Applied Soft Computing 111 (2021) 107679

Contents lists available at ScienceDirect

Applied Soft Computing

journal homepage: www.elsevier.com/locate/asoc

Dynamic fuzzy neighborhood rough set approach for interval-valued information systems with fuzzy decision

Lei Yang [a,*], Keyun Qin [a], Binbin Sang [b], Weihua Xu [c]

[a] School of Mathematics, Southwest Jiaotong University, Chengdu, 611756, PR China
[b] School of Computing and Artificial Intelligence, Southwest Jiaotong University, Chengdu, 611756, PR China
[c] College of Artificial Intelligence, Southwest University, Chongqing, 400715, PR China

ARTICLE INFO

Article history:
Received 3 October 2020
Received in revised form 17 June 2021
Accepted 29 June 2021
Available online 7 July 2021

Keywords:
Interval-valued information system with fuzzy decision
δ-fuzzy neighborhood relation
Fuzzy neighborhood rough set
Incremental mechanisms
Dynamic algorithms

ABSTRACT

Nowadays, many extended rough set models are proposed to acquire valuable knowledge from interval-valued information system. These existing models mainly focus on different forms of similarity relations. However, most of these similarity relations are qualitative rather than quantitative, which is not reasonable in some practical cases. In addition, with the arrival of new objects and the removal of obsolete objects, the interval-valued information system with fuzzy decision (IvIS_FD) is always changing with time. Therefore, how to efficiently mining knowledge from dynamic IvIS_FD is a meaningful topic. Motivated by these two issues, we study the dynamic fuzzy neighborhood rough set approach for IvIS_FD, aiming to effectively update the rough approximations when the IvIS_FD evolves over time. Firstly, δ-fuzzy neighborhood relation is defined to describe the similarity relation between objects quantitatively. Secondly, we introduce a novel fuzzy neighborhood rough set model and its matrix representation suitable for IvIS_FD. On this basis, we discuss the incremental mechanisms to update fuzzy neighborhood approximations when multiple objects are added to or deleted from an IvIS_FD, respectively. Meanwhile, corresponding dynamic algorithms are designed and explained. Finally, experiments are performed on nine public data sets to evaluate the performance of the dynamic fuzzy neighborhood rough set model. Experimental results verify that the proposed model is effective and efficient for updating rough approximations in dynamic IvIS_FD.

1. Introduction

At the information era, data is an important carrier of knowledge. With the advancement of science and technology, single-valued data cannot satisfy people's demands for knowledge representation in production and life. Consequently, some generalized and complicated data representation types have emerged, among which interval-valued data is a typical instance.

In practical applications, interval-valued data is widely used to describe the imprecise and vague concepts, such as temperature changes, stock price fluctuations, blood pressure ranges and so on. Furthermore, in some decision analysis problems, interval-valued data is always accompanied by fuzzy decision attributes, and the sample set evolves over time (i.e., dynamic interval-valued data). For example, the data for climate description is a typical dynamic interval-valued data with fuzzy decision. A specific application scenario is introduced as follows. The interval-valued data is used to record the temperature, humidity, and ultraviolet intensity (i.e., attributes) of each day (i.e., object), and the evaluation of the quality of each day's weather is described by a fuzzy concept (i.e., fuzzy decision). Over time, new data records will be gradually added to this data, and some old data will be gradually removed from this data (i.e., the data will dynamically evolve over time). How to mine such data sets effectively and efficiently to obtain useful information or rules? Motivated by this issue, this study focuses on investigating efficient knowledge discovery methods for dynamic interval-valued data with fuzzy decision.

1.1. Comparative analysis of related works

As a natural data mining or knowledge discovery method, rough set theory (RST) [1] proposed by Pawlak has been widely used in pattern recognition [2], decision analysis [3,4] and other fields. The RST can handle various incomplete, inaccuracy and inconsistent information without any prior knowledge. However, the original RST only applies to discrete data sets. To make the RST more widely applicable to different types of data sets, many extended rough set models have been proposed successively [5–11]. Moreover, the research on knowledge discovery of interval-valued data sets based on the RST has received more and more attention [12–15].

* Corresponding author.
E-mail address: cqyanglei@yeah.net (L. Yang).

https://doi.org/10.1016/j.asoc.2021.107679
1568-4946/

In recent years, some extended rough set models are proposed to acquire valuable knowledge from interval-valued data. For an interval-valued decision system (IvDS), Dai et al. constructed the similarity measure between two intervals based on possible degree, and defined the θ-similarity classes of objects by using a given similarity rate. Then, the rough approximation operators, the θ-accuracy and θ-roughness measures are presented for IvDS [16]. In an interval-valued information system (IvIS), Leung et al. proposed the α-tolerance relations based on misclassification rate, and presented a rough set model by using α-tolerance classes [17]. At the same time, the α-classification reduction method for IvIS was presented to obtain all classification rules. For the IvDS, Zhang et al. constructed a similarity relation between objects by using multi-confidence, and then introduced a related rough set model [18]. For an incomplete IvIS, Dai et al. constructed α-weak similarity degree and α-weak similarity relation, and then proposed the notions of upper and lower approximations based on the α-weak similarity relation [19]. Ma studied the kernel similarity relation based on kernel function to compute the similarity between objects in the IvDS, and proposed the notions of kernel upper and lower approximation operators [20]. Subsequently, the kernel accuracy measure, kernel roughness measure, and kernel approximation accuracy measure were proposed and examined. The comparative among the extended models in the above literatures is shown in Table 1.

For different types of interval-valued data, many rough set models based on different similarity relations were explored to discover knowledge in the above researches. However, most of these similarity relations are qualitative relations rather than quantitative relations in the sense that, these similarity relations are classical binary relation constructed by using some thresholds on similarity degrees. For example, in [16], the similarity relation is defined by the magnitude between the similarity degree and threshold θ. If the similarity degree between two objects is greater than or equal to threshold δ, a certain relation is considered between the two objects. This definition is unreasonable for numerical data sets in some cases, because there is no essential difference between real number 0.799 and real number 0.8. However, if the threshold is set to 0.8, then the two objects with similarity of 0.8 are related, while the two objects with similarity of 0.799 are considered not related. Therefore, qualitative similarity relations are not desirable in most application scenarios. In addition, the determination of the threshold values is subjective in many cases. As a consequence, one of the motivations of this study is to realize the quantitative description of similarity relation in the context of an interval-valued information system with fuzzy decision (IvIS_FD).

In practical application, with the collection of new data and the deletion of obsolete data, the original data sets will change. The changes here include the changes of objects [21–23], attributes [24,25], attribute values [26,27] and multi-dimensional changes [28,29]. When data sets change over time, the original knowledge will not be applicable. Therefore, we need to make some strategies to update the original knowledge. Traditional algorithms for renovating knowledge need to calculate the whole updated data set from the beginning, which is inefficient or even infeasible in practical application. The incremental learning method provides a good idea for us to acquire knowledge efficiently in dynamic environment.

In recent years, many scholars have proposed substantial incremental knowledge discovery approaches based on RST according to the change of different dimension of data sets [30–37]. Some recent research results on incremental knowledge discovery are introduced in Table 2. Through analysis, we can know that scholars have done a lot of works on the dynamic updating rough approximations methods for different types of information systems. Some methods study the change of approximations according to the change of single element, while most methods use the matrix representation of rough set model to study the update of approximations when multiple elements change. It is worth noting that these existing methods are not suitable for the dynamic IvIS_FD. Hence, another motivation of this paper is to realize the dynamic updating of approximations under the background of dynamic IvIS_FD with time-evolving objects.

1.2. Our work

Based on all the above discussions, we propose dynamic updating approach of rough approximations for fuzzy neighborhood rough set in IvIS_FD with time-evolving objects. The main contributions of this study are summarized as follows.

- We present a novel fuzzy neighborhood rough set model suitable for IvIS_FD, then its matrix representation method and matrix-based static calculation approximations algorithm is introduced.
- We construct incremental mechanisms to update the approximations of the proposed model when the objects change. Meanwhile, corresponding matrix-based dynamic algorithms are designed.
- Comparative experiments are executed on nine public data sets, and the results show that the proposed model and dynamic algorithms are effective and efficient.

The remaining content of this paper is arranged as follows. The preliminary knowledge related to this study is briefly reviewed in Section 2. In Section 3, the fuzzy neighborhood rough set model suitable for IvIS_FD and the matrix representation of the proposed model are constructed. Section 4 presents dynamic update mechanisms of fuzzy neighborhood rough set and two kind of dynamic algorithms. Experimental design and analysis are presented in Section 5. On the basis of summarizing this work, the theoretical and practical implications of the proposed method, the weaknesses of the proposed method, and future work are given in Section 6.

2. Preliminaries

In order to make the article more fluent, some essential concepts of IvIS and fuzzy rough set are briefly reviewed in this section. More detailed descriptions can be found in [21,46–48].

2.1. Some fundamental concepts about IvIS

In general, a closed interval on the real numbers set $\mathbb{R}$ is a set of real numbers in the form of $I = [l, r] = \{x \in \mathbb{R} \mid l \leq x \leq r\}$, $l \leq r$, where l and r are called the left and right endpoint of closed interval I, respectively.

Assume $I_1 = [l_1, r_1]$ and $I_2 = [l_2, r_2]$ are two intervals. The intersection and union operation of I_1 and I_2 are denoted as follows:

$$I_1 \cap I_2 = \begin{cases} [max\{l_1, l_2\}, min\{r_1, r_2\}], & max\{l_1, l_2\} \leq min\{r_1, r_2\}, \\ \emptyset, & otherwise. \end{cases} \tag{1}$$

$$I_1 \cup I_2 = \begin{cases} [min\{l_1, l_2\}, max\{r_1, r_2\}], & max\{l_1, l_2\} \leq min\{r_1, r_2\}, \\ [l_1, r_1] \cup [l_2, r_2], & otherwise. \end{cases} \tag{2}$$

An information system is usually represented by a quadruple (U, A, V, f), where $U = \{x_1, x_2, \ldots, x_m\}$ is a non-empty finite

Table 1
Comparison between different extension models for IvIS.

Year	Authors	Data background	Specific methods	Binary relation	Reference
2008	Leung et al.	IvIS	α-tolerance relation based on misclassification rate was introduced to propose a rough set model	Qualitative	[17]
2013	Dai et al.	IvDS	Similarity measure based θ-similarity relation and rough approximation operators	Qualitative	[16]
2014	Zhang et al.	IvDS	Multi-confidence based similarity relation and the related rough set model	Qualitative	[18]
2017	Dai et al.	Incomplete IvIS	α-weak similarity degree, α-weak similarity relation, and the relevant approximations	Qualitative	[19]
2017	Ma	IvDS	kernel upper and lower approximation operators based on kernel similarity relation	Qualitative	[20]

Table 2
The review of some incremental methods for updating approximations.

Year	Authors	Incremental feature selection methods	Reference
2016	Zhang et al.	Incremental approaches for updating approximations of the similarity-based rough set model in an IvIS with time-evolving attributes	[38]
2017	Hu et al.	Incremental updating approximations when objects are added into or deleted from the fuzzy information system over two universes	[39]
2017	Zeng et al.	Incremental mechanism of updating a novel Gaussian kernel fuzzy rough set when the attribute values changed	[40]
2017	Hu et al.	Dynamic mechanisms to update approximations of multi-granulation rough sets while refining or coarsening attribute values	[41]
2017	Huang et al.	Calculating rough approximations of fuzzy concepts in dynamic fuzzy decision systems with simultaneous variation of objects and features	[42]
2020	Guo et al.	Dynamic updating approximations approach for double-quantitative decision-theoretic rough sets with the variation of objects	[43]
2020	Huang et al.	Incremental approaches for maintaining approximations in dynamic composite dominance rough set with the change of attributes	[44]
2020	Wang et al.	Updating approximations approach for dynamic ordered information systems when attributes increase and attribute values vary simultaneously	[45]

Table 3
An interval-valued information system with fuzzy decision.

U	a_1	a_2	a_3	a_4	a_5	$\tilde{d}$
x_1	[2.17,2.86]	[2.45,2.96]	[5.32,7.23]	[3.21,3.95]	[2.54,3.12]	0.5
x_2	[3.37,4.75]	[3.43,4.85]	[7.24,10.47]	[4.00,5.77]	[3.24,4.70]	0.8
x_3	[1.83,2.70]	[1.78,2.98]	[7.23,10.27]	[2.96,4.07]	[2.06,2.79]	0.2
x_4	[1.35,2.12]	[1.42,2.09]	[2.59,3.93]	[1.87,2.62]	[1.67,2.32]	0.2
x_5	[3.46,5.35]	[3.37,5.11]	[6.37,10.28]	[3.76,5.70]	[3.41,5.28]	1.0
x_6	[2.29,3.43]	[2.60,3.48]	[6.71,8.81]	[3.30,4.23]	[3.01,3.84]	0.8
x_7	[2.22,3.07]	[2.43,3.32]	[4.37,7.05]	[2.66,3.68]	[2.39,3.20]	0.9

set of objects, $A = \{a_1, a_2, \dots, a_n\}$ is a non-empty finite set of attributes, $V = \cup_{a\in A} V_a$ and V_a is the domain of attribute a, $f : U \times A \to V$ is the information function of information system, $f(x, a) \in V_a$ ($\forall\ x \in U, a \in A$). If for any $x \in U, a \in A, f(x, a) = [a^l(x), a^r(x)]$, then (U, A, V, f) is called an IvIS and V_a is a set of intervals. When IvIS is accompanied by fuzzy decision, we call it interval-valued information system with fuzzy decision (IvIS_FD). Generally, we use $(U, A \cup \{\tilde{d}\}, V, f)$ to represent an IvIS_FD. This study will take IvIS_FD as the background. An example (Table 3) shown in [49] is used to illustrate the concept of IvIS_FD.

Chen and Qin indicated that since the attribute value of an object in an IvIS is an interval, and the similarity of two intervals depends on the length of the interval where they intersect [46], the following definition can be given in an IvIS.

Definition 1. Let (U, A, V, f) be an IvIS. For any $x_i, x_j \in U$ and $a_k \in A$, the similarity degree between x_i and x_j under the attribute a_k is defined as

$$\mathcal{S}_{a_k}(x_i, x_j) = \begin{cases} 0, & f(x_i, a_k) \cap f(x_j, a_k) = \emptyset, \\ 1, & f(x_i, a_k) \cap f(x_j, a_k) \neq \emptyset \\ & \wedge |f(x_i, a_k) \cup f(x_j, a_k)| = 0, \\ \frac{|f(x_i,a_k)\cap f(x_j,a_k)|}{|f(x_i,a_k)\cup f(x_j,a_k)|}, & otherwise, \end{cases} \tag{3}$$

where $|\cdot|$ expresses the length of interval. In this paper, we stipulate that the length of a single point is zero.

Given an IvIS, for any $x_i, x_j \in U$, there are many definitions of the similarity degree between x_i and x_j under the attribute set A. This paper adopts the definition proposed by Yu and Xu in [21]. The specific definition is described below.

Definition 2. Let (U, A, V, f) be an IvIS. For any $x_i, x_j \in U$ and $a_k \in A$, $\mathcal{S}_{a_k}(x_i, x_j)$ is the similarity degree between x_i and x_j under the attribute a_k. The similarity degree between x_i and x_j under the attribute set A is defined as

$$\mathcal{S}_A(x_i, x_j) = \min\{\mathcal{S}_{a_k}(x_i, x_j) | a_k \in A\}. \tag{4}$$

Example 1. Table 3 shows an IvIS_FD, where $U = \{x_1, x_2, \dots, x_7\}$. From Eqs. (3) and (4), the $\mathcal{S}_A$ is computed as

$$\mathcal{S}_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0.0846 & 0.3643 \\ 0 & 1 & 0 & 0 & 0.6324 & 0.0222 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.6324 & 0 & 0 & 1 & 0 & 0 \\ 0.0846 & 0.0222 & 0 & 0 & 0 & 1 & 0.0766 \\ 0.3643 & 0 & 0 & 0 & 0 & 0.0766 & 1 \end{bmatrix}_{7\times 7}.$$

2.2. Fuzzy rough set

Dubois and Prade proposed fuzzy rough set model [47] by combining fuzzy set and RST. The fuzzy rough set model can deal with numerical or continuous data. In this subsection, we introduce some basic concepts of fuzzy set, fuzzy similarity relation and fuzzy approximation operators.

Let U be a non-empty universe, mapping $\mathcal{A} : U \to [0.1]$ is called a fuzzy set on U. The set of all fuzzy sets on U is denoted as $\mathcal{F}(U)$. A fuzzy binary relation R on U is called fuzzy similarity relation if R satisfies

(1) Reflexivity: $R(x, x) = 1, \forall x \in U$;
(2) Symmetry: $R(x, y) = R(y, x), \forall x, y \in U$.

For any $x \in U$, every fuzzy similarity relation R can induce a fuzzy similarity class $[x]_R = \{(y, R(x, y)) : y \in U\}$.

Definition 3. Given a non-empty universe U and a fuzzy similarity relation R. For any $\mathcal{A} \in \mathcal{F}(U)$, the fuzzy lower and upper approximations of $\mathcal{A}$ are defined as

$$\underline{R}\mathcal{A}(x) = \inf_{y\in U} \max\{1 - R(x, y), \mathcal{A}(y)\},\ \ x \in U, \tag{5}$$

$$\overline{R}\mathcal{A}(x) = \sup_{y\in U} \min\{R(x, y), \mathcal{A}(y)\},\ \ x \in U. \tag{6}$$

$\underline{R}\mathcal{A}$ and $\overline{R}\mathcal{A}$ are called the fuzzy lower and upper approximation operators of $\mathcal{A}$, respectively.

3. Fuzzy neighborhood rough set based on IvIS_FD

In this section, the fuzzy neighborhood rough set model for IvIS_FD is proposed. Considering that matrix operation is a significant part in mathematics, the matrix representation of fuzzy neighborhood rough set is also studied in this section.

3.1. Fuzzy neighborhood rough set based on IvIS_FD

Firstly, according to the similarity degree between objects in the IvIS, the δ-fuzzy neighborhood relation in the IvIS_FD is defined. Furthermore, the definition of δ-fuzzy neighborhood information granule is proposed. Then, a fuzzy neighborhood rough set model is proposed by combining δ-fuzzy neighborhood relation with fuzzy decision. Meanwhile, some related properties of the proposed model are studied. Finally, the fuzzy neighborhood approximation accuracy is introduced.

Definition 4. Let $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD, $\forall x, y \in U$ and $\delta \in (0, 1]$. For any $B \subseteq A$, the δ-fuzzy neighborhood relation with respect to B is defined as

$$R_B^\delta = \{((x, y), \mathcal{S}_B(x, y)) : \mathcal{S}_B(x, y) \geq \delta, (x, y) \in U^2\}, \tag{7}$$

where $\mathcal{S}_B(x, y)$ is the similarity degree between x and y under B. Threshold δ, named neighborhood radius, is used to control the similarity degree between objects.

Obviously, R_B^δ is reflexive and symmetric.

Definition 5. Let $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD, $\forall x, y \in U$ and $\delta \in (0, 1]$. For any $B \subseteq A$, R_B^δ is the δ-fuzzy neighborhood relation with respect to B. For any $x \in U$, the δ-fuzzy neighborhood information granule related to x is defined as

$$[x]_B^\delta(y) = \begin{cases} \mathcal{S}_B(x, y), & \mathcal{S}_B(x, y) \geq \delta; \\ 0, & \mathcal{S}_B(x, y) < \delta. \end{cases} \tag{8}$$

Normally, $[x]_B^\delta$ is called δ-fuzzy neighborhood class induced by R_B^δ.

Property 1. *Let $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD, $\delta, \delta_1, \delta_2 \in (0, 1]$, $B, B_1, B_2 \subseteq A$. R_B^δ is the δ-fuzzy neighborhood relation induced by B. The following properties hold.*
(1) If $B_1 \subseteq B_2$, then $R_{B_2}^\delta \subseteq R_{B_1}^\delta$;
(2) If $\delta_1 \leq \delta_2$, then $[x]_B^{\delta_2} \subseteq [x]_B^{\delta_1}$ for any $x \in U$.

Proof. (1) If $B_1 \subseteq B_2$, it is easy to discover $\mathcal{S}_{B_2}(x, y) \leq \mathcal{S}_{B_1}(x, y)$ for any $(x, y) \in U^2$ according to Definition 2. For any $(x, y) \in R_{B_2}^\delta$, we can obtain $\mathcal{S}_{B_2}(x, y) \geq \delta$ according to Definition 4, so $\mathcal{S}_{B_1}(x, y) \geq \delta$. Furthermore, $(x, y) \in R_{B_1}^\delta$. To sum up, $R_{B_2}^\delta \subseteq R_{B_1}^\delta$.
(2) According to Definition 5, if $\delta_1 \leq \delta_2$, it is obvious that $[x]_B^{\delta_2}(y) \leq [x]_B^{\delta_1}(y)$ for any $x, y \in U$. So $[x]_B^{\delta_2} \subseteq [x]_B^{\delta_1}$ for any $x \in U$. $\square$

Definition 6. Let $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD, $\delta \in (0, 1]$. For any $B \subseteq A$, R_B^δ is the δ-fuzzy neighborhood relation induced by B. The fuzzy neighborhood lower and upper approximations of the fuzzy decision $\tilde{d}$ with respect to B are defined as

$$\underline{R_B^\delta}\tilde{d}(x) = \inf_{y\in U} \max\{1 - R_B^\delta(x, y), \tilde{d}(y)\}, x \in U, \tag{9}$$

$$\overline{R_B^\delta}\tilde{d}(x) = \sup_{y\in U} \min\{R_B^\delta(x, y), \tilde{d}(y)\}, x \in U, \tag{10}$$

where $R_B^\delta(x, y) = [x]_B^\delta(y)$.

Property 2. *Let $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD, $\delta, \delta_1, \delta_2 \in (0, 1]$. $\underline{R_A^\delta}\tilde{d}$ and $\overline{R_A^\delta}\tilde{d}$ are the fuzzy neighborhood lower and upper approximation operators of $\tilde{d}$. The following properties hold.*
(1) $\underline{R_A^\delta}\tilde{d} \subseteq \tilde{d} \subseteq \overline{R_A^\delta}\tilde{d}$;
(2) $B \subseteq A \Rightarrow \underline{R_B^\delta}\tilde{d} \subseteq \underline{R_A^\delta}\tilde{d}$, $\overline{R_A^\delta}\tilde{d} \subseteq \overline{R_B^\delta}\tilde{d}$;
(3) $\delta_1 \leq \delta_2 \Rightarrow \underline{R_A^{\delta_1}}\tilde{d} \subseteq \underline{R_A^{\delta_2}}\tilde{d}$, $\overline{R_A^{\delta_2}}\tilde{d} \subseteq \overline{R_A^{\delta_1}}\tilde{d}$.

Proof. (1) For any $x \in U$, we can discover $\mathcal{S}_A(x, x) = 1$ according to Definitions 1 and 2. By Definition 4, $R_A^\delta(x, x) = 1$. Furthermore, $\max\{1 - R_A^\delta(x, x), \tilde{d}(x)\} = \tilde{d}(x)$. It is obvious that $\inf_{y\in U} \max\{1 - R_A^\delta(x, y), \tilde{d}(y)\} \leq \tilde{d}(x)$ from Definition 6, namely, $\underline{R_A^\delta}\tilde{d}(x) \leq \tilde{d}(x)$. So $\underline{R_A^\delta}\tilde{d} \subseteq \tilde{d}$. Similarly, it is easy to prove $\tilde{d} \subseteq \overline{R_A^\delta}\tilde{d}$. To summarize, $\underline{R_A^\delta}\tilde{d} \subseteq \tilde{d} \subseteq \overline{R_A^\delta}\tilde{d}$.
(2) According to Property 1, if $B \subseteq A$, then $R_A^\delta \subseteq R_B^\delta$. So for any $(x, y) \in U^2$, $R_A^\delta(x, y) \leq R_B^\delta(x, y)$. Obviously, $\inf_{y\in U} \max\{1 - R_A^\delta(x, y), \tilde{d}(y)\} \geq \inf_{y\in U} \max\{1 - R_B^\delta(x, y), \tilde{d}(y)\}$, namely, $\underline{R_A^\delta}\tilde{d}(x) \geq \underline{R_B^\delta}\tilde{d}(x)$. So $\underline{R_B^\delta}\tilde{d} \subseteq \underline{R_A^\delta}\tilde{d}$. Similarly, it is easy to prove $\overline{R_A^\delta}\tilde{d} \subseteq \overline{R_B^\delta}\tilde{d}$.
(3) If $\delta_1 \leq \delta_2$, by Definition 4, we can get $R_A^{\delta_2} \subseteq R_A^{\delta_1}$. Similar to the proof process in (2), we can easily prove that $\underline{R_A^{\delta_1}}\tilde{d} \subseteq \underline{R_A^{\delta_2}}\tilde{d}$ and $\overline{R_A^{\delta_2}}\tilde{d} \subseteq \overline{R_A^{\delta_1}}\tilde{d}$. $\square$

Definition 7. Let $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD, $\delta \in (0, 1]$. For any $B \subseteq A$, the fuzzy neighborhood approximation accuracy of $\tilde{d}$ with respect to B is defined as

$$Acc = \frac{\sum_{x\in U} \underline{R_B^\delta}\tilde{d}(x)}{\sum_{x\in U} \overline{R_B^\delta}\tilde{d}(x)}. \tag{11}$$

Apparently, $0 \leq Acc \leq 1$.

3.2. Matrix representation of the fuzzy neighborhood approximations in IvIS_FD

From Definition 6, the calculation of fuzzy neighborhood approximation sets is mainly related to δ-fuzzy neighborhood relation and fuzzy decision. Therefore, this subsection first studies the matrix representation of the δ-fuzzy neighborhood relation and the fuzzy decision. Then the fuzzy neighborhood approximation sets of IvIS_FD are defined by matrix operation. Finally, a matrix-based static algorithm is designed to calculate the fuzzy neighborhood approximations in IvIS_FD.

Definition 8. Let $U = \{x_1, x_2, \ldots, x_n\}$ is a finite nonempty object set. If $\mathcal{A} \subseteq \mathcal{F}(U)$, then $\mathcal{A}$ can be written as a characteristic vector shown as $G(\mathcal{A}) = [g_i]_{n\times 1}$, where

$$g_i = \mathcal{A}(x_i),\ \ 1 \leq i \leq n. \tag{12}$$

Example 2. From Table 3, $U = \{x_1, x_2, \ldots, x_7\}$, the fuzzy decision attribute $\tilde{d}$ can induce a fuzzy set on U such as

$$\tilde{d} = \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{1.0}{x_5} + \frac{0.8}{x_6} + \frac{0.9}{x_7}.$$

According to Definition 8, $\tilde{d}$ can be written as $G(\tilde{d}) = [0.5, 0.8, 0.2, 0.2, 1.0, 0.8, 0.9]^T$, where "$T$" indicates the transpose operation.

Definition 9. Let $(U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD. For any $B \subseteq A$, R_B^δ is δ-fuzzy neighborhood relation induced by B. The δ-fuzzy neighborhood relation matrix with respect to R_B^δ is defined as $\mathbb{M}_U^{R_B^\delta} = [m_{ij}^{R_B^\delta}]_{n\times n}$, where

$$m_{ij}^{R_B^\delta} = \begin{cases} R_B^\delta(x_i, x_j), & (x_i, x_j) \in R_B^\delta, \\ 0, & otherwise, \end{cases} \tag{13}$$

and n is the number of objects in U.

Example 3 (*Continuation of Example 1*)**.** Let $\delta = 0.2$, by using Definition 9, the δ-fuzzy neighborhood relation matrix $\mathbb{M}_U^{R_A^\delta}$ with respect to R_A^δ is computed as

$$\mathbb{M}_U^{R_A^\delta} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0.3643 \\ 0 & 1 & 0 & 0 & 0.6324 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.6324 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0.3643 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{7\times 7}.$$

Definition 10. Let $(U, A \cup \{\tilde{d}\}, V, f)$ be an IvIS_FD. For any $B \subseteq A$, R_B^δ is δ-fuzzy neighborhood relation induced by B, $\mathbb{M}_U^{R_B^\delta}$ is the δ-fuzzy neighborhood relation matrix with respect to R_B^δ. The fuzzy decision $\tilde{d}$ can be written as a characteristic vector shown as $G(\tilde{d})$. Then the fuzzy neighborhood lower and upper vectors of $\tilde{d}$ can be defined as

$$\Phi_{R_B^\delta}(\tilde{d}) = (1 - \mathbb{M}_U^{R_B^\delta}) \circ_{(\vee,\wedge)} G(\tilde{d}), \tag{14}$$

$$\Psi_{R_B^\delta}(\tilde{d}) = \mathbb{M}_U^{R_B^\delta} \circ_{(\wedge,\vee)} G(\tilde{d}), \tag{15}$$

where $\circ_{(\vee,\wedge)}$ is min–max composite operator and $\circ_{(\wedge,\vee)}$ is max–min composite operator.

Note: If $R = [r_{ij}]_{m\times n}$, $S = [s_{ij}]_{n\times l}$, $P = R \circ_{(\vee,\wedge)} S = [p_{ij}]_{m\times l}$ and $Q = R \circ_{(\wedge,\vee)} S = [q_{ij}]_{m\times l}$ are four fuzzy matrices, then R, S, P and Q satisfy the operation rules as

$p_{ij} = \wedge_{k=1}^{n}(r_{ik} \vee s_{kj}), i = 1, 2, \ldots, m, j = 1, 2, \ldots, l;$
$q_{ij} = \vee_{k=1}^{n}(r_{ik} \wedge s_{kj}), i = 1, 2, \ldots, m, j = 1, 2, \ldots, l.$

Obviously, Definitions 6 and 10 are equivalent.

Example 4. Given an IvIS_FD as shown in Table 3, $\mathbb{M}_U^{R_A^\delta}$ is the δ-fuzzy neighborhood relation matrix. The fuzzy neighborhood lower and upper vectors of $\tilde{d}$ can be calculated as

$$\Phi_{R_A^\delta}(\tilde{d}) = (1 - \mathbb{M}_U^{R_A^\delta}) \circ_{(\vee,\wedge)} G(\tilde{d})$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0.6357 \\ 1 & 0 & 1 & 1 & 0.3676 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0.3676 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0.6357 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}_{7\times 7} \circ_{(\vee,\wedge)} \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 1.0 \\ 0.8 \\ 0.9 \end{bmatrix}_{7\times 1} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 0.8 \\ 0.8 \\ 0.6357 \end{bmatrix}_{7\times 1},$$

$$\Psi_{R_A^\delta}(\tilde{d}) = \mathbb{M}_U^{R_A^\delta} \circ_{(\wedge,\vee)} G(\tilde{d})$$

Table 4
The time complexity of Algorithm 1.

Steps	Time complexity
1–3	$O(\|U\|)$
4–8	$O(\|U\|^2)$
9	$O(\|U\|^3)$
10	$O(\|U\|^3)$
Total	$O(\|U\| + \|U\|^2 + \|U\|^3)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0.3643 \\ 0 & 1 & 0 & 0 & 0.6324 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.6324 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0.3643 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{7\times 7} \circ_{(\wedge,\vee)} \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 1.0 \\ 0.8 \\ 0.9 \end{bmatrix}_{7\times 1} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 1.0 \\ 0.8 \\ 0.9 \end{bmatrix}_{7\times 1}.$$

Then, the fuzzy neighborhood lower and upper approximations of $\tilde{d}$ are shown as

$$\underline{R_A^\delta}\tilde{d} = \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.8}{x_5} + \frac{0.8}{x_6} + \frac{0.6357}{x_7},$$
$$\overline{R_A^\delta}\tilde{d} = \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{1.0}{x_5} + \frac{0.8}{x_6} + \frac{0.9}{x_7}.$$

It can be seen from Definition 10 that if we want to obtain the fuzzy neighborhood lower and upper approximations in IvIS_FD, we must first acquire the δ-fuzzy neighborhood relation matrix and characteristic vector. Next, according to the above discussion, we design a matrix-based static algorithm (Algorithm 1) to calculate the fuzzy neighborhood approximations in IvIS_FD.

Algorithm 1 Matrix-based static algorithm for calculating the fuzzy neighborhood approximations in IvIS_FD (MSFN).

Input: An $\mathcal{I} = (U, A \cup \{\tilde{d}\}, V, f)$ and threshold δ.
Output: $\underline{R_A^\delta}\tilde{d}$, $\overline{R_A^\delta}\tilde{d}$.
1: **for** $i = 1$ to $|U|$ **do**
2: Calculate $G(\tilde{d}) = [g_i]_{|U|\times 1}$ according to Definition 8;
3: **end for**
4: **for** $i = 1$ to $|U|$ **do**
5: **for** $j = 1$ to $|U|$ **do**
6: Calculate $\mathbb{M}_U^{R_A^\delta} = [m_{ij}^{R_A^\delta}]_{|U|\times|U|}$ according to Definition 9;
7: **end for**
8: **end for**
9: Compute: $\Phi_{R_A^\delta}(\tilde{d}) = (1 - \mathbb{M}_U^{R_A^\delta}) \circ_{(\vee,\wedge)} G(\tilde{d})$;
10: Compute: $\Psi_{R_A^\delta}(\tilde{d}) = \mathbb{M}_U^{R_A^\delta} \circ_{(\wedge,\vee)} G(\tilde{d})$;
11: **return** $\underline{R_A^\delta}\tilde{d}$, $\overline{R_A^\delta}\tilde{d}$.

The steps in Algorithm 1 are explained as follows. Steps 1–3 is to generate the characteristic vector of fuzzy decision in IvIS_FD. Steps 4–8 is to calculate the δ-fuzzy neighborhood relation matrix. Step 9 is to calculate the fuzzy neighborhood lower vector. Step 10 is to calculate the fuzzy neighborhood upper vector. Step 11 is to return the fuzzy neighborhood approximation sets. Furthermore, the time complexity of the main steps in this algorithm are listed in Table 4.

4. Updating mechanisms of fuzzy neighborhood rough set when multiple objects change

With the continuous development of the society, data is constantly updated. We adopt the Algorithm 1 to discover knowledge is very time-consuming when multiple objects in IvIS_FD evolve with time. Because this algorithm needs to recalculate knowledge from scratch. In order to improve the efficiency of knowledge discovery in dynamic IvIS_FD, we study the updating mechanisms of fuzzy neighborhood rough set. Through the research in Section 3, it is found that the characteristic vector and the δ-fuzzy neighborhood relation matrix play important roles in obtaining fuzzy neighborhood approximation sets. Therefore, we mainly study the updating mechanisms of characteristic vector and δ-fuzzy neighborhood relation matrix when objects in IvIS_FD changes with time.

4.1. Updating mechanisms of fuzzy neighborhood rough set when adding objects

This subsection introduces the updating strategies of $G(\tilde{d})$ and $\mathbb{M}_U^{R_B^\delta}$ when multiple objects are added to IvIS_FD. In order to explain the theory more clearly, an illustrative example is given. At the same time, corresponding dynamic algorithm are designed and explained.

Proposition 1. *Let $\mathcal{I}^{t_0} = (U^{t_0}, A \cup \{\tilde{d}\}^{t_0}, V^{t_0}, f^{t_0})$ be an IvIS_FD at time t_0. At time t_1, the IvIS_FD has been changed to $\mathcal{I}^{t_1} = (U^{t_1}, A \cup \{\tilde{d}\}^{t_1}, V^{t_1}, f^{t_1})$, where $U^{t_1} = U^{t_0} + \Delta U$ and $\Delta U = \{x_{n+1}, x_{n+2}, \dots, x_{n+n'}\}$. $G(\tilde{d}^{t_0})$ is the characteristic vector at time t_0. The updated characteristic vector at time t_1 is denoted as $G(\tilde{d}^{t_1}) = [g_i']_{(n+n')\times 1}$, where*

$$g_i' = \begin{cases} g_i, & 1 \le i \le n; \\ \tilde{d}(x_i), & n+1 \le i \le n+n'. \end{cases} \tag{16}$$

Proof. According to Definition 8, $G(\tilde{d}^{t_0})$ should be updated to a new characteristic vector $G(\tilde{d}^{t_1}) = [g_i']_{(n+n')\times 1}$ when $\Delta U = \{x_{n+1}, x_{n+2}, \dots, x_{n+n'}\}$ is added to $\mathcal{I}^{t_0}$. Obviously, the fuzzy decision of $\mathcal{I}^{t_0}$ cannot be changed when ΔU is added to $\mathcal{I}^{t_0}$. So $g_i' = g_i$ always holds for any $1 \le i \le n$. Furthermore, for any $x_i \in \Delta U$, there is $x_i \notin U$, so the corresponding value in the characteristic vector should be calculated according to Definition 8. In other words, $g_i' = \tilde{d}(x_i)$ for any $n+1 \le i \le n+n'$. □

Proposition 2. *Let $\mathcal{I}^{t_0} = (U^{t_0}, A \cup \{\tilde{d}\}^{t_0}, V^{t_0}, f^{t_0})$ be an IvIS_FD at time t_0. At time t_1, the IvIS_FD has been changed to $\mathcal{I}^{t_1} = (U^{t_1}, A \cup \{\tilde{d}\}^{t_1}, V^{t_1}, f^{t_1})$, where $U^{t_1} = U^{t_0} + \Delta U$ and $\Delta U = \{x_{n+1}, x_{n+2}, \dots, x_{n+n'}\}$. For any $B \subseteq A$, $\mathbb{M}_{U^{t_0}}^{R_B^\delta} = [m_{ij}^{R_B^\delta}]_{n\times n}$ is the δ-fuzzy neighborhood relation matrix with respect to R_B^δ at time t_0. The updated δ-fuzzy neighborhood relation matrix at time t_1 is denoted as $\mathbb{M}_{U^{t_1}}^{R_B^\delta} = [\hat{m}_{ij}^{R_B^\delta}]_{(n+n')\times(n+n')}$, where*

$$\hat{m}_{ij}^{R_B^\delta} = \begin{cases} m_{ij}^{R_B^\delta}, & 1 \le i, j \le n; \\ R_B^\delta(x_i, x_j), & (x_i, x_j) \in R_B^\delta, (n+1 \le i \le n+n') \\ & \vee (n+1 \le j \le n+n'); \\ 0, & otherwise. \end{cases} \tag{17}$$

Proof. According to Definition 9, $\mathbb{M}_{U^{t_0}}^{R_B^\delta}$ should be updated to a new fuzzy neighborhood relation matrix $\mathbb{M}_{U^{t_1}}^{R_B^\delta} = [\hat{m}_{ij}^{R_B^\delta}]_{(n+n')\times(n+n')}$ when $\Delta U = \{x_{n+1}, x_{n+2}, \dots, x_{n+n'}\}$ is added to $\mathcal{I}^{t_0}$. It is not difficult to find that $\mathbb{M}_{U^{t_1}}^{R_B^\delta}$ can be divided into four parts, that is, $\mathbb{M}_{U^{t_1}}^{R_B^\delta} = \left[\begin{array}{c|c} [\hat{m}_{ij}^{1R_B^\delta}]_{n\times n} & [\hat{m}_{ij}^{2R_B^\delta}]_{n\times n'} \\ \hline [\hat{m}_{ij}^{3R_B^\delta}]_{n'\times n} & [\hat{m}_{ij}^{4R_B^\delta}]_{n'\times n'} \end{array}\right]$. Next, we make a brief analysis of the four parts.

(1) The $[\hat{m}_{ij}^{1R_B^\delta}]_{n\times n}$ is the fuzzy neighborhood relation matrix of $U \times U$ under B, that is to say, $[\hat{m}_{ij}^{1R_B^\delta}]_{n\times n} = [m_{ij}^{R_B^\delta}]_{n\times n}$.

(2) The $[\hat{m}_{ij}^{2R_B^\delta}]_{n\times n'}$ is the fuzzy neighborhood relation matrix of $U \times \Delta U$ under B, where

$$\hat{m}_{ij}^{2R_B^\delta} = \begin{cases} R_B^\delta(x_i, x_j), & (x_i, x_j) \in R_B^\delta, (1 \le i \le n) \\ & \wedge (n+1 \le j \le n+n'); \\ 0, & otherwise. \end{cases}$$

(3) The $[\hat{m}_{ij}^{3R_B^\delta}]_{n'\times n}$ is the fuzzy neighborhood relation matrix of $\Delta U \times U$ under B, where

$$\hat{m}_{ij}^{3R_B^\delta} = \begin{cases} R_B^\delta(x_i, x_j), & (x_i, x_j) \in R_B^\delta, (n+1 \le i \le n+n') \\ & \wedge (1 \le j \le n); \\ 0, & otherwise. \end{cases}$$

(4) The $[\hat{m}_{ij}^{4R_B^\delta}]_{n'\times n'}$ is the fuzzy neighborhood relation matrix of $\Delta U \times \Delta U$ under B, where

$$\hat{m}_{ij}^{4R_B^\delta} = \begin{cases} R_B^\delta(x_i, x_j), & (x_i, x_j) \in R_B^\delta, (n+1 \le i \le n+n') \\ & \wedge (n+1 \le j \le n+n'); \\ 0, & otherwise. \end{cases}$$

To sum up the above discussion, we can unify $[\hat{m}_{ij}^{2R_B^\delta}]_{n\times n'}$, $[\hat{m}_{ij}^{3R_B^\delta}]_{n'\times n}$ and $[\hat{m}_{ij}^{4R_B^\delta}]_{n'\times n'}$ into the following forms, namely,

$$\hat{m}_{ij}^{R_B^\delta} = \begin{cases} R_B^\delta(x_i, x_j), & (x_i, x_j) \in R_B^\delta, (n+1 \le i \le n+n') \\ & \vee (n+1 \le j \le n+n'); \\ 0, & otherwise. \end{cases}$$

Thus, Eq. (17) has been confirmed. □

Table 5
A new IvIS_FD after adding objects.

U	a_1	a_2	a_3	a_4	a_5	$\tilde{d}$
x_1	[2.17,2.86]	[2.45,2.96]	[5.32,7.23]	[3.21,3.95]	[2.54,3.12]	0.5
x_2	[3.37,4.75]	[3.43,4.85]	[7.24,10.47]	[4.00,5.77]	[3.24,4.70]	0.8
x_3	[1.83,2.70]	[1.78,2.98]	[7.23,10.27]	[2.96,4.07]	[2.06,2.79]	0.2
x_4	[1.35,2.12]	[1.42,2.09]	[2.59,3.93]	[1.87,2.62]	[1.67,2.32]	0.2
x_5	[3.46,5.35]	[3.37,5.11]	[6.37,10.28]	[3.76,5.70]	[3.41,5.28]	1.0
x_6	[2.29,3.43]	[2.60,3.48]	[6.71,8.81]	[3.30,4.23]	[3.01,3.84]	0.8
x_7	[2.22,3.07]	[2.43,3.32]	[4.37,7.05]	[2.66,3.68]	[2.39,3.20]	0.9
x_8	[2.51,4.04]	[2.52,4.12]	[7.12,11.26]	[4.44,6.91]	[3.06,4.65]	0.7
x_9	[1.24,2.00]	[1.35,1.91]	[3.83,5.31]	[2.13,3.01]	[1.72,2.34]	0.0
x_{10}	[1.00,1.72]	[1.10,1.82]	[3.58,5.65]	[1.67,2.53]	[1.10,1.84]	0.4

Example 5 (*Continuation of Example 4*). We add $\Delta U = \{x_8, x_9, x_{10}\}$ to Table 3, as shown in Table 5. Then, a new object set is denoted as $U' = \{x_1, x_2, \dots, x_{10}\}$. First, from Eq. (17), the δ-fuzzy neighborhood relation matrix $\mathbb{M}_{U^{t_0}}^{R_A^\delta}$ is updated as in Box I. Then, according to Eq. (16), the characteristic vector at time t_1 is computed as $G(\tilde{d}^{t_1}) = [0.5, 0.8, 0.2, 0.2, 1.0, 0.8, 0.9, 0.7, 0.0, 0.4]^T$. Finally, from Definition 10, the fuzzy neighborhood lower and upper vectors of $\tilde{d}^{t_1}$ can be calculated as given in Box II. From Definition 6, the fuzzy neighborhood lower and upper approximations of $\tilde{d}^{t_1}$ are shown as

$$\underline{R_A^\delta}\tilde{d}^{t_1} = \frac{0.5}{x_1} + \frac{0.7039}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.7958}{x_5} + \frac{0.8}{x_6} + \frac{0.6357}{x_7}$$

$$\mathbb{M}_{U^{t_1}}^{R_A^{\delta}} = \left[\begin{array}{c|c} [\hat{m}_{ij}^{1R_A^{\delta}}]_{n\times n} & [\hat{m}_{ij}^{2R_A^{\delta}}]_{n\times n'} \\ \hline [\hat{m}_{ij}^{3R_A^{\delta}}]_{n'\times n} & [\hat{m}_{ij}^{4R_A^{\delta}}]_{n'\times n'} \end{array}\right]$$

$$= \left[\begin{array}{ccccccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0.3643 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0.6324 & 0 & 0 & 0.2961 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.6324 & 0 & 0 & 1 & 0 & 0 & 0.2042 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0.3643 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0.2961 & 0 & 0 & 0.2024 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]_{10\times 10}.$$

Box I.

$$\Phi_{R_A^{\delta}}(\tilde{d}^{t_1}) = (1 - \mathbb{M}_{U^{t_1}}^{R_A^{\delta}}) \circ_{(\vee,\wedge)} G(\tilde{d}^{t_1}) = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 0.6357 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0.3676 & 1 & 1 & 0.7039 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0.3676 & 1 & 1 & 0 & 1 & 1 & 0.7958 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0.6357 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0.7039 & 1 & 1 & 0.7958 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}_{10\times 10} \circ_{(\vee,\wedge)} \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 1.0 \\ 0.8 \\ 0.9 \\ 0.7 \\ 0.0 \\ 0.4 \end{bmatrix}_{10\times 1}$$

$$= \begin{bmatrix} 0.5 \\ 0.7039 \\ 0.2 \\ 0.2 \\ 0.7958 \\ 0.8 \\ 0.6357 \\ 0.7 \\ 0.0 \\ 0.4 \end{bmatrix}_{10\times 1},$$

$$\Psi_{R_A^{\delta}}(\tilde{d}^{t_1}) = \mathbb{M}_{U^{t_1}}^{R_A^{\delta}} \circ_{(\wedge,\vee)} G(\tilde{d}^{t_1}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0.3643 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0.6324 & 0 & 0 & 0.2961 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.6324 & 0 & 0 & 1 & 0 & 0 & 0.2042 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0.3643 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0.2961 & 0 & 0 & 0.2024 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}_{10\times 10} \circ_{(\wedge,\vee)} \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 1.0 \\ 0.8 \\ 0.9 \\ 0.7 \\ 0.0 \\ 0.4 \end{bmatrix}_{10\times 1}$$

$$= \begin{bmatrix} 0.5 \\ 0.8 \\ 0.2 \\ 0.2 \\ 1.0 \\ 0.8 \\ 0.9 \\ 0.7 \\ 0.0 \\ 0.4 \end{bmatrix}_{10\times 1}.$$

Box II.

Table 6
The time complexity of Algorithm 2.

Steps	Time complexity
2–4	$O(\lvert\Delta U\rvert)$
5–15	$O(\lvert U\rvert\lvert\Delta U\rvert+\lvert\Delta U\rvert^2)$
16	$O(\lvert U+\Delta U\rvert^3)$
17	$O(\lvert U+\Delta U\rvert^3)$
Total	$O(\lvert\Delta U\rvert+\lvert U\rvert\lvert\Delta U\rvert+\lvert\Delta U\rvert^2+\lvert U+\Delta U\rvert^3)$

$$
+\frac{0.7}{x_8}+\frac{0.0}{x_9}+\frac{0.4}{x_{10}},
$$

$$
\overline{R_A^\delta}\tilde{d}^{t_1}=\frac{0.5}{x_1}+\frac{0.8}{x_2}+\frac{0.2}{x_3}+\frac{0.2}{x_4}+\frac{1.0}{x_5}+\frac{0.8}{x_6}+\frac{0.9}{x_7}+\frac{0.7}{x_8}+\frac{0.0}{x_9}+\frac{0.4}{x_{10}}.
$$

According to the updating strategies of $G(\tilde{d})$ and $\mathbb{M}_U^{R_B^\delta}$ in this subsection when multiple objects are added to IvIS_FD, we design a matrix-based dynamic algorithm for updating the fuzzy neighborhood approximations as shown in Algorithm 2.

Algorithm 2 Matrix-based dynamic algorithm for updating the fuzzy neighborhood approximations when adding multiple objects to IvIS_FD (MDFNA).

Input:
(1) An $\mathcal{I}^{t_0}=(U^{t_0},A\cup\{\tilde{d}\}^{t_0},V^{t_0},f^{t_0})$ at time t_0 and threshold δ;
(2) The $\Delta U=\{x_{n+1},x_{n+2},\ldots,x_{n+n'}\}$ as the added object set;
(3) The characteristic vector $G(\tilde{d}^{t_0})$ at time t_0, the δ-fuzzy neighborhood relation matrix $\mathbb{M}_{U^{t_0}}^{R_A^\delta}$ at time t_0.

Output: $\underline{R_A^\delta}\tilde{d}^{t_1}$, $\overline{R_A^\delta}\tilde{d}^{t_1}$.

1: Initialize $\mathbb{M}_{U^{t_1}}^{R_A^\delta}\leftarrow\mathbb{M}_{U^{t_0}}^{R_A^\delta}$, $G(\tilde{d}^{t_1})\leftarrow G(\tilde{d}^{t_0})$;
2: **for** $i=n+1$ to $n+n'$ **do**
3: Calculate $G(\tilde{d}^{t_1})=[g_i']_{(n+n')\times 1}$ according to Proposition 1;
4: **end for**
5: **for** $i=n+1$ to $n+n'$ **do**
6: **for** $j=1$ to n **do**
7: Calculate $\mathbb{M}_{U^{t_1}}^{R_A^\delta}=[\hat{m}_{ij}^{R_A^\delta}]_{(n+n')\times(n+n')}$ according to Proposition 2;
8: $\hat{m}_{ji}^{R_A^\delta}=\hat{m}_{ij}^{R_A^\delta}$;
9: **end for**
10: **end for**
11: **for** $i=n+1$ to $n+n'$ **do**
12: **for** $j=n+1$ to $n+n'$ **do**
13: Calculate $\mathbb{M}_{U^{t_1}}^{R_A^\delta}=[\hat{m}_{ij}^{R_A^\delta}]_{(n+n')\times(n+n')}$ according to Proposition 2;
14: **end for**
15: **end for**
16: Compute: $\Phi_{R_A^\delta}(\tilde{d}^{t_1})=(1-\mathbb{M}_{U^{t_1}}^{R_A^\delta})\circ_{(\vee,\wedge)}G(\tilde{d}^{t_1})$;
17: Compute: $\Psi_{R_A^\delta}(\tilde{d}^{t_1})=\mathbb{M}_{U^{t_1}}^{R_A^\delta}\circ_{(\wedge,\vee)}G(\tilde{d}^{t_1})$;
18: **return** $\underline{R_A^\delta}\tilde{d}^{t_1}$, $\overline{R_A^\delta}\tilde{d}^{t_1}$.

The steps in Algorithm 2 are explained as follows. Step 1 is to initialize δ-fuzzy neighborhood relation matrix and characteristic vector. Steps 2–4 is to update the characteristic vector after adding objects. Steps 5–15 is to update the δ-fuzzy neighborhood relation matrix after adding objects. Step 16 is to calculate the fuzzy neighborhood lower vector. Step 17 is to calculate the fuzzy neighborhood upper vector. Step 18 is to return the fuzzy neighborhood approximation sets. Moreover, the time complexity of the main steps in this algorithm are listed in Table 6.

4.2. Updating mechanisms of fuzzy neighborhood rough set when deleting objects

Similar to Section 4.1, we introduce the updating strategies of $G(\tilde{d})$ and $\mathbb{M}_U^{R_B^\delta}$ when multiple objects are deleted from IvIS_FD in this subsection. An illustrative example is given to explain the related theory more clearly. What is more, corresponding dynamic algorithm is designed to update the fuzzy neighborhood approximations.

Table 7
An IvIS_FD after deleting object set.

U	a_1	a_2	a_3	a_4	a_5	$\tilde{d}$
x_1	[2.17,2.86]	[2.45,2.96]	[5.32,7.23]	[3.21,3.95]	[2.54,3.12]	0.5
~~x_2~~	~~[3.37,4.75]~~	~~[3.43,4.85]~~	~~[7.24,10.47]~~	~~[4.00,5.77]~~	~~[3.24,4.70]~~	~~0.8~~
x_3	[1.83,2.70]	[1.78,2.98]	[7.23,10.27]	[2.96,4.07]	[2.06,2.79]	0.2
x_4	[1.35,2.12]	[1.42,2.09]	[2.59,3.93]	[1.87,2.62]	[1.67,2.32]	0.2
x_5	[3.46,5.35]	[3.37,5.11]	[6.37,10.28]	[3.76,5.70]	[3.41,5.28]	1.0
~~x_6~~	~~[2.29,3.43]~~	~~[2.60,3.48]~~	~~[6.71,8.81]~~	~~[3.30,4.23]~~	~~[3.01,3.84]~~	~~0.8~~
x_7	[2.22,3.07]	[2.43,3.32]	[4.37,7.05]	[2.66,3.68]	[2.39,3.20]	0.9

Proposition 3. *Let $\mathcal{I}^{t_0}=(U^{t_0},A\cup\{\tilde{d}\}^{t_0},V^{t_0},f^{t_0})$ be an IvIS_FD at time t_0. At time t_1, the IvIS_FD has been changed to $\mathcal{I}^{t_1}=(U^{t_1},A\cup\{\tilde{d}\}^{t_1},V^{t_1},f^{t_1})$, where $U^{t_1}=U^{t_0}-\Delta U$ and $\Delta U=\{x_{l_1},x_{l_2},\ldots,x_{l_{n'}}\}$ where $1\le l_1<l_2<\cdots<l_{n'}\le n$. $G(\tilde{d}^{t_0})$ is the characteristic vector at time t_0. The updated characteristic vector at time t_1 is denoted as $G(\tilde{d}^{t_1})=[g_i']_{(n-n')\times 1}$, where*

$$
g_i'=\begin{cases} g_{i+k-1}, & l_{k-1}-k+1<i<l_k-k+1,\ 1\le k\le n';\\ g_{i+n'}, & l_{n'}-n'+1\le i\le n-n'. \end{cases} \tag{18}
$$

Proof. According to Definition 8, $G(\tilde{d}^{t_0})$ should be updated to a new characteristic vector $G(\tilde{d}^{t_1})=[g_i']_{(n-n')\times 1}$ when $\Delta U=\{x_{l_1},x_{l_2},\ldots,x_{l_{n'}}\}$ is removed from $\mathcal{I}^{t_0}$. Obviously, for any $l_{k-1}<i<l_k$, where $1\le k\le n'$, the elements g_i in characteristic vector $G(\tilde{d}^{t_0})$ should be shifted forward by $k-1$ positions when x_{l_k} is deleted from $\mathcal{I}^{t_0}$, namely, $g_i'=g_{i+k-1}$. Furthermore, the elements g_i in characteristic vector $G(\tilde{d}^{t_0})$ will be shifted forward by n' positions for any $l_{n'}-n'+1\le i\le n-n'$. That is to say $g_i'=g_{i+n'}$ for any $l_{n'}-n'+1\le i\le n-n'$. □

Proposition 4. *Let $\mathcal{I}^{t_0}=(U^{t_0},A\cup\{\tilde{d}\}^{t_0},V^{t_0},f^{t_0})$ be an IvIS_FD at time t_0. At time t_1, the IvIS_FD has been changed to $\mathcal{I}^{t_1}=(U^{t_1},A\cup\{\tilde{d}\}^{t_1},V^{t_1},f^{t_1})$, where $U^{t_1}=U^{t_0}-\Delta U$ and $\Delta U=\{x_{l_1},x_{l_2},\ldots,x_{l_{n'}}\}$ where $1\le l_1<l_2<\cdots<l_{n'}\le n$. For any $B\subseteq A$, $\mathbb{M}_{U^{t_0}}^{R_B^\delta}=[m_{ij}^{R_B^\delta}]_{n\times n}$ is the δ-fuzzy neighborhood relation matrix with respect to R_B^δ at time t_0. The updated δ-fuzzy neighborhood relation matrix at time t_1 is denoted as $\mathbb{M}_{U^{t_1}}^{R_B^\delta}=[\hat{m}_{ij}^{R_B^\delta}]_{(n-n')\times(n-n')}$, where*

$$
\hat{m}_{ij}^{R_B^\delta}=\begin{cases} m_{(i+k-1,j+k-1)}^{R_B^\delta}, & l_{k-1}-k+1<i,j<l_k-k+1,\\ & 1\le k\le n';\\ m_{(i+n',j+n')}^{R_B^\delta}, & l_{n'}-n'+1\le i,j\le n-n'. \end{cases} \tag{19}
$$

Proof. The proof process is similar to Proposition 3. □

Example 6 (*Continuation of Example 4*)**.** We delete $\Delta U=\{x_2,x_6\}$ from Table 3, as shown in Table 7. Then, a new object set is denoted as $U'=\{x_1,x_3,x_4,x_5,x_7\}$. First, according to Eq. (19), the δ-fuzzy neighborhood relation matrix $\mathbb{M}_{U^{t_0}}^{R_A^\delta}$ is updated as

$$\mathbb{M}_{U^{t_1}}^{R_A^\delta}=\begin{bmatrix} 1 & \cancel{0} & 0 & 0 & 0 & \cancel{0} & 0.3643\\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0.6324} & \cancel{0} & \cancel{0}\\ 0 & \cancel{0} & 1 & 0 & 0 & \cancel{0} & 0\\ 0 & \cancel{0} & 0 & 1 & 0 & \cancel{0} & 0\\ 0 & \cancel{0.6324} & 0 & 0 & 1 & \cancel{0} & 0\\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0}\\ 0.3643 & \cancel{0} & 0 & 0 & 0 & \cancel{0} & 1 \end{bmatrix}$$
$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0.3643\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0.3643 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5}.$$

Then, according to Eq. (18), the characteristic vector at time t_1 is computed as $G(\tilde{d}^{t_1}) = [0.5, 0.2, 0.2, 1.0, 0.9]^T$. Finally, from Definition 10, the fuzzy neighborhood lower and upper vectors of $\tilde{d}^{t_1}$ can be calculated as

$$\Phi_{R_A^\delta}(\tilde{d}^{t_1}) = (1-\mathbb{M}_{U^{t_1}}^{R_A^\delta})\circ_{(\vee,\wedge)} G(\tilde{d}^{t_1})$$
$$=\begin{bmatrix} 0 & 1 & 1 & 1 & 0.6357\\ 1 & 0 & 1 & 1 & 1\\ 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 1 & 0 & 1\\ 0.6357 & 1 & 1 & 1 & 0 \end{bmatrix}_{5\times 5}\circ_{(\vee,\wedge)}\begin{bmatrix}0.5\\0.2\\0.2\\1.0\\0.9\end{bmatrix}_{5\times 1}$$
$$=\begin{bmatrix}0.5\\0.2\\0.2\\1\\0.6357\end{bmatrix}_{5\times 1},$$

$$\Psi_{R_A^\delta}(\tilde{d}) = \mathbb{M}_{U^{t_1}}^{R_A^\delta}\circ_{(\wedge,\vee)} G(\tilde{d}^{t_1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.3643\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0.3643 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times 5}$$
$$\circ_{(\wedge,\vee)}\begin{bmatrix}0.5\\0.2\\0.2\\1.0\\0.9\end{bmatrix}_{5\times 1}=\begin{bmatrix}0.5\\0.2\\0.2\\1.0\\0.9\end{bmatrix}_{5\times 1}.$$

According to Definition 6, the fuzzy neighborhood lower and upper approximations of $\tilde{d}^{t_1}$ are shown as

$$\underline{R_A^\delta}\tilde{d}^{t_1} = \frac{0.5}{x_1}+\frac{0.2}{x_3}+\frac{0.2}{x_4}+\frac{1.0}{x_5}+\frac{0.6357}{x_7},$$
$$\overline{R_A^\delta}\tilde{d}^{t_1} = \frac{0.5}{x_1}+\frac{0.2}{x_3}+\frac{0.2}{x_4}+\frac{1.0}{x_5}+\frac{0.9}{x_7}.$$

In this subsection, we discuss the updating strategies of $G(\tilde{d})$ and $\mathbb{M}_U^{R_B^\delta}$ when many objects are removed from the original IvIS_FD. Next, we introduce a corresponding incremental algorithm (see Algorithm 3) based on these updating strategies for updating the fuzzy neighborhood approximations.

The steps in Algorithm 3 are explained as follows. Steps 1–6 is to update the δ-fuzzy neighborhood relation matrix and characteristic vector after objects are deleted. Step 7 is to calculate the fuzzy neighborhood lower vector after objects are deleted. Step 8 is to calculate the fuzzy neighborhood upper vector after objects are deleted. Step 9 is to return the fuzzy neighborhood approximation sets. In addition, the time complexity of the main steps in this algorithm are listed in Table 8.

Algorithm 3 Matrix-based dynamic algorithm for updating the fuzzy neighborhood approximations when deleting multiple objects from IvIS_FD (MDFND).

Input:
(1) An $\mathcal{I}^{t_0} = (U^{t_0}, A\cup\{\tilde{d}\}^{t_0}, V^{t_0}, f^{t_0})$ at time t_0 and threshold δ;
(2) The $\Delta U = \{x_{l_1}, x_{l_2}, \ldots, x_{l_{n'}}\}$ is an deleted object set;
(3) The characteristic vector $G(\tilde{d}^{t_0})$ at time t_0, the δ-fuzzy neighborhood relation matrix $\mathbb{M}_{U^{t_0}}^{R_A^\delta}$ at time t_0.

Output: $\underline{R_A^\delta}\tilde{d}^{t_1}$, $\overline{R_A^\delta}\tilde{d}^{t_1}$.

1: **for** $k=1$ to n' **do**
2: $\quad\mathbb{M}_{U^{t_0}}^{R_A^\delta}(l_k, \ :\ ) = []$;
3: $\quad\mathbb{M}_{U^{t_0}}^{R_A^\delta}(\ :\ , l_k) = []$;
4: $\quad G(\tilde{d}^{t_0})(l_k, \ :\ ) = []$;
5: **end for**
6: $\mathbb{M}_{U^{t_1}}^{R_A^\delta} \leftarrow \mathbb{M}_{U^{t_0}}^{R_A^\delta}$, $G(\tilde{d}^{t_1}) \leftarrow G(\tilde{d}^{t_0})$;
7: Compute: $\Phi_{R_A^\delta}(\tilde{d}^{t_1}) = (1-\mathbb{M}_{U^{t_1}}^{R_A^\delta})\circ_{(\vee,\wedge)} G(\tilde{d}^{t_1})$;
8: Compute: $\Psi_{R_A^\delta}(\tilde{d}^{t_1}) = \mathbb{M}_{U^{t_1}}^{R_A^\delta}\circ_{(\wedge,\vee)} G(\tilde{d}^{t_1})$;
9: **return** $\underline{R_A^\delta}\tilde{d}^{t_1}$, $\overline{R_A^\delta}\tilde{d}^{t_1}$.

Table 8
The time complexity of Algorithm 3.

Steps	Time complexity
1–6	$O(\lvert\Delta U\rvert)$
7	$O(\lvert U-\Delta U\rvert^3)$
8	$O(\lvert U-\Delta U\rvert^3)$
Total	$O(\lvert\Delta U\rvert+\lvert U-\Delta U\rvert^3)$

Table 9
The summary of data set.

No.	Data sets	Abbreviation	Samples	Features
1	Iris	Iris	150	4
2	Ecoli	Ecoli	336	7
3	Housing	Hous	506	13
4	Cloud	Cloud	1024	10
5	Banknote Authentication	Bank	1372	4
6	Yeast	Yeast	1484	8
7	Wine Quality-red	Wred	1599	11
8	Abalone	Abalone	4177	7
9	Wine Quality-white	Wite	4898	11

5. Experimental evaluations

In this section, we prove the performance of the dynamic fuzzy neighborhood rough set model proposed in this paper through a series of comparative experiments. Performance evaluation is mainly explained from two aspects. On the one hand, the effectiveness of the dynamic model is verified by the effectiveness analysis of algorithms MDFNA and MDFND. On the other hand, the efficiency of the dynamic model is demonstrated by the efficiency analysis of algorithms MDFNA and MDFND.

5.1. Preparation before experiment

In this subsection, we mainly introduce the method of constructing IvIS_FD from numerical data sets, the compared algorithms, and the running environment of experiment.

As we all know, very few real interval-valued data sets are publicly available, so we cannot obtain them. In [17,24,38,44,50–52], the interval-valued data sets are obtained from numerical data sets according to different data preprocessing methods. Therefore, we first download nine numerical data sets from UCI. Specific information about the original numerical data sets is shown in Table 9.

Then, we adopt the data preprocessing method in [17]. Let (U, A, V, f) be a numerical information system. For any $x \in U$ and $a \in A$, $f^{-}(x, a) = f(x, a) - 2\theta_a$, $f^{+}(x, a) = f(x, a) + 2\theta_a$, where θ_a is the standard deviation of V_a. Therefore, the original real-valued $f(x, a)$ becomes the current interval-valued $[f^{-}(x, a), f^{+}(x, a)]$. Consequently, the IvIS is generated from the original numerical information system. In addition, the fuzzy decision of IvIS is a fuzzy set about U, which can be generated by Matlab R2014a.

In this paper, in addition to algorithm MSFN, we also adopt three related algorithms proposed by other scholars as comparative algorithms to evaluate the performance of algorithm MDFNA and MDFND. The three related algorithms are introduced as follows.

- Algorithm IVPR. Chen and Qin defined the variable precision tolerance relation based on similarity and proposed a rough set model under the background of IvIS [46]. We design the algorithm corresponding to the model according to the relevant theory, which is called algorithm IVPR. The parameter α involved in the model is preset to 0.9.
- Algorithm IPSD. Dai et al. proposed the similarity measure between two intervals based on possible degree in IvDS, and defined the θ-similarity classes of objects. Then, the expressions of upper and lower approximations were given in IvDS [16]. We design algorithm IPSD according to the relevant information to obtain the corresponding upper and lower approximation sets. The parameter θ is preset to 0.9.
- Algorithm IWSD. For the incomplete IvIS, Dai et al. defined the α-weak similarity relation and α-weak θ similarity class, and proposed a novel rough set model for the incomplete IvIS [19]. We design the algorithm IWSD according to the relevant knowledge. Parameters α and θ in the model are preset to 0.6 and 0.9, respectively.

The parameter δ in each algorithms proposed in this study is set as 0.9. All the algorithms are implemented by Matlab R2014a, and all the comparative experiments are executed on a personal computer with Intel(R) Core(TM) i5-10210U CPU 1.60 GHz, 8.0 GB of memory.

5.2. The effectiveness analysis of algorithms MDFNA and MDFND

For different data sets, we show the effectiveness of algorithms MDFNA and MDFND by comparing the approximation accuracy and running time of different algorithms under a fixed incremental ratio in this section. Approximate accuracy is used as an evaluation index of validity, which is defined by $Acc = \sum_{x \in U} lower(x) / \sum_{x \in U} upper(x)$.

In the experiment of data dynamic increase, we take the first 50% of the objects from each data set in Table 9 as the original object set, and the remaining objects as the newly added objects. In the experiment of data dynamic deletion, the object set of each data set in Table 9 is taken as the original objects, and then 50% of the objects from each data set are randomly selected as the deleted objects. Subsequently, we use different algorithms to compute the upper and lower approximate sets corresponding to different rough set models. Then record the approximate accuracy and running time of each experiment separately. Finally, the experimental results of approximate accuracy and running time are shown in Tables 10 and 11 respectively.

For each data set, Table 10 shows that the approximation accuracy of algorithm MDFNA (MDFND) is not only equal to that of algorithm MSFN, but also higher than that of the other three algorithms. In addition, it can be seen from Table 11 that as the scale of objects in data set continues to increase, the running time of all algorithms also increases. But compared with other algorithms, algorithm MDFNA (MDFND) are always faster than other algorithms when the data changes dynamically. These phenomena fully demonstrate the effectiveness of the dynamic algorithms.

5.3. The efficiency analysis of algorithms MDFNA and MDFND

In this subsection, we compare the running time of dynamic algorithm MDFNA (MDFND) with the other four algorithms under different increment ratios to illustrate the efficiency of dynamic fuzzy neighborhood rough set when objects in the data sets change dynamically. Detailed experimental design and analysis are as follows.

In the experiment of data dynamic increase, we select the first 50% of the objects from each data set in Table 9 as the original data set, and then divide the remaining objects into five equal parts and add them to the original data set in turn, that is, 20%, 40%, …, 100% of the remaining objects are added to the original data set. Subsequently, we run different algorithms and record the running time of each experiment. The experimental results are shown in Fig. 1, where the abscissa represents the ratio of adding objects, and the ordinate represents the calculation time (in seconds) to obtain rough approximations. In the experiment of data dynamic deletion, we take the object set of each data set in Table 9 as the original objects, and then randomly delete 10%, 20%, …, 50% of them. Similar to the experiment of data dynamic increase, we run five different algorithms and record the running time. The experimental results are shown in Fig. 2, where the abscissa represents the ratio of deleting objects, and the ordinate represents the calculation time (in seconds) to obtain the approximation sets. Different colors and marked broken lines are used to represent the running time of different algorithms. Please refer to the legend for corresponding information.

As can be seen from Fig. 1, the running time of all algorithms increases as the ratio of adding objects increases. In addition, the running time of algorithm MDFNA is always lower than that of the other four algorithms. Similarly, we can see from Fig. 2 that the running time of all algorithms decreases as the ratio of deleting objects increases. At the same time, the running time of algorithm MDFND is always lower than that of the other four algorithms. The main reason for this phenomenon is that algorithms MDFNA and MDFND take the δ-fuzzy neighborhood relation matrix and characteristic vector of the initial data set as the initial information, and update the initial information after adding or deleting objects, thus saving a lot of repeated calculation time.

5.4. Summary

The results of experimental evaluations show that algorithms MDFNA and MDFND not only shorten the time of obtaining approximation sets in dynamic environment, but also have higher approximation accuracy than other algorithms. Therefore, we can draw the following conclusion: the proposed dynamic fuzzy neighborhood rough set for IvIS_FD is effective and efficient.

6. Conclusions

This paper proposes a novel fuzzy neighborhood rough set model for mining knowledge from IvIS_FD, and constructs the incremental update approximation mechanisms based on the proposed model for dynamic IvIS_FD. Firstly, a quantified δ-fuzzy neighborhood relation and its induced fuzzy information granule are defined. Secondly, the δ-fuzzy neighborhood rough set model and its matrix representation are investigated. Subsequently, two kind of incremental mechanisms and corresponding dynamic algorithms are introduced to update fuzzy neighborhood approximations when multiple objects are added to or deleted from an IvIS_FD, respectively. Finally, a series of comparative experiments are performed to show the effectiveness and efficiency of the proposed model.

Table 10
The comparison of approximation accuracy of different algorithms in dynamic environment.

Data set	The addition of objects					The deletion of objects				
	IVPR	IPSD	IWSD	MSFN	MDFNA	IVPR	IPSD	IWSD	MSFN	MDFND
Iris	0.3871	0.0385	0.0385	0.7727	0.7727	0.5385	0.0968	0.1200	0.8426	0.8426
Ecoli	0.6765	0.2364	0.1000	0.9228	0.9228	0.7368	0.4211	0.2143	0.9693	0.9693
Hous	0.8750	0.4375	0.3111	0.9478	0.9478	0.7917	0.8065	0.5806	0.9530	0.9530
Cloud	0.3333	0.0217	0.0154	0.7092	0.7092	0.5000	0.0368	0.0187	0.8285	0.8285
Bank	0.0089	0.0008	0.0008	0.2120	0.2120	0.0410	0.0060	0.0026	0.3463	0.3463
Yeast	0.8774	0.2752	0.1851	0.9537	0.9537	0.9341	0.3894	0.2072	0.9633	0.9633
Wred	0.5602	0.4245	0.3712	0.7948	0.7948	0.7065	0.6947	0.5818	0.8845	0.8845
Abalone	0.0285	0.0029	0.0021	0.2454	0.2454	0.0500	0.0044	0.0023	0.3262	0.3262
Wite	0.4514	0.3395	0.2790	0.7280	0.7280	0.6075	0.5413	0.4907	0.8374	0.8374

Table 11
The comparison of running time of different algorithms in dynamic environment (in seconds).

Data set	The addition of objects					The deletion of objects				
	IVPR	IPSD	IWSD	MSFN	MDFNA	IVPR	IPSD	IWSD	MSFN	MDFND
Iris	0.156	0.297	0.141	0.172	0.141	0.031	0.031	0.047	0.047	0.016
Ecoli	1.063	1.063	1.156	1.344	0.750	0.234	0.313	0.266	0.328	0.094
Hous	3.516	3.703	3.984	4.281	2.297	0.813	0.859	0.938	1.031	0.156
Cloud	11.203	11.859	12.453	13.656	8.094	2.734	2.797	3.094	3.391	0.625
Bank	10.797	10.219	10.016	15.156	9.500	2.578	2.453	2.516	3.656	1.109
Yeast	18.328	18.281	20.156	23.313	14.047	4.531	4.625	4.969	5.938	1.219
Wred	27.625	27.563	29.203	32.922	19.453	6.688	6.875	7.516	8.172	1.406
Abalone	130.609	131.172	143.078	169.625	104.703	32.984	32.563	35.813	42.938	9.516
Wite	266.078	262.906	277.375	307.516	180.063	64.594	64.563	69.094	77.047	13.188

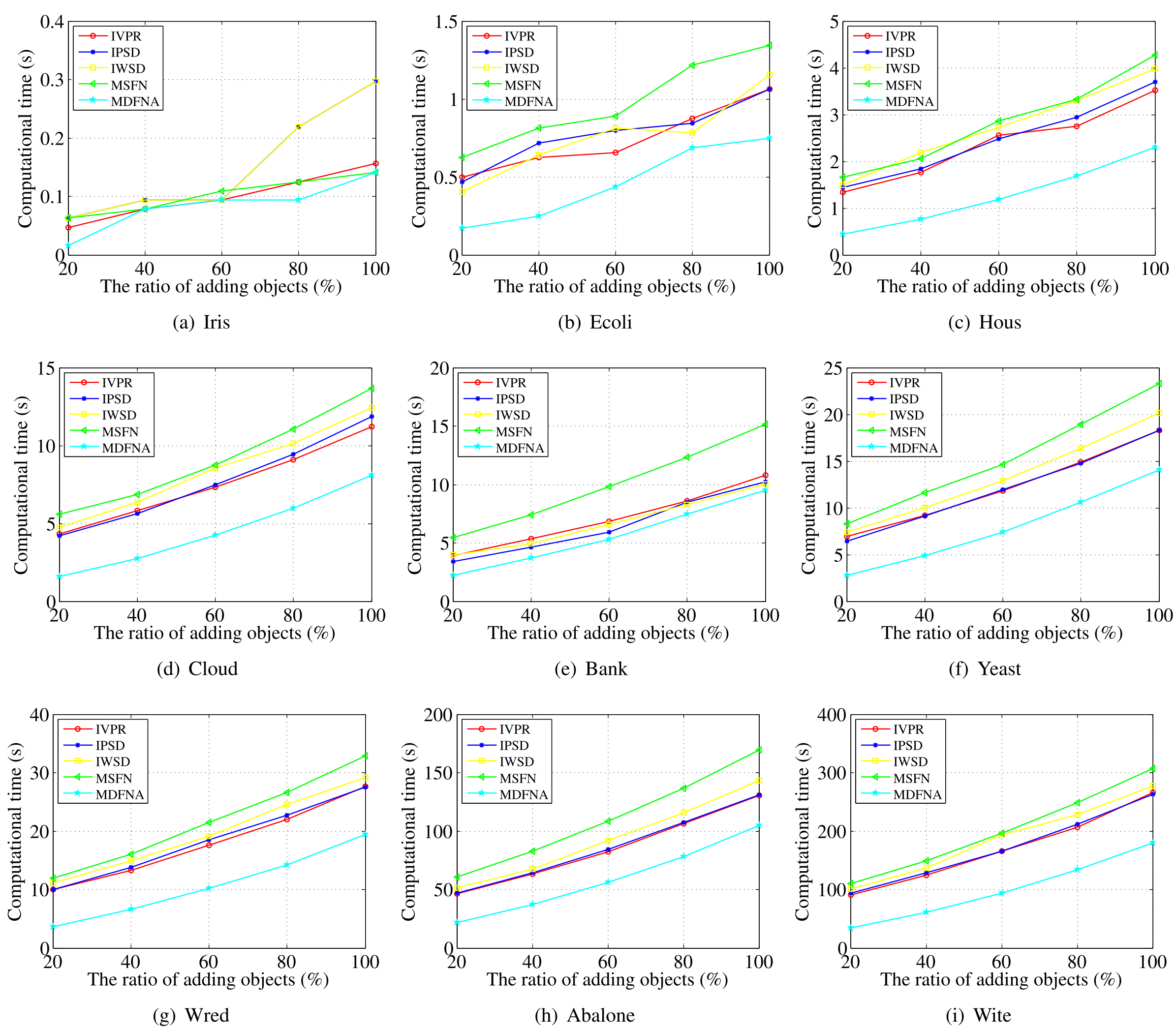

Fig. 1. The computational time of different algorithms versus different ratios of adding objects.

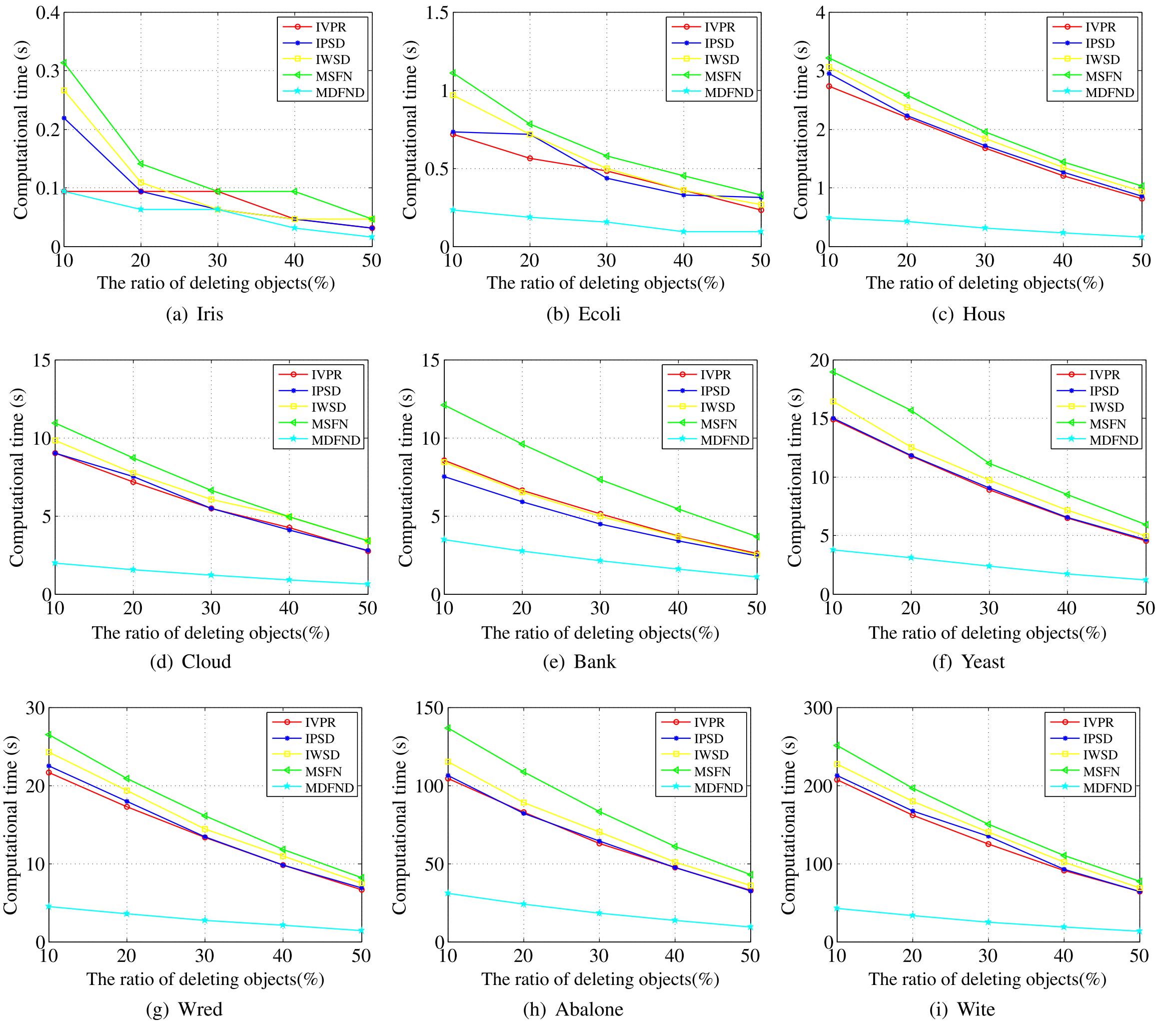

Fig. 2. The computational time of different algorithms versus different ratios of deleting objects.

6.1. The theoretical and practical implications

In Section 3, the δ-fuzzy neighborhood relation has been proposed to realize the quantitative description of binary relation between objects. Compared with the variable precision compatibility relation mentioned in [46], the proposed quantitative relation can accurately describe the degree of similarity between objects from [0, 1], while the former is a Boolean relation that can only describe the relation between objects qualitatively from {0, 1}. On the other hand, the incremental update mechanisms proposed in Section 4 take the δ-fuzzy neighborhood relation matrix and the characteristic vector at time t_0 as the prior knowledge to update rough approximations at time t_1. Theoretically, this can reduce the time for us to obtain approximations, and the time complexity of the three algorithms can confirm this point.

Nine publicly available data sets from UCI are used for numerical comparative experiments. The experimental results have demonstrated the effectiveness and efficiency of incremental algorithms MDFNA and MDFND to update the fuzzy neighborhood approximations as compared to algorithm MSFN. Meanwhile, we have also compared our model with other three related models proposed by other scholars in approximation accuracy. Accuracy results illustrate that our method performs quite better than other methods for acquiring rough approximations from IvIS_FD.

6.2. The weaknesses of the proposed method and future works

Some weaknesses of the proposed rough set model and dynamic algorithms are introduced as follows. First, although we introduce a function (i.e. Eq. (3)) to calculate the similarity degree of objects in interval-valued data, there may be many functions to describe the similarity degree. We should discuss them and study the choice of parameter in the proposed model. Second, the approximations of the proposed model are represented by the matrix method. Although the matrix representation method can simplify the calculation process and intuitively express the construction of the method, the calculation space occupied by the matrix operation is relatively large, which may not be suitable for large-scale data knowledge acquisition. Third, the proposed dynamic algorithm acquires new knowledge on the basis of original knowledge. The dynamic algorithm avoids the repeated calculation of the original knowledge, so it can reduce the time cost. However, the storage of original knowledge requires a certain amount of storage space, which increases the storage space requirements of computing devices.

In future work, on the one hand, the above-mentioned weaknesses will be emphatically studied and overcome. On the other hand, some research gaps will continue to be studied. Firstly, we can further consider the updating mechanisms of fuzzy neighborhood rough set when the attributes or attribute values change with time. Secondly, missing attribute values are inevitable in some actual databases. Therefore, it is necessary to construct an appropriate rough set model for incomplete IvIS_FD to acquire knowledge. Thirdly, in practical applications, the collected data are often inevitably polluted with noise due to the influence of assorted unstable factors in the process of data collection, storage and transmission. Hence, a robust model should be further proposed to combat noise interference.

CRediT authorship contribution statement

Lei Yang: Methodology, Validation, Writing - original draft, Writing - review & editing, Software, Data curation. **Keyun Qin:** Conceptualization, Resources, Visualization, Supervision, Project administration, Funding acquisition. **Binbin Sang:** Supervision, Methodology. **Weihua Xu:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (No. 61976130).

References

[1] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[2] S. Roy, P. Shivakumara, N. Jain, V. Khare, A. Dutta, U. Pal, T. Lu, Rough-fuzzy based scene categorization for text detection and recognition in video, Pattern Recognit. 80 (2018) 64–82.
[3] H.H. Inbarani, A.T. Azar, J. Ganesan, Supervised hybrid feature selection based on PSO and rough sets for medical diagnosis, Comput. Methods Programs Biomed. 113 (1) (2014) 175–185.
[4] J.Y. Shyng, H.M. Shieh, G.H. Tzeng, An integration method combining Rough Set Theory with formal concept analysis for personal investment portfolios, Knowl.-Based Syst. 23 (6) (2010) 586–597.
[5] S. Greco, B. Matarazzo, R. Slowinski, Rough approximation of a preference relation by dominance relations, European J. Oper. Res. 117 (1) (1999) 63–83.
[6] D.W. Pei, A generalized model of fuzzy rough sets, Int. J. Gen. Syst. 34 (5) (2005) 603–613.
[7] Q.H. Hu, D.R. Yu, J.F. Liu, C.X. Wu, Neighborhood rough set based heterogeneous feature subset selection, Inform. Sci. 178 (18) (2008) 3577–3594.
[8] W. Ziarko, Analysis of uncertain information in the framework of variable precision rough sets, Found. Comput. Decis. Sci. 18 (1993) 381–396.
[9] C.Z. Wang, M.W. Shao, Q. He, Y.H. Qian, Y.L. Qi, Feature subset selection based on fuzzy neighborhood rough sets, Knowl.-Based Syst. 111 (2016) 173–179.
[10] B.B. Sang, L. Yang, H.M. Chen, W.H. Xu, Y.T. Guo, Z. Yuan, Generalized multi-granulation double-quantitative decision-theoretic rough set of multi-source information system, Internat. J. Approx. Reason. 115 (2019) 157–179.
[11] J. Ye, J.M. Zhan, W.P. Ding, H. Fujita, A novel fuzzy rough set model with fuzzy neighborhood operators, Inform. Sci. 544 (2021) 266–297.
[12] B.Z. Sun, Z.T. Gong, D.G. Chen, Fuzzy rough set theory for the interval-valued fuzzy information systems, Inform. Sci. 178 (13) (2008) 2794–2815.
[13] D.D. Yang, T. Deng, H. Fujita, Partial-overall dominance three-way decision models in interval-valued decision systems, Internat. J. Approx. Reason. 126 (2020) 308–325.
[14] Y.H. Qian, J.Y. Liang, C.Y. Dang, Interval ordered information systems, Comput. Math. Appl. 56 (8) (2008) 1994–2009.
[15] X.B. Yang, D.J. Yu, J.Y. Yang, L.H. Wei, Dominance-based rough set approach to incomplete interval-valued information system, Data Knowl. Eng. 68 (11) (2009) 1331–1347.
[16] J.H. Dai, W.T. Wang, J.S. Mi, Uncertainty measurement for interval-valued information systems, Inform. Sci. 251 (2013) 63–78.
[17] Y. Leung, M.M. Fischer, W.Z. Wu, J.S. Mi, A rough set approach for the discovery of classification rules in interval-valued information systems, Internat. J. Approx. Reason. 47 (2) (2008) 233–246.
[18] X. Zhang, C.L. Mei, D.G. Chen, J.H. Li, Multi-confidence rule acquisition and confidence-preserved attribute reduction in interval-valued decision systems, Internat. J. Approx. Reason. 55 (8) (2014) 1787–1804.
[19] J.H. Dai, B.J. Wei, X.H. Zhang, Q.L. Zhang, Uncertainty measurement for incomplete interval-valued information systems based on α-weak similarity, Knowl.-Based Syst. 136 (2017) 159–171.
[20] X. Ma, Measures associated with granularity and rough approximations in interval-valued information tables based on kernel similarity relations, Inform. Sci. 538 (2020) 337–357.
[21] J.H. Yu, W.H. Xu, Incremental knowledge discovering in interval-valued decision information system with the dynamic data, Int. J. Mach. Learn. Cybern. 8 (3) (2017) 849–864.
[22] C. Luo, T.R. Li, Z. Yi, H. Fujita, Matrix approach to decision-theoretic rough sets for evolving data, Knowl.-Based Syst. 99 (2016) 123–134.
[23] D. Liu, T.R. Li, J.B. Zhang, A rough set-based incremental approach for learning knowledge in dynamic incomplete information systems, Internat. J. Approx. Reason. 55 (8) (2014) 1764–1786.
[24] J.H. Yu, M.H. Chen, W.H. Xu, Dynamic computing rough approximations approach to time-evolving information granule interval-valued ordered information system, Appl. Soft Comput. 60 (2017) 18–29.
[25] T.R. Li, D. Ruan, G. Wets, J. Song, Y. Xu, A rough sets based characteristic relation approach for dynamic attribute generalization in data mining, Knowl.-Based Syst. 20 (5) (2007) 485–494.
[26] C. Luo, T.R. Li, H.M. Chen, H. Fujita, Z. Yi, Incremental rough set approach for hierarchical multicriteria classification, Inform. Sci. 429 (2018) 72–87.
[27] H.M. Chen, T.R. Li, D. Ruan, J.H. Lin, C.X. Hu, A rough-set-based incremental approach for updating approximations under dynamic maintenance environments, IEEE Trans. Knowl. Data Eng. 25 (2) (2013) 274–284.
[28] S. Wang, T.R. Li, C. Luo, H.M. Chen, H. Fujita, Domain-wise approaches for updating approximations with multi-dimensional variation of ordered information systems, Inform. Sci. 478 (2019) 100–124.
[29] S. Wang, T.R. Li, C. Luo, H. Fujita, Efficient updating rough approximations with multi-dimensional variation of ordered data, Inform. Sci. 372 (2016) 690–708.
[30] H.M. Chen, T.R. Li, D. Ruan, Maintenance of approximations in incomplete ordered decision systems while attribute values coarsening or refining, Knowl.-Based Syst. 31 (2012) 140–161.
[31] S.Y. Li, T.R. Li, D. Liu, Incremental updating approximations in dominance-based rough sets approach under the variation of the attribute set, Knowl.-Based Syst. 40 (2013) 17–26.
[32] J.B. Zhang, T.R. Li, D. Ruan, D. Liu, Rough sets based matrix approaches with dynamic attribute variation in set-valued information systems, Internat. J. Approx. Reason. 53 (4) (2012) 620–635.
[33] Q.H. Zhang, G.X. Lv, Y.H. Chen, G.Y. Wang, A dynamic three-way decision model based on the updating of attribute values, Knowl.-Based Syst. 142 (2018) 71–84.
[34] C.X. Hu, S.X. Liu, G.X. Liu, Matrix-based approaches for dynamic updating approximations in multigranulation rough sets, Knowl.-Based Syst. 122 (2017) 51–63.
[35] C. Luo, T.R. Li, H.M. Chen, H. Fujita, Z. Yi, Efficient updating of probabilistic approximations with incremental objects, Knowl.-Based Syst. 109 (2016) 71–83.
[36] C.X. Hu, L. Zhang, Incremental updating probabilistic neighborhood three-way regions with time-evolving attributes, Internat. J. Approx. Reason. 120 (2020) 1–23.
[37] C.X. Hu, L. Zhang, B.J. Wang, Z. Zhang, F.Z. Li, Incremental updating knowledge in neighborhood multigranulation rough sets under dynamic granular structures, Knowl.-Based Syst. 163 (2019) 811–829.
[38] Y.Y. Zhang, T.R. Li, C. Luo, J.B. Zhang, H.M. Chen, Incremental updating of rough approximations in interval-valued information systems under attribute generalization, Inform. Sci. 373 (2016) 461–475.
[39] J. Hu, T.R. Li, C. Luo, H. Fujita, S.Y. Li, Incremental fuzzy probabilistic rough sets over two universes, Internat. J. Approx. Reason. 81 (2017) 28–48.
[40] A.P. Zeng, T.R. Li, J. Hu, H.M. Chen, C. Luo, Dynamical updating fuzzy rough approximations for hybrid data under the variation of attribute values, Inform. Sci. 378 (2017) 363–388.
[41] C.X. Hu, S.X. Liu, X.L. Huang, Dynamic updating approximations in multigranulation rough sets while refining or coarsening attribute values, Knowl.-Based Syst. 130 (2017) 62–73.
[42] Y.Y. Huang, T.R. Li, C. Luo, H. Fujita, S. Horng, Matrix-based dynamic updating rough fuzzy approximations for data mining, Knowl.-Based Syst. 119 (2017) 273–283.
[43] Y.T. Guo, E.C.C. Tsang, M. Hu, X.X. Lin, D.G. Chen, W.H. Xu, B.B. Sang, Incremental updating approximations for double-quantitative decision-theoretic rough sets with the variation of objects, Knowl.-Based Syst. 189 (2020) http://dx.doi.org/10.1016/j.knosys.2019.105082.
[44] Q.Q. Huang, T.R. Li, Y.Y. Huang, X. Yang, H. Fujita, Dynamic dominance rough set approach for processing composite ordered data, Knowl.-Based Syst. 187 (2020) 104829.

[45] S. Wang, T.R. Li, C. Luo, J. Hu, H. Fujita, T.Q. Huang, A novel approach for efficient updating approximations in dynamic ordered information systems, Inform. Sci. 507 (2020) 197–219.
[46] Z.C. Chen, K.Y. Qin, Attribute reduction of interval-valued information system based on variable precision tolerance relation, Comput. Sci. 36 (3) (2009) 163–166, (in Chinese).
[47] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (2–3) (1990) 191–209.
[48] R.E. Moore, R.B. Kearfott, M.J. Cloud, Introduction to Interval Analysis, Society for Industrial and Applied Mathematics, Philadelphia, 2009.
[49] X.B. Yang, Y. Qi, D.J. Yu, H.L. Yu, J.Y. Yang, α-Dominance relation and rough sets in interval-valued information systems, Inform. Sci. 294 (2015) 334–347.
[50] Y.Y. Huang, T.R. Li, C. Luo, H. Fujita, S.J. Horng, Dynamic fusion of multisource interval-valued data by fuzzy granulation, IEEE Trans. Fuzzy Syst. 26 (6) (2018) 3403–3417.
[51] W.S. Du, B.Q. Hu, Approximate distribution reducts in inconsistent interval-valued ordered decision tables, Inform. Sci. 271 (2014) 93–114.
[52] B.Y. Lin, X.Y. Zhang, W.H. Xu, Y.X. Wu, Dynamically updating approximations based on multi-threshold tolerance relation in incomplete interval-valued decision information systems, Knowl. Inf. Syst. 62 (3) (2020) 1063–1087.