to $[0, 1]$, which is $\mathcal{A} : U \to [0, 1]$, then $\mathcal{A}$ is called the fuzzy set on $U$. For any $x_i \in U$, $\mathcal{A}(x_i)$ is called the membership function of $\mathcal{A}$, or the membership of $x_i$ for $\mathcal{A}$. The fuzzy set is denoted as $\mathcal{A} = \frac{\mathcal{A}(x_1)}{x_1} + \frac{\mathcal{A}(x_2)}{x_2} + \cdots + \frac{\mathcal{A}(x_n)}{x_n}$ or $\mathcal{A} = \sum_{i=1}^{n} \frac{\mathcal{A}(x_i)}{x_i}$. Note that a crisp set $A$ can be regarded as a special fuzzy set, it can also be denoted as $A = \sum_{i=1}^{n} \frac{A(x_i)}{x_i}$, where $\forall A(x_i) \in \{0, 1\}$.

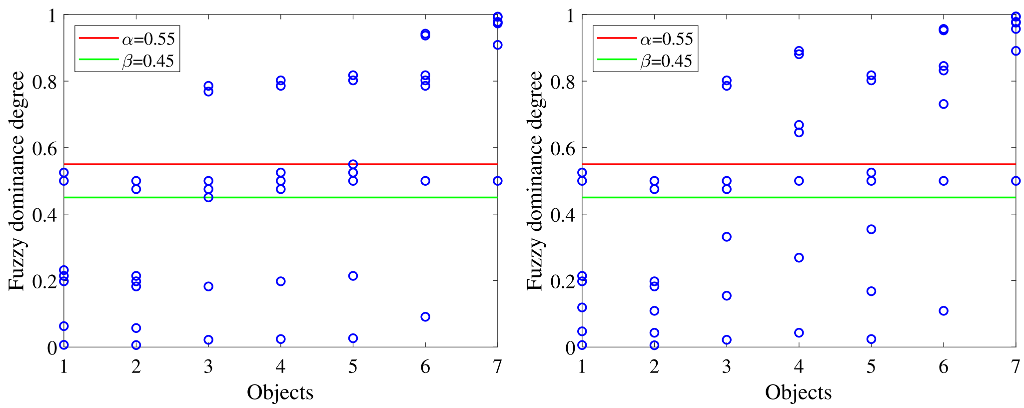
Let $\mathcal{A}, \mathcal{B}$ are two fuzzy sets, for any $x \in U$, some operations of fuzzy set are defined as (1) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathcal{A}(x) = \mathcal{B}(x)$; (2) $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \mathcal{A}(x) \leq \mathcal{B}(x)$; (3) $(\mathcal{A} \cup \mathcal{B})(x) = max\{\mathcal{A}(x), \mathcal{B}(x)\} = \mathcal{A}(x) \vee \mathcal{B}(x)$; (4) $(\mathcal{A} \cap \mathcal{B})(x) = min\{\mathcal{A}(x), \mathcal{B}(x)\} = \mathcal{A}(x) \wedge \mathcal{B}(x)$; (5) $|\mathcal{A}| = \sum_{i=1}^{n} \mathcal{A}(x_i)$; (6) $\emptyset$ is also a fuzzy set, $\emptyset(x) = 0$.

**Definition 3.1.** Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V \rangle$, $\forall a_k \in A$ and $x_i, x_j \in U$, the fuzzy dominance degree between $x_i$ and $x_j$ on $a_k$ is defined as

$$\mathcal{D}_{a_k}^{\prec}(x_i, x_j) = \frac{1}{2}(\mathcal{LD}_{a_k}^{\prec}(x_i, x_j) + \mathcal{RD}_{a_k}^{\prec}(x_i, x_j)), \tag{10}$$

where $\mathcal{LD}_{a_k}^{\prec}(x_i, x_j) = \frac{1}{1+e^{-p(v_{a_k}^{l}(x_j) - v_{a_k}^{l}(x_i))}}$ is called the left fuzzy dominance degree, $\mathcal{RD}_{a_k}^{\prec}(x_i, x_j) = \frac{1}{1+e^{-p(v_{a_k}^{r}(x_j) - v_{a_k}^{r}(x_i))}}$ is called the right fuzzy dominance degree, and $p$ is a positive constant.

For convenience, $\mathcal{D}_{a_k}^{\prec}(x_i, x_j)$, $\mathcal{LD}_{a_k}^{\prec}(x_i, x_j)$, and $\mathcal{RD}_{a_k}^{\prec}(x_i, x_j)$ can be simplified to $\mathcal{D}_{(i,j)}^{\prec a_k}$, $\mathcal{LD}_{(i,j)}^{\prec a_k}$, and $\mathcal{RD}_{(i,j)}^{\prec a_k}$, respectively. The $\mathcal{LD}_{(i,j)}^{\prec a_k}$ indicates the extent of the left boundary of $x_j$ better than that of $x_i$ on $a_k$. Similarly, the $\mathcal{RD}_{(i,j)}^{\prec a_k}$ indicates the extent of the




(a) The distribution of $\mathcal{LD}_{(i,j)}^{\prec a_1}$ (b) The distribution of $\mathcal{RD}_{(i,j)}^{\prec a_1}$

**Fig. 1.** The distributions of left and right fuzzy dominance degrees on attribute $a_1$.

right boundary of $x_j$ better than that of $x_i$ on $a_k$. The values of fuzzy dominance degree embody the preference degree between interval numbers.

From Definition 3.1, it is easy to find that the characteristics of the calculation formula of $\mathcal{LD}_{(i,j)}^{\prec a_k}$ as follows. If $v_{a_k}^l(x_j) > v_{a_k}^l(x_i)$, then $0.5 < \mathcal{LD}_{(i,j)}^{\prec a_k} < 1$; if $v_{a_k}^l(x_j) = v_{a_k}^l(x_i)$, then $\mathcal{LD}_{(i,j)}^{\prec a_k} = 0.5$; if $v_{a_k}^l(x_j) < v_{a_k}^l(x_i)$, then $0 < \mathcal{LD}_{(i,j)}^{\prec a_k} < 0.5$. The calculation formula of $\mathcal{RD}_{(i,j)}^{\prec a_k}$ has the same characteristics. Fig. 1 shows the distributions of left and right fuzzy dominance degrees among objects on attribute $a_1$ in Table 5.

From Fig. 1, we can easily find that the values of left (right) fuzzy dominance degree in the area between $\alpha$ and $\beta$ are very close to 0.5. This indicates that the left (right) boundary of these objects under attribute $a_1$ can be regarded as no difference, because this case may be caused by noise. To avoid the influence of noise, we draw on the idea of neighborhood, and then define a fuzzy dominance neighborhood relation in IvODS.

**Definition 3.2.** Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, $\forall a_k \in B \subseteq A$ and $x_i, x_j \in U$, the fuzzy dominance neighborhood relation between $x_i$ and $x_j$ on $a_k$ is defined as

$$\mathcal{N}_{a_k}^{\prec}(x_i, x_j) = \begin{cases} 0.5, & (\beta \le \mathcal{LD}_{(i,j)}^{\prec a_k} \le \alpha) \wedge (\beta \le \mathcal{RD}_{(i,j)}^{\prec a_k} \le \alpha); \\ \frac{1}{2}(0.5 + \mathcal{RD}_{(i,j)}^{\prec a_k}), & (\beta \le \mathcal{LD}_{(i,j)}^{\prec a_k} \le \alpha) \wedge ((\mathcal{RD}_{(i,j)}^{\prec a_k} < \beta) \vee (\mathcal{RD}_{(i,j)}^{\prec a_k} > \alpha)); \\ \frac{1}{2}(\mathcal{LD}_{(i,j)}^{\prec a_k} + 0.5), & ((\mathcal{LD}_{(i,j)}^{\prec a_k} < \beta) \vee (\mathcal{LD}_{(i,j)}^{\prec a_k} > \alpha)) \wedge (\beta \le \mathcal{RD}_{(i,j)}^{\prec a_k} \le \alpha); \\ \mathcal{D}_{(i,j)}^{\prec a_k}, & otherwise, \end{cases} \tag{11}$$

where $\beta \in [0.4, 0.5)$, $\alpha \in (0.5, 0.6]$. Moreover, the fuzzy dominance neighborhood relation on attribute subset $B$ is defined as

$$\mathcal{N}_B^{\prec}(x_i, x_j) = \min_{a_k \in B} \mathcal{N}_{a_k}^{\prec}(x_i, x_j). \tag{12}$$

Analogously, $\mathcal{N}_B^{\prec}(x_i, x_j)$ can be simplified to $\mathcal{N}_{(i,j)}^{\prec B}$, which can derive a fuzzy dominance neighborhood relation matrix, i.e., $\widetilde{\mathbb{N}}_U^{\prec B} = [\mathcal{N}_{(i,j)}^{\prec B}]_{n\times n}$.

**Definition 3.3.** Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, $\forall B \subseteq A$, the fuzzy dominating neighborhood set and fuzzy dominated neighborhood set of $x_i \in U$ in term of $B$ are defined as

$$\mathcal{N}_B^{+}(x_i) = \frac{\mathcal{N}_{(i,1)}^{\prec B}}{x_1} + \frac{\mathcal{N}_{(i,2)}^{\prec B}}{x_2} + \cdots + \frac{\mathcal{N}_{(i,n)}^{\prec B}}{x_n}; \tag{13}$$

$$\mathcal{N}_B^{-}(x_i) = \frac{\mathcal{N}_{(1,i)}^{\prec B}}{x_1} + \frac{\mathcal{N}_{(2,i)}^{\prec B}}{x_2} + \cdots + \frac{\mathcal{N}_{(n,i)}^{\prec B}}{x_n}, \tag{14}$$

which are called the fuzzy knowledge granules induced by $\mathcal{N}_{(i,j)}^{\prec B}$.

Obviously, $\mathcal{N}_B^{+}(x_i)$ and $\mathcal{N}_B^{-}(x_i)$ are two fuzzy sets, then $\mathcal{N}_B^{+}(x_i)(x_j) = \mathcal{N}_{(i,j)}^{\prec B}$, $|\mathcal{N}_B^{+}(x_i)| = \sum_{j=1}^{n} \mathcal{N}_{(i,j)}^{\prec B}$, $\mathcal{N}_B^{-}(x_i)(x_j) = \mathcal{N}_{(j,i)}^{\prec B}$, and $|\mathcal{N}_B^{-}(x_i)| = \sum_{j=1}^{n} \mathcal{N}_{(j,i)}^{\prec B}$.

**Property 3.1.** *For any $B_1, B_2 \subseteq A$ and $\forall x_i \in U$, the following properties hold.*

(1) *If $B_1 \subseteq B_2$, then $\mathcal{N}_{B_2}^{+}(x_i) \subseteq \mathcal{N}_{B_1}^{+}(x_i)$ and $\mathcal{N}_{B_2}^{-}(x_i) \subseteq \mathcal{N}_{B_1}^{-}(x_i)$.*
(2) *$\mathcal{N}_{B_1}^{+}(x_i) \cap \mathcal{N}_{B_2}^{+}(x_i) = \mathcal{N}_{B_1 \cup B_2}^{+}(x_i)$ and $\mathcal{N}_{B_1}^{-}(x_i) \cap \mathcal{N}_{B_2}^{-}(x_i) = \mathcal{N}_{B_1 \cup B_2}^{-}(x_i)$.*

**Proof.** (1) For any $x_j \in U$, known $B_1 \subseteq B_2$, according to Definition 3.2, we have $\mathcal{N}_{B_2}^{\prec}(x_i, x_j) = \mathcal{N}_{B_1 \cup (B_2 - B_1)}^{\prec}(x_i, x_j) = \min\{\mathcal{N}_{B_1}^{\prec}(x_i, x_j), \mathcal{N}_{B_2 - B_1}^{\prec}(x_i, x_j)\} \le \mathcal{N}_{B_1}^{\prec}(x_i, x_j)$, i.e., $\mathcal{N}_{B_2}^{\prec}(x_i, x_j) \le \mathcal{N}_{B_1}^{\prec}(x_i, x_j)$. Then, according to Definition 3.3, we can naturally determine that $\mathcal{N}_{B_2}^{+}(x_i)(x_j) \le \mathcal{N}_{B_1}^{+}(x_i)(x_j)$ hold. Thus, we can obtain $\mathcal{N}_{B_2}^{+}(x_i) \subseteq \mathcal{N}_{B_1}^{+}(x_i)$. Analogously, the $\mathcal{N}_{B_2}^{-}(x_i) \subseteq \mathcal{N}_{B_1}^{-}(x_i)$ can be proved. (2) It can be proved immediately based on Definitions 3.2 and 3.3. □

Property 3.1 shows that fuzzy knowledge granules based on fuzzy dominance neighborhood relation are monotonic.

### *3.2. Approximations of FDNRS*

In this subsection, the approximations of the upward and downward unions are defined by comprehensively considering fuzzy dominance neighborhood relation in IvODS. Then, some related properties are presented and proved.

**Definition 3.4.** Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, $\forall B \subseteq A$ and $t \in \{1, \ldots, T\}$, the fuzzy lower and upper approximations of the upward union $Cl_t^{\succeq}$ and downward union $Cl_t^{\preceq}$ under $B$ are respectively defined as

$$\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \inf_{x_j \in U} \max(1 - \mathcal{N}_B^{+}(x_i)(x_j), Cl_t^{\succeq}(x_j)), \tag{15}$$

$$\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \sup_{x_j \in U} \min(\mathcal{N}_B^{-}(x_i)(x_j), Cl_t^{\succeq}(x_j)); \tag{16}$$

$$\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\preceq})(x_i) = \inf_{x_j \in U} \max(1 - \mathcal{N}_B^{-}(x_i)(x_j), Cl_t^{\preceq}(x_j)), \tag{17}$$

$$\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\preceq})(x_i) = \sup_{x_j \in U} \min(\mathcal{N}_B^{+}(x_i)(x_j), Cl_t^{\preceq}(x_j)). \tag{18}$$

Next, we simplify the four fuzzy approximation operators in Definition 3.4.

- For Eq. (15), $x_j \in U$ can be divided into two cases, i.e., $x_j \in Cl_t^{\succeq}$ and $x_j \notin Cl_t^{\succeq}$ ($x_j \in Cl_{t-1}^{\preceq}$). If $x_j \in Cl_t^{\succeq}$, then $Cl_t^{\succeq}(x_j) = 1$. Due to $1-\mathcal{N}_B^{+}(x_i)(x_j) \leq 1$, we have $\max(1-\mathcal{N}_B^{+}(x_i)(x_j), Cl_t^{\succeq}(x_j)) = 1$; If $x_j \notin Cl_t^{\succeq}$, then $Cl_t^{\succeq}(x_j) = 0$. Due to $1-\mathcal{N}_B^{+}(x_i)(x_j) \geq 0$, we have $\max(1-\mathcal{N}_B^{+}(x_i)(x_j), Cl_t^{\succeq}(x_j)) = 1-\mathcal{N}_B^{+}(x_i)(x_j)$. Obviously, $1 - \mathcal{N}_B^{+}(x_i)(x_j) \leq 1$, so we can easily get $\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \inf_{x_j \notin Cl_t^{\succeq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)$ ($\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \inf_{x_j \in Cl_{t-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)$).
- For Eq. (16), $x_j \in U$ can be divided into two cases, i.e., $x_j \in Cl_t^{\succeq}$ and $x_j \notin Cl_t^{\succeq}$. If $x_j \in Cl_t^{\succeq}$, then $Cl_t^{\succeq}(x_j) = 1$. Because $\mathcal{N}_B^{-}(x_i)(x_j) \leq 1$, we can get $\min(\mathcal{N}_B^{-}(x_i)(x_j), Cl_t^{\succeq}(x_j)) = \mathcal{N}_B^{-}(x_i)(x_j)$. If $x_j \notin Cl_t^{\succeq}$, then $Cl_t^{\succeq}(x_j) = 0$. Because $\mathcal{N}_B^{-}(x_i)(x_j) \geq 0$, we can get $\min(\mathcal{N}_B^{-}(x_i)(x_j), Cl_t^{\succeq}(x_j)) = 0$. Obviously, $\mathcal{N}_B^{-}(x_i)(x_j) \geq 0$, so we can easily get $\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \sup_{x_j \in Cl_t^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j)$.

Similarly, we can also simplify Eqs. (17) and (18). The following we give the simplified forms of these four fuzzy approximation operators respectively.

$$\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \inf_{x_j \notin Cl_t^{\succeq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j), \tag{19}$$

$$\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = \sup_{x_j \in Cl_t^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j); \tag{20}$$

$$\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\preceq})(x_i) = \inf_{x_j \notin Cl_t^{\preceq}} 1 - \mathcal{N}_B^{-}(x_i)(x_j), \tag{21}$$

$$\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\preceq})(x_i) = \sup_{x_j \in Cl_t^{\preceq}} \mathcal{N}_B^{+}(x_i)(x_j). \tag{22}$$

Subsequently, we give the reasonable explanations of these four approximation operators.

- From Eq. (19), we can intuitively find that for any $x_i \in U$, the membership of $x_i$ to fuzzy set $\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})$ depends on the best object that does not belong to class $Cl_t^{\succeq}$. The greater the degree that this object is better than $x_i$, the smaller the membership of $x_i$ to the fuzzy set $\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})$, and vice versa. From a semantic perspective, $\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i)$ reflects the degree to which object $x_i$ must belong to class $Cl_t^{\succeq}$. That is, the greater the magnitude of the best object in $(Cl_t^{\succeq})^c$ is better than $x_i$, the smaller the membership of $x_i$ to class $Cl_t^{\succeq}$. For example, when $x_j$ ($x_j \in (Cl_t^{\succeq})^c$) is better than $x_i$ (i.e., $\mathcal{N}_B^{+}(x_i)(x_j) > 0.5$), if $x_i \in Cl_t^{\succeq}$, then the decision-making of $x_i$ violates the monotonic consistency principle, so $\underline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) < 0.5$ is inevitable. On the contrary, if the objects that do not belong to class $Cl_t^{\succeq}$ are much smaller than $x_i$, then $x_i$ must belong to class $Cl_t^{\succeq}$ to a large extent. In addition, Eq. (21) can be interpreted similarly. Therefore, the fuzzy lower approximations follow the monotonic consistency principle.
- From Eq. (20), we can intuitively find that for any $x_i \in U$, the membership of $x_i$ to fuzzy set $\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})$ depends on the worst object that belongs to class $Cl_t^{\succeq}$. The greater the degree that $x_i$ is better than this object, the greater the membership of $x_i$ to the fuzzy set $\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})$, and vice versa. From a semantic perspective, $\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i)$ reflects the degree to which object $x_i$ may belong to class $Cl_t^{\succeq}$. In other words, if $x_i$ is much larger than the objects that belong to class $Cl_t^{\succeq}$, then $x_i$ may belong to class $Cl_t^{\succeq}$ to a large extent. Moreover, Eq. (22) can be interpreted similarly.

The above explanation is consistent with our intuition. Therefore, these four fuzzy approximation operators are reasonable. To facilitate understanding, subsequently, we use an example to demonstrate the calculation of fuzzy knowledge granules and approximations in FDNRS.

**Example 2.** Continuing from Example 1. According to Eqs. (13) and (14), the fuzzy dominating neighborhood set and fuzzy dominated neighborhood set of each object are calculated as

$\mathcal{N}_A^{+}(x_1) = \frac{0.5000}{x_1} + \frac{0.1208}{x_2} + \frac{0.0032}{x_3} + \frac{0.0235}{x_4} + \frac{0.0029}{x_5} + \frac{0.8792}{x_6} + \frac{0.0075}{x_7}$,
$\mathcal{N}_A^{-}(x_1) = \frac{0.5000}{x_1} + \frac{0.1436}{x_2} + \frac{0.2228}{x_3} + \frac{0.1667}{x_4} + \frac{0.1978}{x_5} + \frac{0.0049}{x_6} + \frac{0.0064}{x_7}$,
$\mathcal{N}_A^{+}(x_2) = \frac{0.1436}{x_1} + \frac{0.5000}{x_2} + \frac{0.0115}{x_3} + \frac{0.0115}{x_4} + \frac{0.0194}{x_5} + \frac{0.9498}{x_6} + \frac{0.0268}{x_7}$,
$\mathcal{N}_A^{-}(x_2) = \frac{0.1208}{x_1} + \frac{0.5000}{x_2} + \frac{0.2060}{x_3} + \frac{0.1535}{x_4} + \frac{0.1824}{x_5} + \frac{0.0115}{x_6} + \frac{0.0058}{x_7}$,
$\mathcal{N}_A^{+}(x_3) = \frac{0.2228}{x_1} + \frac{0.2060}{x_2} + \frac{0.5000}{x_3} + \frac{0.0268}{x_4} + \frac{0.0493}{x_5} + \frac{0.8316}{x_6} + \frac{0.7105}{x_7}$,
$\mathcal{N}_A^{-}(x_3) = \frac{0.0032}{x_1} + \frac{0.0115}{x_2} + \frac{0.5000}{x_3} + \frac{0.0115}{x_4} + \frac{0.1436}{x_5} + \frac{0.0002}{x_6} + \frac{0.0219}{x_7}$,
$\mathcal{N}_A^{+}(x_4) = \frac{0.1667}{x_1} + \frac{0.1535}{x_2} + \frac{0.0115}{x_3} + \frac{0.5000}{x_4} + \frac{0.0616}{x_5} + \frac{0.7666}{x_6} + \frac{0.0268}{x_7}$,
$\mathcal{N}_A^{-}(x_4) = \frac{0.0235}{x_1} + \frac{0.0115}{x_2} + \frac{0.0268}{x_3} + \frac{0.5000}{x_4} + \frac{0.2895}{x_5} + \frac{0.0004}{x_6} + \frac{0.0021}{x_7}$,
$\mathcal{N}_A^{+}(x_5) = \frac{0.1978}{x_1} + \frac{0.1824}{x_2} + \frac{0.1436}{x_3} + \frac{0.2895}{x_4} + \frac{0.5000}{x_5} + \frac{0.8089}{x_6} + \frac{0.2895}{x_7}$,
$\mathcal{N}_A^{-}(x_5) = \frac{0.0029}{x_1} + \frac{0.0194}{x_2} + \frac{0.0493}{x_3} + \frac{0.0616}{x_4} + \frac{0.5000}{x_5} + \frac{0.0004}{x_6} + \frac{0.0049}{x_7}$,
$\mathcal{N}_A^{+}(x_6) = \frac{0.0049}{x_1} + \frac{0.0115}{x_2} + \frac{0.0002}{x_3} + \frac{0.0004}{x_4} + \frac{0.0004}{x_5} + \frac{0.5000}{x_6} + \frac{0.0004}{x_7}$,
$\mathcal{N}_A^{-}(x_6) = \frac{0.8792}{x_1} + \frac{0.9498}{x_2} + \frac{0.8316}{x_3} + \frac{0.7666}{x_4} + \frac{0.8089}{x_5} + \frac{0.5000}{x_6} + \frac{0.1001}{x_7}$,
$\mathcal{N}_A^{+}(x_7) = \frac{0.0064}{x_1} + \frac{0.0058}{x_2} + \frac{0.0219}{x_3} + \frac{0.0021}{x_4} + \frac{0.0049}{x_5} + \frac{0.1001}{x_6} + \frac{0.5000}{x_7}$,
$\mathcal{N}_A^{-}(x_7) = \frac{0.0075}{x_1} + \frac{0.0268}{x_2} + \frac{0.7105}{x_3} + \frac{0.0268}{x_4} + \frac{0.2895}{x_5} + \frac{0.0004}{x_6} + \frac{0.5000}{x_7}$.

Then, according to Definition 3.4, the fuzzy lower and upper approximations of the upward and downward unions are calculated as

$\underline{\mathcal{N}_B^{\prec}}(Cl_1^{\succeq}) = \frac{1.0000}{x_1} + \frac{1.0000}{x_2} + \frac{1.0000}{x_3} + \frac{1.0000}{x_4} + \frac{1.0000}{x_5} + \frac{1.0000}{x_6} + \frac{1.0000}{x_7}$,
$\overline{\mathcal{N}_B^{\prec}}(Cl_1^{\succeq}) = \frac{0.5000}{x_1} + \frac{0.5000}{x_2} + \frac{0.5000}{x_3} + \frac{0.5000}{x_4} + \frac{0.5000}{x_5} + \frac{0.9498}{x_6} + \frac{0.7105}{x_7}$,
$\underline{\mathcal{N}_B^{\prec}}(Cl_1^{\preceq}) = \frac{0.7772}{x_1} + \frac{0.5000}{x_2} + \frac{0.5000}{x_3} + \frac{0.5000}{x_4} + \frac{0.9384}{x_5} + \frac{0.0502}{x_6} + \frac{0.2895}{x_7}$,
$\overline{\mathcal{N}_B^{\prec}}(Cl_1^{\preceq}) = \frac{0.5000}{x_1} + \frac{0.1436}{x_2} + \frac{0.7105}{x_3} + \frac{0.1667}{x_4} + \frac{0.5000}{x_5} + \frac{0.0049}{x_6} + \frac{0.5000}{x_7}$,
$\underline{\mathcal{N}_B^{\prec}}(Cl_2^{\succeq}) = \frac{0.5000}{x_1} + \frac{0.8564}{x_2} + \frac{0.2895}{x_3} + \frac{0.8333}{x_4} + \frac{0.5000}{x_5} + \frac{0.9951}{x_6} + \frac{0.5000}{x_7}$,
$\overline{\mathcal{N}_B^{\prec}}(Cl_2^{\succeq}) = \frac{0.2228}{x_1} + \frac{0.5000}{x_2} + \frac{0.5000}{x_3} + \frac{0.5000}{x_4} + \frac{0.0616}{x_5} + \frac{0.9498}{x_6} + \frac{0.7105}{x_7}$,
$\underline{\mathcal{N}_B^{\prec}}(Cl_2^{\preceq}) = \frac{0.8333}{x_1} + \frac{0.5000}{x_2} + \frac{0.9885}{x_3} + \frac{0.5000}{x_4} + \frac{0.9384}{x_5} + \frac{0.0502}{x_6} + \frac{0.9732}{x_7}$,
$\overline{\mathcal{N}_B^{\prec}}(Cl_2^{\preceq}) = \frac{0.8792}{x_1} + \frac{0.9498}{x_2} + \frac{0.8316}{x_3} + \frac{0.7666}{x_4} + \frac{0.8089}{x_5} + \frac{0.5000}{x_6} + \frac{0.5000}{x_7}$,
$\underline{\mathcal{N}_B^{\prec}}(Cl_3^{\succeq}) = \frac{0.1208}{x_1} + \frac{0.0502}{x_2} + \frac{0.1684}{x_3} + \frac{0.2334}{x_4} + \frac{0.1911}{x_5} + \frac{0.5000}{x_6} + \frac{0.5000}{x_7}$,
$\overline{\mathcal{N}_B^{\prec}}(Cl_3^{\succeq}) = \frac{0.1667}{x_1} + \frac{0.5000}{x_2} + \frac{0.0115}{x_3} + \frac{0.5000}{x_4} + \frac{0.0616}{x_5} + \frac{0.9498}{x_6} + \frac{0.0268}{x_7}$,
$\underline{\mathcal{N}_B^{\prec}}(Cl_3^{\preceq}) = \frac{1.0000}{x_1} + \frac{1.0000}{x_2} + \frac{1.0000}{x_3} + \frac{1.0000}{x_4} + \frac{1.0000}{x_5} + \frac{1.0000}{x_6} + \frac{1.0000}{x_7}$,
$\overline{\mathcal{N}_B^{\prec}}(Cl_3^{\preceq}) = \frac{0.8792}{x_1} + \frac{0.9498}{x_2} + \frac{0.8316}{x_3} + \frac{0.7666}{x_4} + \frac{0.8089}{x_5} + \frac{0.5000}{x_6} + \frac{0.5000}{x_7}$.

**Property 3.2.** *For any $B \subseteq A$ and $\forall p, q \in \{1, \ldots, T\}$, the following properties hold.*

(1) $\underline{\mathcal{N}_B^{\prec}}(Cl_1^{\succeq}) = U$, $\underline{\mathcal{N}_B^{\prec}}(Cl_T^{\preceq}) = U$; $\overline{\mathcal{N}_B^{\prec}}(Cl_{T+1}^{\succeq}) = \emptyset$, $\overline{\mathcal{N}_B^{\prec}}(Cl_0^{\preceq}) = \emptyset$.

(2) $\underline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c) = (\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c$, $\underline{\mathcal{N}_B^{\prec}}((Cl_p^{\preceq})^c) = (\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}))^c$; $\overline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c) = (\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c$, $\overline{\mathcal{N}_B^{\prec}}((Cl_p^{\preceq})^c) = (\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}))^c$.

(3) $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq}) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$, $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq} \cap Cl_q^{\preceq}) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$;
$\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$, $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq} \cup Cl_q^{\preceq}) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$.

(4) *If $Cl_p^{\succeq} \subseteq Cl_q^{\succeq}$, then $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ and $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$;*
*If $Cl_p^{\preceq} \subseteq Cl_q^{\preceq}$, then $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \subseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$ and $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$.*

(5) $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) \supseteq \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$, $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq} \cup Cl_q^{\preceq}) \supseteq \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \cup \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$;

$\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$, $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq} \cap Cl_q^{\preceq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \cap \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$.

**Proof.**

(1) It is straightforward according to Definition 3.4.

(2) First, we prove $\underline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c) = (\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c$. From Eqs. (20) and (21), for any $x_i \in U$, we have $\underline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c)(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_{p-1}^{\preceq})(x_i) = \inf_{x_j \notin Cl_{p-1}^{\preceq}} 1 - \mathcal{N}_B^{-}(x_i)(x_j) = \inf_{x_j \in Cl_p^{\succeq}} 1 - \mathcal{N}_B^{-}(x_i)(x_j) = 1 - \sup_{x_j \in Cl_p^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j) = 1 - \overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq})(x_i) = (\overline{\mathcal{N}_B^{\prec}}(Cl_t^{\succeq}))^c(x_i)$. Thus, $\underline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c) = (\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c$ holds. Similarly, $\underline{\mathcal{N}_B^{\prec}}((Cl_p^{\preceq})^c) = (\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}))^c$ can also be proved. Second, we prove $\overline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c) = (\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c$. From Eqs. (19) and (22), for any $x_i \in U$, we have $\overline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c)(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_{p-1}^{\preceq})(x_i) = \sup_{x_j \in Cl_{p-1}^{\preceq}} \mathcal{N}_B^{+}(x_i)(x_j) = 1 - 1 + \sup_{x_j \in Cl_{p-1}^{\preceq}} \mathcal{N}_B^{+}(x_i)(x_j) = 1 - (1 - \sup_{x_j \in Cl_{p-1}^{\preceq}} \mathcal{N}_B^{+}(x_i)(x_j)) = 1 - (\inf_{x_j \notin Cl_p^{\succeq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)) = 1 - \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) = (\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c(x_i)$. Thus, $\overline{\mathcal{N}_B^{\prec}}((Cl_p^{\succeq})^c) = (\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}))^c$ holds. Similarly, $\overline{\mathcal{N}_B^{\prec}}((Cl_p^{\preceq})^c) = (\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}))^c$ can also be proved.

(3) First, we prove $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq}) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. When $p = q$, this equation obviously holds. When $p > q$, we have $Cl_p^{\succeq} \subset Cl_q^{\succeq}$, then $Cl_p^{\succeq} \cap Cl_q^{\succeq} = Cl_p^{\succeq}$. Thus, for any $x_i \in U$, we have $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq})(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i)$. The other side, we have $(\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq}))(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) \wedge \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) = (\inf_{x_j \in Cl_{p-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)) \wedge (\inf_{x_j \in Cl_{q-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j))$. Due to $Cl_{q-1}^{\preceq} \subset Cl_{p-1}^{\preceq}$, $(\inf_{x_j \in Cl_{p-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)) \wedge (\inf_{x_j \in Cl_{q-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)) = \inf_{x_j \in Cl_{p-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j) = \inf_{x_j \notin Cl_p^{\succeq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i)$. So we can get $(\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq}))(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq})(x_i)$. Analogously, when $p < q$, we can also get $(\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq}))(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq})(x_i)$. Thus, $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cap Cl_q^{\succeq}) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ holds. Similarly, $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq} \cap Cl_q^{\preceq}) = \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \cap \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$ can also be proved. Second, we prove $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. When $p = q$, this equation obviously holds. When $p > q$, we have $Cl_p^{\succeq} \subset Cl_q^{\succeq}$, then $Cl_p^{\succeq} \cup Cl_q^{\succeq} = Cl_q^{\succeq}$. Thus, for any $x_i \in U$, we have $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq})(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i)$. The other side, we have $(\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq}))(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) \vee \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) = (\sup_{x_j \in Cl_p^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j)) \vee (\sup_{x_j \in Cl_q^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j))$. Because $Cl_p^{\succeq} \subset Cl_q^{\succeq}$, $(\sup_{x_j \in Cl_p^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j)) \vee (\sup_{x_j \in Cl_q^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j)) = \sup_{x_j \in Cl_q^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j) = \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i)$. So we can get $(\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq}))(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq})(x_i)$. Analogously, when $p < q$, we can also get $(\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq}))(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq})(x_i)$. Thus, $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ holds. Similarly, $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq} \cup Cl_q^{\preceq}) = \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \cup \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$ can also be proved.

(4) First, we prove that if $Cl_p^{\succeq} \subseteq Cl_q^{\succeq}$, then $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ and $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. From Eq. (19), for any $x_i \in U$, we have $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) = \inf_{x_j \in Cl_{p-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)$ and $\underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) = \inf_{x_j \in Cl_{q-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j)$. Because $Cl_p^{\succeq} \subseteq Cl_q^{\succeq} \Rightarrow Cl_{p-1}^{\preceq} \subseteq Cl_{q-1}^{\preceq}$, we can get $\inf_{x_j \in Cl_{p-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j) \leq \inf_{x_j \in Cl_{q-1}^{\preceq}} 1 - \mathcal{N}_B^{+}(x_i)(x_j) \Rightarrow \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) \leq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) \Rightarrow \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. From Eq. (20), for any $x_i \in U$, we have $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) = \sup_{x_j \in Cl_p^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j)$ and $\overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) = \sup_{x_j \in Cl_q^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j)$. Because $Cl_p^{\succeq} \subseteq Cl_q^{\succeq}$, we can get $\sup_{x_j \in Cl_p^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j) \leq \sup_{x_j \in Cl_q^{\succeq}} \mathcal{N}_B^{-}(x_i)(x_j) \Rightarrow \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})(x_i) \leq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})(x_i) \Rightarrow \overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. In summary, if $Cl_p^{\succeq} \subseteq Cl_q^{\succeq}$, then $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ and $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. Similarly, if $Cl_p^{\preceq} \subseteq Cl_q^{\preceq}$, then $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \subseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$ and $\overline{\mathcal{N}_B^{\prec}}(Cl_p^{\preceq}) \subseteq \overline{\mathcal{N}_B^{\prec}}(Cl_q^{\preceq})$ can also be proved.

(5) First, we prove $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) \supseteq \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$. Due to $Cl_p^{\succeq} \subseteq Cl_p^{\succeq} \cup Cl_q^{\succeq}$ and $Cl_q^{\succeq} \subseteq Cl_p^{\succeq} \cup Cl_q^{\succeq}$, we have $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) \supseteq \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq})$ and $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) \supseteq \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ according to Property 3.2 (4). Naturally, we can get that $\underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq} \cup Cl_q^{\succeq}) \supseteq \underline{\mathcal{N}_B^{\prec}}(Cl_p^{\succeq}) \cup \underline{\mathcal{N}_B^{\prec}}(Cl_q^{\succeq})$ holds. Similarly, the other three formulas can also be proved. □

## 4. Conditional entropy based on FDNRS and non-monotonic feature selection in IvODS

As a common uncertainty measure, information entropy is widely used in feature selection tasks [27,28,39]. In this section, we first propose a conditional entropy based on FDNRS, called FDNCE, and analyze its monotonicity. Afterwards, we define a non-monotonic reduct search strategy using FDNCE. Finally, we introduce a heuristic feature selection algorithm with the non-monotone reduct search strategy.

### 4.1. Fuzzy dominance neighborhood conditional entropy to IvODS

In [26], Hu et al. successively proposed dominance conditional entropy (DCE) and fuzzy dominance conditional entropy (FDCE) for evaluating the consistency degree of the ranking of objects under conditional attributes and decisions in an ODS. Obviously, the DCE follows the dominance relation, which only reflects the dominance relation between objects from the qualitative perspectives. The FDCE follows the fuzzy dominance relation, which reflects the dominance relation between objects from both qualitative and quantitative perspectives. Naturally, these two metrics can be applied to IvODS by simply changing the preference relation between single values to that of interval values. However, as we mentioned earlier, the fuzzy dominance relation does not consider the effects of noise. To make up for this defect, the following we define the FDNCE in an IvODS.

**Definition 4.1.** Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V \rangle$, $\forall B \subseteq A$, the FDNCE of $B$ relative to $d$ is defined as

$$\mathcal{NE}_{d|B}^{\prec}(U) = -\frac{1}{|U|} \sum_{i=1}^{n} \log \frac{|\mathcal{N}_B^{+}(x_i) \cap D_d^{+}(x_i)|}{|\mathcal{N}_B^{+}(x_i)|}, \tag{23}$$

where $|*|$ represents the cardinality of set $*$, $\mathcal{N}_B^{+}(x_i)$ is the fuzzy dominating neighborhood set of $x_i$ under $B$, and $D_d^{+}(x_i)$ is the dominating set of $x_i$ under $d$.

In Eq. (23), $\frac{|\mathcal{N}_B^{+}(x_i) \cap D_d^{+}(x_i)|}{|\mathcal{N}_B^{+}(x_i)|}$ can be regarded as a variable, which is the core part of $\mathcal{NE}_{d|B}^{\prec}(U)$. Intuitively, this variable measures the consistency degree of the objects ranking in terms of the conditional attribute set $B$ and the decision $d$. It is easy to find that the value of FDNCE is inversely proportional to this variable, and $\mathcal{NE}_{d|B}^{\prec}(U)$ is non-negative. When using FDNCE to evaluate an attribute subset, we expect that the ranking information provided by this attribute subset for the objects in IvODS is the same as the decision. Therefore, the smaller $\mathcal{NE}_{d|B}^{\prec}(U)$ (or the larger the variable $\frac{|\mathcal{N}_B^{+}(x_i) \cap D_d^{+}(x_i)|}{|\mathcal{N}_B^{+}(x_i)|}$) indicates that the attribute subset $B$ is more meaningful. Next, we prove that FDNCE is non-monotonicity.

**Proposition 4.1.** *Let $C \subseteq B \subseteq A$, then $\mathcal{NE}^{\prec}_{d|C}(U) \leq \mathcal{NE}^{\prec}_{d|B}(U)$ or $\mathcal{NE}^{\prec}_{d|C}(U) \geq \mathcal{NE}^{\prec}_{d|B}(U)$ is indeterminate, namely, FDNCE is non-monotonic.*

**Proof.** From Eq. (23), we have

$$\triangle = \mathcal{NE}^{\prec}_{d|B}(U) - \mathcal{NE}^{\prec}_{d|C}(U) = \frac{1}{|U|}\sum_{i=1}^{n}(\log\frac{|\mathcal{N}_C^+(x_i)\cap D_d^+(x_i)|}{|\mathcal{N}_C^+(x_i)|} - \log\frac{|\mathcal{N}_B^+(x_i)\cap D_d^+(x_i)|}{|\mathcal{N}_B^+(x_i)|}).$$

Assuming that $g_1(x_i) = \frac{|\mathcal{N}_C^+(x_i)\cap D_d^+(x_i)|}{|\mathcal{N}_C^+(x_i)|}$ and $g_2(x_i) = \frac{|\mathcal{N}_B^+(x_i)\cap D_d^+(x_i)|}{|\mathcal{N}_B^+(x_i)|}$. It can be obtained that $\triangle = \frac{1}{|U|}\sum_{i=1}^{n}(\log g_1(x_i) - \log g_2(x_i)) = \frac{1}{|U|}\sum_{i=1}^{n}\log\frac{g_1(x_i)}{g_2(x_i)}$. Since $|\mathcal{N}_C^+(x_i)\cap D_d^+(x_i)| < |\mathcal{N}_C^+(x_i)|$ and $|\mathcal{N}_B^+(x_i)\cap D_d^+(x_i)| < |\mathcal{N}_B^+(x_i)|$ hold, then $0 < g_1(x_i), g_2(x_i) < 1$ holds. Hence, $\frac{g_1(x_i)}{g_2(x_i)} > 1$ ($\frac{g_1(x_i)}{g_2(x_i)} < 1$) is uncertain. So $\triangle > 0$ ($\triangle < 0$) is indeterminate. Therefore, FDNCE is non-monotonic. □

### 4.2. The evaluation of attributes in IvODS

In this subsection, we introduce a non-monotonic reduct search strategy using FDNCE in IvODS.

**Definition 4.2.** Given an IvODS $IS^{\preceq} = \langle U, A\cup\{d\}, V\rangle$, $\forall Q \subseteq A$, we say $Q$ is a reduct of $A$ relative to $d$ if $Q$ satisfies
(1) $\mathcal{NE}^{\prec}_{d|Q}(U) \leq \mathcal{NE}^{\prec}_{d|A}(U)$,
(2) $\forall a_k \in Q$, $\mathcal{NE}^{\prec}_{d|(Q-\{a_k\})}(U) > \mathcal{NE}^{\prec}_{d|Q}(U)$.

The first item guarantees that the selected attribute subset $Q$ can provide correct objects ranking information that is not worse than that of raw attribute set $A$. The second item requires that no redundant attributes in the selected attribute subset $Q$.

According to Definition 4.2, we define the inner and outer significance of an attribute as follows.

**Definition 4.3.** Given an IvODS $IS^{\preceq} = \langle U, A\cup\{d\}, V\rangle$, $\forall B \subseteq A$ and $\forall a \in B$, the inner significance of $a$ relative to $B$ is defined as

$$sig^U_{inner}(a, B, d) = \mathcal{NE}^{\prec}_{d|(B-\{a\})}(U) - \mathcal{NE}^{\prec}_{d|B}(U). \tag{24}$$

**Definition 4.4.** Given an IvODS $IS^{\preceq} = \langle U, A\cup\{d\}, V\rangle$, $\forall B \subseteq A$ and $\forall a \in (C-B)$, the outer significance of $a$ relative to $B$ is defined as

$$sig^U_{outer}(a, B, d) = \mathcal{NE}^{\prec}_{d|B}(U) - \mathcal{NE}^{\prec}_{d|(B\cup\{a\})}(U). \tag{25}$$

The matrix representation of knowledge is an intuitive and effective way for processing complex data, and the calculation of the matrix can be easily implemented using a computer. In particular, the relation between objects is usually expressed and stored in the form of a matrix in the computer. Thence, it is necessary to present a method for computing FDNCE by using relation matrices. In what follows, we define some operations on relation matrices.

**Definition 4.5.** Let $B_1, B_2 \subseteq A\cup\{d\}$, $\mathbb{R}^{B_1}_U = [r^{B_1}_{(i,j)}]_{n\times n}$ and $\mathbb{R}^{B_2}_U = [r^{B_2}_{(i,j)}]_{n\times n}$ are two relation matrices under attribute sets $B_1$ and $B_2$, respectively, then the "$\wedge$" and "$*$" operations between them are defined as

$$\mathbb{R}^{B_1}_U \wedge \mathbb{R}^{B_2}_U = [\min\{r^{B_1}_{(i,j)}, r^{B_2}_{(i,j)}\}]_{n\times n} = \mathbb{R}^{B_1\cup B_2}_U, \tag{26}$$

$$\mathbb{R}^{B_1}_U * \mathbb{R}^{B_2}_U = [r^{B_1}_{(i,j)} \times r^{B_2}_{(i,j)}]_{n\times n}. \tag{27}$$

**Definition 4.6.** Let $B \subseteq A\cup\{d\}$, $\mathbb{R}^B_U = [r^B_{(i,j)}]_{n\times n}$ be a relation matrix, and its diagonal matrix is defined as $\widehat{\mathbb{R}^B_U} = [\widehat{r}^B_{(i,j)}]_{n\times n}$, where

$$\widehat{r}^B_{(i,j)} = \begin{cases} \sum_{l=1}^{n} r^B_{(i,l)}, & i,j\in[1,n], i=j; \\ 0, & i,j\in[1,n], i\neq j. \end{cases} \tag{28}$$

Moreover, the determinant and inverse matrix of $\widehat{\mathbb{R}^B_U}$ are denoted as $|\widehat{\mathbb{R}^B_U}| = \Pi^n_{i=j=1}\widehat{r}^B_{(i,j)}$ and $(\widehat{\mathbb{R}^B_U})^{-1} = [1/\widehat{r}^B_{(i,j)}]_{n\times n}$, respectively.

**Corollary 4.1.** *Given an IvODS $IS^{\preceq} = \langle U, A\cup\{d\}, V\rangle$, $\forall B \subseteq A$, the formula for calculating FDNCE using matrices is expressed as*

$$\mathcal{NE}^{\prec}_{d|B}(U) = -\frac{1}{|U|}\log|\widehat{\widetilde{\mathbb{N}}^{\prec B\cup d}_U} * (\widehat{\widetilde{\mathbb{N}}^{\prec B}_U})^{-1}|, \tag{29}$$

*where $\widetilde{\mathbb{N}}^{\prec B\cup d}_U = \widetilde{\mathbb{N}}^{\prec B}_U \wedge \mathbb{D}^{\preceq d}_U = [\mathcal{N}^{\prec B\cup d}_{(i,j)}]_{n\times n}$, $\mathbb{D}^{\preceq d}_U$ is a dominance relation matrix derived by dominance relation $D^{\preceq}_d$.*

**Proof.** According to Eq. (29), we can get that

$$\begin{aligned}\mathcal{NE}^{\prec}_{d|B}(U) &= -\frac{1}{|U|}\log \Pi^n_{i=j=1}\frac{\widehat{\mathcal{N}}^{\prec B\cup d}_{(i,j)}}{\widehat{\mathcal{N}}^{\prec B}_{(i,j)}} = -\frac{1}{|U|}\log\frac{\Pi^n_{i=j=1}\widehat{\mathcal{N}}^{\prec B\cup d}_{(i,j)}}{\Pi^n_{i=j=1}\widehat{\mathcal{N}}^{\prec B}_{(i,j)}} \\ &= -\frac{1}{|U|}\log\frac{\Pi^n_{i=1}(\sum^n_{l=1}\mathcal{N}^{\prec B\cup d}_{(i,l)})}{\Pi^n_{i=1}(\sum^n_{l=1}\mathcal{N}^{\prec B}_{(i,l)})} = -\frac{1}{|U|}\log\frac{\Pi^n_{i=1}|\mathcal{N}^+_{B\cup d}(x_i)|}{\Pi^n_{i=1}|\mathcal{N}^+_B(x_i)|} \\ &= -\frac{1}{|U|}\log\frac{\Pi^n_{i=1}|\mathcal{N}^+_B(x_i)\cap D^+_d(x_i)|}{\Pi^n_{i=1}|\mathcal{N}^+_B(x_i)|} \\ &= -\frac{1}{|U|}\sum_{l=1}^{n}\log\frac{|\mathcal{N}^+_B(x_i)\cap D^+_d(x_i)|}{|\mathcal{N}^+_B(x_i)|}.\end{aligned}$$

From this we can conclude that the results of computing FDNCE by Eqs. (23) and (29) are equal. □

Next, we use an example to demonstrate the process of calculating FDNCE by using relation matrices.

**Example 3.** Continuing from Example 2. First, we calculate the fuzzy dominance neighborhood relation matrix $\widetilde{\mathbb{N}}^{\prec A}_U$ and the dominance relation matrix $\mathbb{D}^{\preceq d}_U$ as $\widetilde{\mathbb{N}}^{\prec A}_U =$

$$\begin{bmatrix} 0.5000 & 0.1208 & 0.0032 & 0.0235 & 0.0029 & 0.8792 & 0.0075 \\ 0.1436 & 0.5000 & 0.0115 & 0.0115 & 0.0194 & 0.9498 & 0.0268 \\ 0.2228 & 0.2060 & 0.5000 & 0.0268 & 0.0493 & 0.8316 & 0.7105 \\ 0.1667 & 0.1535 & 0.0115 & 0.5000 & 0.0616 & 0.7666 & 0.0268 \\ 0.1978 & 0.1824 & 0.1436 & 0.2895 & 0.5000 & 0.8089 & 0.2895 \\ 0.0049 & 0.0115 & 0.0002 & 0.0004 & 0.0004 & 0.5000 & 0.0004 \\ 0.0064 & 0.0058 & 0.0219 & 0.0021 & 0.0049 & 0.1001 & 0.5000 \end{bmatrix}_{7\times 7},$$

$$\mathbb{D}^{\preceq d}_U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{7\times 7}$$

. Then, we calculate matrix $\widetilde{\mathbb{N}}^{\prec A\cup d}_U$ by Eq. (26) as

$$\widetilde{\mathbb{N}}^{\prec A\cup d}_U = \widetilde{\mathbb{N}}^{\prec A}_U \wedge \mathbb{D}^{\preceq d}_U$$

$$= \begin{bmatrix} 0.5000 & 0.1208 & 0.0032 & 0.0235 & 0.0029 & 0.8792 & 0.0075 \\ 0 & 0.5000 & 0 & 0.0115 & 0 & 0 & 0 \\ 0 & 0.2060 & 0.5000 & 0.0268 & 0 & 0.8316 & 0 \\ 0 & 0.1535 & 0 & 0.5000 & 0 & 0 & 0 \\ 0.1978 & 0.1824 & 0.1436 & 0.2895 & 0.5000 & 0.8089 & 0.2895 \\ 0 & 0.0115 & 0.0002 & 0.0004 & 0 & 0.5000 & 0 \\ 0.0064 & 0.0058 & 0.0219 & 0.0021 & 0.0049 & 0.1001 & 0.5000 \end{bmatrix}_{7\times 7}.$$

Subsequently, the matrices $\widetilde{\mathbb{N}}_U^{\prec A}$ and $\widetilde{\mathbb{N}}_U^{\prec A\cup d}$ are diagonalized by Eq. (28) as

$$\widehat{\mathbb{N}}_U^{\prec A} = \begin{bmatrix} 1.5372 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.6625 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.5469 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.6867 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.4117 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5178 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6412 \end{bmatrix}_{7\times 7},$$

$$\widehat{\mathbb{N}}_U^{\prec A\cup d} = \begin{bmatrix} 1.5372 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5115 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.5643 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6535 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.4117 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5120 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6412 \end{bmatrix}_{7\times 7}.$$

Finally, the FDNCE $\mathcal{NE}^{\prec}_{\widehat{d|A}}(U)$ is calculated by Eq. (29) as $\mathcal{NE}^{\prec}_{\widehat{d|A}}(U) = -\frac{1}{7}\log|\widehat{\mathbb{N}}_U^{\prec A\cup d} * (\widehat{\mathbb{N}}_U^{\prec A})^{-1}| = 0.5411$.

### 4.3. Heuristic feature selection algorithm based on FDNCE to IvODS

In this subsection, we design a FDNCE based heuristic feature selection algorithm to IvODS (HFS-IvO) according to Definition 4.2, and then analyze its time complexity.

**Algorithm 1** HFS-IvO algorithm

**Input:** An IvODS $IS^{\preceq} = \langle U, A\cup\{d\}, V\rangle$, parameters $\alpha$, and $\beta$.
**Output:** A reduct $Red_U$.

```
1: Initialize Red_U ← ∅;
2: Calculate FDNCE NE^≺_{d|A}(U) by Eq. (29);
3: for k = 1 to |A| do
4:     Calculate sig^U_inner(a_k, A, d) by Definition 4.3;
5:     if sig^U_inner(a_k, A, d) > 0 then
6:         Red_U ← Red_U ∪ {a_k};
7:     end if
8: end for
9: Let Q ← Red_U;
10: while NE^≺_{d|Q}(U) > NE^≺_{d|A}(U) do
11:     for l = 1 to |A − Q| do
12:         Calculate sig^U_outer(a_l, Q, d) by Definition 4.4;
13:     end for
14:     Select a_0 = max{sig^U_outer(a_l, Q, d), a_l ∈ (A − Q)};
15:     Q ← Q ∪ {a_0};
16: end while
17: for each a ∈ Q do
18:     Calculate FDNCE NE^≺_{d|(Q−{a})}(U) by Eq. (29);
19:     if NE^≺_{d|(Q−{a})}(U) ≤ NE^≺_{d|Q}(U) then
20:         Q ← Q − {a};
21:     end if
22: end for
23: Red_U ← Q;
24: return Red_U;
```

Next, we explain the steps in Algorithm 1. Step 2 is to calculate FDNCE under raw attribute set $A$. Steps 3–9 is to add attributes with inner significance greater than zero to $Red_U$, and let $Q = Red_U$. Steps 10–16 is to search the attribute with the highest outer significance from remaining attribute subset $A-Q$ to $Q$ until Step 10 does not hold. Steps 17–22 is to delete redundant attributes from attribute subset $Q$. Steps 23–24 is to output the final reduct. The time complexity of the main steps in this algorithm are listed in Table 6.

**Table 6**
The time complexity of HFS-IvO algorithm.

| Steps | Time complexity | Steps | Time complexity |
|---|---|---|---|
| 2 | $O(\lvert A\rvert\lvert U\rvert^2)$ | 10–16 | $O(\lvert A\rvert^2\lvert U\rvert^2)$ |
| 3–9 | $O(\lvert A\rvert^2\lvert U\rvert^2)$ | 17–22 | $O(\lvert Q\rvert^2\lvert U\rvert^2)$ |

## 5. Incremental feature selection for dynamic IvODS with the variation of multiple objects

For dynamic IvODS, employing the HFS-IvO algorithm to compute a reduct is very time-consuming, especially in large data. Because this algorithm retrains the changed IvODS as a new one, which needs to recalculate knowledge from scratch. To improve efficiency, this section presents two incremental algorithms for feature selection on the basis of HFS-IvO algorithm.

### 5.1. Incremental feature selection for adding object set

This subsection first presents the updating mechanism of FDNCE when adding object set to an IvODS. Then, on this basis, a corresponding incremental feature selection algorithm is proposed.

#### 5.1.1. Updating mechanism of FDNCE

Uncertainty metric is an important part of feature selection algorithms, and its calculation speed determines the efficiency of the algorithms. Thence, this subsection present an incremental update mechanism that is used to quickly compute the new FDNCE when adding objects to an IvODS. From Eq. (29), we can easily find that the pivotal step in the process of updating FDNCE is to calculate the corresponding diagonal matrix in an incremental manner. In what follows, the principle for updating the diagonal matrix is presented.

**Proposition 5.1.** *Given an IvODS $IS^{\preceq} = \langle U, A\cup\{d\}, V\rangle$, adding object set $U_{ad} = \{x_{n+1}, x_{n+2}, \ldots, x_{n+n'}\}$ to $IS^{\preceq}$, then the changed object set is $U' = U \cup U_{ad}$. Let $\forall B \subseteq A$, known the previous diagonal matrix is $\widehat{\mathbb{N}}_U^{\prec B} = [\widehat{\mathcal{N}}^{\prec B}_{(i,j)}]_{n\times n}$, which is updated to $\widehat{\mathbb{N}}_{U'}^{\prec B} = [\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}]_{(n+n')\times(n+n')}$ after adding objects, where*

$$\widehat{\mathcal{N}}'^{\prec B}_{(i,j)} = \begin{cases} \widehat{\mathcal{N}}^{\prec B}_{(i,j)} + \sum\limits_{l=n+1}^{n+n'} \mathcal{N}^{\prec B}_{(i,l)}, & i,j\in[1,n], i=j; \\ \sum\limits_{l=1}^{n+n'} \mathcal{N}^{\prec B}_{(i,l)}, & i,j\in[n+1,n+n'], i=j; \\ 0, & i,j\in[1,n+n'], i\neq j, \end{cases} \tag{30}$$

*where $\widehat{\mathcal{N}}^{\prec B}_{(i,j)}$ is known, $\sum_{l=n+1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)}$ and $\sum_{l=1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)}$ need to be calculated by Definition 3.2.*

**Proof.** According to Definition 4.6, we know that all non-diagonal elements in matrix $\widehat{\mathbb{N}}_{U'}^{\prec B}$ are zero, that is, $\forall i,j\in[1,n+n']$ and $i\neq j$, $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)} = 0$ always holds. Then $\forall i,j\in[1,n]$ and $i=j$, we have $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)} = \sum_{l=1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)} = \sum_{l=1}^{n}\mathcal{N}^{\prec B}_{(i,l)} + \sum_{l=n+1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)} = \widehat{\mathcal{N}}^{\prec B}_{(i,j)} + \sum_{l=n+1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)}$, where $\widehat{\mathcal{N}}^{\prec B}_{(i,j)}$ is known, and $\sum_{l=n+1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)}$ needs to be calculated by Definition 3.2. Furthermore, $\forall i,j\in[n+1,n+n']$ and $i=j$, $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)} = \sum_{l=1}^{n+n'}\mathcal{N}^{\prec B}_{(i,l)}$ also needs to be calculated by Definition 3.2. In summary, based on the previous diagonal matrix $\widehat{\mathbb{N}}_U^{\prec B}$, we calculate new knowledge to obtain an updated diagonal matrix $\widehat{\mathbb{N}}_{U'}^{\prec B}$, where $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}$ is denoted as Eq. (30). □

Analogously, the diagonal matrix $\widehat{\mathbb{N}}_{U'}^{\prec B\cup d}$ can also be updated by Proposition 5.1. Therefore, according to Eq. (29), we can directly

**Table 7**
A new IvODS after adding object set.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $d$ |
|---|---|---|---|---|---|
| $x_1$ | [0.28, 0.30] | [0.33, 0.40] | [0.54, 0.66] | [0.53, 0.65] | 1 |
| $x_2$ | [0.27, 0.29] | [0.49, 0.60] | [0.36, 0.44] | [0.41, 0.50] | 3 |
| $x_3$ | [0.40, 0.43] | [0.41, 0.50] | [0.27, 0.33] | 0 | 2 |
| $x_4$ | [0.41, 0.50] | [0.08, 0.10] | [0.20, 0.24] | [0.41, 0.50] | 3 |
| $x_5$ | [0.42, 0.44] | [0.16, 0.20] | 0 | [0.16, 0.20] | 1 |
| $x_6$ | [0.55, 0.60] | [0.82, 1.00] | [0.72, 0.88] | [0.82, 1.00] | 2 |
| $x_7$ | [0.78, 0.81] | [0.65, 0.80] | [0.36, 0.44] | [0.08, 0.10] | 1 |
| $x_8$ | [0.75, 0.77] | [0.25, 0.30] | [0.40, 0.48] | [0.45, 0.55] | 1 |
| $x_9$ | [0.83, 0.84] | [0.90, 1.00] | [0.90, 1.00] | [0.90, 1.00] | 3 |
| $x_{10}$ | [0.85, 0.88] | 0 | [0.34, 0.42] | [0.08, 0.10] | 3 |

compute the new FDNCE using the updated matrices $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B}$ and $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B\cup d}$. Subsequently, according to Proposition 5.1, we use an example to demonstrate the updating process of FDNCE.

**Example 4.** Continuing from Example 3, adding object set $U_{ad} = \{x_8, x_9, x_{10}\}$ to Table 5, then the new IvODS is shown in Table 7, where the new object set is denoted as $U' = \{x_1, x_2, \ldots, x_{10}\}$. First, we update the diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}$ and $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}$ according to Proposition 5.1 as given in Box I. Then, based on the updated diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}$ and $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}$, the new FDNCE $\mathcal{NE}_{d|A}^{\prec}(U')$ is calculated by Eq. (29) as $\mathcal{NE}_{d|A}^{\prec}(U') = 0.3316$.

*5.1.2. The incremental feature selection algorithm*

This subsection introduces a FDNCE based incremental feature selection algorithm when adding object set to IvODS (IFSA-IvO), and then analyze its time complexity.

**Algorithm 2** IFSA-IvO algorithm

**Input:** An original IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, and its reduct $Q$, parameters $\alpha$, $\beta$, original diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec A}$, $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec A\cup d}$, $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec Q}$, $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec Q\cup d}$, and $U_{ad} = \{x_{n+1}, x_{n+2}, \ldots, x_{n+n'}\}$;
**Output:** A new reduct $Red_{U'}$ on $U \cup U_{ad}$.

1: Add object set $U' \leftarrow U \cup U_{ad}$;
2: Update the diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec A} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}$, $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec A\cup d} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}$, $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec Q} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec Q}$, $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec Q\cup d} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec Q\cup d}$ by Proposition 5.1;
3: Calculate the new FDNCE $\mathcal{NE}_{d|A}^{\prec}(U')$ and $\mathcal{NE}_{d|Q}^{\prec}(U')$ by Eq. (29);
4: **if** $\mathcal{NE}_{d|Q}^{\prec}(U') > \mathcal{NE}_{d|A}^{\prec}(U')$ **then**
5:   **for** each $a \in (A - Q)$ **do**
6:     Calculate $sig_{outer}^{U'}(a, Q, d)$ by Eq. (25), then ranking these attributes w.r.t descending order of their outer significance, and record the results as $\{a'_1, a'_2, \ldots, a'_{|A-Q|}\}$;
7:   **end for**
8:   **while** $\mathcal{NE}_{d|Q}^{\prec}(U') > \mathcal{NE}_{d|A}^{\prec}(U')$ **do**
9:     **for** $h = 1$ to $|A - Q|$ **do**
10:       Select $Q \leftarrow Q \cup \{a'_h\}$ and calculate $\mathcal{NE}_{d|Q}^{\prec}(U')$;
11:     **end for**
12:   **end while**
13: **end if**
14: **for** each $a \in Q$ **do**
15:   Calculate FDNCE $\mathcal{NE}_{d|(Q-\{a\})}^{\prec}(U')$ by Eq. (29);
16:   **if** $\mathcal{NE}_{d|(Q-\{a\})}^{\prec}(U') \le \mathcal{NE}_{d|Q}^{\prec}(U')$ **then**
17:     $Q \leftarrow Q - \{a\}$;
18:   **end if**
19: **end for**
20: $Red_{U'} \leftarrow Q$;
21: **return** $Red_{U'}$;

**Table 8**
The time complexity of IFSA-IvO algorithm.

| Steps | Time complexity | Steps | Time complexity |
|---|---|---|---|
| 2–3 | $O(\|A\|\|U_{ad}\|\|U'\|)$ | 14–19 | $O(\|Q\|^2\|U'\|^2)$ |
| 5–12 | $O((\|A\| - \|Q\|)\|U'\|^2)$ | | |

**Table 9**
The comparison of the time complexity of algorithms HFS-IvO and IFSA-IvO.

| Algorithms | Time complexity |
|---|---|
| HFS-IvO | $O(\|A\|\|U'\|^2 + \|A\|^2\|U'\|^2 + \|A\|^2\|U'\|^2 + \|Q\|^2\|U'\|^2)$ |
| IFSA-IvO | $O(\|A\|\|U_{ad}\|\|U'\| + (\|A\| - \|Q\|)\|U'\|^2 + \|Q\|^2\|U'\|^2)$ |

In Algorithm 2, Step 1 is to add the object set to the original IvODS. Step 2 is to update the original diagonal matrices by Proposition 5.1. Step 3 is to calculate the new FDNCE by Eq. (29). Step 4 is to determine whether the new FDNCE under the previous reduct $Q$ is greater than that of under the raw attribute set $A$, if not, then keep the previous reduct unchanged. Steps 5–7 is to construct a descending sequence for the remaining attributes. Steps 8–12 is to incrementally update the selected attribute subset until Step 8 does not hold. Steps 14–19 is to remove redundant attributes from the selected attribute subset. Steps 20–21 is to output the final reduct. The time complexity of the main steps in this algorithm are listed in Table 8. Subsequently, we collect the time complexity of algorithms HFS-IvO and IFSA-IvO to Table 9 for intuitive comparison.

From Table 9, we can easily find that the time complexity of IFSA-IvO algorithm is usually much less than that of HFS-IvO algorithm. Because HFS-IvO algorithm computes a new reduct from scratch, it ignores the previously acquired knowledge. By contrast, IFSA-IvO algorithm uses the previous knowledge for accelerating the acquisition of a new reduct. Thence, compared with HFS-IvO algorithm, IFSA-IvO algorithm saves time cost.

*5.2. Incremental feature selection for deleting object set*

In this subsection, we first introduce an incremental update mechanism for calculating the new FDNCE when object set is deleted from an IvODS. Then, on this basis, a corresponding incremental feature selection algorithm is proposed.

*5.2.1. Updating mechanism of FDNCE*

To update FDNCE, below we present the principle for updating the diagonal matrix when deleting objects set.

**Proposition 5.2.** *Given an IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, deleting object set $U_{de} = \{x_{q_1}, x_{q_2}, \ldots, x_{q_{n'}}\}$ from $IS^{\preceq}$, then the changed object set is $U' = U - U_{de}$. Let $\forall B \subseteq A$, known the previous relation matrix $\widetilde{\mathbb{N}}_U^{\prec B} = [\mathcal{N}_{(i,j)}^{\prec B}]_{n\times n}$ and its diagonal matrix $\widehat{\widetilde{\mathbb{N}}}_U^{\prec B} = [\widehat{\mathcal{N}}_{(i,j)}^{\prec B}]_{n\times n}$, where the diagonal matrix is updated to $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B} = [\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}]_{(n-n')\times(n-n')}$ after deleting objects, where*

$$\widehat{\mathcal{N}}'^{\prec B}_{(i,j)} = \begin{cases} \widehat{\mathcal{N}}^{\prec B}_{(i+k-1,j+k-1)} - \sum_{t=1}^{n'} \mathcal{N}^{\prec B}_{(i+k-1,q_t)}, & i,j \in [q_{k-1}-k+2, q_k-k+1), i=j; \\ \widehat{\mathcal{N}}^{\prec B}_{(i+n',j+n')} - \sum_{t=1}^{n'} \mathcal{N}^{\prec B}_{(i+n',q_t)}, & i,j \in [q_{n'}-n'+1, n-n'], i=j; \\ 0, & i,j \in [1, n-n'], i \neq j, \end{cases} \tag{31}$$

*where $1 \le k \le n'$.*

**Proof.** When the object set $U_{de}$ is deleted, the raw object set becomes $U' = \{x_1, x_2, \ldots, x_{n-n'}\}$. In $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B}$, the elements on the

$$\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}=\left[\begin{array}{ccccccc|ccc}
1.5372 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1698 & 0.9706 & 0.0075\\
0 & 1.6625 & 0 & 0 & 0 & 0 & 0 & 0.0653 & 0.9829 & 0.0049\\
0 & 0 & 2.5469 & 0 & 0 & 0 & 0 & 0.1436 & 0.9852 & 0.0115\\
0 & 0 & 0 & 1.6867 & 0 & 0 & 0 & 0.6106 & 0.9765 & 0.0268\\
0 & 0 & 0 & 0 & 2.4117 & 0 & 0 & 0.7210 & 0.9829 & 0.1436\\
0 & 0 & 0 & 0 & 0 & 0.5178 & 0 & 0.0021 & 0.5950 & 0.0002\\
0 & 0 & 0 & 0 & 0 & 0 & 0.6412 & 0.0123 & 0.5612 & 0.0009\\
\hline
0.0090 & 0.0082 & 0.0075 & 0.0476 & 0.0131 & 0.1368 & 0.0176 & 0.5000 & 0.6791 & 0.0176\\
0.0029 & 0.0039 & 0.0001 & 0.0002 & 0.0001 & 0.0702 & 0.0002 & 0.0012 & 0.5000 & 0.0001\\
0.0032 & 0.0029 & 0.0110 & 0.0170 & 0.0128 & 0.0524 & 0.3318 & 0.2593 & 0.5000 & 0.5000
\end{array}\right]_{10\times 10}$$

$$=\left[\begin{array}{cccccccccc}
2.6851 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2.7156 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 3.6872 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 3.3005 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 4.2592 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1.1150 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1.2157 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4364 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5789 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6903
\end{array}\right]_{10\times 10},$$

$$\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}=\left[\begin{array}{ccccccc|ccc}
1.5372 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1698 & 0.9706 & 0.0075\\
0 & 0.5115 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9829 & 0.0049\\
0 & 0 & 1.5643 & 0 & 0 & 0 & 0 & 0 & 0.9852 & 0.0115\\
0 & 0 & 0 & 0.6535 & 0 & 0 & 0 & 0 & 0.9765 & 0.0268\\
0 & 0 & 0 & 0 & 2.4117 & 0 & 0 & 0.7210 & 0.9829 & 0.1436\\
0 & 0 & 0 & 0 & 0 & 0.5120 & 0 & 0 & 0.5950 & 0.0002\\
0 & 0 & 0 & 0 & 0 & 0 & 0.6412 & 0.0123 & 0.5612 & 0.0009\\
\hline
0.0090 & 0.0082 & 0.0075 & 0.0476 & 0.0131 & 0.1368 & 0.0176 & 0.5000 & 0.6791 & 0.0176\\
0 & 0.0039 & 0 & 0.0002 & 0 & 0 & 0 & 0 & 0.5000 & 0.0001\\
0 & 0.0029 & 0 & 0.0170 & 0 & 0 & 0 & 0 & 0.5000 & 0.5000
\end{array}\right]_{10\times 10}$$

$$=\left[\begin{array}{cccccccccc}
2.6851 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1.4993 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2.5610 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1.6567 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 4.2592 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1.1072 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1.2157 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4364 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5042 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0199
\end{array}\right]_{10\times 10}.$$

**Box I.**

off-diagonal lines are all zero, i.e., $\forall i,j\in[1,n-n']$ and $i\neq j$, $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}=0$ always holds. According to Definition 4.6, for elements on the diagonal, we have $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}=\sum_{l=1}^{n}\mathcal{N}^{\prec B}_{(i,l)}-\sum_{t=1}^{n'}\mathcal{N}^{\prec B}_{(i,t)}=\widehat{\mathcal{N}}^{\prec B}_{(i,j)}-\sum_{t=1}^{n'}\mathcal{N}^{\prec B}_{(i,t)}$, and its position has two changes in $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B}$. One for any $i,j\in[q_{k-1},q_k)$ and $i=j$, the row and column coordinates of $\widehat{\mathcal{N}}^{\prec B}_{(i,j)}$ should be shifted forward by $k-1$ positions at the same time. After that, we can get that for any $i,j\in[q_{k-1}-k+2,q_k-k+1)$ and $i=j$, $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}=\widehat{\mathcal{N}}^{\prec B}_{(i+k-1,j+k-1)}-\sum_{t=1}^{n'}\mathcal{N}^{\prec B}_{(i+k-1,q_t)}$ holds. On the other hand, for any $i,j\in[q_{n'}-n'+1,n-n']$ and $i=j$, the row and column coordinates of $\widehat{\mathcal{N}}^{\prec B}_{(i,j)}$ should be shifted forward by $n'$ positions simultaneously. Then, we have $\widehat{\mathcal{N}}'^{\prec B}_{(i,j)}=\widehat{\mathcal{N}}^{\prec B}_{(i+n',j+n')}-\sum_{t=1}^{n'}\mathcal{N}^{\prec B}_{(i+n',q_t)}$ holds. To sum up, based on the previous relation matrix $\widetilde{\mathbb{N}}_{U}^{\prec B}$ and its diagonal matrix $\widehat{\widetilde{\mathbb{N}}}_{U}^{\prec B}$, we delete the corresponding knowledge to obtain an updated diagonal matrix $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B}$. □

Analogously, the diagonal matrix $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B\cup d}$ can also be updated by Proposition 5.2. Hence, according to Eq. (29), we can directly compute the new FDNCE using the updated matrices $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B}$ and $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec B\cup d}$. Subsequently, according to Proposition 5.2, we use an example to demonstrate the updating process of FDNCE.

**Example 5.** Continuing from Example 3, deleting object set $U_{ad}=\{x_2,x_4\}$ from Table 5, then the new IvODS is shown in Table 10, where the new object set is denoted as $U'=\{x_1,x_3,x_5,x_6,x_7\}$. First, we update the diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}$ and $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}$ according to Proposition 5.2 as given in Box II. Then, based on the updated diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}$ and $\widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}$, the new FDNCE $\mathcal{NE}^{\prec}_{d|A}(U')$ is calculated using Eq. (29) as $\mathcal{NE}^{\prec}_{d|A}(U')=0.1628$.

$$
\widehat{\mathbb{N}^{\prec A}_{U'}} = \begin{bmatrix}
1.5372-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(1,t)} & \cancel{0} & 0 & \cancel{0} & 0 & 0 & 0 \\
\cancel{0} & \cancel{1.6625} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\
0 & \cancel{0} & 2.5469-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(3,t)} & \cancel{0} & 0 & 0 & 0 \\
\cancel{0} & \cancel{0} & \cancel{0} & \cancel{1.6867} & \cancel{0} & \cancel{0} & \cancel{0} \\
0 & \cancel{0} & 0 & \cancel{0} & 2.4117-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(5,t)} & 0 & 0 \\
0 & \cancel{0} & 0 & \cancel{0} & 0 & 0.5178-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(6,t)} & 0 \\
0 & \cancel{0} & 0 & \cancel{0} & 0 & 0 & 0.6412-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(7,t)}
\end{bmatrix}_{7\times 7}
$$

$$
= \begin{bmatrix}
1.3929 & 0 & 0 & 0 & 0 \\
0 & 2.3142 & 0 & 0 & 0 \\
0 & 0 & 1.9398 & 0 & 0 \\
0 & 0 & 0 & 0.5059 & 0 \\
0 & 0 & 0 & 0 & 0.6333
\end{bmatrix}_{5\times 5},
$$

$$
\widehat{\mathbb{N}^{\prec A\cup d}_{U'}} = \begin{bmatrix}
1.5372-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(1,t)} & \cancel{0} & 0 & \cancel{0} & 0 & 0 & 0 \\
\cancel{0} & \cancel{0.5115} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\
0 & \cancel{0} & 1.5643-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(3,t)} & \cancel{0} & 0 & 0 & 0 \\
\cancel{0} & \cancel{0} & \cancel{0} & \cancel{0.6535} & \cancel{0} & \cancel{0} & \cancel{0} \\
0 & \cancel{0} & 0 & \cancel{0} & 2.4117-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(5,t)} & 0 & 0 \\
0 & \cancel{0} & 0 & \cancel{0} & 0 & 0.5120-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(6,t)} & 0 \\
0 & \cancel{0} & 0 & \cancel{0} & 0 & 0 & 0.6412-\sum\limits_{t=2,4}\mathcal{N}^{\prec A}_{(7,t)}
\end{bmatrix}_{7\times 7}
$$

$$
= \begin{bmatrix}
1.3929 & 0 & 0 & 0 & 0 \\
0 & 1.3316 & 0 & 0 & 0 \\
0 & 0 & 1.9398 & 0 & 0 \\
0 & 0 & 0 & 0.5002 & 0 \\
0 & 0 & 0 & 0 & 0.6333
\end{bmatrix}_{5\times 5}.
$$

**Box II.**

**Table 10**
A new IvODS after deleting object set.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $d$ |
|---|---|---|---|---|---|
| $x_1$ | [0.28, 0.30] | [0.33, 0.40] | [0.54, 0.66] | [0.53, 0.65] | 1 |
| ~~$x_2$~~ | ~~[0.27, 0.29]~~ | ~~[0.49, 0.60]~~ | ~~[0.36, 0.44]~~ | ~~[0.41, 0.50]~~ | ~~3~~ |
| $x_3$ | [0.40, 0.43] | [0.41, 0.50] | [0.27, 0.33] | 0 | 2 |
| ~~$x_4$~~ | ~~[0.41, 0.50]~~ | ~~[0.08, 0.10]~~ | ~~[0.20, 0.24]~~ | ~~[0.41, 0.50]~~ | ~~3~~ |
| $x_5$ | [0.42, 0.44] | [0.16, 0.20] | 0 | [0.16, 0.20] | 1 |
| $x_6$ | [0.55, 0.60] | [0.82, 1.00] | [0.72, 0.88] | [0.82, 1.00] | 2 |
| $x_7$ | [0.78, 0.81] | [0.65, 0.80] | [0.36, 0.44] | [0.08, 0.10] | 1 |

### 5.2.2. The incremental feature selection algorithm

This subsection introduces a FDNCE based incremental feature selection algorithm when deleting object set from IvODS (IFSD-IvO), and then analyze its time complexity.

In Algorithm 3, Step 1 is to delete the object set. Step 2 is to update the original diagonal matrices by Proposition 5.2. Step 3 is to compute the new FDNCE by Eq. (29). Step 4 is to determine whether the new FDNCE under the original reduct is not higher than that of under the entire attribute set, if so, then keep the original reduct unchanged. Steps 5–7 is to construct a descending sequence for the remaining attributes. Steps 8–12 is to incrementally update the selected feature subset until Step 8 does not hold. Steps 14–19 is to remove redundant attributes from the selected attribute subset. Steps 20–21 is to output the final reduct. The time complexity of the main steps in this algorithm are listed in Table 11. Subsequently, the time complexity of algorithms HFS-IvO and IFSD-IvO are collected into Table 12 for intuitive comparison. Obviously, the time complexity of IFSD-IvO algorithm is much lower than that of HFS-IvO algorithm. The main reason is that IFSD-IvO algorithm uses the previous knowledge when calculating the new reduct, while HFS-IvO algorithm calculates a new reduct from scratch, which does not use the previous knowledge. So HFS-IvO algorithm is very time consuming for calculating a new reduct.

**Table 11**
The time complexity of IFSD-IvO algorithm.

| Steps | Time complexity | Steps | Time complexity |
|---|---|---|---|
| 2–3 | $O(\|U_{de}\|\|U\|)$ | 14–19 | $O(\|Q\|^2\|U'\|^2)$ |
| 5–12 | $O((\|A\|-\|Q\|)\|U'\|^2)$ | | |

**Table 12**
The comparison of the time complexity of algorithms HFS-IvO and IFSD-IvO.

| Algorithms | Time complexity |
|---|---|
| HFS-IvO | $O(\|A\|\|U'\|^2+\|A\|^2\|U'\|^2+\|A\|^2\|U'\|^2+\|Q\|^2\|U'\|^2)$ |
| IFSD-IvO | $O(\|U_{de}\|\|U\|+(\|A\|-\|Q\|)\|U'\|^2+\|Q\|^2\|U'\|^2)$ |

**Algorithm 3** IFSD-IvO algorithm

**Input:** An original IvODS $IS^{\preceq} = \langle U, A \cup \{d\}, V\rangle$, and its reduct $Q$, parameters $\alpha$, $\beta$, original relation matrices $\widetilde{\mathbb{N}}_U^{\prec A}$, $\widetilde{\mathbb{N}}_U^{\prec A\cup d}$, $\widetilde{\mathbb{N}}_U^{\prec Q}$, $\widetilde{\mathbb{N}}_U^{\prec Q\cup d}$, and their diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_U^{\prec A}$, $\widehat{\widetilde{\mathbb{N}}}_U^{\prec A\cup d}$, $\widehat{\widetilde{\mathbb{N}}}_U^{\prec Q}$, $\widehat{\widetilde{\mathbb{N}}}_U^{\prec Q\cup d}$, and $U_{de} = \{x_{q_1}, x_{q_2}, \ldots, x_{q_{n'}}\}$;
**Output:** A new reduct $Red_{U'}$ on $U - U_{de}$.

Delete object set $U' \leftarrow U - U_{de}$;
Update the diagonal matrices $\widehat{\widetilde{\mathbb{N}}}_U^{\prec A} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A}$, $\widehat{\widetilde{\mathbb{N}}}_U^{\prec A\cup d} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec A\cup d}$, $\widehat{\widetilde{\mathbb{N}}}_U^{\prec Q} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec Q}$, $\widehat{\widetilde{\mathbb{N}}}_U^{\prec Q\cup d} \to \widehat{\widetilde{\mathbb{N}}}_{U'}^{\prec Q\cup d}$ by Proposition 5.2;
Calculate the new FDNCE $\mathcal{NE}^{\prec}_{d|A}(U')$ and $\mathcal{NE}^{\prec}_{d|Q}(U')$ by Eq. (29);
**if** $\mathcal{NE}^{\prec}_{d|Q}(U') > \mathcal{NE}^{\prec}_{d|A}(U')$ **then**
  **for** each $a \in (A - Q)$ **do**
    Calculate $sig^{U'}_{outer}(a, Q, d)$ by Eq. (25), then construct a descending sequence of attributes, and record the results as $\{a'_1, a'_2, \ldots, a'_{|A-Q|}\}$;
  **end for**
  **while** $\mathcal{NE}^{\prec}_{d|Q}(U') > \mathcal{NE}^{\prec}_{d|A}(U')$ **do**
    **for** $h = 1$ to $|A - Q|$ **do**
      Select $Q \leftarrow Q \cup \{a'_h\}$ and calculate $\mathcal{NE}^{\prec}_{d|Q}(U')$;
    **end for**
  **end while**
**end if**
**for** each $a \in Q$ **do**
  Compute FDNCE $\mathcal{NE}^{\prec}_{d|(Q-\{a\})}(U')$ by Eq. (29);
  **if** $\mathcal{NE}^{\prec}_{d|(Q-\{a\})}(U') \le \mathcal{NE}^{\prec}_{d|Q}(U')$ **then**
    $Q \leftarrow Q - \{a\}$;
  **end if**
**end for**
$Red_{U'} \leftarrow Q$;
**return** $Red_{U'}$;

**Table 13**
The summary of datasets.

| No. | Datasets | Abbreviation | Objects | Attributes | Classes |
|---|---|---|---|---|---|
| 1 | Wisconsin Prognostic Breast Cancer | WPBC | 198 | 32 | 2 |
| 2 | Auto MPG | Auto | 398 | 7 | 3 |
| 3 | Housing | Hous | 506 | 13 | 5 |
| 4 | Australian Credit | Aust | 690 | 14 | 2 |
| 5 | Credit Approval | Cred | 690 | 14 | 2 |
| 6 | Wine Quality-red | Wred | 1599 | 11 | 10 |
| 7 | Car Evaluation | Car | 1728 | 6 | 4 |
| 8 | Cardiotocography | Card | 2126 | 21 | 3 |
| 9 | Wine Quality-white | Wite | 4898 | 11 | 10 |

## 6. Experiments and analysis

In this section, we perform a series of experiments to test the robustness of the proposed metric and evaluate the performance of the proposed incremental feature selection algorithms. The configuration of computer used for experiments is as follows. CPU is Intel(R) Core(TM) i7-8700. Clock Speed is 3.20 GHz. Memory is 16.0 GB. Operation System is 64-bit Windows 10. The algorithms are coded in Java and run in Java platform. The code of algorithms can be downloaded from the GitHub homepage.[1] We downloaded nine datasets from the UCI machine learning repository, and a summary of them is provided in Table 13.

However, very few real interval-valued datasets are publicly available. In [5,6,36,54,58–63], the interval-value datasets are obtained through different data preprocessing methods, which convert the single-value datasets into the interval-value datasets. Before performing the experiments, we use a similar data preprocessing method to obtain the interval-valued datasets. First, for categorical attributes, we use integers instead of symbols, and define order relation of the integers in accordance with semantics of the attributes. Then, these datasets are normalized using

$$\hat{v}_{ik} = \frac{v_{ik} - min(V_{a_k})}{max(V_{a_k}) - min(V_{a_k})}. \tag{32}$$

Finally, this normalized single value $\hat{v}_{ik}$ is constructed as an interval number $[\hat{v}^l_{ik}, \hat{v}^r_{ik}]$, where

$$\hat{v}^l_{ik} = (1-\alpha) \times \hat{v}_{ik}, \tag{33}$$

$$\hat{v}^r_{ik} = (1+\alpha) \times \hat{v}_{ik}. \tag{34}$$

In Eqs. (33) and (34), the $\alpha$ represents error precision. In this experiment, we stipulate that $\alpha = 0.05$ and if $\hat{v}^r_{ik} > 1$, then $\hat{v}^r_{ik} = \hat{v}_{ik}$.

### 6.1. Evaluation on the robustness of metric FDNCE in IvODS

In this subsection, we randomly select four datasets in Table 13 to test the robustness of metrics DCE, FDCE, and FDNCE in IvODS. For each preprocessed dataset, we choose different proportions of data to add random noise. These datasets with noise are obtained by

$$[\hat{v}^l_{ik}, \hat{v}^r_{ik}] = \begin{cases} [\hat{v}^l_{ik} + r^l_{ik}, \hat{v}^r_{ik} + r^r_{ik}], & 0 \le (\hat{v}^l_{ik} + r^l_{ik}) \le (\hat{v}^r_{ik} + r^r_{ik}) \le 1; \\ [\hat{v}^l_{ik}, \hat{v}^r_{ik}], & \text{otherwise}, \end{cases} \tag{35}$$

where $r^l_{ik}, r^r_{ik} \in [0, 1]$. Then, these three metrics are calculated for different levels noise datasets. The experimental results are shown in Fig. 2.

From Fig. 2, we can find that the fluctuation of FDNCE curve is relatively small as the noise level increases. Moreover, in each sub-figure, we also show the standard deviation (STDEV) of the calculation result of each metric. From these histograms, we can intuitively observe that the STDEV of FDNCE is minimal. Therefore, we can conclude that the robustness of metric FDNCE is the best one compared with other two metrics.

### 6.2. Performance evaluations of incremental algorithms IFSA-IvO and IFSD-IvO

The performance of the proposed incremental algorithms are evaluated from the perspective of effectiveness and efficiency. In this subsection, we introduce the compared algorithms, experimental design, and experimental results and analysis.

#### 6.2.1. Compared algorithms

Four feature selection (attribute reduction) algorithms for interval-valued data are adopted as comparison algorithms, as shown below.

- Algorithm DRSQR. Du et al. proposed a DRSA based QuickReduct algorithm for ordered data [64]. We replace the single-valued dominance relation in this algorithm with the interval-valued dominance relation (as indicated by Definition 2.4), and then naturally use this algorithm for attribute reduction of interval-valued ordered data.
- Algorithm RDAR. Dai et al. proposed several uncertainty measures for interval-valued data, where the measure $\theta-$rough degree is used in the attribute reduction algorithm of interval-valued data [30]. This algorithm is written as RDAR, where the parameter $\theta$ is preset to 0.5.

[1] https://github.com/binbinsang/Experimental-source-code.git.

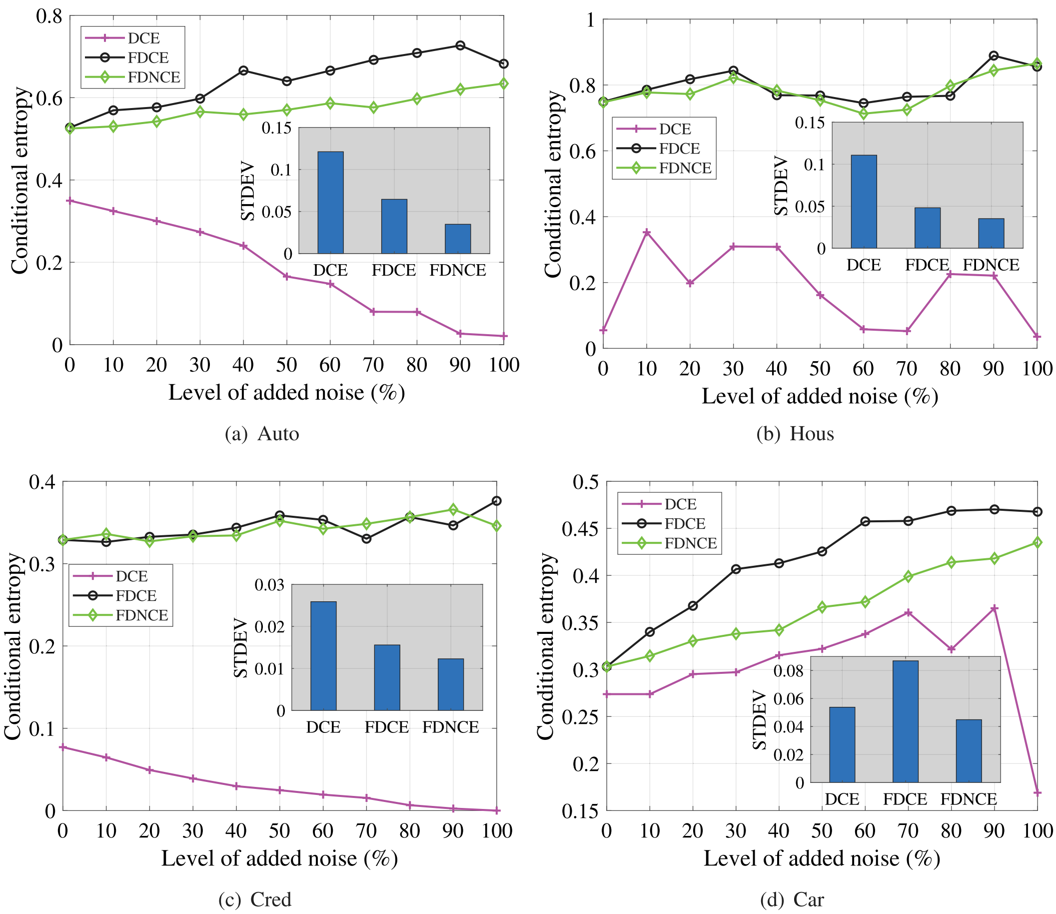



**Fig. 2.** The comparison of robustness of metrics at different noise levels.

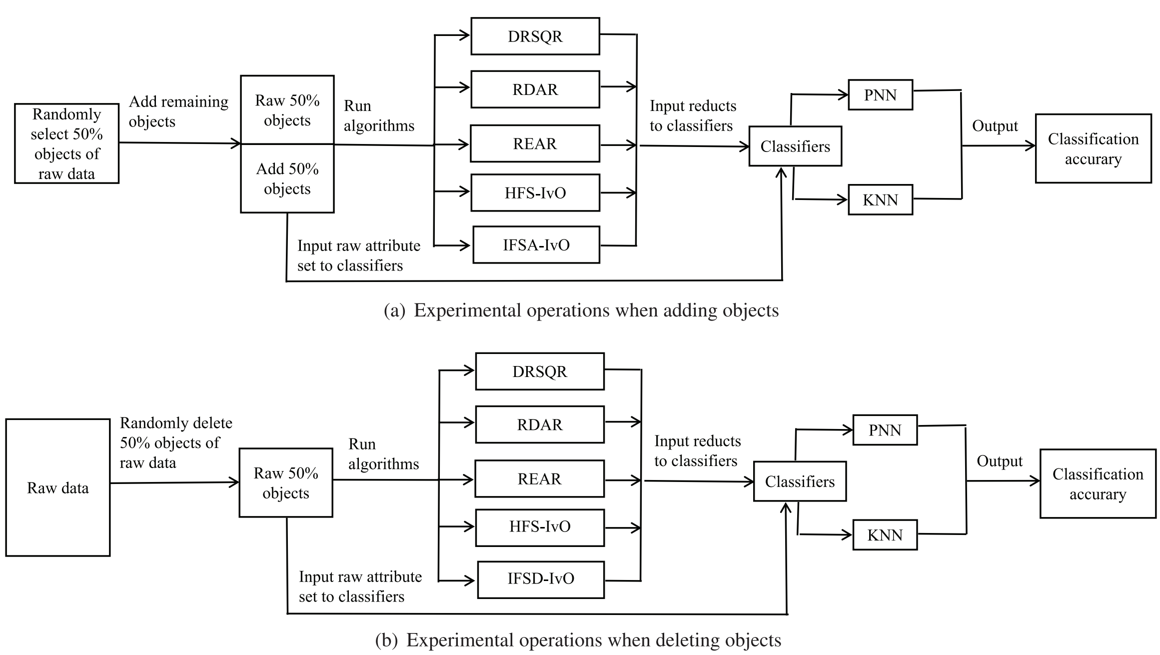



**Fig. 3.** Experimental operations for evaluating the effectiveness of incremental algorithms.

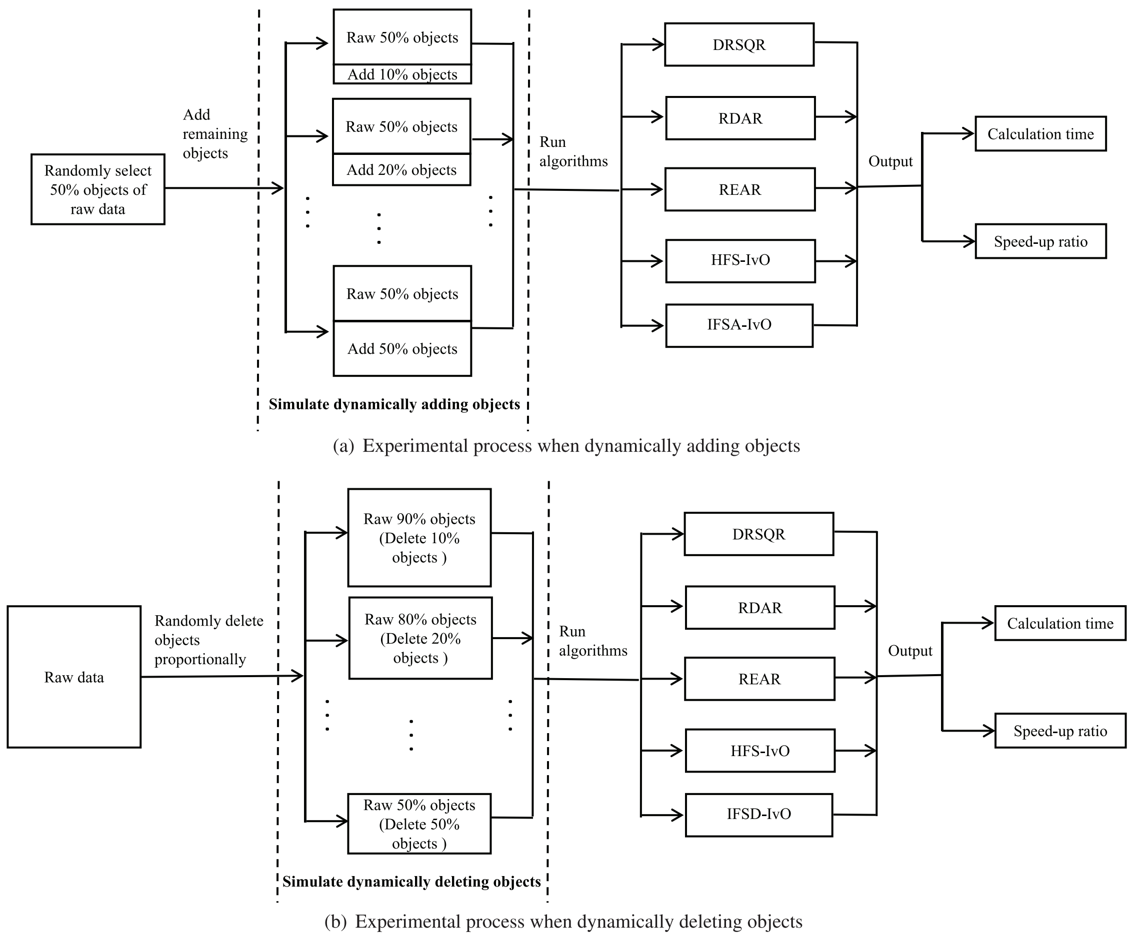



**Fig. 4.** Experimental schemes of evaluating the efficiency of incremental algorithms.

- Algorithm REAR. Xie et al. presented a new uncertainty measure for interval-valued data, called $\theta-$rough entropy, which is used in the attribute reduction algorithm of interval-valued data [33]. This algorithm is written as REAR, where the parameter $\theta$ is preset to 0.4.
- Algorithm HFS-IvO. It is a FDNCE based heuristic feature selection algorithm for interval-valued ordered data given in Algorithm 1.

### 6.2.2. Experimental design

In this experiment, the classification accuracy of the reduct generated by the feature selection algorithm is used to show the effectiveness of this algorithm, the time and speed-up ratio calculated by the feature selection algorithm show the efficiency of this algorithm.

(1) Evaluation indexes

The evaluation index of effectiveness is classification accuracy, and that of efficiency is calculation time and speed-up ratio.

Currently, most classifiers cannot handle interval-valued data [30]. For this purpose, Dai et al. extended two commonly used classifiers Probabilistic Neural Network (PNN) and K-Nearest Neighbor (KNN), which are used to measure the classification effect of the attribute subsets of interval-valued data [30]. In this experiment, we use these two classifiers to evaluate the effectiveness of feature selection algorithms. 10-fold cross-validation is adopted in classification. Here, the percentage of correctly classified instances is used as an evaluation indicator, and it can be obtained via running classifiers. Moreover, the speed-up ratio is calculated as $S = T_{Comparison-algorithm}/T_{Incremental-algorithm}$, where $T_*$ is the computational time of $*$ algorithm.

(2) The scheme of effectiveness evaluations

In order to compare the effectiveness of the two incremental algorithms with the other four algorithms, we design the corresponding experimental schemes as shown in Fig. 3, where Figs. 3(a) & 3(b) is used to compare the incremental algorithm IFSA-IvO & IFSD-IvO, respectively, with the other four algorithms.

(3) The scheme of efficiency evaluations

We record the calculation time and speed-up ratio of feature selection algorithms in the dynamic adding and deleting data environments, respectively. The less calculation time of an algorithm, the faster the calculation speed is, which means that the efficiency of the algorithms is higher, and vice versa. Therefore, the efficiency of algorithms are measured by comparing the calculation time of the algorithms. The experimental schemes are shown in Fig. 4.

### 6.2.3. Experimental results and analysis

(1) Experimental results of effectiveness evaluation

The experimental results evaluating the effectiveness of the incremental algorithms and the other four algorithms are provided in Tables 14 and 15. In Tables 14 and 15, the "raw" is the classification accuracy of the raw attribute set, the optimal classification accuracies are in boldface, and the number in bracket after each classification accuracy result indicates the size of the generated reduct.

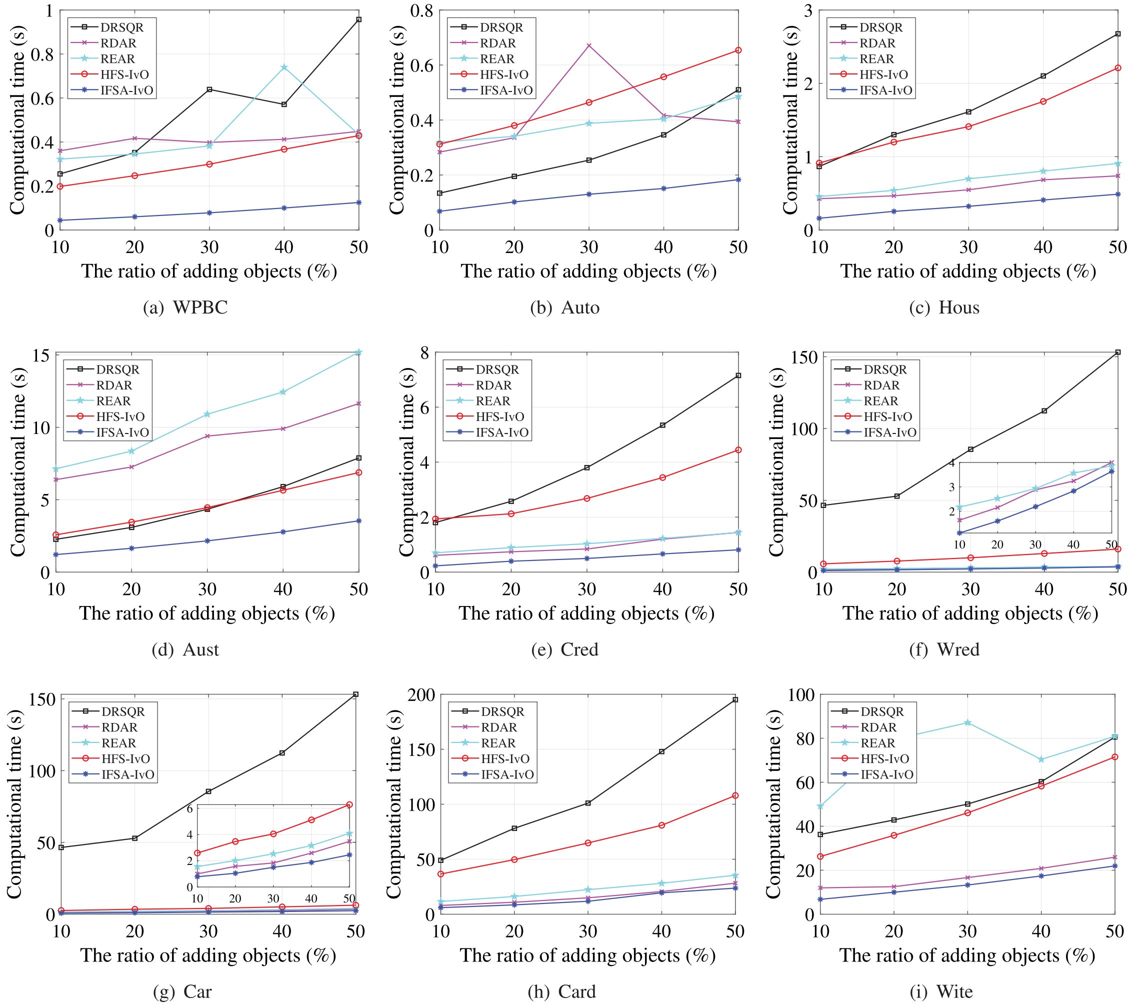



**Fig. 5.** The computational time of different algorithms versus different ratios of adding objects.

**Table 14**
The comparison of classification accuracy of different algorithms when adding objects (%).

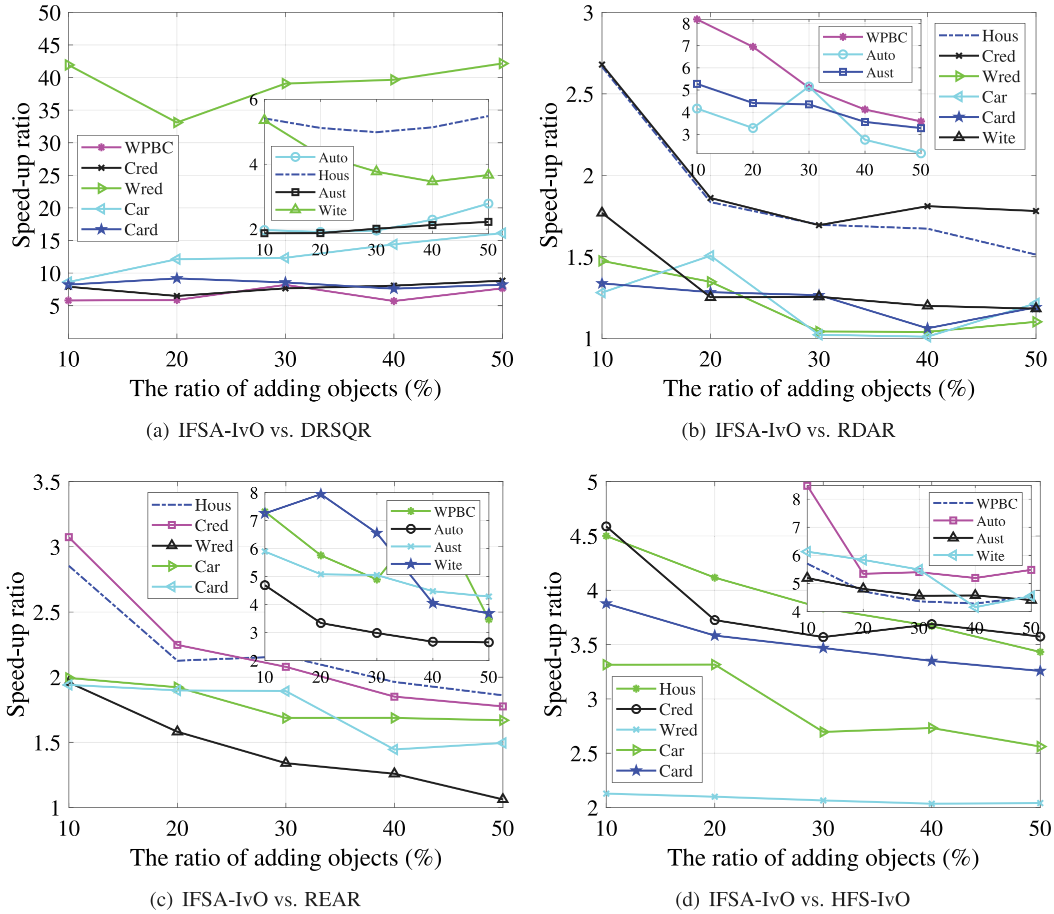
| Datasets | PNN | | | | | | KNN | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | All attributes | DRSQR | RDAR | REAR | HFS-IvO | IFSA-IvO | All attributes | DRSQR | RDAR | REAR | HFS-IvO | IFSA-IvO |
| WPBC | 46.70 | 48.73 (12) | 47.72 (2) | 47.72 (4) | 45.69 (6) | **53.30 (8)** | 50.76 | 50.25 (12) | 51.33 (2) | 49.75 (4) | 45.69 (6) | **52.28 (8)** |
| Auto | **67.00** | 63.48 (5) | 66.75 (5) | 66.00 (4) | 64.23 (4) | 64.23 (4) | 72.80 | 70.28 (5) | 66.00 (5) | 71.54 (4) | **74.81 (4)** | **74.81 (4)** |
| Hous | 43.37 | 43.96 (10) | 38.42 (4) | 43.76 (7) | **46.12 (2)** | **46.12 (2)** | 66.93 | 67.72 (10) | 64.16 (4) | 67.92 (7) | **69.77 (2)** | **69.77 (2)** |
| Aust | 84.33 | 84.33 (12) | 73.15 (4) | 84.76 (7) | 84.62 (7) | **85.34 (5)** | 83.16 | **84.18 (12)** | 70.10 (4) | 83.89 (7) | 82.87 (7) | 80.84 (5) |
| Cred | 60.96 | 61.83 (12) | 43.25 (4) | 61.68 (7) | 40.06 (3) | **62.39 (7)** | 67.20 | 66.62 (12) | 64.44 (4) | 68.21 (7) | 60.81 (3) | **68.36 (7)** |
| Wred | 22.59 | 22.59 (10) | 23.19 (9) | 22.63 (9) | 20.84 (2) | **23.77 (1)** | **50.69** | 49.19 (10) | 47.87 (9) | 22.63 (9) | 46.06 (2) | 46.31 (1) |
| Car | 47.83 | 47.83 (5) | 36.13 (2) | 47.65 (3) | 70.01 (1) | **79.17 (1)** | 67.17 | 67.75 (5) | 69.89 (2) | 67.69 (3) | 70.01 (1) | **79.17 (1)** |
| Card | 76.60 | 45.76 (13) | 66.85 (6) | 77.12 (11) | 80.04 (3) | **80.74 (4)** | 87.10 | 83.95 (13) | 86.16 (6) | 87.01 (11) | 86.35 (3) | **88.75 (4)** |
| Wite | 54.25 | 54.89 (7) | 51.90 (5) | 45.41 (4) | 47.86 (5) | **55.13 (7)** | 48.74 | 48.81 (7) | 48.03 (5) | 48.41 (4) | 46.62 (5) | **49.40 (7)** |

From Tables 14 and 15, we find that for most datasets, the classification effect of the proposed incremental algorithms are not only slightly higher than the overall attribute set, but also slightly higher than the other four comparison algorithms. From the perspective of the size of the reduct, the size of the reducts generated by the proposed incremental algorithms and the algorithm HFS-IvO are equal or very close in most datasets, and the size of the reducts generated by the incremental algorithms is smaller than that of the algorithms DRSQR, RDAR, and REAR in most datasets. Therefore, it can be concluded that the proposed incremental algorithms can effectively delete redundant attributes and improve classification accuracy. This fully shows that our incremental algorithms are effective.

(2) Experimental results of efficiency evaluation

First, we compare the computational efficiency of the incremental algorithm IFSA-IvO with the other four comparison algorithms. The detailed experimental operation is shown in Fig. 4(a), and the experimental results are shown in Figs. 5 and 6.

From Fig. 5, we find that for most datasets, the computational time of IFSA-IvO algorithm is less than that of other four algorithms. In particular, for all datasets, the calculation time of the algorithm IFSA-IvO is significantly lower than that of algorithms DRSQR and HFS-IvO. Furthermore, as the size of the added object




(a) IFSA-IvO vs. DRSQR (b) IFSA-IvO vs. RDAR (c) IFSA-IvO vs. REAR (d) IFSA-IvO vs. HFS-IvO

**Fig. 6.** The speed-up ratios that algorithm IFSA-IvO relates to different algorithms.

**Table 15**
The comparison of classification accuracy of different algorithms when deleting objects (%).

| Datasets | PNN | | | | | | KNN | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | All attributes | DRSQR | RDAR | REAR | HFS-IvO | IFSD-IvO | All attributes | DRSQR | RDAR | REAR | HFS-IvO | IFSD-IvO |
| WPBC | 54.55 | 47.47 (11) | 50.51 (2) | 48.48 (4) | **57.58 (1)** | **57.58 (1)** | **58.59** | 51.52 (11) | 57.58 (2) | 48.48 (4) | **58.59 (1)** | **58.59 (1)** |
| Auto | 73.37 | 74.87 (5) | 77.89 (1) | 72.86 (4) | 68.34 (1) | **78.89 (2)** | 73.37 | 72.36 (5) | 70.85 (1) | 73.37 (4) | 68.34 (1) | **73.87 (2)** |
| Hous | 34.39 | 35.97 (9) | 25.30 (4) | 34.39 (7) | 34.78 (9) | **37.57 (7)** | 62.45 | 64.03 (9) | 64.03 (4) | 63.64 (7) | 62.45 (9) | **67.98 (7)** |
| Aust | 84.35 | **85.80 (11)** | 73.91 (4) | 81.74 (6) | 84.93 (13) | 85.22 (7) | 83.19 | **84.35 (11)** | 68.99 (4) | 82.03 (6) | 84.06 (13) | 83.19 (7) |
| Cred | 59.13 | **61.74 (12)** | 56.81 (4) | 55.65 (6) | 46.96 (2) | 46.96 (2) | **62.32** | 62.03 (12) | 60.87 (4) | 56.81 (6) | 45.80 (2) | 45.80 (2) |
| Wred | 55.52 | 56.27 (10) | 49.26 (3) | 55.38 (1) | 56.27 (7) | **56.65 (6)** | 53.82 | 53.32 (10) | 49.31 (3) | 50.38 (1) | 53.44 (7) | **54.82 (6)** |
| Car | 64.47 | 64.00 (3) | 75.00 (2) | 64.12 (3) | 58.10 (4) | **79.17 (1)** | 72.22 | **79.17 (3)** | 78.94 (2) | 72.69 (3) | **79.17 (4)** | **79.17 (1)** |
| Card | **76.08** | 45.76 (12) | 62.15 (7) | 74.01 (10) | 67.70 (6) | 67.70 (6) | 80.41 | 77.87 (12) | 83.15 (7) | 81.17 (10) | **87.38 (6)** | **87.38 (6)** |
| Wite | 54.13 | 54.09 (7) | 56.00 (2) | 55.57 (1) | **57.00 (2)** | **57.00 (2)** | 47.49 | **48.00 (7)** | 43.00 (2) | 40.57 (1) | 43.16 (2) | 43.16 (2) |

set increases, the growth trend of the time consumed using IFSA-IvO algorithm is slower than that using other four algorithms. Moreover, Fig. 6 shows that the incremental algorithm is at least nearly one times or more faster than other four algorithms on all the datasets. For most datasets, the algorithm IFSA-IvO is on average four times faster than the other four algorithms. Therefore, the experimental results prove that the incremental algorithm IFSA-IvO can efficiently obtain a reduct when adding objects.

Second, we compare the computational efficiency of the incremental algorithm IFSD-IvO with the other four comparison algorithms. The detailed experimental operation is shown in Fig. 4(b), and the experimental results are shown in Figs. 7 and 8.

Fig. 7 shows that on each dataset, the calculation time of these five algorithms decreases as the amount of deleted data increases, where the running time of incremental algorithm IFSD-IvO is the least one. The time consumed by these five algorithms is roughly arranged in descending order as IFSD-IvO $\prec$ RDAR $\prec$ REAR $\prec$ HFS-IvO $\prec$ DRSQR, which can be viewed from Fig. 8. In Fig. 8, for most datasets, the calculation speed of algorithm IFSD-IvO is several times of other algorithms. In particular, Figs. 8(a) and 8(d) show that algorithm IFSD-IvO is dozens of times faster than algorithms DRSQR and HFS-IvO. Accordingly, we can conclude that the incremental algorithm IFSD-IvO can efficiently obtain a reduct when deleting objects.

(3) Summary

After experimental analysis, it can be concluded that incremental algorithms IFSA-IvO and IFSD-IvO not only decreases the computational time, but also improve the classification performance. Accordingly, compared with other four algorithms,

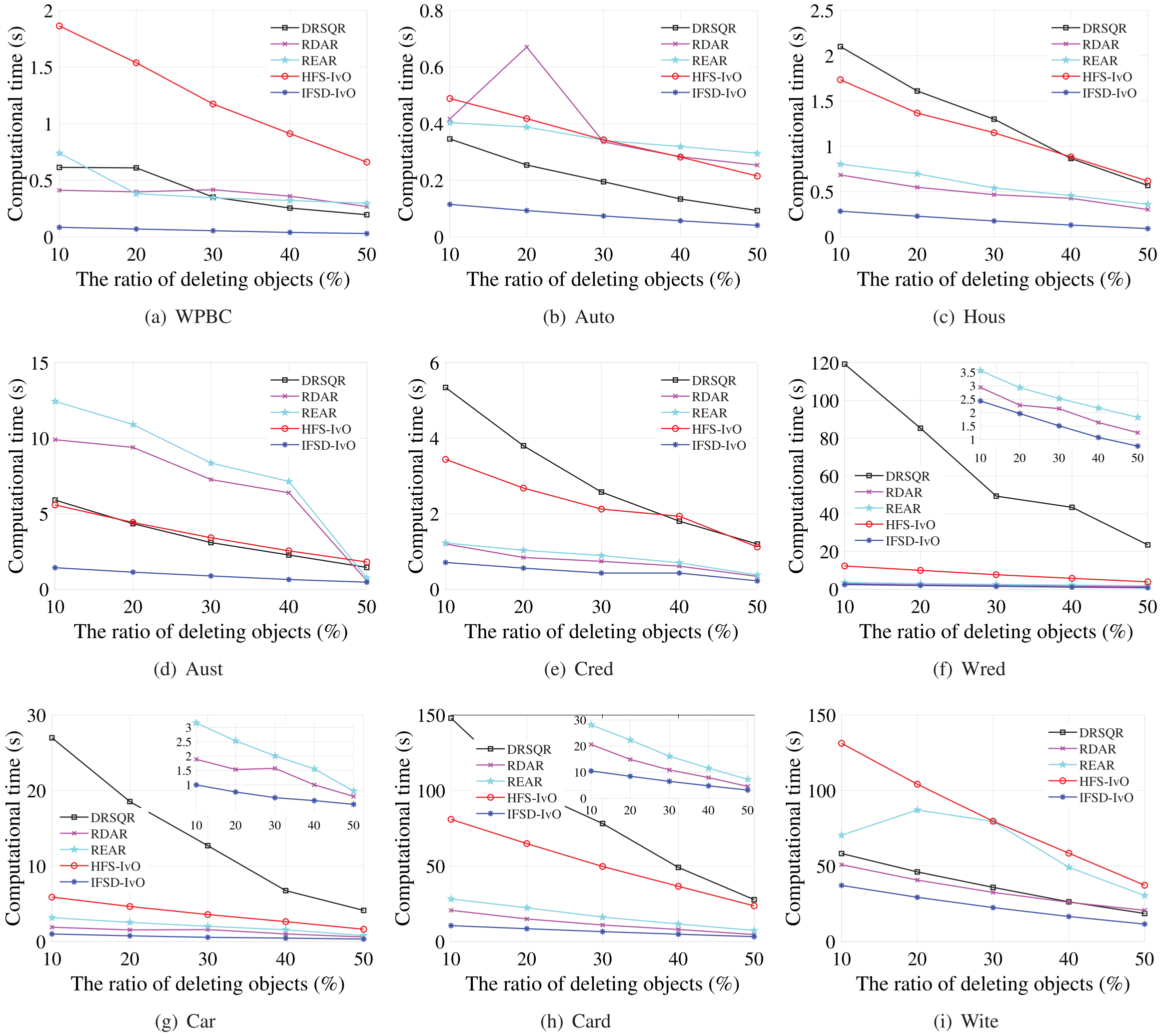



**Fig. 7.** The computational time of different algorithms versus different ratios of deleting objects.

incremental algorithms can quickly generate a satisfying reduct when multiple objects are added to or deleted from an IvODS.

## 7. Conclusion and future work

In this study, we propose incremental feature selection methods based on FDNRS for dynamic interval-valued ordered data. The main works are as follows: (1) We propose a FDNRS model for IvODS and present its relevant properties. (2) Based on the proposed model, a robust conditional entropy (i.e., FDNCE) is proposed for attribute reduction of IvODS. (3) For dynamically adding objects to or deleting objects from an IvODS, we develop two incremental feature selection algorithms accordingly. Experiments are performed on nine public datasets. The result of the experiment proves the robustness of the metric FDNCE and the effectiveness and efficiency of the proposed incremental algorithms.

This work studies incremental feature selection algorithms for dynamic interval-value ordered data with object set changes. Nevertheless, dynamic data with the variation of multi-sided is closer to reality, which inspire our further research. In future work, we will investigate incremental feature selection approaches for dynamic IvODS with the variation of multi-sided.

## CRediT authorship contribution statement

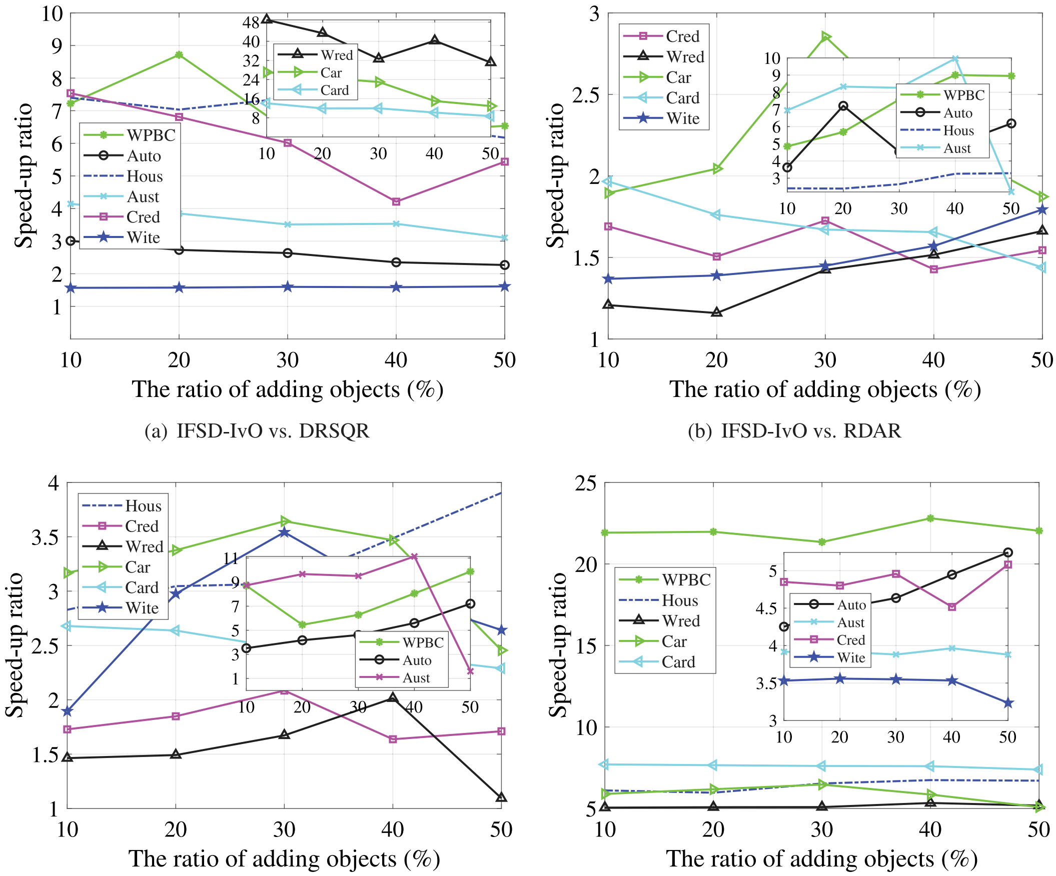
**Binbin Sang:** Methodology, Validation, Writing - original draft, Writing - review & editing. **Hongmei Chen:** Conceptualization, Resources, Visualization, Supervision, Project administration, Funding acquisition. **Lei Yang:** Formal analysis, Data curation. **Tianrui Li:** Resources, Supervision, Funding acquisition. **Weihua Xu:** Resources, Funding acquisition. **Chuan Luo:** Resources, Supervision.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 61976182, 61572406, 62076171, 61602327, 61876157, 61976245), Sichuan Key R&D project, China (2020 YFG0035), Key program for International S&T Cooperation of Sichuan Province, China (2019YFH0097).




(a) IFSD-IvO vs. DRSQR (b) IFSD-IvO vs. RDAR

(c) IFSD-IvO vs. REAR (d) IFSD-IvO vs. HFS-IvO

**Fig. 8.** The speed-up ratios that algorithm IFSD-IvO relates to other different algorithms.

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