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# Two types of coverings based multigranulation rough fuzzy sets and applications to decision making

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**Abstract**
Covering based multigranulation rough fuzzy set, as a generalization of granular computing and covering based rough fuzzy set theory, is a vital tool for dealing with the vagueness and multigranularity in artificial intelligence and management sciences. By means of neighborhoods, we introduce two types of coverings based (optimistic, pessimistic and variable precision) multigranulation rough fuzzy set models, respectively. Some axiomatic systems are also obtained. The relationships between two types of coverings based (optimistic, pessimistic and variable precision) multigranulation rough fuzzy set models are established. Based on the theoretical discussion for the covering based multigranulation rough fuzzy set models, we present an approach to multiple criteria group decision making problem. These two types of basic models and the procedure of decision making methods as well as the algorithm for the new approach are given in detail. By comparative analysis, the ranking results based on two different models have a highly consensus. Although there exist some different ranking results on these two methods, the optimal selected alternative is the same.

**Keyword** Multigranulation rough set · Covering based (optimistic, pessimistic and variable precision) multigranulation rough fuzzy set · Neighborhood · Covering based rough fuzzy set · Multiple criteria group decision making

## 1 Introduction

Rough set theory (briefly, RST) was firstly mentioned by Pawlak (1982) as a useful approach to copy with uncertain information. Until now, this theory has been applied to many different research fields, such as control, knowledge discovery, data analysis, pattern recognition, information process, granular computing, and so on (for examples, see Abu-Donia 2012; Greco et al. 2001; Jensen and Shen 2007; Yao 2010, 2016). It is well known that Pawklak's

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model builds on an equivalence relation that characterizes a partition formed by classes of indiscernible object which causes the limited applications. Due to the reason, many generalized rough set models has been established by many researchers in these years, such as, rough fuzzy set and fuzzy rough set models, general binary relations, neighborhood rough sets, probabilistic rough sets, decision-theoretic rough sets, soft rough sets, rough soft sets, and so on (see Chen et al. 2014; Dubois and Prade 1990; Qian et al. 2014b; Sun and Ma 2015b, 2017; Wu and Zhang 2004; Yao 1998; Yeung et al. 2015; Zhan et al. 2017a, b). Now it is widely accepted that these general rough set models provide very useful and flexible environments to cope with roughness and vagueness in a more precise manner.

As one of the extended models, covering based rough set model (briefly, CRS-model) is a vital research topic of generalized RST. CRS is a very powerful tool that enables the researcher to study data mining in a more general manner. Nowadays, many researchers investigated this topic in recent thirty years.

After Dubois and Prade (1990) proposed two classes of important concepts of rough fuzzy sets (briefly, RFSs) and fuzzy rough sets (briefly, FRSs) by combining rough sets and fuzzy sets, many researchers proposed many generalized FRS-models. At the same time, some scholars generalized CRS-models to fuzzy covering-based rough sets (briefly, FRC) by combining fuzzy sets and CRSs.

Zadeh (1997) introduced the concept of granular computing and analysed fuzzy information granulation. Nowadays, this theory plays an important role in soft computing, data mining, information systems, machine learning, and so on, for examples, see Yao (2005), Pedrycz (2013), Pedrycz et al. (2008) and Zhang et al. (2016). Qian et al. (2010b) extended Pawlak's single granular rough set models to multigranulation rough set models (briefly, MGRS-models). For more details, see Qian et al. (2010a, 2014a, b). Nowadays, MGRS has become an attractive topic in AI and management sciences and has attracted many researchers to engage in this area from theoretical and applied aspects, for examples, see Huang et al. (2014), Lin et al. (2012, 2013), Sun and Ma (2015a), She and He (2012), Xu and Leung (1998), Xu et al. (2012, 2013), Yang et al. (2012a, b) and Yao and She (2015) and Zhang et al. (2017, 2018).

Group decision-making (briefly, GDM) attempts to provide solutions to solve such complex real-world problems. However in real life, many decision-making problems in fields like industrial engineering, management science or operational research usually require multiple criteria decision making (briefly, MCDM). Consequently, to tackle both settings multiple criteria group decision making (briefly, MCGDM) or pursues the selection or ranking of the feasible alternatives by a group, in the presence of conflicting and interactive criteria.

In continuation of these research efforts we proceed to investigate the MCGDM problem. By viewing existing studies, it appears that there is a lack of investigation on the applications in multiple criteria group decision making methods by covering based multigranulation rough fuzzy set models. This motivates the present paper on covering based multigranulation rough fuzzy set models, as well as their applications in multiple criteria group decision making. Based on the theoretical discussion for the covering based multigranulation rough fuzzy set models, we present an approach to multiple criteria group decision making problem. These two types of basic models and the procedure of decision making methods as well as the algorithm for the new approach are given in detail. By comparative analysis, the ranking results based on two different models have a highly consensus. Although there exist some different ranking results on these two methods, the optimal selected alternative is the same.

In view of this reason, in the present paper, we investigate two types of coverings based multigranulation rough fuzzy set models by means of the neighborhoods, which is different from the existing literatures. It is listed as follows: Sect. 2 gives detailedly literature review on rough sets and extenshions and decision making methods. Section 3 is devoted to recall some basic concepts and fundamental properties related to RST and CRSs. In Sects. 4 and 5, by means of neighborhoods, we introduce two types of coverings based (optimistic, pessimistic and variable precision) multigranulation rough fuzzy set models, respectively. Some axiomatic systems are also obtained. The relationships between two types of coverings based (optimistic, pessimistic and variable precision) multigranulation rough fuzzy set models are established in Sect. 6. Finally, we give an application of multiple criteria group decision making methods by covering based multigranulation rough fuzzy set models in Sect. 7. Section 8 gives the discussion and points out some future work in this topic.

## 2 Literature review

This section gives a necessarily short overview of rough set theory and their extensions and covering based fuzzy rough sets. We also include some comments on the problem of decision making in an uncertain environment as refers to these specific frameworks.

### 2.1 Rough set theory and extensions

Covering based rough set theory is an important extension of rough sets. Pomykala (1987) defined two dual approximation operators. Bonikowski et al. (1998) proposed another CRS-model by minimal description. Further, Tsang et al. (2008) proposed the third CRS-model. Zhu (2007, 2009a, b) and Zhu and Wang (2003, 2007, 2012) proposed some kinds of CRS-models and discussed the relationships among them. Xu and Zhang (2007) proposed a new CRS-model and discussed measuring roughness. Further, Liu and Sai (2009) compared the relationships between Zhu's CRS-models and Xu's CRS-models. Further important properties related to CRS-models are proved in Ma (2012, 2015), Yao and Yao (2012) and Zhu (2011).

Deng et al. (2007) defined a fuzzy coverings by a fuzzy relation. Li et al. (2008) discussed fuzzy covering based generalized fuzzy rough operators. Ma (2016) investigated two kinds of fuzzy rough coverings by means of a fuzzy $\beta$-neighborhood. Further, Yang and Hu (2016) discussed some types of fuzzy $\beta$-coverings based on rough sets, which are generalizations of Ma (2016). Recently, D'eer et al. (2016, 2017) discussed fuzzy neighborhoods based on FRCs. In view of the above literatures, they would constitute a foundation of FRC-models. These studies have aroused interest in rough covering theory, which has quickly become an important and useful research topic in uncertainty theory.

Xu et al. (2014, 2011) put forth the concept of multigranulation rough fuzzy set models (according to the view of Dubios and Prade) by means of an equivalent relation. In fact, it should be called a multigranulation rough fuzzy set based on Dubois and Prade's idea in Dubois and Prade (1990). Liu et al. (2014); Liu and Pedrycz (2016) proposed covering-based multigranulation fuzzy rough sets.

### 2.2 Decision making methods

Mardani et al. (2015) reviewed multiple criteria group decision making methods based on fuzzy set theory from 1994 to 2014. Regarding decision making methods based on rough set theory, many researchers put forth new procedures and techniques too [the reader is addressed to Sun and Ma (2015b, 2017) for examples].

Recently, Sun et al. (2017b) put forward three-way group decision making based on multi-granulation fuzzy decision. At the same time, Sun et al. (2017a) proposed multigranulation fuzzy rough set models and corresponding decision making applications.

Our research method is especially motivated by these interesting proposals.

## 3 Basic terminologies and results

This section reviews some terminologies and results related to RST, CRS and MGRSs. Throughout this article $U$ denotes a finite non-empty set and $\rho$ is an equivalence relation of $U \times U$. The equivalence relation $\rho$ induces a partition of $U$, denoted by $[x]_\rho$, and $U/\rho = \{[x]_\rho | \forall x \in U\}$ stands for the equivalence classes of $x$. Let $\mu$ denote a fuzzy set of $U$. For any set $U$, $\mathcal{F}(U)$ denotes its all fuzzy sets.

### 3.1 Covering based rough sets

Let $\mathbb{C}$ be a family of non-empty subsets $U_i$ of $U$. If $\bigcup_{U_i \in \mathbb{C}} U_i = U$, then $\mathbb{C}$ is called a covering of $U$. The pair $(U, \mathbb{C})$ is called a covering approximation space (briefly, CAS).

Let $(U, \mathbb{C})$ be a CAS and $x \in U$, the neighborhood of $x$ is defined by $N_{\mathbb{C}}(x) = \bigcap\{U_i \in \mathbb{C} | x \in U_i\}$.

Let $\Omega = \{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m\}$ be a family of coverings of $U$, where $\mathbb{C}_i = \{\mathbb{C}_{i1}, \mathbb{C}_{i2}, \ldots, \mathbb{C}_{il_i}\}$, for all $i = 1, 2, \ldots, m$. We can define the neighborhood of $x$ of $\Omega$ is defined by $N_{\mathbb{C}_i}(x) = \bigcap\{C_{ij} \in \mathbb{C}_i | x \in C_{ij}, j = 1, 2, \ldots, l\}$, for all $i = 1, 2, \ldots, m$.

**Definition 3.1** (Zhu 2009a) Let $(U, \mathbb{C})$ be a CAS. For any $X \subseteq U$, the lower and upper approximations of $X$ w.r.t. $\mathbb{C}$ are defined

$$\underline{\mathbb{C}}(X) = \{x \in U | N_{\mathbb{C}}(x) \subseteq X\}$$

and

$$\overline{\mathbb{C}}(X) = \{x \in U | N_{\mathbb{C}}(x) \cap X \neq \emptyset\}.$$

If $\underline{\mathbb{C}}(X) = \overline{\mathbb{C}}(X)$, then $X$ is called exact, otherwise $X$ is a covering based rough set (briefly, CRS).

### 3.2 Covering based rough fuzzy sets

**Definition 3.2** Let $(U, \mathbb{C})$ be a CAS. For any $\mu \in \mathcal{F}(U)$, the lower and upper approximations of $\mu$ w.r.t. $\mathbb{C}$ are defined

$$\underline{\mathbb{C}}(\mu)(x) = \bigwedge\{\mu(y) | y \in N_{\mathbb{C}}(x)\}$$

and

$$\overline{\mathbb{C}}(\mu)(x) = \bigvee\{\mu(y) | y \in N_{\mathbb{C}}(x)\},$$

for all $x \in U$.

If $\underline{\mathbb{C}}(\mu) = \overline{\mathbb{C}}(\mu)$, then $\mu$ is called fuzzy definable, otherwise $\mu$ is a covering based rough fuzzy set (briefly, CRFS).

**Theorem 3.3** *Let* $(U, \mathbb{C})$ *be a CAS and* $\mu, \nu \in \mathcal{F}(U)$, *then*

*(1)* $\underline{\mathbb{C}}(\mu \cap \nu) = \underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu)$;
*(2)* $\overline{\mathbb{C}}(\mu \cup \nu) = \overline{\mathbb{C}}(\mu) \cup \overline{\mathbb{C}}(\nu)$.

***Proof*** We only prove that (1) holds and (2) is similar.

For any $x \in U$, then $\underline{\mathbb{C}}(\mu \cap \nu) = \bigwedge\{\mu(y) \wedge \nu(y) | y \in N_{\mathbb{C}}(x)\}$, which implies, $\underline{\mathbb{C}}(\mu \cap \nu) \leq \mu(y) \cap \nu(y) \leq \nu(y)$ for all $y \in N_{\mathbb{C}}(x)$, that is, $\underline{\mathbb{C}}(\mu \cap \nu)(x) \leq \underline{\mathbb{C}}(\nu)(x)$. Similarly, $\underline{\mathbb{C}}(\mu \cap \nu)(x) \leq \underline{\mathbb{C}}(\mu)(x)$. Hence, $\underline{\mathbb{C}}(\mu \cap \nu)(x) \leq \underline{\mathbb{C}}(\mu)(x) \wedge \underline{\mathbb{C}}(\nu)(x)$, that is, $\underline{\mathbb{C}}(\mu \cap \nu) \subseteq \underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu)$. (*)

Conversely, $(\underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu))(x) = \underline{\mathbb{C}}(\mu)(x) \wedge \underline{\mathbb{C}}(\nu)(x) \leq \underline{\mathbb{C}}(\mu)(x) \leq \mu(y)$ for all $y \in N_{\mathbb{C}}(x)$. Similarly, $(\underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu))(x) \leq \nu(y)$ for all $y \in N_{\mathbb{C}}(x)$. Hence, $(\underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu))(x) \leq \mu(y) \wedge \nu(y)$ for all $y \in N_{\mathbb{C}}(x)$. This means that $(\underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu))(x) \leq \bigwedge\{\mu(y) \wedge \nu(y) | y \in N_{\mathbb{C}}(x)\} = \underline{\mathbb{C}}(\mu \cap \nu)(x)$, that is, $\underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu) \subseteq \underline{\mathbb{C}}(\mu \cap \nu)$. (*)

By (*) and (**), we have $\underline{\mathbb{C}}(\mu \cap \nu) = \underline{\mathbb{C}}(\mu) \cap \underline{\mathbb{C}}(\nu)$. □

### 3.3 Multigranulation rough (fuzzy) sets

**Definition 3.4** (Qian et al. 2010b) (1) Let $I = (U, AT, f)$ be an information system and $A_1, A_2, \ldots, A_m \subseteq AT$, $m \leq 2^{|AT|}$. The optimistic lower and upper approximations of $X \subseteq U$ w.r.t. $\{A_i\}_{i \in \mathbb{N}^+}$ are defined

$$\underline{R}^{o}_{\sum_{i=1}^{m} A_i}(X) = \left\{ x \in U | \bigvee_{i=1}^{m} [x]_{A_i} \subseteq X \right\}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m} A_i}(X) = \left\{ x \in U | \bigwedge_{i=1}^{m} [x]_{A_i} \cap X \neq \emptyset \right\}.$$

If $\underline{R}^{o}_{\sum_{i=1}^{m} A_i}(X) \neq \overline{R}^{o}_{\sum_{i=1}^{m} A_i}(X)$, then $X$ is called an optimistic multigranulation rough set (briefly, OMGRS), otherwise it is optimistic definable.

(2) Let $I = (U, AT, f)$ be an information system and $A_1, A_2, \ldots, A_m \subseteq AT, m \leq 2^{|AT|}$. The pessimistic lower and upper approximations of $X \subseteq U$ w.r.t. $\{A_i\}_{i \in \mathbb{N}^+}$ are defined

$$\underline{R}^{p}_{\sum_{i=1}^{m} A_i}(X) = \left\{ x \in U | \bigwedge_{i=1}^{m} [x]_{A_i} \subseteq X \right\}$$

and

$$\overline{R}^{p}_{\sum_{i=1}^{m} A_i}(X) = \left\{ x \in U | \bigvee_{i=1}^{m} [x]_{A_i} \cap X \neq \emptyset \right\}.$$

If $\underline{R}^{p}_{\sum_{i=1}^{m} A_i}(X) \neq \overline{R}^{p}_{\sum_{i=1}^{m} A_i}(X)$, then $X$ is called a pessimistic multigranulation rough set (briefly, PMGRS), otherwise it is pessimistic definable.

**Definition 3.5** (Xu et al. 2014) (1) Let $I = (U, AT, f)$ be an information system and $A_1, A_2, \ldots, A_m \subseteq AT, m \leq 2^{|AT|}$. For any $\mu \in \mathcal{F}(U)$, denote

$$\underline{R}^{o}_{\sum_{i=1}^{m} A_i}(\mu)(x) = \bigvee_{i=1}^{m} \bigwedge \{\mu(y) | y \in [x]_{A_i}\}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m} A_i}(\mu)(x) = \bigwedge_{i=1}^{m} \bigvee \{\mu(y) | y \in [x]_{A_i}\},$$

for all $x \in U$.

Then $\underline{R}^{o}_{\sum_{i=1}^{m} A_i}(\mu)$ and $\overline{R}^{o}_{\sum_{i=1}^{m} A_i}(\mu)$ are respectively called the optimistic multigranulation lower approximation operator and optimistic multigranulation upper approximation operator of $\mu$ w.r.t. $\{A_i\}_{i \in \mathbb{N}^+}$ .

If $\underline{R}^{o}_{\sum_{i=1}^{m} A_i}(\mu) \neq \overline{R}^{o}_{\sum_{i=1}^{m} A_i}(\mu)$, then $\mu$ is called an optimistic multigranulation rough fuzzy set (briefly, OMGRFS), otherwise it is optimistic fuzzy definable.

(2) Let $I = (U, AT, f)$ be an information system and $A_1, A_2, \ldots, A_m \subseteq AT, m \leq 2^{|AT|}$. For any $\mu \in \mathcal{F}(U)$, denote

$$\underline{R}^{p}_{\sum_{i=1}^{m} A_i}(\mu)(x) = \bigwedge_{i=1}^{m} \bigwedge \{\mu(y) | y \in [x]_{A_i}\}$$

and

$$\overline{R}^{p}_{\sum_{i=1}^{m} A_i}(\mu)(x) = \bigvee_{i=1}^{m} \bigvee \{\mu(y) | y \in [x]_{A_i}\},$$

for all $x \in U$.

Then $\underline{R}^{p}_{\sum_{i=1}^{m} A_i}(\mu)$ and $\overline{R}^{p}_{\sum_{i=1}^{m} A_i}(\mu)$ are respectively called the pessimistic multigranulation lower approximation operator and pessimistic multigranulation upper approximation operator of $\mu$ w.r.t. $\{A_i\}_{i \in \mathbb{N}^+}$ .

If $\underline{R}^{p}_{\sum_{i=1}^{m} A_i}(\mu) \neq \overline{R}^{p}_{\sum_{i=1}^{m} A_i}(\mu)$, then $\mu$ is called a pessimistic multigranulation rough fuzzy set (briefly, PMGRFS), otherwise it is pessimistic fuzzy definable.

## 4 Type-1 covering based multigranulation rough fuzzy sets

In this section, we divide into three parts. In Sect. 4.1, we investigate type-1 covering based optimistic multigranulation rough fuzzy sets. In Sect. 4.2, we describe type-1 covering based pessimistic multigranulation rough fuzzy sets. In Sect. 4.3, we discuss type-1 covering based variable precision multigranulation rough fuzzy sets. And, type-1 covering based multigranulation rough fuzzy sets are based on the block view.

### 4.1 Type-1 covering based optimistic multigranulation rough fuzzy sets

In this subsection, we introduce the concept of type-1 covering based optimistic multigranulation rough fuzzy sets and investigate some related properties.

**Definition 4.1** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, we denote

$$\underline{R}^{o}_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)(x) = \bigvee_{i=1}^{m} \bigwedge \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} \tag{4.1}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)(x) = \bigwedge_{i=1}^{m} \bigvee \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\}, \tag{4.2}$$

for all $x \in U$, where "$\bigvee$" means "max" and "$\bigwedge$" means "min".

Then $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ and $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ are respectively called the type-1 optimistic multigranulation lower approximation operator and type-1 optimistic multigranulation upper approximation operator of $\mu$.

If $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \neq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$, then $\mu$ is called a type-1 covering based optimistic multigranulation rough fuzzy set (briefly, 1-COMGRFS), otherwise it is optimistic fuzzy definable.

From the view of risk decision making, $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ can be regarded as "max–min" rule, and $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ can be regarded as "min–max" rule.

***Remark 4.2*** (1) If $\mathbb{C}_1 = \mathbb{C}_2 = \cdots = \mathbb{C}_m$, then the above formulas become as follows:

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigwedge\{\mu(y)|y \in N_{\mathbb{C}}(x)\} \tag{4.3}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigvee\{\mu(y)|y \in N_{\mathbb{C}}(x)\}. \tag{4.4}$$

for all $x \in U$.

This means that $(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into a covering based rough fuzzy set.

In particular, if $N_{\mathbb{C}}(x) = [x]_R$ for all $x \in U$, then the above formulas (4.3) and (4.4) become

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigwedge\{\mu(y)|y \in [x]_R\} \tag{4.5}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigvee\{\mu(y)|y \in [x]_R\}. \tag{4.6}$$

for all $x \in U$.

Hence, $(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into a rough fuzzy set proposed by Dubois and Prade (1990).

(2) If $\mu$ is a crisp set of $U$, then $\mu(y) = 0$ or $\mu(y) = 1$ for all $y \in U$, and the formulas (4.1) and (4.2) become as follows:

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \left\{x \in U | \bigvee_{i=1}^{m} N_{\mathbb{C}_i}(x) \subseteq \mu\right\} \tag{4.7}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \left\{x \in U | \bigwedge_{i=1}^{m} N_{\mathbb{C}_i}(x) \cap \mu \neq \emptyset\right\}. \tag{4.8}$$

This means that $(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into the covering based optimistic multigranulation rough set model proposed by Yang et al. (2012a).

Further, if $\mathbb{C}_1 = \mathbb{C}_2 = \cdots = \mathbb{C}_m$, then the above formulas (4.7) and (4.8) become as follows:

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \{x \in U | N_{\mathbb{C}}(x) \subseteq \mu\} \tag{4.9}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \{x \in U | N_{\mathbb{C}}(x) \cap \mu \neq \emptyset\}. \tag{4.10}$$

Hence, $(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into a CRS proposed by Zhu (2009a) and Xu and Zhang (2007), respectively.

In particular, if $N_{\mathbb{C}}(x) = [x]_R$, then the above formulas (4.9) and (4.10) become as follows:

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \{x \in U | [x]_R \subseteq \mu\} \tag{4.11}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \{x \in U | [x]_R \cap \mu \neq \emptyset\}. \tag{4.12}$$

Thus, $(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into a rough set proposed by Pawlak (1982).

(3) If $\mu$ is a crisp set of $U$ and $N_{\mathbb{C}_i}(x) = [x]_{R_i}$, then the formulas (4.7) and (4.8) become as follows:

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \left\{x \in U | \bigvee_{i=1}^{m}[x]_{R_i} \subseteq \mu\right\} \tag{4.13}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \left\{x \in U | \bigwedge_{i=1}^{m}[x]_{R_i} \cap \mu \neq \emptyset\right\}. \tag{4.14}$$

This means that $(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into the optimistic multigranulation rough set model proposed by Qian et al. (2010b).

***Example 4.3*** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2 \in \Omega$, where $U = \{x_1, \ldots, x_4\}$ and $\mathbb{C}_1 = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_3, x_4\}\}$, and $\mathbb{C}_2 = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_2, x_4\}, \{x_4\}\}$. Then

$$N_{\mathbb{C}_1}(x_1) = \{x_1, x_2\}, N_{\mathbb{C}_1}(x_2) = \{x_2\}, N_{\mathbb{C}_1}(x_3) = \{x_3, x_4\}, N_{\mathbb{C}_1}(x_4) = \{x_3, x_4\}$$

and

$$N_{\mathbb{C}_2}(x_1) = \{x_1\}, N_{\mathbb{C}_2}(x_2) = \{x_2, x_4\}, N_{\mathbb{C}_2}(x_3) = \{x_1, x_3\}, N_{\mathbb{C}_2}(x_4) = \{x_4\}.$$

Define $\mu = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4}$. Some simple calculations show $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4}$ and $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4}$. Thus, $\mu$ is a 1-COMGRFS.

From Definitions 4.1 and 3.2, we can obtain the following results.

**Proposition 4.4** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \cdots, \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, then*

*(1)* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \bigcup_{i=1}^{m}\underline{\mathbb{C}_i}(\mu)$;

*(2)* $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \bigcap_{i=1}^{m}\overline{\mathbb{C}_i}(\mu)$.

**Theorem 4.5** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, then*

*(1LH)* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \mu \subseteq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$;

*(2L) If* $\mu \subseteq \nu$, *then* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$;

*(2H) If* $\mu \subseteq \nu$, *then* $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$;

*(3L) (1)* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu \cap \nu) \subseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \cap \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$;

*(2)* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu \cup \nu) \supseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \cup \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$;

*(3H)* *(1)* $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cup\nu)\supseteq\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\cup\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$;

*(2)* $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cap\nu)\subseteq\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\cap\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$;

*(4LH)* *(1)* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\sim\mu)=\sim\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\sim\mu)$;

*(2)* $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\sim\mu)=\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\sim\mu)$;

*(5L)* $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$;

*(5H)* $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$.

***Proof*** (1LH) Obviously.

(2L) Let $\mu,\nu\in\mathcal{F}(U)$ such that $\mu\subseteq\nu$ and $x\in U$. Then $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x)=\bigvee_{i=1}^{m}\bigwedge\{\mu(y)\mid y\in N_{\mathbb{C}_i}(x)\}\le\bigvee_{i=1}^{m}\bigwedge\{\nu(y)\mid y\in N_{\mathbb{C}_i}(x)\}=\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)(x)$.

(2H) The proof is similar to (2L).

(3L) (1) By Proposition 4.4(1) and Theorem 3.3, we have

$$\begin{aligned}\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cap\nu)&=\bigcup_{i=1}^{m}\underline{\mathbb{C}_i}(\mu\cap\nu)\\&=\bigcup_{i=1}^{m}(\underline{\mathbb{C}_i}(\mu)\cap\underline{\mathbb{C}_i}(\nu))\\&\subseteq\bigcup_{i=1}^{m}\underline{\mathbb{C}_i}(\mu)\cap\bigcup_{i=1}^{m}\underline{\mathbb{C}_i}(\nu)\\&=\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\cap\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu).\end{aligned}$$

(2) By (2L), $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cup\nu)\supseteq\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ since $\mu\cup\nu\supseteq\mu$. Similarly, $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cup\nu)\supseteq\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$. Hence, $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cup\nu)\supseteq\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\cup\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)$.

(3H) (1) By Proposition 4.4(2) and Theorem 3.3, we have

$$\begin{aligned}\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu\cup\nu)&=\bigcap_{i=1}^{m}\overline{\mathbb{C}_i}(\mu\cup\nu)\\&=\bigcap_{i=1}^{m}(\overline{\mathbb{C}_i}(\mu)\cup\overline{\mathbb{C}_i}(\nu))\\&\supseteq\bigcap_{i=1}^{m}\overline{\mathbb{C}_i}(\mu)\cup\bigcap_{i=1}^{m}\overline{\mathbb{C}_i}(\nu)\\&=\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\cup\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu).\end{aligned}$$

(2) The proof is similar to (3L)(2).

(4LH) For any $x\in U$,

$$\begin{aligned}\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\sim\mu)(x)&=\bigvee_{i=1}^{m}\bigwedge\{1-\mu(y)|y\in N_{\mathbb{C}_i}(x)\}\\&=\bigvee_{i=1}^{m}(1-\bigvee\{\mu(y)|y\in N_{\mathbb{C}_i}(x)\}\\&=1-\bigwedge_{i=1}^{m}\bigvee\{1-\mu(y)|y\in N_{\mathbb{C}_i}(x)\}\end{aligned}$$

$$= 1 - \overline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$$
$$= \sim \overline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x).$$

Similarly, we can prove that $\overline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu)(x) =\sim \underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$ holds.

(5L) For any $x \in U$,

$$\begin{aligned}
\underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}\left(\underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}\mu\right)(x) &= \bigvee_{i=1}^m \bigwedge \left\{\underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(y) | y \in N_{\mathbb{C}_i}(x)\right\} \\
&= \bigvee_{i=1}^m \bigwedge \left\{\bigvee_{i=1}^m \bigwedge \{\mu(z) | z \in N_{\mathbb{C}_i}(y)\} | y \in N_{\mathbb{C}_i}(x)\right\} \\
&= \bigvee_{i=1}^m \bigwedge \{\mu(z) | z \in N_{\mathbb{C}_i}(y) \wedge y \in N_{\mathbb{C}_i}(x)\} \\
&= \bigvee_{i=1}^m \bigwedge \{\mu(z) | z \in N_{\mathbb{C}_i}(y) \subseteq N_{\mathbb{C}_i}(x)\} \\
&= \bigvee_{i=1}^m \bigwedge \{\mu(z) | z \in N_{\mathbb{C}_i}(x)\} \\
&= \underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x).
\end{aligned}$$

(5H) It is similar to (5L). □

***Remark 4.6*** The inclusions of Theorem 4.5 are proper as proven by Example 4.7 as follows.

***Example 4.7*** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2 \in \Omega$, where $U = \{x_1, \ldots, x_6\}$. Now, we define $\mathbb{C}_1 = \{\{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_4, x_5, x_6\}\}$ and $\mathbb{C}_2 = \{\{x_1, x_2, x_3\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_5, x_6\}\}$. Then

$$N_{\mathbb{C}_1}(x_1) = \{x_1, x_2\}, N_{\mathbb{C}_1}(x_2) = \{x_2\}, N_{\mathbb{C}_1}(x_3) = \{x_2, x_3\}, N_{\mathbb{C}_1}(x_4) = \{x_4\},$$
$$N_{\mathbb{C}_1}(x_5) = N_{\mathbb{C}_1}(x_6) = \{x_4, x_5, x_6\}$$

and

$$N_{\mathbb{C}_2}(x_1) = \{x_1, x_2, x_3\}, N_{\mathbb{C}_2}(x_2) = \{x_2\}, N_{\mathbb{C}_2}(x_3) = \{x_3\}, N_{\mathbb{C}_2}(x_4) = \{x_4, x_5\},$$
$$N_{\mathbb{C}_2}(x_5) = \{x_5\}, N_{\mathbb{C}_2}(x_6) = \{x_5, x_6\}.$$

Define $\mu = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.3}{x_5} + \frac{0.8}{x_6}$ and $\nu = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.8}{x_4} + \frac{0.4}{x_5} + \frac{0.6}{x_6}$. Then $\mu \cup \nu = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6}$ and $\mu \cap \nu = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0.3}{x_5} + \frac{0.6}{x_6}$. By calculations, we have

$$\underline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6},$$
$$\overline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.3}{x_5} + \frac{0.8}{x_6},$$
$$\underline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.8}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6},$$
$$\overline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) = \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.8}{x_4} + \frac{0.4}{x_5} + \frac{0.6}{x_6},$$

$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)=\frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.4}{x_3}+\frac{0.2}{x_4}+\frac{0.3}{x_5}+\frac{0.3}{x_6},$$
$$\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)=\frac{0.2}{x_1}+\frac{0.1}{x_2}+\frac{0.4}{x_3}+\frac{0.2}{x_4}+\frac{0.3}{x_5}+\frac{0.6}{x_6},$$
$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cup\nu)=\frac{0.3}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.8}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6},$$
$$\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cup\nu)=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.8}{x_4}+\frac{0.4}{x_5}+\frac{0.8}{x_6}.$$

From the above discussions, we know that

$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cup\nu)\supsetneq\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cup\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)$$
$$\text{and }\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)\subsetneq\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cap\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu).$$

***Example 4.8*** Let $(U,\Omega)$ be a CAS and $\mathbb{C}_1,\mathbb{C}_2\in\Omega$, where $U=\{x_1,\ldots,x_6\}$. Define $\mathbb{C}_1=\{\{x_1,x_2\},\{x_3,x_4\},\{x_5\},\{x_6\}\}$ and $\mathbb{C}_2=\{\{x_1\},\{x_2,x_3,x_4\},\{x_5,x_6\}\}$. Then

$$N_{\mathbb{C}_1}(x_1)=N_{\mathbb{C}_1}(x_2)=\{x_1,x_2\},N_{\mathbb{C}_1}(x_3)=\{x_3,x_4\},$$
$$N_{\mathbb{C}_1}(x_5)=\{x_5\},N_{\mathbb{C}_1}(x_6)=\{x_6\}$$

and

$$N_{\mathbb{C}_2}(x_1)=\{x_1\},N_{\mathbb{C}_2}(x_2)=N_{\mathbb{C}_1}(x_3)=N_{\mathbb{C}_1}(x_4)=\{x_2,x_3,x_4\},$$
$$N_{\mathbb{C}_2}(x_5)=N_{\mathbb{C}_2}(x_6)=\{x_5,x_6\}.$$

Define $\mu=\frac{0.5}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.1}{x_4}+\frac{0.8}{x_5}+\frac{0.4}{x_6}$ and $\nu=\frac{0.1}{x_1}+\frac{0.8}{x_2}+\frac{0.6}{x_3}+\frac{0.2}{x_4}+\frac{0.3}{x_5}+\frac{0.9}{x_6}$. Then $\mu\cap\nu=\frac{0.1}{x_1}+\frac{0.3}{x_2}+\frac{0.6}{x_3}+\frac{0.1}{x_4}+\frac{0.3}{x_5}+\frac{0.4}{x_6}$ and $\mu\cup\nu=\frac{0.5}{x_1}+\frac{0.8}{x_2}+\frac{0.7}{x_3}+\frac{0.2}{x_4}+\frac{0.8}{x_5}+\frac{0.9}{x_6}$. By calculations, we have

$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)=\frac{0.5}{x_1}+\frac{0.3}{x_2}+\frac{0.1}{x_3}+\frac{0.1}{x_4}+\frac{0.8}{x_5}+\frac{0.4}{x_6},$$
$$\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)=\frac{0.5}{x_1}+\frac{0.5}{x_2}+\frac{0.7}{x_3}+\frac{0.7}{x_4}+\frac{0.8}{x_5}+\frac{0.4}{x_6},$$
$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.1}{x_1}+\frac{0.2}{x_2}+\frac{0.2}{x_3}+\frac{0.2}{x_4}+\frac{0.3}{x_5}+\frac{0.9}{x_6},$$
$$\overline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.1}{x_1}+\frac{0.8}{x_2}+\frac{0.6}{x_3}+\frac{0.6}{x_4}+\frac{0.3}{x_5}+\frac{0.9}{x_6},$$
$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)=\frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.1}{x_3}+\frac{0.1}{x_4}+\frac{0.3}{x_5}+\frac{0.4}{x_6}.$$

From the above discussions, we know that

$$\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)\subsetneq\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cap\underline{R}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu).$$

Now, we introduce the level set of 1-COMGRFSs.

**Definition 4.9** Let $(U,\Omega)$ be a CAS and $\mathbb{C}_1,\mathbb{C}_2,\ldots\mathbb{C}_m\in\Omega$. For any $\mu\in\mathcal{F}(U)$ and $0<\beta\leq\alpha\leq 1$, the $\alpha$-level set and $\beta$-level set of $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ and $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$, respectively are defined as follows:

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\alpha=\left\{x\in U\,|\,\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x)\geq\alpha\right\}\tag{4.15}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\beta = \left\{x \in U | \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) \geq \beta\right\}. \tag{4.16}$$

The proofs of the following theorem are clear and we omit the proofs.

**Theorem 4.10** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu, \nu \in \mathcal{F}(U)$ and $0 < \beta \leq \alpha \leq 1$, then*

*(1) If $\mu \subseteq \nu$, then $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\alpha \subseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)_\alpha$;*
*(2) If $\mu \subseteq \nu$, then $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\beta \subseteq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)_\beta$;*
*(3) $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu \cap \nu)_\alpha \subseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\alpha \cap \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)_\alpha$;*
*(4) $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu \cup \nu)_\beta \supseteq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\beta \cup \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)_\beta$;*
*(5) $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu \cup \nu)_\alpha \supseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\alpha \cup \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)_\alpha$;*
*(6) $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu \cap \nu)_\beta \subseteq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\beta \cap \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\nu)_\beta$.*

**Definition 4.11** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $0 < \beta \leq \alpha \leq 1$. Then the accuracy and roughness of $\mu$ are as follows:

$$R^{o}_{\mu}(\alpha, \beta) = \frac{|\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\alpha|}{|\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)_\beta|}.$$

and

$$\sigma^{o}_{\mu}(\alpha, \beta) = 1 - R^{o}_{\mu}(\alpha, \beta).$$

***Example 4.12*** (Continued from Example 4.3) Let $\alpha = 0.6$ and $\beta = 0.4$. Then $R^{o}_{\mu}(\alpha, \beta) = \frac{1}{2}$ and $\sigma^{o}_{\mu}(\alpha, \beta) = \frac{1}{2}$.

## 4.2 Type-1 covering based pessimistic multigranulation rough fuzzy sets

In this subsection, we introduce the concept of type-1 covering based pessimistic multigranulation rough fuzzy sets and investigate some related properties. Due to the same method, we only give the concept. Some similar basic properties are omitted.

**Definition 4.13** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, we denote

$$\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigwedge_{i=1}^{m} \bigwedge \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} \tag{4.17}$$

and

$$\overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigvee_{i=1}^{m} \bigvee \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\}, \tag{4.18}$$

for all $x \in U$.

Then $\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ and $\overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ are respectively called the type-1 pessimistic multigranulation lower approximation operator and type-1 pessimistic multigranulation upper approximation operator of $\mu$.

If $\underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \neq \overline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$, then $\mu$ is called a type-1 covering based pessimistic multigranulation rough fuzzy set (briefly, 1-CPMGRFS), otherwise it is pessimistic fuzzy definable.

From the view of risk decision making, $\underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ can be regarded as "min-min" rule, and $\overline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ can be regarded as "max–max" rule.

***Remark 4.14*** (1) If $\mathbb{C}_1 = \mathbb{C}_2 = \cdots = \mathbb{C}_m$, then the above formulas (4.17) and (4.18) become (4.3) and (4.4). This means that $(\underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu), \overline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$ degenerates into a covering based rough fuzzy set.

(2) If $\mu$ is a crisp set of $U$, then the formulas (4.17) and (4.18) become as follows:

$$\underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \left\{x \in U | \bigwedge_{i=1}^{m} N_{\mathbb{C}_i}(x) \subseteq \mu\right\} \tag{4.19}$$

and

$$\overline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \left\{x \in U | \bigvee_{i=1}^{m} N_{\mathbb{C}_i}(x) \cap \mu \neq \emptyset\right\}. \tag{4.20}$$

This means that $(\underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu), \overline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$ degenerates into the covering based pessimistic multigranulation rough set model proposed by Yang et al. (2012a).

## 4.3 Type-1 covering based variable precision multigranulation rough fuzzy sets

From Sects. 4.1 and 4.2, we can see that 1-COMGRFSs and 1-CPMGRFSs only consider two extremely cases of a decision making process: completely risk-preferring and completely risk-averse. In fact, in real world, there are many uncertain complicated problems which we can not deal with only by these two extremely cases. In view of this reason, in this subsection, we introduce two kinds of type-1 covering based variable precision multigranulation rough fuzzy sets and investigate some related properties.

First, we introduce the concept of type-1 covering based I-variable precision multigranulation rough fuzzy sets.

**Definition 4.15** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $t \in [0, 1]$, we denote

$$I\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigvee_{i=1}^{m} \bigwedge\{\mu(y) \vee t | y \in N_{\mathbb{C}_i}(x)\} \tag{4.21}$$

and

$$I\overline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge_{i=1}^{m} \bigvee\{\mu(y) \wedge (1-t) | y \in N_{\mathbb{C}_i}(x)\}, \tag{4.22}$$

for all $x \in U$.

Then $I\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $I\overline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ are respectively called the type-1 I-variable precision multigranulation lower approximation operator and type-1 I-variable precision multigranulation upper approximation operator of $\mu$.

If $I\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \neq I\overline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$, then $\mu$ is called a type-1 covering based I-variable precision multigranulation rough fuzzy set (briefly, 1-ICPMGRFC), otherwise it is 1-I-variable precision fuzzy definable.

Next, we introduce the concept of type-1 covering based II-variable precision multigranulation rough fuzzy sets.

**Definition 4.16** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $t \in [0, 1]$, we denote

$$II\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge_{i=1}^m \bigwedge \{\mu(y) \vee t | y \in N_{\mathbb{C}_i}(x)\} \tag{4.23}$$

and

$$II\overline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigvee_{i=1}^m \bigvee \{\mu(y) \wedge (1-t) | y \in N_{\mathbb{C}_i}(x)\}, \tag{4.24}$$

for all $x \in U$.

Then $II\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $II\overline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ are respectively called the type-1 II-variable precision multigranulation lower approximation operator and type-1 II-variable precision multigranulation upper approximation operator of $\mu$.

If $II\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \neq II\overline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$, then $\mu$ is called a type-1 covering based II-variable precision multigranulation rough fuzzy set (briefly, 1-IICPMGRFS), otherwise it is 1-II-variable precision fuzzy definable.

***Remark 4.17*** If $t = 0$, then

(1)

$$I\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigvee_{i=1}^m \bigwedge \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} = \underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$$

and

$$I\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge_{i=1}^m \bigvee \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} = \underline{R}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x).$$

(2)

$$II\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge_{i=1}^m \bigwedge \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} = \underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$$

and

$$II\underline{R}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigvee_{i=1}^m \bigvee \{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} = \underline{R}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x).$$

This means that the 1-IVCPMGRFS and 1-IICVPMGRFS models will degenerate into 1-COMGRFS and 1-CPMGRFS models, respectively.

***Example 4.18*** (Continued from Example 4.3) Let $t = 0.5$, then

(1) $I\underline{R}^{0.5}_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4}$

and

$I\overline{R}^{0.5}_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4}$.

(2) $II\underline{R}^{0.5}_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$

and

$II\overline{R}^{0.5}_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$.

From the above example, we can observe that $I\underline{R}^{t}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \nsubseteq I\overline{R}^{t}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ and $II\underline{R}^{t}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \nsubseteq II\overline{R}^{t}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$.

## 5 Type-2 covering based multigranulation rough fuzzy sets

In this section, we consider another type of covering based multigranulation rough fuzzy sets. In Sect. 5.1, we investigate type-2 covering based optimistic multigranulation rough fuzzy sets. In Sect. 5.2, we describe type-2 covering based pessimistic multigranulation rough fuzzy sets. In Sect. 5.3, we discuss type-2 covering based variable precision multigranulation rough fuzzy sets. And, type-2 covering based multigranulation rough fuzzy sets are based on the point view.

### 5.1 Type-2 covering based optimistic multigranulation rough fuzzy sets

In this subsection, we introduce the concept of type-2 covering based optimistic multigranulation rough fuzzy sets and investigate some related properties.

**Definition 5.1** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, we denote

$$\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigwedge \left\{ \mu(y) | y \in \bigcup_{i=1}^{m} N_{\mathbb{C}_i}(x) \right\} \tag{5.1}$$

and

$$\overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) = \bigvee \left\{ \mu(y) | y \in \bigcup_{i=1}^{m} N_{\mathbb{C}_i}(x) \right\}, \tag{5.2}$$

for all $x \in U$.

Then $\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ and $\overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ are respectively called the type-2 optimistic multigranulation lower approximation operator and type-2 optimistic multigranulation upper approximation operator of $\mu$.

If $\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \neq \overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$, then $\mu$ is called a type-2 covering based optimistic multigranulation rough fuzzy set (briefly, 2-COMGRFS), otherwise it is 2-optimistic fuzzy definable.

***Remark 5.2*** (1) If $\mathbb{C}_1 = \mathbb{C}_2 = \cdots = \mathbb{C}_m$, then the above formulas become (4.3) and (4.4). This means that $(\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu))$ degenerates into a covering based rough fuzzy set.

(2) If $\mu$ is a crisp set of $U$, then the formulas (5.1) and (5.2) become (4.7) and (4.8). This means that $(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu), \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$ degenerates into the covering based optimistic multigranulation rough set model proposed by Yang et al. (2012a).

***Example 5.3*** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2 \in \Omega$, where $U = \{x_1, \ldots, x_6\}$. Define $\mathbb{C}_1 = \{\{x_1\}, \{x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_6\}\}$ and $\mathbb{C}_2 = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_6\}\}$. Then

$$N_{\mathbb{C}_1}(x_1) = \{x_1\}, N_{\mathbb{C}_1}(x_2) = \{x_2, x_3\}, N_{\mathbb{C}_1}(x_3) = \{x_3, x_4, x_5\}, N_{\mathbb{C}_1}(x_4) = \{x_3, x_4\},$$
$$N_{\mathbb{C}_1}(x_5) = \{x_4, x_5\}, N_{\mathbb{C}_1}(x_6) = \{x_4, x_5, x_6\}$$

and

$$N_{\mathbb{C}_1}(x_1) = \{x_1, x_2\}, N_{\mathbb{C}_1}(x_2) = \{x_2\}, N_{\mathbb{C}_1}(x_3) = \{x_2, x_3, x_4\}, N_{\mathbb{C}_1}(x_4) = \{x_4\},$$
$$N_{\mathbb{C}_1}(x_5) = \{x_5\}, N_{\mathbb{C}_1}(x_6) = \{x_5, x_6\}.$$

Define $\mu = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6}$. Some simple calculations show $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6}$ and $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4} + \frac{0.7}{x_5} + \frac{0.8}{x_6}$. Thus, $\mu$ is a 2-COMGRFS.

**Proposition 5.4** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $x \in U$, then*

*(1)* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \leq \bigcup_{i=1}^m \underline{\mathbb{C}_i}(\mu)(x)$;

*(2)* $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \geq \bigcup_{i=1}^m \overline{\mathbb{C}_i}(\mu)(x)$.

***Proof*** (1) For any $x \in U$, then $N_{\mathbb{C}_i}(x) \subseteq \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$, and so, $\underline{\mathbb{C}_i}(\mu)(x) = \wedge\{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} \geq \wedge\{\mu(y) | y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$. Thus, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \leq \bigcup_{i=1}^m \underline{\mathbb{C}_i}(\mu)(x)$.

(2) For any $x \in U$, then $N_{\mathbb{C}_i}(x) \subseteq \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$, and so, $\overline{\mathbb{C}_i}(\mu)(x) = \vee\{\mu(y) | y \in N_{\mathbb{C}_i}(x)\} \leq \vee\{\mu(y) | y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} = \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$. Thus, $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \geq \bigcup_{i=1}^m \overline{\mathbb{C}_i}(\mu)(x)$. □

**Theorem 5.5** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, then*

*(1LH)* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \subseteq \mu \subseteq \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$;

*(2L)* *If* $\mu \subseteq \nu$, *then* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \subseteq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$;

*(2H)* *If* $\mu \subseteq \nu$, *then* $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \subseteq \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$;

*(3L)* *(1)* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$;

*(2)* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cup \nu) \supseteq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cup \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$;

*(3H)* *(1)* $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cup \nu) = \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cup \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$;

*(2)* $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu) \subseteq \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$;

*(4LH)* *(1)* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu) = \sim \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu)$;

*(2)* $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu)$;

*(5L)* $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$;

*(5H)* $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$.

***Proof*** (1LH) Clearly.

(2L) Let $\mu, \nu \in \mathcal{F}(U)$ such that $\mu \subseteq \nu$ and $x \in U$. Then $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge\{\mu(y) \mid y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} \leq \mu(y) \leq \nu(y)$ for all $y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$. Hence, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \leq \wedge\{\nu(y)|y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x)$.

(2H) The proof is similar to (2L).

(3L) (1) $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)(x) = \bigwedge\{\mu(y) \wedge \nu(y) \mid y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} \leq \mu(y) \wedge \nu(y) \leq \nu(y)$ for all $y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$, that is, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)(x) \leq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x)$. Similarly, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)(x) \leq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$. Hence, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)(x) \leq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \wedge \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x) = (\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu))(x)$. This means that $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu) \subseteq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$. (*)

On the other hand, $(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu))(x) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \wedge \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x) \leq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x) \leq \mu(y)$ for all $y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$. Similarly, $(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu))(x) \leq \nu(y)$ for all $y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$. Hence, $(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu))(x) \leq \mu(y) \wedge \nu(y)$ for all $y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$. This means that $(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu))(x) \leq \bigwedge\{\mu(y) \wedge \nu(y) | y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)(x)$, that is, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu) \subseteq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)$. (**)

By (*) and (**), we have $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \cap \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cap \nu)$.

(2) $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge\{\mu(y) \mid y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} \leq \mu(y) \leq \mu(y) \vee \nu(y)$ for all $y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)$. Hence, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cup \nu)(x) = \bigwedge\{\mu(y) \vee \nu(y) \mid y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\} \geq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$. Similarly, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cup \nu)(x) \geq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x)$. Hence, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cup \nu)(x) \geq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x) \vee \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)(x)$. This means that $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu \cup \nu) \supseteq \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu) \cup \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\nu)$.

(3H) It is similar to (3L).

(4LH) For any $x \in U$,

$$\begin{aligned}
\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu)(x) &= \bigwedge\left\{1 - \mu(y)|y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\right\} \\
&= 1 - \bigvee_{i=1}^m \bigvee\left\{\mu(y)|y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\right\} \\
&= 1 - \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \\
&= \sim \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x).
\end{aligned}$$

Similarly, we can prove that $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\sim \mu)(x) =\sim \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)$ holds.

(5L) For any $x \in U$,

$$\begin{aligned}
\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}\left(\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}\mu\right)(x) &= \bigwedge_{i=1}^m \left\{\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(y)|y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\right\} \\
&= \bigwedge\left\{\bigwedge\left\{\mu(z)|z \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(y)\right\}|y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\right\}
\end{aligned}$$

$$= \bigwedge\left\{\mu(z)|z\in\bigcup_{i=1}^{m}N_{\mathbb{C}_i}(y)\wedge y\in\bigcup_{i=1}^{m}N_{\mathbb{C}_i}(x)\right\}$$
$$= \bigwedge\left\{\mu(z)|z\in\bigcup_{i=1}^{m}N_{\mathbb{C}_i}(y)\subseteq\bigcup_{i=1}^{m}N_{\mathbb{C}_i}(x)\right\}$$
$$= \bigwedge\left\{\mu(z)|z\in\bigcup_{i=1}^{m}N_{\mathbb{C}_i}(x)\right\}$$
$$= \underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x).$$

(5H) It is similar to (5L). □

***Remark 5.6*** The inclusions (3L)(2) and (3H)(1) of Theorem 5.5 are proper as proven by Example 5.7 below.

***Example 5.7*** (Continued from Example 5.3)
Define $\mu=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.1}{x_3}+\frac{0.7}{x_4}+\frac{0.4}{x_5}+\frac{0.8}{x_6}$ and $\nu=\frac{0.7}{x_1}+\frac{0.2}{x_2}+\frac{0.6}{x_3}+\frac{0.5}{x_4}+\frac{0.8}{x_5}+\frac{0.6}{x_6}$.
Then $\mu\cup\nu=\frac{0.7}{x_1}+\frac{0.3}{x_2}+\frac{0.6}{x_3}+\frac{0.7}{x_4}+\frac{0.8}{x_5}+\frac{0.8}{x_6}$ and $\mu\cap\nu=\frac{0.6}{x_1}+\frac{0.2}{x_2}+\frac{0.1}{x_3}+\frac{0.5}{x_4}+\frac{0.4}{x_5}+\frac{0.6}{x_6}$.
By calculations, we have

$$\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)=\frac{0.3}{x_1}+\frac{0.1}{x_2}+\frac{0.1}{x_3}+\frac{0.4}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6},$$
$$\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.7}{x_4}+\frac{0.7}{x_5}+\frac{0.8}{x_6},$$
$$\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.2}{x_1}+\frac{0.2}{x_2}+\frac{0.2}{x_3}+\frac{0.5}{x_4}+\frac{0.5}{x_5}+\frac{0.5}{x_6},$$
$$\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.7}{x_1}+\frac{0.6}{x_2}+\frac{0.8}{x_3}+\frac{0.8}{x_4}+\frac{0.8}{x_5}+\frac{0.8}{x_6},$$
$$\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)=\frac{0.2}{x_1}+\frac{0.1}{x_2}+\frac{0.1}{x_3}+\frac{0.4}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6},$$
$$\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cap\nu)=\frac{0.6}{x_1}+\frac{0.2}{x_2}+\frac{0.5}{x_3}+\frac{0.5}{x_4}+\frac{0.5}{x_5}+\frac{0.6}{x_6},$$
$$\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cup\nu)=\frac{0.3}{x_1}+\frac{0.3}{x_2}+\frac{0.3}{x_3}+\frac{0.7}{x_4}+\frac{0.7}{x_5}+\frac{0.7}{x_6},$$
$$\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu\cup\nu)=\frac{0.7}{x_1}+\frac{0.6}{x_2}+\frac{0.8}{x_3}+\frac{0.8}{x_4}+\frac{0.8}{x_5}+\frac{0.8}{x_6},$$
$$\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cap\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.2}{x_1}+\frac{0.1}{x_2}+\frac{0.1}{x_3}+\frac{0.4}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6},$$
$$\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cup\underline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.2}{x_1}+\frac{0.1}{x_2}+\frac{0.1}{x_3}+\frac{0.4}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6},$$
$$\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cap\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.7}{x_4}+\frac{0.7}{x_5}+\frac{0.8}{x_6},$$
$$\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\mu)\cup\overline{C}^{o}_{\mathbb{C}_1+\mathbb{C}_2}(\nu)=\frac{0.7}{x_1}+\frac{0.6}{x_2}+\frac{0.8}{x_3}+\frac{0.8}{x_4}+\frac{0.8}{x_5}+\frac{0.8}{x_6}.$$

From the above discussions, we know that

$$\underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu \cup \nu) \supsetneq \underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) \cup \underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu)$$
$$\text{and } \overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu \cap \nu) \subsetneq \overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) \cap \overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu).$$

Now, we introduce the level set of 2-COMGRFSs.

**Definition 5.8** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $0 < \beta \leq \alpha \leq 1$, the $\alpha$-level set and $\beta$-level set of $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$, respectively are defined as follows:

$$\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)_\alpha = \left\{x \in U | \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \geq \alpha\right\} \tag{5.3}$$

and

$$\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)_\beta = \left\{x \in U | \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) \geq \beta\right\}. \tag{5.4}$$

By using the above concept, we can define the approximate precise of 2-COMGRFSs.

**Definition 5.9** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $0 < \beta \leq \alpha \leq 1$. Then the accuracy and roughness of $\mu$ are as follows:

$$C^o_\mu(\alpha, \beta) = \frac{|\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)_\alpha|}{|\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)_\beta|}.$$

and

$$\tau^o_\mu(\alpha, \beta) = 1 - C^o_\mu(\alpha, \beta).$$

***Example 5.10*** (Continued from Example 4.3) Let $\alpha = 0.6$ and $\beta = 0.4$, it follows from Example 4.3 that $C^o_\mu(0.6, 0.4) = \frac{1}{2}$ and $\tau^o_\mu(0.6, 0.4) = \frac{1}{2}$.

By calculations, we have

$$\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} \text{ and } \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4}.$$

Hence, $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)_{0.6} = \{x_2\}$ and $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)_{0.4} = \{x_1, x_2, x_3, x_4\}$, and so, $C^o_\mu(0.6, 0.4) = \frac{1}{4}$ and $\tau^o_\mu(0.6, 0.4) = \frac{3}{4}$.

By comparing Definitions 4.11 and 5.9, we can obtain the following result.

**Theorem 5.11** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $0 < \beta \leq \alpha \leq 1$, then $C^o_\mu(\alpha, \beta) \leq R^o_\mu(\alpha, \beta)$.*

### 5.2 Type-2 covering based pessimistic multigranulation rough fuzzy sets

In this subsection, we introduce the concept of type-2 covering based pessimistic multigranulation rough fuzzy sets and investigate some related properties. Due to the same method of Sect. 5.1, we only give the basic concept and omit some similar basic properties.

**Definition 5.12** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, we denote

$$\underline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge \left\{ \mu(y) | y \in \bigcap_{i=1}^m N_{\mathbb{C}_i}(x) \right\} \tag{5.5}$$

and

$$\overline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigvee \left\{ \mu(y) | y \in \bigcap_{i=1}^m N_{\mathbb{C}_i}(x) \right\}, \tag{5.6}$$

for all $x \in U$.

Then $\underline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $\overline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ are respectively called the type-2 pessimistic multigranulation lower approximation operator and type-2 pessimistic multigranulation upper approximation operator of $\mu$.

If $\underline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \neq \overline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$, then $\mu$ is called a type-2 covering based pessimistic multigranulation rough fuzzy set (briefly, 2-CPMGRFS), otherwise it is 2-pessimistic fuzzy definable.

***Remark 5.13*** (1) If $\mathbb{C}_1 = \mathbb{C}_2 = \cdots = \mathbb{C}_m$, then the above formulas become (4.3) and (4.4). This means that $(\underline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu), \overline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$ degenerates into a covering based rough fuzzy set.

(2) If $\mu$ is a crisp set of $U$, then the formulas become (4.7) and (4.8). This means that $(\underline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu), \overline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu))$ degenerates into the covering based pessimistic multigranulation rough set model proposed by Yang et al. (2012a).

### 5.3 Type-2 covering based variable precision multigranulation rough fuzzy sets

From Sects. 5.1 and 5.2, we can see that 2-COMGRFSs and 2-CPMGRFSs only consider two extremely cases of the decision making process: completely risk-preferring and completely risk-averse. In fact, in real world, there are many uncertain complicated problems which we can deal with thee two extremely cases. In view of this reason, in this subsection, we introduce 2-CVPMGRFSs and investigate some related properties.

**Definition 5.14** Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$ and $t \in [0, 1]$, we denote

$$\underline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigwedge \left\{ \mu(y) | \frac{\sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y)}{m} \geq t \right\} \tag{5.7}$$

and

$$\overline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) = \bigvee \left\{ \mu(y) | \frac{\sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y)}{m} \geq t \right\}, \tag{5.8}$$

for all $x \in U$, where

$$\chi_{N_{\mathbb{C}_i}(x)}(y) = \begin{cases} 1 & \text{if } y \in N_{\mathbb{C}_i}(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\underline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $\overline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ are respectively called the 2-VPMGLAO and 2-VPMGUAO of $\mu$.

If $\underline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu) \neq \overline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$, then $\mu$ is called a type-2 covering based variable precision multigranulation rough fuzzy set (briefly, 2-CPMGRFS), otherwise it is 2-variable precision fuzzy definable.

***Remark 5.15*** If $t = \frac{1}{m}$, then for any $x \in U$,

$$\begin{aligned}\underline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) &= \bigwedge\left\{\mu(y) | \frac{\sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y)}{m} \geq \frac{1}{m}\right\}\\ &= \bigwedge\left\{\mu(y) | \sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y) \geq 1\right\}\\ &= \bigwedge\left\{\mu(y) | y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\right\}\\ &= \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)\end{aligned}$$

and

$$\begin{aligned}\overline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) &= \bigvee\left\{\mu(y) | \frac{\sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y)}{m} \geq \frac{1}{m}\right\}\\ &= \bigvee\left\{\mu(y) | \sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y) \geq 1\right\}\\ &= \bigvee\left\{\mu(y) | y \in \bigcup_{i=1}^m N_{\mathbb{C}_i}(x)\right\}\\ &= \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)\end{aligned}$$

This means that 2-CVPMGRFS model degenerates into a 2-COMGRFS model. Moreover, we know that $\underline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $\overline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu) = \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ for all $t \in (0, \frac{1}{m}]$.

***Remark 5.16*** If $t = 1$, then for any $x \in U$,

$$\begin{aligned}\underline{C}^t_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x) &= \bigwedge\left\{\mu(y) | \frac{\sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y)}{m} \geq 1\right\}\\ &= \bigwedge\left\{\mu(y) | \sum_{i=1}^m \chi_{N_{\mathbb{C}_i}(x)}(y) \geq m\right\}\\ &= \bigwedge\left\{\mu(y) | y \in \bigcap_{i=1}^m N_{\mathbb{C}_i}(x)\right\}\\ &= \underline{C}^p_{\sum_{i=1}^m \mathbb{C}_i}(\mu)(x)\end{aligned}$$

and

$$\begin{aligned}\overline{C}^{t}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x) &= \bigvee\left\{\mu(y)|\frac{\sum_{i=1}^{m}\chi_{N_{\mathbb{C}_i}(x)}(y)}{m}\geq 1\right\}\\ &= \bigvee\left\{\mu(y)|\sum_{i=1}^{m}\chi_{N_{\mathbb{C}_i}(x)}(y)\geq m\right\}\\ &= \bigvee\left\{\mu(y)|y\in\bigcap_{i=1}^{m}N_{\mathbb{C}_i}(x)\right\}\\ &= \overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x)\end{aligned}$$

This means that 2-CVPMGRFS model degenerates into a 2-CPMGRFS model.

***Remark 5.17*** From the above discussions, we can see that 2-COMGRFS and 2-CPMGRFS models are the special cases of 2-CVPMGRFS model. When the value of parameter $t$ changes from $[\frac{1}{m}, 1]$, the 2-CVPMGRFS model describes the gradually changing process from 2-COMGRFS to 2-CPMGRFS models. Hence, 2-CVPMGRFS models can solve any kind of decision making problems with uncertainty in artificial intelligence and management sciences.

***Example 5.18*** (Continued from Example 5.3) Let $t = 0.6$, then

$$\underline{C}^{0.6}_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.1}{x_3}+\frac{0.7}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6}$$

and

$$\overline{C}^{0.6}_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.7}{x_4}+\frac{0.4}{x_5}+\frac{0.8}{x_6}.$$

## 6 The relationships between two types of CMGRFSs

In this section, we investigate the relationships between two types of CMGRFSs.

By Propositions 4.4 and 5.4, we can obtain the relationships between 1-COMGRFSs and 2-COMGRFSs.

**Theorem 6.1** *Let $(U, \Omega)$ be a CAS and $\mathbb{C}_1, \mathbb{C}_2, \ldots \mathbb{C}_m \in \Omega$. For any $\mu \in \mathcal{F}(U)$, then*

*(1) $\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$;*
*(2) $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$.*

**Corollary 6.2** $\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \mu \subseteq \overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subseteq \overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$.

This means that 1-COMGRFS model is more accurate than 2-COMGRFS model.

***Remark 6.3*** The inclusions of Corollary 6.2 are proper as follows.

***Example 6.4*** (Continued from Example 5.3) By calculations, we have

$$\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.1}{x_3}+\frac{0.7}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6}$$

and

$$\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.7}{x_4}+\frac{0.4}{x_5}+\frac{0.8}{x_6}.$$

In Example 5.3, we know that

$$\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\frac{0.3}{x_1}+\frac{0.1}{x_2}+\frac{0.1}{x_3}+\frac{0.4}{x_4}+\frac{0.4}{x_5}+\frac{0.4}{x_6}$$

and

$$\overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\frac{0.6}{x_1}+\frac{0.3}{x_2}+\frac{0.7}{x_3}+\frac{0.7}{x_4}+\frac{0.7}{x_5}+\frac{0.8}{x_6}.$$

Hence,

$$\underline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subsetneq\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subsetneq\mu\subsetneq\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subsetneq\overline{C}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu).$$

Next, we investigate the relationships between 1-CPMGRFS models and 2-CPMGRFS models.

**Lemma 6.5** *Let $(U,\Omega)$ be a CAS and $\mathbb{C}_1,\mathbb{C}_2,\ldots\mathbb{C}_m\in\Omega$. For any $\mu\in\mathcal{F}(U)$, then*

*(1)* $\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\bigcap_{i=1}^{m}\underline{\mathbb{C}_i(\mu)}$;

*(2)* $\overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)=\bigcup_{i=1}^{m}\overline{\mathbb{C}_i(\mu)}$.

***Proof*** It is easily obtained from Definitions 3.2 and 4.13. □

**Lemma 6.6** *Let $(U,\Omega)$ be a CAS and $\mathbb{C}_1,\mathbb{C}_2,\ldots\mathbb{C}_m\in\Omega$. For any $\mu\in\mathcal{F}(U)$ and $x\in U$, then*

*(1)* $\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\geq\bigcap_{i=1}^{m}\underline{\mathbb{C}_i(\mu)}(x)$;

*(2)* $\overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\leq\bigcap_{i=1}^{m}\overline{\mathbb{C}_i(\mu)}(x)$.

***Proof*** (1) For any $x\in U$, then $\bigcap_{i=1}^{m}N_{\mathbb{C}_i}(x)\subseteq N_{\mathbb{C}_i}(x)$, and so, $\underline{\mathbb{C}_i(\mu)}(x)=\wedge\{\mu(y)|y\in N_{\mathbb{C}_i}(x)\}\leq\wedge\{\mu(y)|y\in\bigcap_{i=1}^{m}N_{\mathbb{C}_i}(x)\}=\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x)$. Hence, $\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\geq\bigcap_{i=1}^{m}\underline{\mathbb{C}_i(\mu)}(x)$.

(2) For any $x\in U$, then $\bigcap_{i=1}^{m}N_{\mathbb{C}_i}(x)\subseteq N_{\mathbb{C}_i}(x)$, and so, $\overline{\mathbb{C}_i(\mu)}(x)=\vee\{\mu(y)|y\in N_{\mathbb{C}_i}(x)\}\geq\vee\{\mu(y)|y\in\bigcap_{i=1}^{m}N_{\mathbb{C}_i}(x)\}=\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)(x)$. Hence, $\overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\leq\bigcup_{i=1}^{m}\overline{\mathbb{C}_i(\mu)}(x)$. □

By Lemmas 6.5 and 6.6, we have

**Theorem 6.7** *Let $(U,\Omega)$ be a CAS and $\mathbb{C}_1,\mathbb{C}_2,\ldots\mathbb{C}_m\in\Omega$. For any $\mu\in\mathcal{F}(U)$, then*

*(1)* $\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subseteq\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$;

*(2)* $\overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subseteq\overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$;

**Corollary 6.8** $\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subseteq\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subseteq\mu\subseteq\overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)\subseteq\overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$.

This means that 2-CPMGRFS model is more accurate than 1-CPMGRFS model.

***Remark 6.9*** The inclusions of Corollary 6.8 are proper as follows.

***Example 6.10*** (Continued from Example 5.3) By calculations, we have

$$\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6},$$
$$\overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4} + \frac{0.7}{x_5} + \frac{0.8}{x_6},$$
$$\underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6},$$
$$\overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6}.$$

Hence,

$$\underline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subsetneq \underline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subsetneq \mu \subsetneq \overline{C}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) \subsetneq \overline{R}^{p}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu).$$

## 7 An approach to MCGDM method based on CMGRFS models

In this section, we give an application of MCGDM method based on CMGRFS models. For the sake of simplicity, we use 1-COMNGRFS models and 2-COMNGRFS models to present the MCGDM method, respectively. In fact, all methods of CMRGRFS models also can be used to discuss the MCGDM problems. By means of different methods, the obtained results are also different. In order to achieve the most accurate results, further diagnosis is necessary in combination with other hybrid methods in our real world. In this paper, the data originate in opinions from experts. In order to improve the reliability, we depend on CMGRFS models to analyze the opinions of the experts.

### 7.1 An application of MCGDM method based on 1-COMGRFS models

In this subsection, we use 1-COMGRFS models to present the MCGDM method in order to determine whether a person is ill or not.

#### 7.1.1 Decision making methodology based on 1-COMGRFS models

Firstly, in order to determine whether a person is ill or not, we establish the multigranulation fuzzy decision information systems based on the considered MCGDM problems.

Secondly, we construct the covering approximation space and compute the neighborhoods.

Thirdly, we calculate the $\underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ and $\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu)$ for the given fuzzy set $\mu$. Then we can obtain the ranking of the given degree of sicken by experts as their judgements for every patients. Denote by

$$\sum_{i=1}^{m} R_i(\mu) = \lambda \underline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu) + (1-\lambda)\overline{R}^{o}_{\sum_{i=1}^{m}\mathbb{C}_i}(\mu), \text{ where } \lambda \in [0, 1].$$

Finally, based on the value of $\sum_{i=1}^{m} R_i(\mu)$, we give the ranking for the given degree of sicken by using the principle of maximum membership in Zadeh's fuzzy set theory.

***Remark 7.1*** (1) If $\lambda = 1$, then $\sum_{i=1}^{m} R_i(\mu) = \underline{R}^o_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)$.
(2) If $\lambda = 0$, then $\sum_{i=1}^{m} R_i(\mu) = \overline{R}^o_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)$.

From the point of view of risk decision making with uncertainty, $\sum_{i=1}^{m} R_i(\mu)$ can be regarded as the compromise rule with a right weight $\lambda$. In general, $\lambda$ reflects the preference of decision maker for the risk of decision making problems. The larger the value of $\lambda$ when decision maker is risk-preferring. The smaller the value of $\lambda$ when decision maker is risk-averse. Hence the decision maker can adjust $\lambda$ according to the goal in real life.

### 7.1.2 Algorithm for the proposed MCGDM method based on 1-COMGRFS models

Now, we put forth an algorithm for the proposed MCGDM method.

**Algorithm I (Based on 1-COMGRFS models):**

**Step 1** Input the object $U = \{x_1, \ldots, x_n\}$, the condition attributes $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m\}$ and a fuzzy set $\mu$ of $U$.

**Step 2** Compute the neighborhood $N_{\mathbb{C}_i}(x)$, for all $x \in U$, $i = 1, 2, \ldots, m$.

**Step 3** Compute $\underline{R}^o_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)$ and $\overline{R}^o_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)$ according to Definition 4.1.

**Step 4** Determine the right weight value of $\lambda$, where $\lambda \in [0, 1]$.

**Step 5** Compute $\sum\limits_{i=1}^{m} R_i(\mu) = \lambda \underline{R}^o_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu) + (1-\lambda)\overline{R}^o_{\sum_{i=1}^{m} \mathbb{C}_i}(\mu)$.

**Step 6** Obtain the ranking according to the decision principle.

### 7.1.3 An applied example

Assume that $u = \{x_1, \ldots, x_6\}$ is a set of patients. According to the patients' symptoms, the doctors will construct the covering set $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2\}$ which consists of two condition attributes, where $\mathbb{C}_1 = \{\{x_1\}, \{x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_6\}\}$ and $\mathbb{C}_2 = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_6\}\}$ are two kinds of symptoms of sicken.

Let $\mu$ and $\nu$ denote the degree of sicken by two doctors $A$ and $B$ as their judgements for every patients, respectively, as follows:

$$\mu = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6}$$

and

$$\nu = \frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.9}{x_6}.$$

By calculation,

$$\underline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.1}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6}$$

and

$$\overline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.8}{x_6}.$$

Let $\lambda = 0.2$, then the MCGDM of $\mu$ can be obtained as follows:

$$\sum_{i=1}^{2} R_i(\mu) = 0.2\underline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) + (1-0.2)\overline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu)$$
$$= \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.58}{x_3} + \frac{0.7}{x_4} + \frac{0.4}{x_5} + \frac{0.72}{x_6}.$$

According to the principle of maximum membership in Zadeh's fuzzy set theory, the order of the degree of sicken as follows: $x_6 \succ x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$. Hence, doctor $A$ thinks the patient $x_6$ is more likely to be sicken.

Similarly, we can calculate

$$\underline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) = \frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}$$

and

$$\overline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) = \frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.9}{x_6}.$$

Hence, then the MCGDM of $\nu$ can be obtained as follows:

$$\sum_{i=1}^{2} R_i(\nu) = 0.2\underline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) + (1-0.2)\overline{R}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu)$$
$$= \frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.46}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.84}{x_6}.$$

According to the principle of maximum membership in Zadeh's fuzzy set theory, the order of the degree of sicken as follows: $x_6 \succ x_1 \succ x_5 \succ x_4 \succ x_3 \succ x_2$. Hence doctor $B$ also thinks the patient $x_6$ is more likely to be sicken.

***Remark 7.2*** (1) In the above MCGDM method, we use 1-COMGRFS models.
(2) From the point of view of risk decision making with uncertainty, in general, $\lambda$ reflects the preference of decision maker for the risk of decision making problems. The larger the value of $\lambda$ when decision maker is risk-preferring. The smaller the value of $\lambda$ when decision maker is risk-averse. Hence the decision maker can adjust $\lambda$ according to the goal in our real life.

## 7.2 An application of MCGDM method based on 2-COMGRFS models

In this subsection, we use 2-COMGRFS models to present the MCGDM method in order to determine whether a person is ill or not.

### 7.2.1 Decision making methodology based on 2-COMGRFS models

Firstly, in order to determine whether a person is ill or not we establish the multigranulation fuzzy decision information systems based on the considered MCGDM problems.

Secondly, we construct the covering approximation space and compute the neighborhoods.

Thirdly, we calculate the $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ for the given fuzzy set $\mu$. Then we can obtain the ranking of the given degree of sicken by experts as their judgements for every patients. Denote by

$$\sum_{i=1}^{m} C_i(\mu) = \lambda \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) + (1-\lambda)\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu), \text{ where } \lambda \in [0, 1].$$

Finally, based on the value of $\sum_{i=1}^m C_i(\mu)$, we give the ranking for the given degree of sicken by using the principle of maximum membership in Zadeh's fuzzy set theory.

***Remark 7.3*** (1) If $\lambda = 1$, then $\sum_{i=1}^m C_i(\mu) = \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$.
(2) If $\lambda = 0$, then $\sum_{i=1}^m C_i(\mu) = \overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$.

From the point of view of risk decision making with uncertainty, $\sum_{i=1}^m C_i(\mu)$ can be regarded as the compromise rule with a right weight $\lambda$. In general, $\lambda$ reflects the preference of decision maker for the risk of decision making problems. The larger the value of $\lambda$ when decision maker is risk-preferring. The smaller the value of $\lambda$ when decision maker is risk-averse. Hence the decision maker can adjust $\lambda$ according to the goal in our real world.

### 7.2.2 Algorithm for the proposed MCGDM method based on 2-COMGRFS models

Now, we put forth an algorithm for the proposed MCGDM method.
**Algorithm II (Based on 2-COMGRFS models):**

**Step 1** Input the object $U = \{x_1, \ldots, x_n\}$, the condition attributes $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_m\}$ and a fuzzy set $\mu$ of $U$.

**Step 2** Compute the neighborhood $N_{\mathbb{C}_i}(x)$, for all $x \in U$, $i = 1, 2, \ldots, m$.

**Step 3** Compute $\underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ and $\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$ according to Definition 5.1.

**Step 4** Determine the right weight value of $\lambda$, where $\lambda \in [0, 1]$.

**Step 5** Compute $\sum_{i=1}^m C_i(\mu) = \lambda \underline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu) + (1-\lambda)\overline{C}^o_{\sum_{i=1}^m \mathbb{C}_i}(\mu)$.

**Step 6** Obtain the ranking according to the decision principle.

### 7.2.3 An applied example

Consider the examples as in Sect. 7.1. By calculation, we have

$$\underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6}$$

and

$$\overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4} + \frac{0.7}{x_5} + \frac{0.8}{x_6}.$$

Let $\lambda = 0.2$, then the MCGDM of $\mu$ can be obtained as follows:

$$\sum_{i=1}^{2} C_i(\mu) = 0.2\underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu) + (1-0.2)\overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\mu)$$
$$= \frac{0.54}{x_1} + \frac{0.26}{x_2} + \frac{0.58}{x_3} + \frac{0.64}{x_4} + \frac{0.64}{x_5} + \frac{0.72}{x_6}.$$

According to the principle of maximum membership in Zadeh's fuzzy set theory, the order of the degree of sicken as follows: $x_6 \succ x_4 = x_5 \succ x_3 \succ x_1 \succ x_2$. Hence, doctor $A$ thinks the patient $x_6$ is more likely to be sicken.

Similarly, we can calculate

$$\underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6}$$

and

$$\overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) = \frac{0.7}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.9}{x_6}.$$

Hence, then the MCGDM of $\nu$ can be obtained as follows:

$$\begin{aligned}\sum_{i=1}^{2} C_i(\nu) &= 0.2\underline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu) + (1-0.2)\overline{C}^o_{\mathbb{C}_1+\mathbb{C}_2}(\nu)\\ &= \frac{0.6}{x_1} + \frac{0.28}{x_2} + \frac{0.52}{x_3} + \frac{0.58}{x_4} + \frac{0.58}{x_5} + \frac{0.82}{x_6}.\end{aligned}$$

According to the principle of maximum membership in Zadeh's fuzzy set theory, the order of the degree of sicken as follows: $x_6 \succ x_1 \succ x_4 = x_5 \succ x_3 \succ x_2$. Hence doctor $B$ also thinks the patient $x_6$ is more likely to be sicken.

***Remark 7.4*** (1) In the above MCGDM method, we use 2-COMGRFS models. Certainly, we also can use other CMGRFS models to discuss this topic.

(2) By means of different methods, the obtained results are also different. In order to achieve the most accurate results, further diagnosis is necessary in combination with other hybrid methods.

***Remark 7.5*** (1) In **Algorithms I** and **II**, we know that the time complexities of computing $\mathbb{C}_i$ partitions are both $\mathbf{O}(nm|\mathbb{C}_i|)$. Thus, the time complexities are both

$$\begin{aligned}\mathbf{O}(nm|\mathbb{C}_1| + nm|\mathbb{C}_2| + \cdots + nm|\mathbb{C}_m|) &= \mathbf{O}(nm(|\mathbb{C}_1| + |\mathbb{C}_2| + \cdots + |\mathbb{C}_m|))\\ &= \mathbf{O}\left(nm\sum_{i=1}^{m}|\mathbb{C}_i|\right).\end{aligned}$$

(2) In **Algorithms I** and **II**, shortcomings of the algorithms: $(a)$ its calculating quantity is large, $(b)$ The value of $\lambda$ has a great impact on the decision result, which requires the decision maker to choose $\lambda$. If the value of $\lambda$ is misvalued, the decision result will be affected.

Any external comparative analysis suffers from the limitation that traditional MAGDM problems with fuzzy information are mainly focused on using fuzzy aggregation operators w.r.t. a fuzzy binary relation. That is, the preference evaluations of different decision-makers are melded together by using the selected fuzzy aggregation operator, and then the key issue of traditional MAGDM problems arises as to how to define effective fuzzy aggregation operators. But especially complicated real-world problems cannot rely on a (fuzzy) binary relation alone. Covering based rough (fuzzy) set models go beyond that point because they do not depend on a (fuzzy) binary relation, but on generalizations of that concept. In particular, covering based multigranulation rough fuzzy rough sets, which generalize both covering based rough fuzzy sets and multigranulation rough sets, are better suited to deal with more

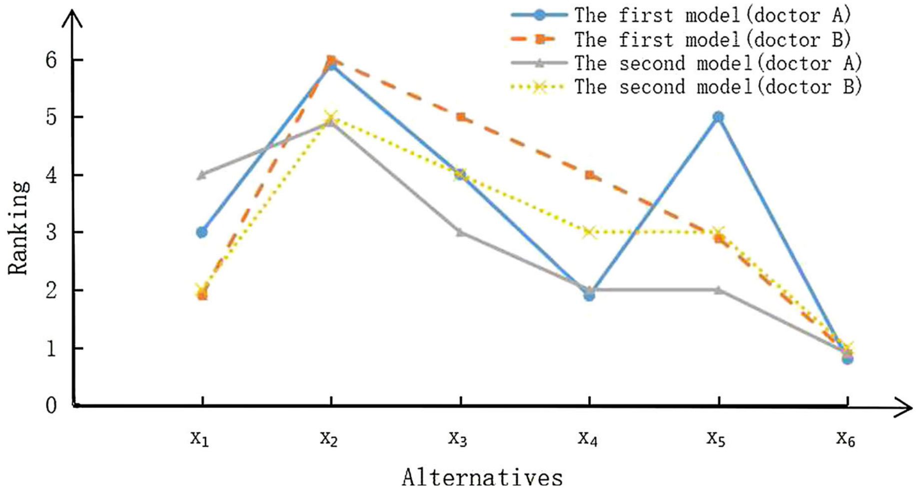


**Fig. 1** The representations of the rankings of all alternatives based four types of cases

subtle problems as described in this section. We have made a preliminary attempt to explore the methods and models that apply to MAGDM problems based on CMGRFS theory, which are comparatively little studied in the broad field of MAGDM problems.

### 7.3 Comparative analysis

From the above analysis, we observe that the ranking results obtained the decision making methods based two different models have a high degree of consensus. To put forth a better perspective of the comparison results, we give the results of the rankings of the alternatives obtained by above four types of cases, see Fig. 1.

It can be easily seen from the Fig. 1 that the ranking of the six candidate alternatives from the above four cases are quite similar. The optimal results are the same. As there exist some minor different places on other alternatives, the reason is that we choose two different models. In real-world decision making process, any preference between the results may be a surjective choice by the decision-maker.

## 8 Discussion and the future work

It is well known that covering rough set model is a vital research topic of generalized rough set theory. Covering rough set model is a very powerful tool that enables the researcher to study data mining in a more general manner. Many researchers proposed many generalized fuzzy rough set models. At the same time, some scholars generalized covering based rough set models to covering based fuzzy rough set models by combining fuzzy sets and covering based rough set models. MGRS model is an important topic as a generalization of rough sets and granular computing which is a tool for AI and management sciences. Mardani et al. (2015) reviewed MCGDMMs based on fuzzy set theory from 1994 to 2014. Regarding decision making methods based on rough set theory, many researchers put forward new procedures and techniques too (the reader is addressed to Sun and Ma 2015b, 2017 for examples). By

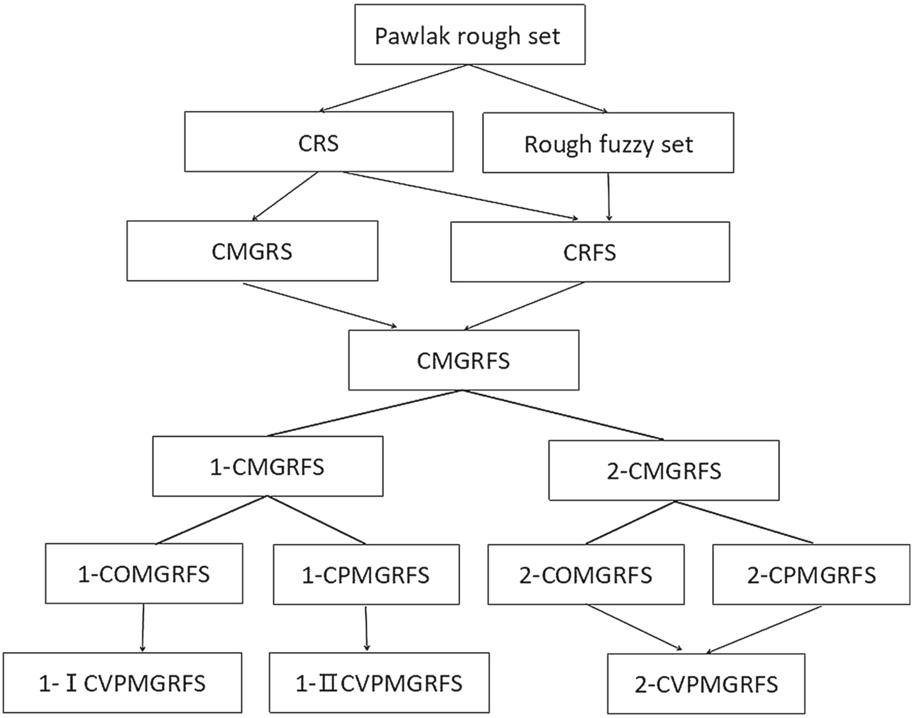


**Fig. 2** The relationships of generalized covering based rough set models. In the Fig. 2, $A \rightarrow B$ means that $A$ is a particular case of $B$ and $A - B$ means that $A$ includes $B$

viewing existing studies, it appears that there is a lack of investigation on the applications in MCGDM method by CMGRFS models. This motivates the present paper on CMGRFS models, as well as their applications in multiple criteria group decision making. In view of this reason, in the present paper, we investigate two types of CMGRFS models by means of the neighborhoods.

The main contributions and the future work in this paper are listed as follows:

(1) Two types of CMGRFS models are investigated.
(2) The relationships between two types of CMGRFS models are established. These studies have aroused interest in covering based rough set theory, which has quickly become an important and useful research topic in uncertainty theory.
(3) Based on the theoretical discussion for the combination of CMGRFS models, we have presented two new approaches to MCGDM problem. The basic model and the procedure of decision making as well as the algorithm for the new approach are given.
(4) The relationships among these kinds of generalized covering based rough set models are investigated, see the following Fig. 2.
(5) There are some issues in this topic deserving further investigation. For instances, the study of parameter reductions of multigranulation rough fuzzy covering models; topological properties and matroidal structures of CMGRFS models; the other decision making applications of CMGRFS models; the research of covering based multigranulation fuzzy rough set models; the design of other hybrid uncertain models and the analysis of their inter-

actions with existing models; the relationships between CMGRFS models and covering based multigranulation fuzzy rough set models.

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