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# The dynamic update method of attribute-induced three-way granular concept in formal contexts

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ABSTRACT

Granular computing is becoming a very hot research field, which has received extensive attention in recent years. It helps us to analyze and solve problems better by dividing complex problems into several simpler ones. Three-way granular concept is an important concept proposed by combining granular computing, formal concept analysis and three-way decision. Using traditional updating methods of three-way granular concepts, a lot of time and space resources are needed when multiple attributes or objects are deleted in formal context. In order to improve the efficiency and flexibility of obtaining three-way concepts, this paper discusses a novel dynamic update method of three-way granular concepts. In this paper, we firstly introduce the related knowledge of three-way granular concepts. Secondly, the update rules of the extension and connotation of attribute-induced three-way granular concepts are discussed in the dynamic formal context to construct three-way granular concepts. Moreover, we develop a method for establishing attribute-induced three-way granular concept by dynamic changes in the case of deleting multiple objects and attributes in the formal context. Furthermore, we design four algorithms to compare between the proposed approaches and traditional updating ways of three-way granular concepts. Finally, the validity of dynamic update method of attribute-induced three-way granular concept is verified through the experimental evaluation using six datasets coming from the University of California-Irvine (UCI) repository.



## 1. Introduction

In 1982, German mathematician Wille put forward Formal Concept Analysis (FCA) theory, which is a powerful tool for data analysis and processing. A classical concept lattice is constructed by a binary relation between a set of objects and a set of attributes according to the contexts. The FCA theory is widely used in data mining, information retrieval, machine learning, artificial intelligence and other fields [1,2].

By using the classical formal concept analysis theory, we can only get two kind concepts from a context, which are ones when all attributes are owned by an object or all objects are of one attribute, respectively. In fact, we can get all the attributes that an object does not possess or all the objects that do not possess one attribute in the complement formal context. However, this information is not reflected in classical formal concepts. To overcome this limitation, Qi, Wei, Yao and others introduce three-way decision ideology to formal concept analysis theory and put forward three-way concept analysis

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[3]. From three-way decision, we can know that the positive domain, the negative domain and the boundary domain are a division of universal set. The extension of the attribute-induced three-way concept consists of positive domain and negative domain, which represent the object set with all attributes in the connotation and the object set without any attributes in the connotation respectively in the formal context. The connotation of the object-induced three-way concept is also composed of positive and negative domains, which represent the set of attributes shared by all objects in the extension and the set of attributes that each object does not have in the extension respectively in the formal context [4–6].

In a formal context, we can notice that three-way concept contains richer semantics than classic concept [7–9]. Moreover, three-way concept theory plays a more and more important role in data mining and rule extraction [10–13]. It has obtained a lot of theoretical results and been widely used. Ren et al. addressed four attribute reductions based on the different criteria produced by the attribute-induced three-way concept and discussed the relationship between these concepts [14]. Yao et al. studied the formal concept analysis based on interval value in the incomplete formal context in reference [15]. Huang et al. proposed a three-way concept learning method based on multi-source data for large data [16]. Yu et al. studied the characteristics of three-way concept lattice and three-way rough concept lattice, and discussed the properties of some special elements [17]. Nowadays, the three-way decision are more and more widely used. Jia et al. combined the characteristics of Chinese and social networks, and proposed a feature fusion method for microblog irony detection based on three-way decision. The method can get a significant improvement than one-stage classification method [18]. Based on the sequential multi-level granular features, a cost-sensitive sequential three-way decision strategy is presented that considers the misclassification cost and test cost in different decision phases from reference [19].

In addition, granular computing was first proposed by Zadeh in his paper “fuzzy sets and information granularity” published in 1979 [20]. He believes that information granules are ubiquitous in many fields, and they have different manifestations in different fields [21]. The key idea of granulation is that we decompose the whole into several relatively simple parts according to different characteristics and standards, each part can be regarded as a grain. Granulation of an object produces a series of granules, which can be used to solve complex practical problems [22–26]. In recent years, granular computing has been favored by researchers and has become a hot research field. Han et al. learned a set of rules for each class and treated each class as a granularity in granularity computing settings by studying the prism algorithm that follows separate and conquer strategy [27]. Ouyang et al. described large-scale digital data by establishing a limited set of representative information granules to obtain the structure of the original data [28]. Hu et al. studied the communication between information systems in granular computing and some properties of information systems in homomorphism [29]. Hu introduced three-way decision spaces through an axiomatic method, established the corresponding three-way decisions, and proposed two open problems on the changes of decision parameters in definition of three-way decisions in order to explore a unified theory of three-way decisions proposed by Yao [30]. Yao combined granular computing with three-way decision to discuss the interaction between the two fields. The integration of the two gives rise to three-way granular computing, that is, thinking, problem solving, and information processing in threes [31]. Li et al. defined a multi-objective attribute reduction based on rough set model of ternary decision theory, and the attribute reduction method proposed in the paper can achieve robust classification performance [32]. In order to meet the special requirements of looking for potential partners in international import and export transactions, Zhi proposed a new concept lattice based on dual concept analysis and discussed the relationship between three-way dual concept lattices and classical dual concept lattices [33].

With the arrival of the era of big data, many disciplines have changed from data-poor to data-rich research fields [34–38]. In order to further improve the efficiency of data representation and data mining, many scholars have combined the basic idea of granular computing with three-way decision and applied it to concept analysis [39–46]. However, there are few studies on the dynamic update of data. No scholars have studied the dynamic update rules of attribute-induced three-way concepts and object-oriented three-way concepts. So, we come up with the dynamic update method of three-way granular concept in this paper to quickly and accurately obtain the three-way granular concept from the complex dynamic formal context, and greatly simplify the time complexity and space complexity. Attribute-induced three-way concepts are dual to the object-oriented three-way concepts, we mainly study the problem of attribute-induced three-way granular concepts renewal and puts forward the corresponding solution algorithm. And then, we verify the effectiveness of the algorithm through the numerical experiment.

The structure of this paper is as follows. In second section, some basic notions are introduced to facilitate our discussion. The rules of updating attribute-induced three-way granular concepts are considered when an attribute or an object is deleted in the contexts in Section 3. In Section 4, the law of renewing attribute-induced three-way granular concepts are investigated when multiple attributes or objects from the context. Section 5 exhibits four algorithms for attribute-induce three-way granular concepts in context, respectively. In addition, several UCI datasets are used to verify the correctness of proposed method in Section 6. Finally, a simple conclusion and future works are given in Section 7.

## 2. Preliminary

In this section, Set Operation, FCA theory, Three-way Concept, Granular Computing and the involved notions are briefly introduced to make the paper self-contained. A detailed description of them can be found in reference [12,46,47]

Let $G$ be a non-empty finite set, $P(G)$ be a power set of the domain $G$, $\mathcal{DP}(G)$ be a cartesian product $\mathcal{P}(G) \times \mathcal{P}(G)$. For $(A, B), (C, D) \in \mathcal{DP}(G)$, we have

(1) $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C, B \subseteq D$
(2) $(A, B) \cap (C, D) = (A \cap C, B \cap D)$
(3) $(A, B) \cup (C, D) = (A \cup C, B \cup D)$

A formal context $(G, M, I)$ consists of $G$, $M$ and the relation $I$ between $G$ and $M$. The elements of $G = \{x_1, x_2, \cdots x_n\}$ are called the objects and the elements of $M = \{a_1, a_2, \cdots, a_m\}$ are called the attributes in the context. For $x \in G$ and $a \in M$, object $x$ has attribute $a$ or attribute $a$ is owned by object $x$ if and only if $xIa$ or $(x, a) \in I$.

We can see that the intersection and union of two cartesian products is the intersection and union of their corresponding elements.

Let $(G, M, I)$ be a formal context. For $X \subseteq G$ and $A \subseteq M$, a pair of operators $* : \mathcal{P}(M) \to \mathcal{P}(G)$ and $* : \mathcal{P}(G) \to \mathcal{P}(M)$ are defined by

$$\begin{aligned} X^* &= \{a \in M | \forall x \in X, (x, a) \in I\}, \\ A^* &= \{x \in X | \forall a \in M, (x, a) \in I\}. \end{aligned} \tag{1}$$

Where the pair of operators "*" express the meaning of jointly possessing. The properties of the two operators can be found in [46], so we do not describe here.

In general, the pair of operators $*$ is called a positive operator. And another pair of operators $\bar{*}$ are given in following, named negative operators.

Let $(G, M, I)$ be a formal context. For $X \subseteq G$ and $A \subseteq M$, a pair of negative operators $\bar{*} : \mathcal{P}(M) \to \mathcal{P}(G)$ and $\bar{*} : \mathcal{P}(G) \to \mathcal{P}(M)$ are defined by

$$\begin{aligned} X^{\bar{*}} &= \{a \in M | \forall x \in X, (x, a) \notin I\}, \\ A^{\bar{*}} &= \{x \in X | \forall a \in M, (x, a) \notin I\}. \end{aligned} \tag{2}$$

The pair of operators $\bar{*}$ represent attributes that all objects do not possess. The operators $*$ and $\bar{*}$ construct the three-way operators $\lessdot$ and $\gtrdot$.

If $(G, M, I)$ is a formal context. For $X, Y \subseteq G$, $A \subseteq M$, a pair of attribute-induced three-way operators $\lessdot : \mathcal{P}(M) \to \mathcal{DP}(G)$ and $\gtrdot : \mathcal{DP}(G) \to \mathcal{P}(M)$ are defined by

$$\begin{aligned} A^{\lessdot} &= (X^*, X^{\bar{*}}), \\ (X, Y)^{\gtrdot} &= \{a \in M | a \in X^*, a \in Y^{\bar{*}}\} \\ &= X^* \cap Y^{\bar{*}}. \end{aligned} \tag{3}$$

The operator represents the objects pair (X,Y) that objects X possess attributes A and objects Y don't possess attributes A. The operator $\gtrdot$ represents attributes $A$ that object set X possess and object set Y do not possess.

For $X, Y, Z, W \subseteq G$ and $A, B \subseteq M$, a pair of attribute-induced three-way operators $\lessdot$ and $\gtrdot$ has the following properties in a formal context $(G, M, I)$.

(1) $A \subseteq A^{\lessdot\gtrdot}, (X, Y) \subseteq (X, Y)^{\gtrdot\lessdot}$
(2) $A \subseteq B \Rightarrow A^{\lessdot} \supseteq B^{\lessdot}$, $(X, Y) \subseteq (Z, W) \Rightarrow (X, Y)^{\gtrdot} \supseteq (Z, W)^{\gtrdot}$
(3) $A^{\lessdot\gtrdot\lessdot} = A^{\lessdot}, (X, Y)^{\gtrdot\lessdot\gtrdot} = (X, Y)^{\gtrdot}$
(4) $A \subseteq (X, Y)^{\gtrdot} \Leftrightarrow (X, Y) \subseteq A^{\lessdot}$
(5) $(A \cup B)^{\lessdot} = A^{\lessdot} \cap B^{\lessdot}, (A \cap B)^{\lessdot} \supseteq A^{\lessdot} \cup B^{\lessdot}$
(6) $((X, Y) \cup (Z, W))^{\gtrdot} = (X, Y)^{\gtrdot} \cap (Z, W)^{\gtrdot}$
(7) $((X, Y) \cap (Z, W))^{\gtrdot} = (X, Y)^{\gtrdot} \cup (Z, W)^{\gtrdot}$

Let $(G, M, I)$ be a formal context. For $X, Y \subseteq G$, $A \subseteq M$, a pair $((X, Y), A)$ is called an attribute-induced three-way concept if and only if $A^{\lessdot} = (X, Y)$ and $(X, Y)^{\gtrdot} = A$. For convenience, we abbreviate it as AE-concept. A pair $(X, Y)$ is called the extension of AE-concept and the attribute set $A$ is called the intension of AE-concept.

In the paper, we use $AEL(G, M, I)$ represents the set of all AE-concepts which are generated by the formal context $(G, M, I)$. For $((X, Y), A)$, $((W, Z), B) \in AEL(G, M, I)$, the partial order relationship are defined as follows:

$$((X, Y), A) \leq ((W, Z), B) \Leftrightarrow (X, Y) \subseteq (W, Z) \Leftrightarrow A \subseteq B. \tag{4}$$

The pair $((X, Y), A)$ is called the subconcept of $((W, Z), B)$, and $((W, Z), B)$ is called the superconcept of $((X, Y), A)$. So, we can get $AEL(G, M, I)$ is a complete lattice under the partial order relation $\leq$ defined above. Moreover, this lattice is called AE-concept lattice. The infimum and supremum operator are given by:

$$\begin{aligned} ((X, Y), A) \vee ((W, Z), B) &= ((X, Y) \cap (W, Z), (A \cup B)^{\lessdot\gtrdot}), \\ ((X, Y), A) \wedge ((W, Z), B) &= ((X, Y) \cup (W, Z), (A \cup B)^{\lessdot\gtrdot}). \end{aligned} \tag{5}$$

**Table 1**
A formal context $(G, M, I)$.

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $x_1$ | 0 | 0 | 1 | 0 | 1 |
| $x_2$ | 1 | 0 | 0 | 1 | 0 |
| $x_3$ | 0 | 1 | 0 | 1 | 0 |
| $x_4$ | 0 | 0 | 1 | 0 | 1 |
| $x_5$ | 0 | 1 | 0 | 1 | 1 |

Let $(G, M, I)$ be a formal context, $\lessdot$: and $\gtrdot$: be a pair of three-way operators. For $X \subseteq G$ and $A, B \subseteq M$, a pair $((X, (A, B))$ is called an object-induced three-way concept, if and only if $X^{\lessdot} = (A, B)$ and $(A, B)^{\gtrdot} = X$. Here, we abbreviate it as OE-concept. Then, the object set $X$ is called the extension, and a pair $(A, B)$ is called the intension of the OE-concept $(X, (A, B))$.

If $(X, (A, B))$ and $(Y, (C, D))$ are two OE-concepts, then they can be ordered by

$$(X, (A, B)) \leq (Y, (C, D)) \Leftrightarrow X \subseteq Y \Leftrightarrow (C, D) \subseteq (A, B). \tag{6}$$

From the above, we can know that all OE-concepts formal a complete lattice, which is called the object-induced three-way concept lattice of $(G, M, I)$, and is written as $OEL(G, M, I)$. The infimum and supremum operators are given by

$$\begin{aligned}(X, (A, B)) \wedge (Y, (C, D)) &= (X \cap Y, ((A, B) \cup (C, D))^{\gtrdot\lessdot}),\\ (X, (A, B)) \vee (Y, (C, D)) &= ((X \cup Y)^{\lessdot\gtrdot}, (A, B) \cap (C, D)).\end{aligned} \tag{7}$$

Let $\lessdot$ and $\gtrdot$ be a pair of three-way operators. If the set of all object-induced three-way concept is denoted by $\mathcal{Q}_1(G, M, I)$ and the set of all attribute-induced three-way concept is represented by $\mathcal{Q}_2(G, M, I)$ in the formal context $(G, M, I)$. Then we have the following properties for any $(X, (A, B)) \in \mathcal{Q}_1(G, M, I)$ and $((X, Y), A) \in \mathcal{Q}_2(G, M, I)$:

$$\begin{aligned}(X, (A, B)) &= (\bigvee_{x \in X} x^{\lessdot\gtrdot}, \bigwedge_{x \in X} x^{\lessdot}),\\ ((X, Y), A) &= (\bigwedge_{a \in A} a^{\lessdot}, \bigvee_{a \in A} a^{\lessdot\gtrdot}).\end{aligned} \tag{8}$$

If $(G, M, I)$ is a formal context, $\lessdot$ and $\gtrdot$ be a pair of three-way operators. For $x \in G$ and $a \in M$, pairs $(a^{\lessdot}, a^{\lessdot\gtrdot})$ and $(x^{\lessdot\gtrdot}, x^{\lessdot})$ are referred to as attribute-induced three-way granular concept and object-induced three-way granular concept, respectively.

By the definition of three-way concept, we can get $(X, Y)^{\lessdot} = A \Leftrightarrow (a, x) \in I$ and $(a, y) \notin I$ for $a \in A$, $x \in X$, $y \in Y$ if the triple $(G, M, I)$ is a formal context. For $\forall x \in X$, $y \in Y$, we have $(m, x) \in I$ and $(m, y) \notin I$ because of $m^{\lessdot} = (X, Y)$. For any $m' \in \{A - m\} \subseteq A$, $x \in X$, $y \in Y$, we can have $(m', x) \in I$ and $(m', y) \notin I$. So, then the following properties hold:
(1) $(m^{\lessdot}, m^{\lessdot\gtrdot}) = ((X, Y), A)$ is an AE-granular concept (attribute-induced three-way granular concept). For $m^{\lessdot} = (X, Y)$, we can have $m^{\lessdot} = m'^{\lessdot}$ if $\forall m' \in \{A - m\}$.
(2) $(x^{\lessdot}, x^{\lessdot\gtrdot}) = (X, (A, B))$ is an OE-granular concept (object-induced three-way granular concept). For $x^{\lessdot} = (A, B)$, we can have $x^{\lessdot} = x'^{\lessdot}$ if $\forall x' \in \{X - x\}$.

Let $\lessdot$ and $\gtrdot$ be a pair of three-way operators. Then the following properties are true.
(1) An arbitrary AE-concept can be induced by AE-granular concept, and that is

$$((X, Y), A) = \bigcap_{a \in A} (a^{\lessdot}, a^{\lessdot\gtrdot}). \tag{9}$$

(2) An arbitrary OE-concept can be induced by OE-granular concept, and that is

$$(X, (A, B)) = \bigcup_{x \in X} (x^{\lessdot\gtrdot}, x^{\lessdot}). \tag{10}$$

**Example 1.** A formal context $(G, M, I)$ is shown in Table 1, where 1 represents acceptance, 0 represents rejection, $G = \{x_1, x_2, x_3, x_4, x_5\}$ represent objects set and $M = \{a, b, c, d, e\}$ represent attributes set.

So, the AE-granular concepts can be obtained from Table 1, which are as below.

$$\begin{aligned}&\{a^{\lessdot}, a^{\lessdot\gtrdot}\} =< (\{x_2\}, \{x_1, x_3, x_4, x_5\}), \{a\} >,\\ &\{b^{\lessdot}, b^{\lessdot\gtrdot}\} =< (\{x_3\}, \{x_1, x_2, x_4, x_5\}), \{b\} >,\\ &\{c^{\lessdot}, c^{\lessdot\gtrdot}\} =< (\{x_1, x_4\}, \{x_2, x_3, x_5\}), \{c\} >,\\ &\{d^{\lessdot}, d^{\lessdot\gtrdot}\} =< (\{x_2, x_3, x_5\}, \{x_1, x_4\}), \{d\} >,\\ &\{e^{\lessdot}, e^{\lessdot\gtrdot}\} =< (\{x_1, x_4, x_5\}, \{x_2, x_3), \{e\} > .\end{aligned}$$

**Table 2**
The formal context $F_1 = (G, M_1, I_1)$ in Example 3.1.

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $x_1$ | 0 | 0 | 1 | 0 |
| $x_2$ | 1 | 0 | 0 | 1 |
| $x_3$ | 0 | 1 | 0 | 1 |
| $x_4$ | 0 | 0 | 1 | 0 |
| $x_5$ | 0 | 1 | 0 | 1 |

## 3. Dynamic update method of AE-granular concepts when an object or an attribute is deleted

This section focuses on the update of AE-granular concepts in the following three cases: (1) delete an attribute; (2) delete an object; (3) delete an attribute and an object at the same time.

### 3.1. Delete an attribute

Because the dynamic updating method of AE-granular concepts does not need to use the deleted subcontext, the AE-granular concepts of the subcontext is obtained directly from AE-granular concepts in the original context by the changing rule. In this section, we mainly discuss the updating way of AE-granular concepts when an attribute is deleted.

**Proposition 3.1.** *If $F = (G, M, I)$ is a formal context, and $F_1 = (G, M_1, I_1)$ is the subcontext after the attribute $a_j$ is removed from $F = (G, M, I)$, then the following properties hold.*

(1) *If $a \in M_1$, $a_j \notin a^{<>}$, then $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>})$.*
(2) *If $a \in M_1$, $a_j \in a^{<>}$, then $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>} - \{a_j\})$.*

**Proof.** Since the objects set of the subcontext is the same as objects set of the original formal context when only one attribute $a_j$ is deleted, we can have $\overline{a^{<}} = a^{<}$. Furthermore, if $a_j \notin a^{<>}$, then $\overline{a^{<>}} = a^{\overline{<>}} = a^{<>}$ hold.
If $a_j \in a^{<>}$, then $\overline{a^{<>}} = a^{<>} - \{a_j\}$ hold. So, the two properties are true.

We study the change of AE-granular concepts when deleting an attribute based on the original context from Example 1, and Example 3.1 shows the effectiveness of Proposition 3.1.

**Example 3.1.** The following subcontext $F_1 = (G, M_1, I_1)$ in Table 2 which is obtained by deleting the attribute $e$ from the original context $F = (G, M, I)$ in Table 1, where $G = \{x_1, x_2, x_3, x_4, x_5\}$ and $M_1 = \{a, b, c, d\}$.
So, the AE-granular concepts can be obtained from Table 2, as shown below:

$$\{a^{<}, a^{<>}\} =< (\{x_2\}, \{x_1, x_3, x_4, x_5\}), \{a\} >;$$
$$\{b^{<}, b^{<>}\} =< (\{x_3, x_4\}, \{x_1, x_2, x_5\}), \{b\} >;$$
$$\{c^{<}, c^{<>}\} =< (\{x_1, x_4\}, \{x_2, x_3, x_5\}), \{c\} >;$$
$$\{d^{<}, d^{<>}\} =< (\{x_2, x_3, x_5\}, \{x_1, x_4\}), \{d\} > .$$

It can be seen from the AE-granular concepts that $\{a^{<}, a^{<>}\}$ satisfies (1) of Proposition 3.1, $\{b^{<}, b^{<>}\}$ satisfies (1) of Proposition 3.1, $\{c^{<}, c^{<>}\}$ satisfies (2) of Proposition 3.1 and $\{d^{<}, d^{<>}\}$ satisfies (1) of Proposition 3.1.

### 3.2. Delete an object

In the previous subsection, we discussed the dynamic renewal method of AE-granular concepts when we deleted an attribute. In this subsection, we continue to discuss the change way of the AE-granular concepts when an object is deleted.

**Proposition 3.2.** *If $(G, M, I)$ is a formal context, and $F_1 = (G_1, M, I_1)$ is the subcontext after the object $x_i$ is removed from $F = (G, M, I)$, then the following properties hold.*

(1) *If $a \in M$, $c \in M - a$, $a^{<} = (X, Y)$, $c^{<} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $x_i \in X$, $C = \{c \in M - a | G - D = \{x_i\}\}$, then $(\overline{a^{<}}, \overline{a^{<>}}) = (\overline{c^{<}}, \overline{c^{<>}}) = \langle (X - \{x_i\}, Y), a^{<>} \cup C) \rangle, c \in C$.*
(2) *If $a \in M$, $c \in M - a$, $a^{<} = (X, Y)$, $c^{<} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $x_i \in Y$, $C = \{c \in M - a | G - D = \{x_i\}\}$, then $(\overline{a^{<}}, \overline{a^{<>}}) = (\overline{c^{<}}, \overline{c^{<>}}) = \langle (X, Y - \{x_i\}), a^{<>} \cup C) \rangle, c \in C$.*

**Table 3**
The formal context $F_1 = (G_1, M, I_1)$ in Example 3.2.

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $x_1$ | 0 | 0 | 1 | 0 | 1 |
| $x_2$ | 1 | 0 | 0 | 1 | 0 |
| $x_3$ | 0 | 1 | 0 | 1 | 0 |
| $x_4$ | 0 | 0 | 1 | 0 | 1 |

**Proof.**

(1) Due to $a^{\lessdot} = (X, Y)$, where $x_i \in X$, we can get $a^{\overline{\lessdot}} = (X - \{x_i\}, Y)$, $X \cap Y = \phi$ and $X \cup Y = G$. Because of $c^{\lessdot} = \{X_1, Y_1\}$, where $c \in C$, we know $X_1 \cap Y_1 = \phi$ and $X_1 \cup Y_1 = G_1$ according to the nature of the AE-granular concept. So, $(X \cap X_1) \cap (Y \cap Y_1) = \phi$ holds. Let $D = D_1 \cup D_2 = (X \cap X_1) \cup (Y \cap Y_1)$, where $D_1 = X \cap X_1$, $D_2 = Y \cap Y_1$, we can get $D = G - x_i = (X - x_i) \cup Y = D_1 \cup D_2 = (X \cap X_1) \cup (Y \cap Y_1)$ from $C = \{c \in M - a | G - D = \{x_i\}\}$ and $x_i \in X$. Therefore, $X_1 = X - x_i$, $Y_1 = Y$ and $I(c, X - x_i) = 1$, $I(c, Y) = 0$ hold by $I(c, X_1) = 1$, $I(c, Y_1) = 0$. Then, $c \in (X - \{x_i\}, Y)^{\gtrdot}$.

(2) Due to $a^{\lessdot} = (X, Y)$, where $x_i \in Y$, we can get $a^{\overline{\lessdot}} = (X, Y - \{x_i\})$, $X \cap Y = \phi$ and $X \cup Y = G$. Because of $c^{\lessdot} = \{X_1, Y_1\}$, where $c \in C$, we know $X_1 \cap Y_1 = \phi$ and $X_1 \cup Y_1 = G_1$ according to the nature of the AE-granular concept. So, $(X \cap X_1) \cap (Y \cap Y_1) = \phi$ holds. Let $D = D_1 \cup D_2 = (X \cap X_1) \cup (Y \cap Y_1)$, where $D_1 = X \cap X_1$, $D_2 = Y \cap Y_1$, we can get $D = G - x_i = X \cup (Y - x_i) = D_1 \cup D_2 = (X \cap X_1) \cup (Y \cap Y_1)$ from $C = \{c \in M - a | G - D = \{x_i\}\}$ and $x_i \in Y$. Therefore, $X_1 = X$, $Y_1 = Y - x_i$ and $I(c, X) = 1$, $I(c, Y - x_i) = 0$ hold by $I(c, X_1) = 1$, $I(c, Y_1) = 0$. Then, $c \in (X, Y - \{x_i\})^{\gtrdot}$.

We study the change of AE-granular concepts when deleting an object based on the original context from Example 1, and Example 3.2 shows the effectiveness of Proposition 3.2.

**Example 3.2.** The following subcontext $F_1 = (G_1, M, I_1)$ in Table 3 which is obtained by deleting the object $x_5$ from the original context $F = (G, M, I)$ in Table 1, where $G_1 = \{x_1, x_2, x_3, x_4\}$ and $M = \{a, b, c, d, e\}$.

So, the AE-granular concepts can be obtained from Table 3, as shown below:

$$\{a^{\lessdot}, a^{\lessdot\gtrdot}\} =< (\{x_2\}, \{x_1, x_3, x_4\}), \{a\} >,$$

$$\{b^{\lessdot}, b^{\lessdot\gtrdot}\} =< (\{x_3, x_4\}, \{x_1, x_2\}), \{b\} >,$$

$$\{c^{\lessdot}, c^{\lessdot\gtrdot}\} = \{e^{\lessdot}, e^{\lessdot\gtrdot}\} =< (\{x_1, x_4\}, \{x_2, x_3\}), \{c, e\} >,$$

$$\{d^{\lessdot}, d^{\lessdot\gtrdot}\} =< (\{x_2, x_3\}, \{x_1, x_4\}), \{d\} > .$$

It can be seen from the AE-granular concepts that $\{a^{\lessdot}, a^{\lessdot\gtrdot}\}$ satisfies (2) of Proposition 3.2, $\{b^{\lessdot}, b^{\lessdot\gtrdot}\}$ satisfies (2) of Proposition 3.2, $\{c^{\lessdot}, c^{\lessdot\gtrdot}\}$ satisfies (2) of Proposition 3.2, $\{d^{\lessdot}, d^{\lessdot\gtrdot}\}$ satisfies (1) of Proposition 3.2 and $\{e^{\lessdot}, e^{\lessdot\gtrdot}\}$ satisfies (2) of Proposition 3.2.

### *3.3. Delete an object and an attribute*

Sections 3.1 and 3.2 give the updating way of AE-granular concept when an object or an attribute is deleted. Then, the updating way of AE-granular concept is discussed after removing both an object and an attribute in this subsection.

**Proposition 3.3.** *If $(G, M, I)$ is a formal context, and $F_1 = (G_1, M_1, I_1)$ is the subcontext after the attribute $a_i$ and the object $x_i$ are removed from $F = (G, M, I)$, then the following properties hold.*

(1) *If $a \in M_1, a_j \notin a^{\lessdot\gtrdot}, a^{\lessdot} = (X, Y), x_i \in X, c^{\lessdot} = \{X_1, Y_1\}, D = (X \cap X_1) \cup (Y \cap Y_1), x_i \in X, C = \{c \in M_1 - a | G - D = \{x_i\}\}$, then $(a^{\overline{\lessdot}}, a^{\overline{\lessdot\gtrdot}}) = (c^{\overline{\lessdot}}, c^{\overline{\lessdot\gtrdot}}) = \langle (X - \{x_i\}, Y), a^{\lessdot\gtrdot} \cup C) \rangle, c \in C$.*

(2) *If $a \in M_1, a_j \notin a^{\lessdot\gtrdot}, a^{\lessdot} = (X, Y), x_i \in Y, c^{\lessdot} = \{X_1, Y_1\}, D = (X \cap X_1) \cup (Y \cap Y_1), x_i \in X, C = \{c \in M_1 - a | G - D = \{x_i\}\}$, then $(a^{\overline{\lessdot}}, a^{\overline{\lessdot\gtrdot}}) = (c^{\overline{\lessdot}}, c^{\overline{\lessdot\gtrdot}}) = \langle (X, Y - \{x_i\}), a^{\lessdot\gtrdot} \cup C) \rangle, c \in C$.*

(3) *If $a \in M_1, a_j \in a^{\lessdot\gtrdot}, a^{\lessdot} = (X, Y), x_i \in X, c^{\lessdot} = \{X_1, Y_1\}, D = (X \cap X_1) \cup (Y \cap Y_1), x_i \in X, C = \{c \in M_1 - a | G - D = \{x_i\}\}$, then $(a^{\overline{\lessdot}}, a^{\overline{\lessdot\gtrdot}}) = (c^{\overline{\lessdot}}, c^{\overline{\lessdot\gtrdot}}) = \langle (X - \{x_i\}, Y), (a^{\lessdot\gtrdot} - a_j) \cup C) \rangle, c \in C$.*

(4) *If $a \in M_1, a_j \in a^{\lessdot\gtrdot}, a^{\lessdot} = (X, Y), x_i \in Y, c^{\lessdot} = \{X_1, Y_1\}, D = (X \cap X_1) \cup (Y \cap Y_1), x_i \in X, C = \{c \in M_1 - a | G - D = \{x_i\}\}$, then $(a^{\overline{\lessdot}}, a^{\overline{\lessdot\gtrdot}}) = (c^{\overline{\lessdot}}, c^{\overline{\lessdot\gtrdot}}) = \langle (X, Y - \{x_i\}), (a^{\lessdot\gtrdot} - a_j) \cup C) \rangle, c \in C$.*

**Proof.** Since the AE-granular concept obtained by deleting both attribute $a_j$ and object $x_i$ is the same as the AE-granular concept obtained by deleting attribute $a_j$ after object $x_i$. The following is deleted by two steps to prove the theorem.

(1) Let $F_2 = (G_2, M, I_2)$ be the subcontext of the original formal context $F = (G, M, I)$ after an object $x_i$ is deleted. $G_2 = G - \{x_i\}$ and $I_2$ represents the relationship between $G_2$ and $M$. In the formal context, for $a^{\lessdot} = (X, Y)$ and $x_i \in X$, then we get

**Table 4**
The formal context $F_1 = (G_1, M_1, I_1)$ in Example 3.3.

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $x_1$ | 0 | 0 | 1 | 0 |
| $x_2$ | 1 | 0 | 0 | 1 |
| $x_3$ | 0 | 1 | 0 | 1 |
| $x_4$ | 0 | 0 | 1 | 0 |

$(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - \{x_i\}, Y), \overline{a^{\lessdot\gtrdot}} \cup C) \rangle$ from (1) of Proposition 3.2. We know that $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - \{x_i\}, Y), \overline{a^{\lessdot\gtrdot}} \cup C) \rangle$ is established from (1) of Proposition 3.1 after an attribute $a_j$ is deleted on the basis and $a_j \notin a^{\lessdot\gtrdot}$.

(2) Let $F_2 = (G_2, M, I_2)$ be the subcontext of the original formal context $F = (G, M, I)$ after an object $x_i$ is deleted. $G_2 = G - \{x_i\}$ and $I_2$ represents the relationship between $G_2$ and $M$. In the formal context, for $a^{\lessdot} = (X, Y)$ and object $x_i$ satisfies $x_i \in Y$, then we can get $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X, Y - \{x_i\}), \overline{a^{\lessdot\gtrdot}} \cup C) \rangle$ from (2) of Proposition 3.2. We know that $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X, Y - \{x_i\}), \overline{a^{\lessdot\gtrdot}} \cup C) \rangle$ is established from (1) of Proposition 3.1 after an attribute $a_i$ is deleted on the basis and $a_j \notin a^{\lessdot\gtrdot}$.

(3) Let $F_2 = (G_2, M, I_2)$ be the subcontext of the original formal context $F = (G, M, I)$ after an object $x_i$ is deleted. $G_2 = G - \{x_i\}$ and $I_2$ represents the relationship between $G_2$ and $M$. In the formal context, for $a^{\lessdot} = (X, Y)$ and Object $x_i$ satisfies $x_i \in X$, then we get $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - \{x_i\}, Y), \overline{a^{\lessdot\gtrdot}} \cup C) \rangle$ from (1) of Proposition 3.2. We know that $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - \{x_i\}, Y), (a^{\lessdot\gtrdot} - \{a_j\})) \cup C \rangle$ is established from (2) of Proposition 3.1 after an attribute $a_i$ is deleted on the basis and $a_j \in a^{\lessdot\gtrdot}$.

(4) Let $F_2 = (G_2, M, I_2)$ be the subcontext of the original formal context $F = (G, M, I)$ after an object $x_i$ is deleted. $G_2 = G - \{x_i\}$ and $I_2$ represents the relationship between $G_2$ and $M$. In the formal context, for $a^{\lessdot} = (X, Y)$ and Object $x_i$ satisfies $x_i \in Y$, then we get $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X, Y - \{x_i\}), \overline{a^{\lessdot\gtrdot}} \cup C) \rangle$ from (2) of Proposition 3.2. We know that $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X, Y - \{x_i\}), (a^{\lessdot\gtrdot} - \{a_j\})) \cup C \rangle$ is established from (2) of Proposition 3.1 after an attribute $a_i$ is deleted on the basis and $a_j \in a^{\lessdot\gtrdot}$.

We study the change of AE-granular concepts when deleting an object and an object based on the original context from Example 1, and Example 3.3 shows the effectiveness of Proposition 3.3.

**Example 3.3.** The following subcontext $F_1 = (G_1, M_1, I_1)$ in Table 4 which is obtained by deleting the object $x_5$ and attribute $e$ from the original context $F = (G, M, I)$ in Table 1, where $G_1 = \{x_1, x_2, x_3, x_4\}$ and $M = \{a, b, c, d\}$.

So, the AE-granular concepts can be obtained from Table 4 as shown below:

$$\{a^{\lessdot}, a^{\lessdot\gtrdot}\} =< (\{x_2\}, \{x_1, x_3, x_4\}), \{a\} >,$$
$$\{b^{\lessdot}, b^{\lessdot\gtrdot}\} =< (\{x_3, x_4\}, \{x_1, x_2\}), \{b\} >,$$
$$\{c^{\lessdot}, c^{\lessdot\gtrdot}\} =< (\{x_1, x_4\}, \{x_2, x_3\}), \{c\} >,$$
$$\{d^{\lessdot}, d^{\lessdot\gtrdot}\} =< (\{x_2, x_3\}, \{x_1, x_4\}), \{d\} > .$$

It can be seen from the AE-granular concepts that the attribute $a$ satisfies (2) of Proposition 3.3, the attribute $b$ satisfies (2) of Proposition 3.3, the attribute $c$ satisfies (2) of Proposition 3.3, the attribute $d$ satisfies (3) of Proposition 3.3.

## 4. Dynamic update method of attribute AE-granular concept when multiple objects or multiple attributes are deleted

Section 3 shows the rule of updating AE-granular concept when a single attribute or object is deleted. In this section, we continue discuss the rule of concept update when multiple objects and multiple attributes are deleted as follows by three cases: (1) delete multiple attributes; (2) delete multiple objects; (3) delete multiple objects and multiple attributes at the same time.

### 4.1. Delete multiple attributes

Because the dynamic updating of AE-granular concepts does not need to use the subcontext which is obtained by deleting multiple attributes and objects from original context, the AE-granular concepts of subcontext is acquired directly from AE-granular concepts by the changing way. In this section, we mainly discuss the changing way of AE-granular concepts when deleting multiple attributes.

**Proposition 4.1.** *If $(G, M, I)$ is a formal context, and $F_1 = (G, M_1, I_1)$ is the subcontext after the attributes set $M_i = \{a_1, a_2, \cdots, a_n\}$ is removed from $F = (G, M, I)$, then the following properties hold.*

**Table 5**
The formal context $F_1 = (G, M_1, I_1)$ in Example 4.1.

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $x_1$ | 0 | 0 | 1 |
| $x_2$ | 1 | 0 | 0 |
| $x_3$ | 0 | 1 | 0 |
| $x_4$ | 0 | 0 | 1 |
| $x_5$ | 0 | 1 | 0 |

(1) *For any $a_j \in M_i$ and $a \in M_1$, if $a_j \notin a^{<>}$, we can get $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>})$.*
(2) *For any $a_j \in M_i$ and $a \in M_1$, if $a_j \in a^{<>}$, we can get $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>} - M_i)$.*
(3) *If $a \in M_1$, $M_i = M_{i1} \bigcup M_{i2}$, $M_{i1}, M_{i2} \neq \phi$, for any $a_i \in M_{i1}$, $a_j \in M_{i2}$, we can get $a_i \notin a^{<>}$ and $a_j \in a^{<>}$, then $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>} - M_2)$.*

**Proof.**

(1) Firstly, we prove that (1) of Proposition 4.1 is established when an attribute $a_1$ is deleted. $F_1 = (G, M_1, I_1)$ is the subcontext after $F = (G, M, I)$ removes the attribute $a_1$, which $a_1 \notin a^{<>}$. $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>})$ is correct in the formal context $F_1 = (G, M_1, I_1)$ according to (1) of Proposition 3.1. So, (1) of Proposition 4.1 holds when deleting an attribute $\{a_1\}$.
Secondly, we prove that (1) of Proposition 4.1 is established when an attribute set $\{a_1, a_2\}$ is deleted. $F_2 = (G, M_2, I_2)$ is the subcontext after $F_1 = (G, M_1, I_1)$ removes the attributes $a_2$, which $a_2 \notin a^{<>}$. $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>})$ is established in the formal context $F_2 = (G, M_2, I_2)$ according to (1) of Proposition 3.1. So, (1) of Proposition 4.1 holds when the attributes set $\{a_1, a_2\}$ is deleted.
Push the class by this, $(\overline{a^{<}}, \overline{a^{<>}}) = \langle a^{<}, a^{<>})\rangle$ is established when the attributes set $M_i = \{a_1, a_2, \cdots, a_n\}$ is deleted. $F_k = (G, M_k, I_k)$ is the subcontext after $F_{k-1} = (G, M_{k-1}, I_{k-1})$ removes the attribute $a_k$, which $a_k \notin a^{<>}$. So, (1) of Proposition 4.1 holds when the attribute set $M_i = \{a_1, a_2, \cdots, a_n\}$ is deleted.
To sum up, (1) of Proposition 4.1 is established by mathematical induction.
(2) The concrete proof process is similar to (1).
(3) Let $M_i = M_{i1} \bigcup M_{i2}$. $M_{i1}$ satisfies (1) of Proposition 4.1 for $F_1 = (G, M_1, I_1)$ that is the subcontext after $F = (G, M, I)$ removes the attribute set $M_1$. $M_2$ satisfies (2) of Proposition 4.1 for $F_2 = (G, M_2, I_2)$ that is the subcontext after $F_1 = (G, M_1, I_1)$ removes the attribute set $M_{i2}$, then $(\overline{a^{<}}, \overline{a^{<>}}) = (a^{<}, a^{<>} - M_i)$ is established. To sum up, (3) of Proposition 4.1 is established.

We study the change of AE-granular concepts when deleting multiple attributes based on the original context from Example 1, and Example 4.1 shows the effectiveness of Proposition 4.1.

**Example 4.1.** The following subcontext $F_1 = (G, M_1, I_1)$ in Table 5 which is obtained by deleting the attributes $d$ and $e$ from the original context $F = (G, M, I)$ in Table 1, where $G_1 = \{x_1, x_2, x_3, x_4, x_5\}$ and $M = \{a, b, c\}$.

So, the AE-granular concepts can be obtained from Table 5, as shown below:

$$\{a^{<}, a^{<>}\} =< (\{x_2\}, \{x_1, x_3, x_4, x_5\}), \{a\} >,$$
$$\{b^{<}, b^{<>}\} =< (\{x_3, x_4\}, \{x_1, x_2, x_5\}), \{b\} >,$$
$$\{c^{<}, c^{<>}\} =< (\{x_1, x_4\}, \{x_2, x_3, x_5\}), \{c\} > .$$

It can be seen from the AE-granular concepts that $\{a^{<}, a^{<>}\}$ satisfies (1) of Proposition 4.1, $\{b^{<}, b^{<>}\}$ satisfies (1) of Proposition 4.1, $\{c^{<}, c^{<>}\}$ satisfies (3) of Proposition 4.1.

### 4.2. Delete multiple objects

In the previous subsection, we discussed the changing rules of AE-granular concepts when multiple attributes are deleted. In this subsection, we will continue to discuss the updating regularity of AE-granular concept when deleting multiple objects.

**Proposition 4.2.** *If $(G, M, I)$ is a formal context, and $F_1 = (G_1, M, I_1)$ is the subcontext after the objects set $X_i = \{x_1, x_2, \cdots, x_n\}$ is removed from $F = (G, M, I)$, then the following properties hold.*

(1) *If $a \in M_1$, $a^{<} = (X, Y)$, $c^{<} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M - a | G - D \subseteq X_i\}$, for any $x_i \in X_i$, we can get $x_i \in X$, then $(\overline{a^{<}}, \overline{a^{<>}}) = \langle (X - X_i, Y), a^{<>} \cup C)\rangle$.*

**Table 6**
The formal context $F_1 = (G_1, M, I_1)$ in Example 4.2.

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $x_1$ | 0 | 0 | 1 | 0 | 1 |
| $x_2$ | 1 | 0 | 0 | 1 | 0 |
| $x_3$ | 0 | 1 | 0 | 1 | 0 |

(2) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M - a | G - D \subseteq X_i\}$, for any $x_i \in X_i$, we can get $x_i \in Y$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X, Y - X_i), a^{\lessdot\gtrdot} \cup C)\rangle$.*
(3) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M - a | G - D \subseteq X_i\}$, $X_i = X_1 \bigcup X_2$, for any $x_i \in X_1$, $x_j \in X_2$, we can get $x_i \in X$ and $x_j \in Y$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_1, Y - X_2), a^{\lessdot\gtrdot} \cup C)\rangle$.*

**Proof.**

(1) Due to $a^{\lessdot} = (X, Y)$, where $X_i \in X$, we can get $\overline{a^{\lessdot}} = (X - \{X_i\}, Y)$, $X \cap Y = \phi$ and $X \cup Y = G$. Because of $c^{\lessdot} = \{X_1, Y_1\}$, where $c \in C$, we know $X_1 \cap Y_1 = \phi$ and $X_1 \cup Y_1 = G_1$ according to the nature of the AE-granular concept. So, $(X \cap X_1) \cap (Y \cap Y_1) = \phi$ holds. Let $D = D_1 \cup D_2 = (X \cap X_1) \cup (Y \cap Y_1)$, where $D_1 = X \cap X_1$, $D_2 = Y \cap Y_1$, we can get $D = D_1 \cup D_2 = (X \cap X_1) \cup (Y \cap Y_1) \supseteq G - X_1 = (X - X_i) \cup Y$ from $C = \{c \in M - a | G - D \subseteq \{X_i\}\}$ and $X_i \in X$. Therefore, $X_1 \supseteq X - X_1$, $Y_1 = Y$ and $I(c, X - X_i) = 1$, $I(c, Y) = 0$ hold by $I(c, X_1) = 1$, $I(c, Y_1) = 0$. Then, $c \in (X - \{X_i\}, Y)^{\gtrdot}$.
(2) The concrete proof process is similar to (1).
(3) Let $X_i = X_1 \bigcup X_2$, $X_1$ satisfy (1) of Proposition 4.2. For $F_1 = (G_1, M, I_1)$ is the subcontext after $F = (G, M, I)$ removes the attribute set $X_1$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_1, Y), a^{\lessdot\gtrdot} \cup C_1)\rangle$ is established. $X_2$ satisfies (2) of Proposition 4.2. For $F_2 = (G_2, M, I_2)$ is the subcontext after $F_1 = (G_1, M, I_1)$ removes the attribute set $X_2$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_1, Y - X_2), a^{\lessdot\gtrdot} \cup C)\rangle$, $C = C_1 \cup C_2$ is established. To sum up, (3) of Proposition 4.2 is established.

We study the change of AE-granular concepts when deleting multiple objects based on the original context from Example 1, and Example 4.2 shows the effectiveness of Proposition 4.2.

**Example 4.2.** The following subcontext $F_1 = (G_1, M, I_1)$ in Table 5 which is obtained by deleting the objects $x_4, x_5$ from the original context $F = (G, M, I)$ in Table 1, where $G_1 = \{x_1, x_2, x_3\}$ and $M = \{a, b, c, d, e\}$.

So, the AE-granular concepts can be obtained from Table 6, as shown below:

$$\{a^{\lessdot}, a^{\lessdot\gtrdot}\} = <(\{x_2\}, \{x_1, x_3\}), \{a\}>,$$
$$\{b^{\lessdot}, b^{\lessdot\gtrdot}\} = <(\{x_3\}, \{x_1, x_2\}), \{b\}>,$$
$$\{c^{\lessdot}, c^{\lessdot\gtrdot}\} = \{e^{\lessdot}, e^{\lessdot\gtrdot}\} = <(\{x_1\}, \{x_2, x_3\}), \{c, e\}>,$$
$$\{d^{\lessdot}, d^{\lessdot\gtrdot}\} = <(\{x_2, x_3\}, \{x_1\}), \{d\}>.$$

It can be seen from the AE-granular concepts that $\{a^{\lessdot}, a^{\lessdot\gtrdot}\}$ satisfies (2) of Proposition 4.2, $\{b^{\lessdot}, b^{\lessdot\gtrdot}\}$ satisfies (3) of Proposition 4.2, $\{c^{\lessdot}, c^{\lessdot\gtrdot}\}$ satisfies (3) of Proposition 4.2, $\{d^{\lessdot}, d^{\lessdot\gtrdot}\}$ satisfies (3) of Proposition 4.2 and $\{e^{\lessdot}, e^{\lessdot\gtrdot}\}$ satisfies (3) of Proposition 4.2.

### *4.3. Delete multiple objects and multiple attributes*

In sections 4.1 and 4.2, we give the updating rules of AE-granular concept when multiple objects or multiple attributes are deleted. Then, we discuss the updating rules of AE-granular concept when both multiple objects and multiple attributes are removed.

**Proposition 4.3.** *If $F = (G, M, I)$ is a formal context, and $F_1 = (G_1, M_1, I_1)$ is the subcontext after the objects set $X_i = \{x_1, x_2, \cdots, x_n\}$ and the attributes set $A_i = \{a_1, a_2, \cdots, a_n\}$ are removed from $F = (G, M, I)$, then the following properties hold.*

(1) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $\underline{D} = (\underline{X} \cap X_1) \cup (\underline{Y} \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_i$ and $a_i \in A_i$, we can get $x_i \in X$ and $a_i \notin a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = (\overline{c^{\lessdot}}, \overline{c^{\lessdot\gtrdot}}) = \langle (X - \{X_i\}, Y), a^{\lessdot\gtrdot} \cup C)\rangle$.*
(2) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $\underline{D} = (\underline{X} \cap X_1) \cup (\underline{Y} \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_i$ and $a_i \in A_i$, we can get $x_i \in Y$ and $a_i \notin a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = (\overline{c^{\lessdot}}, \overline{c^{\lessdot\gtrdot}}) = \langle (X, Y - \{X_i\}), a^{\lessdot\gtrdot} \cup C)\rangle$.*
(3) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq X_i\}$, $X_i = X_1 \cup X_2$, for any $x_i \in X_1$, $x_j \in X_2$ and $a_i \in A_i$, we can get $x_i \in X$, $x_j \in Y$ and $a_i \notin a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_1, Y - X_2), a^{\lessdot\gtrdot} \cup C)\rangle$.*
(4) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $\underline{D} = (\underline{X} \cap X_1) \cup (\underline{Y} \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_i$ and $a_i \in A_i$, we can get $x_i \in X$ and $a_i \in a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = (\overline{c^{\lessdot}}, \overline{c^{\lessdot\gtrdot}}) = \langle (X - \{X_i\}, Y), a^{\lessdot\gtrdot} \cup C - A_i)\rangle$.*

**Table 7**
The formal context $F_1 = (G_1, M_1, I_1)$ in Example 4.3.

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $x_1$ | 0 | 0 | 1 |
| $x_2$ | 1 | 0 | 0 |
| $x_3$ | 0 | 1 | 0 |

(5) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_i$ and $a_i \in A_i$, we can get $x_i \in Y$ and $a_j \in a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = (\overline{c^{\lessdot}}, \overline{c^{\lessdot\gtrdot}}) = \langle (X, Y - \{X_i\}), a^{\lessdot\gtrdot} \cup C - A_i) \rangle$.*

(6) *If $a \in M_1$, $a^{\lessdot} = (X, Y)$, $c^{\lessdot} = \{X_1, Y_1\}$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq X_i\}$, $X_i = X_1 \cup X_2$, for any $x_i \in X_1$, $x_j \in X_2$ and $a_i \in A_i$, we can get $x_i \in X$, $x_j \in Y$ and $a_j \in a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_1, Y - X_2), a^{\lessdot\gtrdot} \cup C - A_i) \rangle$.*

(7) *If $a \in M_1$, $A_i = A_1 \cup A_2$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_i$, $a_i \in A_1$ and $a_j \in A_2$, we can get $x_i \in X$, $a_i \in a^{\lessdot\gtrdot}$ and $a_j \notin a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_i, Y), a^{\lessdot\gtrdot} \cup C - A_1) \rangle$.*

(8) *If $a \in M_1$, $A_i = A_1 \cup A_2$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_i$, $a_i \in A_1$ and $a_j \in A_2$, we can get $x_i \in Y$, $a_i \in a^{\lessdot\gtrdot}$ and $a_j \notin a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X, Y - X_i), a^{\lessdot\gtrdot} \cup C - A_1) \rangle$.*

(9) *If $a \in M_1$, $A_i = A_1 \cup A_2$, $X_i = X_1 \cup X_2$, $D = (X \cap X_1) \cup (Y \cap Y_1)$, $C = \{c \in M_1 - a | G - D \subseteq \{X_i\}\}$, for any $x_i \in X_1$, $x_j \in X_2$, $a_i \in A_1$ and $a_j \in A_2$, we can get $x_i \in X$, $x_j \in Y$, $a_i \in a^{\lessdot\gtrdot}$ and $a_j \notin a^{\lessdot\gtrdot}$, then $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = \langle (X - X_1, Y - X_2), a^{\lessdot\gtrdot} \cup C - A_1) \rangle$.*

**Proof.**

(1) The AE-granular concept obtained after deleting both the attribute set $M_j$ and the object set $X_i$ at the same time, it is the same as that obtained after deleting the object $X_i$ and then deleting the attribute $M_j$. So, we prove the proposition by two steps. $F_1 = (G_1, M_1, I_1)$ is the subcontext after $F = (G, M, I)$ removes the object set $X_i$ and the attribute set $M_j$. Firstly, attribut set $M_j$ is deleted, we know $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = (a^{\lessdot}, a^{\lessdot\gtrdot})$ in the formal context $F_1 = \overline{(G, M_1, I_1)}$ from (1) of Proposition 4.1. Secondly, we delete the object set $X_i$ on the basis and we know $(\overline{a^{\lessdot}}, \overline{a^{\lessdot\gtrdot}}) = (\overline{c^{\lessdot}}, \overline{c^{\lessdot\gtrdot}}) = \langle (X - \{X_i\}, Y), a^{\lessdot\gtrdot} \cup C) \rangle$ in the formal context $F_2 = (G_2, M_1, I_2)$ by (1) of Proposition 4.2.

(2) The theorem proved by using (1) of Proposition 4.1 and (2) of Proposition 4.2, the process is similar to (1).
(3) The theorem proved by using (1) of Proposition 4.1 and (3) of Proposition 4.2, the process is similar to (1).
(4) The theorem proved by using (2) of Proposition 4.1 and (1) of Proposition 4.2, the process is similar to (1).
(5) The theorem proved by using (2) of Proposition 4.1 and (2) of Proposition 4.2, the process is similar to (1).
(6) The theorem proved by using (2) of Proposition 4.1 and (3) of Proposition 4.2, the process is similar to (1).
(7) The theorem proved by using (3) of Proposition 4.1 and (1) of Proposition 4.2, the process is similar to (1).
(8) The theorem proved by using (3) of Proposition 4.1 and (2) of Proposition 4.2, the process is similar to (1).
(9) The theorem proved by using (3) of Proposition 4.1 and (3) of Proposition 4.2, the process is similar to (1).

We study the change of AE-granular concepts when deleting multiple attributes and objects based on the original context from Example 1, and Example 4.3 shows the effectiveness of Proposition 4.3.

**Example 4.3.** The following subcontext $F_1 = (G_1, M_1, I_1)$ in Table 6, which is obtained by deleting objects $x_4, x_5$ and attributes $d, e$ from the original context $F = (G, M, I)$ in Table 1, where $G_1 = \{x_1, x_2, x_3\}$ and $M_1 = \{a, b, c\}$.

So, the AE-granular concepts can be obtained from Table 7, as shown below:

$$\{a^{\lessdot}, a^{\lessdot\gtrdot}\} = <(\{x_2\}, \{x_1, x_3\}), \{a\}>,$$

$$\{b^{\lessdot}, b^{\lessdot\gtrdot}\} = <(\{x_3\}, \{x_1, x_2\}), \{b\}>,$$

$$\{c^{\lessdot}, c^{\lessdot\gtrdot}\} = <(\{x_1\}, \{x_2, x_3\}), \{c\}>.$$

It can be seen from the AE-granular concepts that $\{a^{\lessdot}, a^{\lessdot\gtrdot}\}$ satisfies (2) of Proposition 4.3, $\{b^{\lessdot}, b^{\lessdot\gtrdot}\}$ satisfies (3) of Proposition 4.3 and $\{c^{\lessdot}, c^{\lessdot\gtrdot}\}$ satisfies (9) of Proposition 4.3.

## 5. Dynamic and static updating algorithms of AE-granular concept when multiple attributes and objects are deleted in a formal context

In this section, we design the dynamic and static updating algorithms of AE-granular concept when multiple attribute and object are deleted in a formal context.

### 5.1. The updating algorithms of subcontext when multiple attributes and objects are randomly deleted in a formal context

The given Algorithm 1 is the updating algorithms of subcontext when multiple attributes and objects are randomly deleted in a formal context. The step 2-10 compute the deleted attributes set and the subcontext when multiple attributes

**Algorithm 1:** The algorithm of obtaining subcontext $(G_1, M_1, I_1)$ when multiple attributes and multiple objects are randomly deleted from the original formal context $(G, M, I)$.

**Input** :
(1) Formal context $(G, M, I)$.
(2) The number of attributes (k1) or objects (k2) deleted.
(3) Type of deletion (l)

**Output** :

(1) Subcontext $(G, M, I)$.
(2) Randomly deleted attribute set (c3) and object set (c4).

```
1  begin
2      1:data ← (G, M, I);
3      2:if l == 1 then
                                                   /* Remove the attributes when l=1 */
4          c3 ← ∅                     /* Matrix c3 stores the attribute set that need to be deleted */
5          for i = 1 : k1 do
6              compute the number of columns(n) of data;
7              c1 = fix(n * rand(1, 1) + 1);                   /* Randomly select an attribute */
8              Delete the columns c1;   /* Delete the attribute c1 */ c3 = c3 ∪ c1;   /* Add attribute c1 to c3 */
9          end
10     end
11     3:if l == 2 then
                                                   /* Remove the objects when l=2 */
12         c4 ← ∅                     /* Matrix c3 stores the object set that need to be deleted */
13         for i = 1 : k2 do
14             data ← compute the number of rows(m) of data;
15             c2 = fix(m * rand(1, 1) + 1);                   /* Randomly select an object */
16             Delete the rows c2;   /* Delete the object set c2 */ c4 = c4 ∪ c2;   /* Add attribute c2 to c4 */
17         end
18     end
       return : data, c3, c4;
19 end
```

**Table 8**
The computational complexity of Algorithm 1.

| | |
|---|---|
| step 1-10 | $O(k1 \times \|G\|)$ |
| step 11-18 | $O(k2 \times \|M\|)$ |
| total | $O(k1 \times \|G\| + k2 \times \|M\|)$ |

are randomly deleted (Discontinuous random deletion). We can get the deleted objects set and the subcontext when multiple objects are randomly deleted by step 11-18. At last, return the results. The computational complexity of Algorithm 1, as shown in Table 8.

### 5.2. The static updating algorithms of AE-granular concept when multiple attributes and objects are deleted in a formal context

The given Algorithm 2 is the static updating algorithms of AE-granular concept when multiple attributes and objects are deleted in a formal context. The step 2-5 compute the subcontext when multiple attributes are randomly deleted, import data and initialize the value of variables. We can get all AE-granular concept from the subcontext by step 6-35. We can see that the step 10-17 compute the objects of positive domain and negative domain, and the step 18-25 compute the attributes of connotation which is related to objects of positive domain and not related to objects of negative domain in a concept. At last, return the results. The computational complexity of Algorithm 2, as shown in Table 9.

As is shown in Table 9, the computational complexity of step 10-17 is $O(|G|)$ and the computational complexity of step 18-25 is $O(|M| \times |G|)$.

### 5.3. The dynamic updating algorithms of AE-granular concept when multiple attributes are deleted in a formal context

The given Algorithm 3 is the dynamic updating algorithms of AE-granular concept when multiple attributes are deleted in a formal context. The steps 1-3 import the original AE-granular concept and the deleted attributes. We can get all AE-granular concept after multiple attributes are deleted by step 4-20. We can see that the step 8-12 is compute the attributes of connotation. At last, return the results. The computational complexity of Algorithm 3, as shown in Table 10 (see also Fig. 1).

**Algorithm 2:** The algorithm of deriving AE-granular concepts from formal context.

**Input** :
(1) Formal context $(G_1, M_1, I_1)$.
(2) Deleted attribute or object.

**Output** : AE-granular concepts (Concepts) in subcontext

```
1  begin
2      data1 ← (G_1, M_1, I_1)
3      data ← the subcontext (G, M, I) is obtained by Algorithm 1 from (G_1, M_1, I_1)
4      Concepts ← ∅
5      A ← φ
6      for each a_i ∈ M do
7          if a_i ∉ A then
8              Pos ← ∅
9              Neg ← ∅
10             PA ← ∅
11             NA ← ∅
12             for each x_i ∈ G do
13                 if every I(x_i, a_i) = 1 then
14                     Pos = Pos ∪ x_i      /* If x_i is related to a_i, then x_i is in the positive domain */
15                 end
16                 if every I(x_i, a_i) = 0 then
17                     Neg = Neg ∪ x_i      /* If x_i is not related to a_i, then x_i is in the negative domain */
18                 end
19             end
20             for each a_i ∈ M do
21                 if every I(x_i, a_i) = 1, x_i ∈ Pos then
22                     PA = PA ∪ a_i
23                 end
24                 if every I(x_i, a_i) = 0, x_i ∈ Neg then
25                     NA = NA ∪ a_i
26                 end
27             end
28             A_i = PA ∩ NA
29             A = A ∩ A_i
30             concept = ((Pos, Neg), A_i)
31             A_i = φ
32             Concepts = Concepts ∪ concept
33         end
34     end
35 end
```

**Table 9**
The computational complexity of Algorithm 2.

| | |
|---|---|
| step 1-5 | $O(k1 \times \lvert G\rvert + k2 \times \lvert M\rvert)$ |
| step 6-34 | $o(\lvert M\rvert \times (\lvert G\rvert + \lvert M\rvert \times \lvert G\rvert))$ |
| total | $O(k1 \times \lvert G\rvert + k2 \times \lvert M\rvert + \lvert M\rvert \times (\lvert G\rvert + \lvert M\rvert \times \lvert G\rvert)$ |

### *5.4. The dynamic updating algorithms of AE-granular concept when multiple objects are deleted in a formal context*

The given Algorithm 4 is the dynamic updating algorithms of AE-granular concept when multiple objects are deleted in a formal context. The steps 1-6 import the original AE-granular concept and the deleted attributes. We can get all AE-granular concept after multiple objects are deleted by step 7-36. We can see that the step 14-22 compute the attributes of connotation and the step 23-30 compute the objects of extension. At last, return the results. The computational complexity of Algorithm 4, as shown in Table 11 (see also Fig. 2).

## 6. Experimental analysis

In section 3 and section 4, we present a dynamic approach to update AE-granular concepts. In this part, we compare the time consumption of updating AE-granular concept for the static update method and the dynamic update method. The static method of updating AE-granular concepts delete attributes or objects from the original formal context. Then, we get the subcontext. Finally, we get all the AE-granular concepts from the subcontext. The dynamic updating method of AE-granular concepts is based on the original AE-granular concepts without calculating the subcontext.

In order to further illustrate the advantage of the dynamic method of updating AE-granular concepts in the subcontext of deleting attributes or objects, some experiments are carried out using six datasets where from the UCI. Therefore, the algorithms under dynamic updating and static updating are compared. We pretreated the data in the experiment in order

**Algorithm 3:** Dynamic updating of the AE-granular concept in the case of deleting attributes.

**Input** :
(1) The AE-granular concepts (H1) in the context of the original formal.
(2) The set of deleted attributes (C).

**Output** : The AE-granular concepts of the formal context (H3) after removing multiples attribute from original formal context.

1 **begin**
2 $C \leftarrow$ The set of deleted attributes by Algorithm 1;
3 $Concept \leftarrow$ The AE-granular concepts in the context of the original formal.
4 **for** *each* $concept_i \in Concept$ **do**
5 $Pos \leftarrow$ positive dominate of $concept_i$;
6 $Neg \leftarrow$ negative dominate of $concept_i$;
7 $A_i \leftarrow$ connotation of $concept_i$;
8 **for** *each* $a_i \in C$ **do**
9 **if** $a_i \notin A_i$ **then**
10 $A_i = A_i - a_i$
11 **end**
12 **end**
13 **if** $A_i \neq \phi$ **then**
14 $concept_i = ((Pos, Neg), A_i)$
15 **end**
16 **if** $A_i = \phi$ **then**
17 $concept_i = \phi$
18 **end**
19 **end**
20 **end**

**Table 10**
The computational complexity of Algorithm 3.

| | |
|---|---|
| step 1-3 | 1 |
| step 7-35 | $O(\|Concept\| \times \|C\|$ |
| total | $O(\|Concept\| \times \|C\|$ |

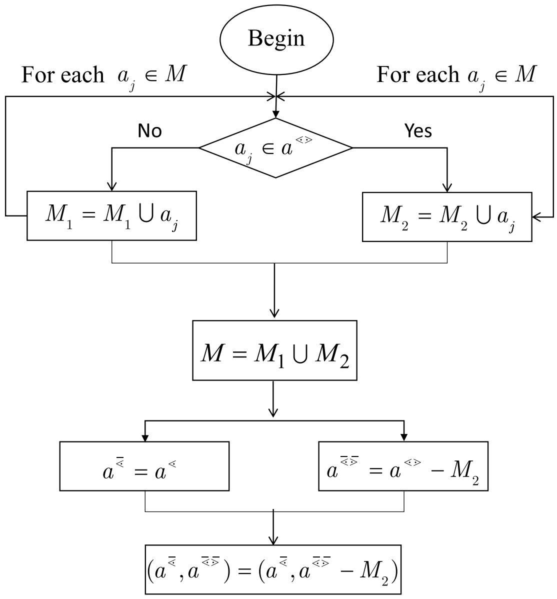



**Fig. 1.** The flow-process diagram of Algorithm 3.

to ensure the validity of the experiment. The basic information of data sets is shown in Table 8. These experiments are implemented by using Matlab R2016b and performed on a personal computer with an Intel Core i6-6700, 3.40 GHz CPU, 16.0 GB of memory, and 64-bit Windows 10.

The experiment includes two parts: the deletion of an object or an attribute and the deletion of multiple objects or attributes. According to the proportion, the objects or attributes of each data set are deleted (see Table 12).

**Algorithm 4:** Dynamic updating of the AE-granular concept in the case of deleting objects.

**Input** :
(1) The AE-granular concepts in the context of the original formal.
(2) The set of deleted objects.

**Output** : The attributes AE-concepts of the formal context (H3) after removing multiple attributes from original formal context.

1 **begin**
2 $OB \leftarrow$ the set of deleted objects by Algorithm 1;
3 $Concept \leftarrow$ the AE-granular concepts in the context of the original formal;
4 $Concept1 = Concept$;
5 $M \leftarrow$ the set of all attributes;
6 $TC \leftarrow \phi$;
7 **for** *each* $concept_i \in Concept$ **do**
8 $A \leftarrow$ connotation of $concept_i$;
9 $Concept1 = Concept1 - concept_i$
10 **if** $A \nsubseteq TC$ **then**
11 $Pos \leftarrow$ positive dominate of $concept_i$;
12 $Neg \leftarrow$ negative dominate of $concept_i$;
13 $C \leftarrow \phi$;
14 **for** *each* $concept1_i \in Concept1$ **do**
15 $Pos1 \leftarrow$ positive dominate of $concept1_i$;
16 $Neg1 \leftarrow$ negative dominate of $concept1_i$;
17 $A1 \leftarrow$ connotation of $concept1_i$;
18 $D = (Pos \cap Pos1) \cup (Neg \cap Neg1)$;
19 **if** $G - D \subseteq OB$ **then**
20 $C = C \cup A1$;
21 **end**
22 **end**
23 **for** *each* $x_i \in OB$ **do**
24 **if** $x_i \in Pos$ **then**
25 $Pos = Pos - x_i$;
26 **end**
27 **if** $x_i \in Neg$ **then**
28 $Neg = Neg - x_i$;
29 **end**
30 **end**
31 $newconcept = ((Pos, Neg), A \cup C)$;
32 $Newconcept = Newconcept \cup newconcept$;
33 $TC = TC \cup C$
34 **end**
35 **end**
36 **end**

**Table 11**
The computational complexity of Algorithm 4.

| | |
|---|---|
| step 1-6 | 1 |
| step 7-35 | $O(\lvert Concept\rvert \times (\lvert Concept\rvert + \lvert OB\rvert)$ |
| total | $O(\lvert Concept\rvert \times (\lvert Concept\rvert + \lvert OB\rvert)$ |

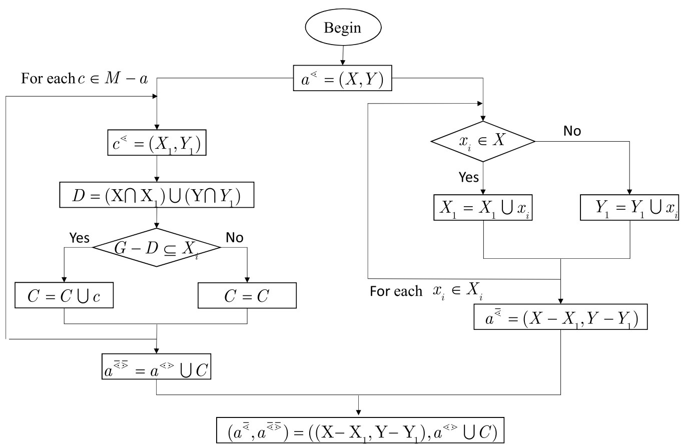
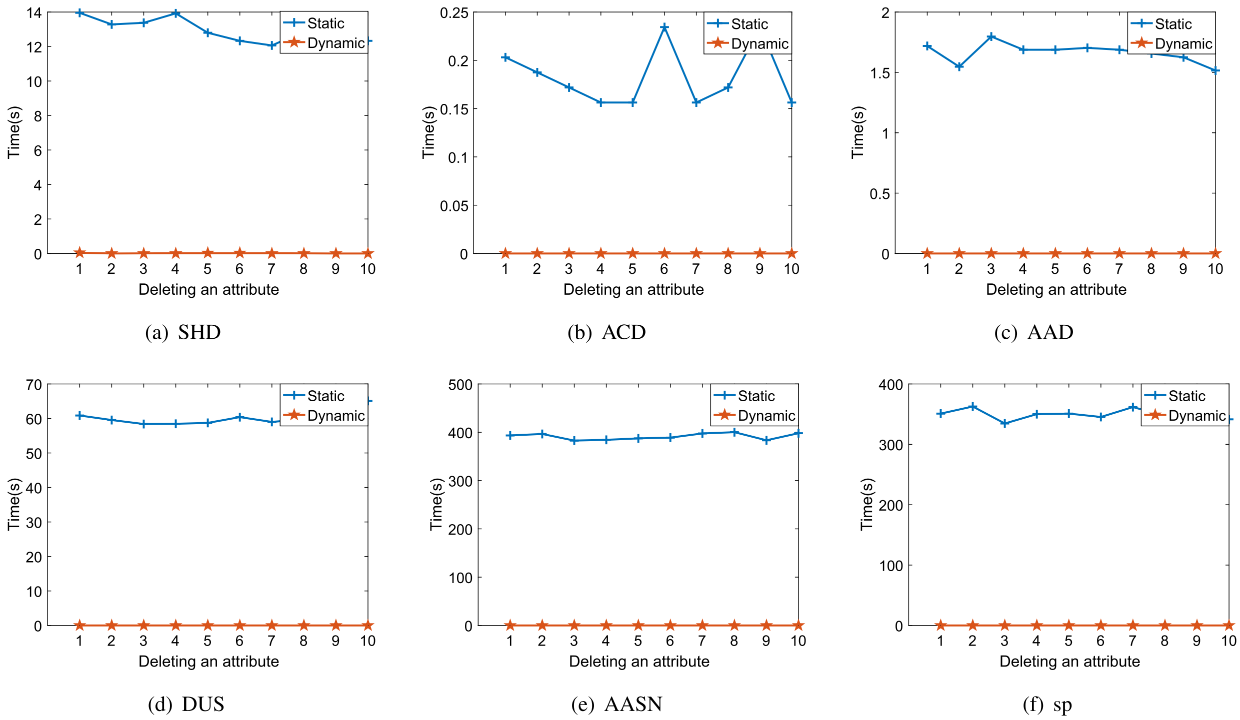



**Fig. 2.** The flow-process diagram of Algorithm 4.

**Table 12**
The basic information of data sets.

| No. | Data set name | Abbreviation | Objects | Attributes |
|---|---|---|---|---|
| 1 | $SemeionHandwrittenDigit$ | SHD | 70 | 204 |
| 2 | $Autism - Child - Data$ | ACD | 292 | 10 |
| 3 | $Autism - Adult - Data$ | AAD | 704 | 10 |
| 4 | $Diabetes130 - UShospitals$ | DUS | 3983 | 20 |
| 5 | $AbscisicAcidSignalingNetwork$ | AASN | 5456 | 43 |
| 6 | $sgemm\ product$ | sp | 67360 | 4 |




(a) SHD (b) ACD (c) AAD (d) DUS (e) AASN (f) sp

**Fig. 3.** The time consumption between static and dynamic update method when deleting an attribute.

### *6.1. In the case of deleting an attribute or an object, the time consumption of the AE-granular concepts is compared*

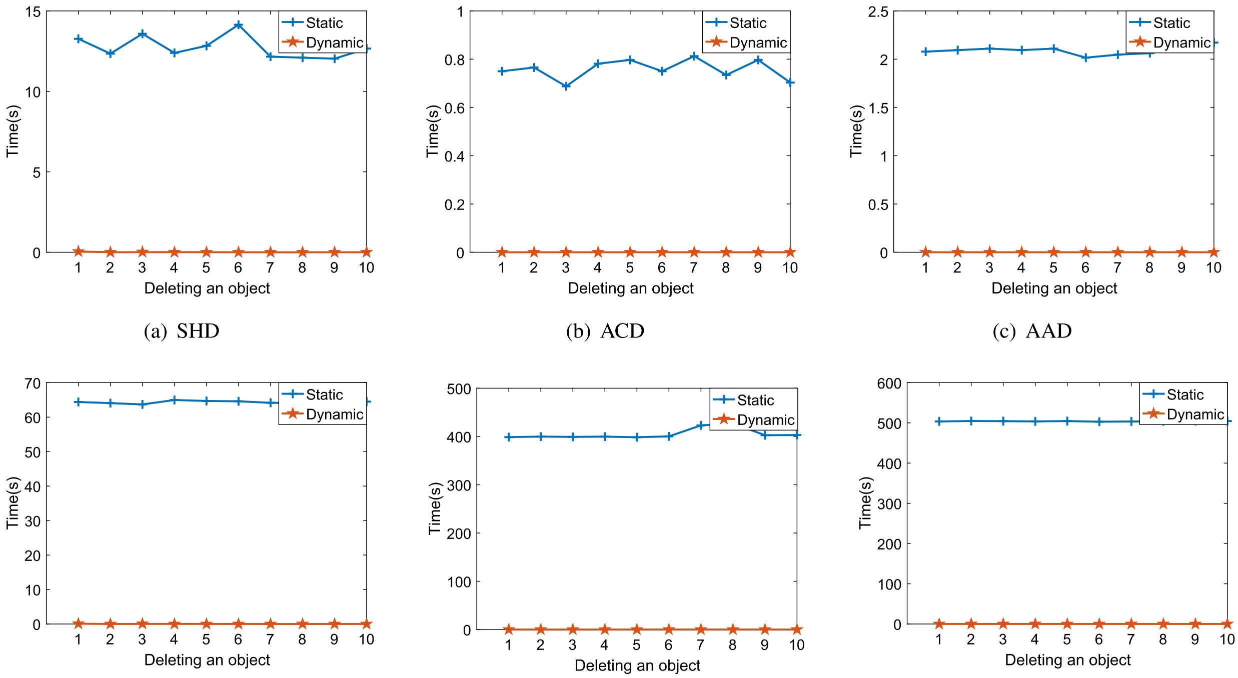
In this subsection, some numerical experiments are conducted to evaluate the performance of the proposed dynamic updating algorithm when an attribute or object is deleted.

Firstly, we randomly delete an attribute from the original context and compare the time of static updating method with dynamic updating method for AE-granular concepts. We repeat 10 experiments in order to ensure the accuracy and objectivity of the experiment. As shown below, Table 9 shows the time of six data sets about updating AE-granular concepts for deleting an attribute. The time consumption of dynamic algorithm and static algorithm for deleting an attribute can be obtained in Fig. 3. The deleted attribute as abscissa and time consumption is used as ordinate. It can be seen from Table 13 that the time computation of dynamic algorithm is much less than the static algorithm when an attribute are deleted. The dynamic updating algorithm does not need to obtain the subcontext, but only needs to consider the change of the granular concept affected by this attribute from the primitive granular concept when deleting an attribute. To sum up, it can be seen that the dynamic algorithm is much better than the static algorithm.

Secondly, we compare the time of updating AE-granular concepts by static and dynamic methods when an object is deleted randomly from the original context. In order to ensure the accuracy and objectivity of the experiment, we repeat 10 experiments as same as the case of deleting an attribute. As shown below, Table 10 shows the time to update the AE-granular concept for six datasets when an object is deleted. The time consumption of dynamic algorithm and static algorithm for deleting an object can be obtained in Fig. 4. The deleted attribute as abscissa and time consumption is used as ordinate. By analyzing and comparing Table 14 and Fig. 4, we can find that the time required for the dynamic update of the AE-granular concepts is close to 0 when only one object is deleted. The time required for the static update way is significantly higher than dynamic update way. When an object is deleted, the dynamic updating method of AE-granular concept only needs to consider the change of the extension and connotation affected by the deleted object from primitive

**Table 13**
Comparison of time consumption between static update method and dynamic update method of AE-granular concept when deleting an attribute (unit is second).

| Attribute | SHD | | ACD | | AAD | | DUS | | AASN | | sp | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic |
| $a_1$ | 13.9531 | 0.0469 | 0.2031 | 0.0000 | 1.7188 | 0.0000 | 60.8594 | 0.0000 | 393.4063 | 0.0000 | 350.9063 | 0.0000 |
| $a_2$ | 13.2813 | 0.0000 | 0.1875 | 0.0000 | 1.5469 | 0.0000 | 59.5469 | 0.0000 | 396.4688 | 0.0000 | 362.6719 | 0.0313 |
| $a_3$ | 13.3750 | 0.0000 | 0.1719 | 0.0000 | 1.7969 | 0.0000 | 58.3906 | 0.0000 | 382.7813 | 0.0000 | 334.5469 | 0.0000 |
| $a_4$ | 13.9219 | 0.0000 | 0.1563 | 0.0000 | 1.6875 | 0.0000 | 58.4531 | 0.0000 | 384.3281 | 0.0000 | 350.0625 | 0.0000 |
| $a_5$ | 12.7969 | 0.0313 | 0.1563 | 0.0000 | 1.6875 | 0.0000 | 58.7188 | 0.0000 | 387.3438 | 0.0000 | 350.9063 | 0.0000 |
| $a_6$ | 12.3281 | 0.0313 | 0.2344 | 0.0000 | 1.7031 | 0.0000 | 60.3906 | 0.0000 | 388.9219 | 0.0000 | 345.3378 | 0.0000 |
| $a_7$ | 12.0625 | 0.0000 | 0.1563 | 0.0000 | 1.6875 | 0.0000 | 59.0000 | 0.0000 | 397.5469 | 0.0000 | 361.7256 | 0.0000 |
| $a_8$ | 12.8750 | 0.0000 | 0.1719 | 0.0000 | 1.6563 | 0.0000 | 59.7813 | 0.0000 | 400.1719 | 0.0000 | 347.3367 | 0.0000 |
| $a_9$ | 12.1563 | 0.0000 | 0.2344 | 0.0000 | 1.6250 | 0.0000 | 60.1563 | 0.0000 | 383.3906 | 0.0000 | 358.3245 | 0.0000 |
| $a_{10}$ | 12.3281 | 0.0000 | 0.1563 | 0.0000 | 1.5156 | 0.0000 | 65.0938 | 0.0000 | 398.0469 | 0.0000 | 341.4127 | 0.0000 |




(a) SHD (b) ACD (c) AAD (d) DUS (e) AASN (f) sp

**Fig. 4.** The time consumption between static and dynamic update method when deleting an object.

**Table 14**
Comparison of time consumption between static update method and dynamic update method of AE-granular concept when deleting an object (unit is second).

| Object | SHD | | ACD | | AAD | | DUS | | AASN | | sp | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic |
| $x_1$ | 13.2656 | 0.0469 | 0.7500 | 0.0000 | 2.0781 | 0.0000 | 64.3906 | 0.0781 | 398.6563 | 0.0625 | 503.5625 | 0.0156 |
| $x_2$ | 12.3438 | 0.0000 | 0.7656 | 0.0000 | 2.0938 | 0.0000 | 64.0625 | 0.0000 | 399.7344 | 0.0000 | 504.625 | 0.0313 |
| $x_3$ | 13.5781 | 0.0156 | 0.6875 | 0.0000 | 2.1094 | 0.0000 | 63.6875 | 0.0625 | 399.1250 | 0.0000 | 504.2969 | 0.0000 |
| $x_4$ | 12.3906 | 0.0000 | 0.7813 | 0.0000 | 2.0938 | 0.0000 | 64.9688 | 0.0000 | 399.6875 | 0.0000 | 503.6719 | 0.0313 |
| $x_5$ | 12.8281 | 0.0000 | 0.7969 | 0.0000 | 2.1094 | 0.0000 | 64.6875 | 0.0000 | 398.4063 | 0.0625 | 504.4688 | 0.0000 |
| $x_6$ | 14.1406 | 0.0000 | 0.7500 | 0.0000 | 2.0156 | 0.0000 | 64.5938 | 0.0000 | 400.2188 | 0.0625 | 502.9219 | 0.0000 |
| $x_7$ | 12.1563 | 0.0000 | 0.8125 | 0.0000 | 2.0469 | 0.0000 | 64.1563 | 0.0000 | 422.9688 | 0.0625 | 503.4844 | 0.0313 |
| $x_8$ | 12.0938 | 0.0000 | 0.7344 | 0.0000 | 2.0625 | 0.0000 | 64.1094 | 0.0156 | 427.0781 | 0.0625 | 503.9688 | 0.0313 |
| $x_9$ | 12.0313 | 0.0000 | 0.7969 | 0.0000 | 2.1250 | 0.0000 | 64.2031 | 0.0000 | 402.6250 | 0.0625 | 504.5313 | 0.0000 |
| $x_{10}$ | 12.6563 | 0.0000 | 0.7031 | 0.0000 | 2.1719 | 0.0000 | 64.5000 | 0.0000 | 402.9531 | 0.0000 | 504.5313 | 0.0313 |

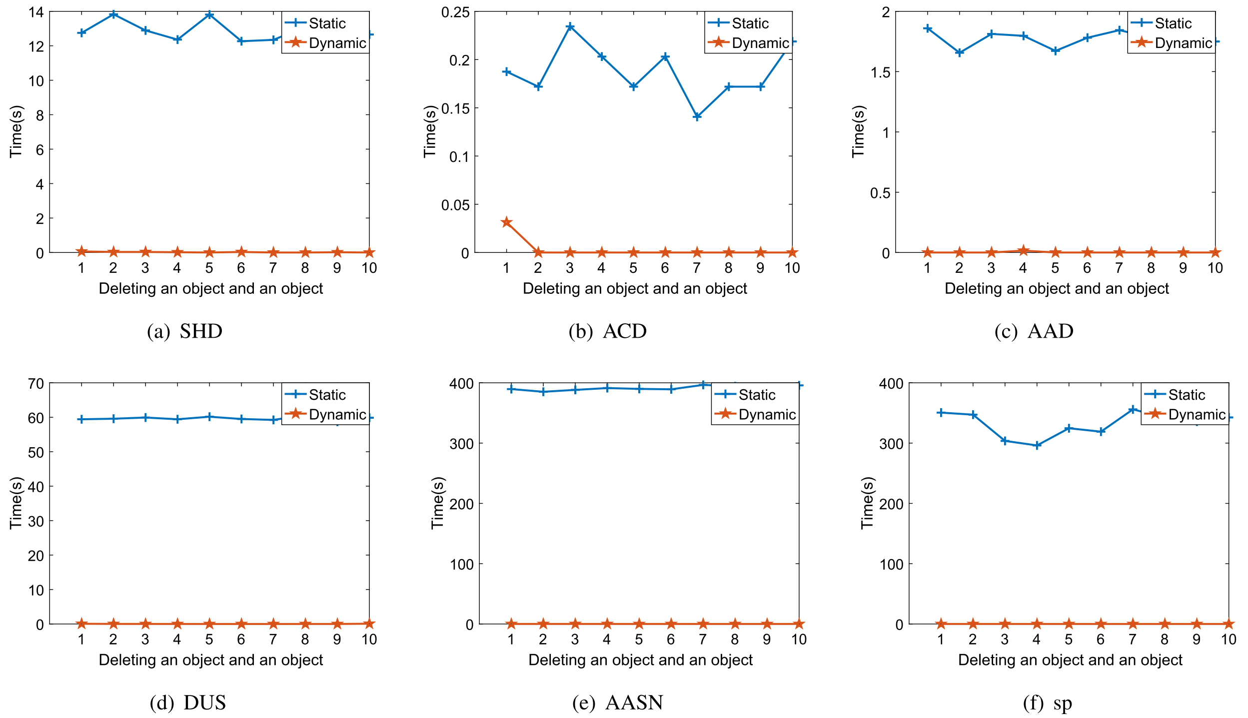



**Fig. 5.** The time consumption between static and dynamic update method when deleting an attribute and object.

**Table 15**
Comparison of time consumption between static update method and dynamic update method of AE-granular concept when deleting an attribute and an object (unit is second).

| AT and OB | SHD | | ACD | | AAD | | DUS | | AASN | | sp | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic |
| $a_1, x_1$ | 12.7500 | 0.0625 | 0.1875 | 0.0313 | 1.8594 | 0.0000 | 59.4219 | 0.0625 | 389.5938 | 0.0469 | 350.5625 | 0 |
| $a_2, x_2$ | 13.8281 | 0.0313 | 0.1719 | 0.0000 | 1.6563 | 0.0000 | 59.5781 | 0.0156 | 385.1406 | 0.0625 | 347.1719 | 0.0156 |
| $a_3, x_3$ | 12.8906 | 0.0313 | 0.2344 | 0.0000 | 1.8125 | 0.0000 | 59.9219 | 0.0000 | 388.2813 | 0.0781 | 303.6406 | 0.0469 |
| $a_4, x_4$ | 12.3594 | 0.0156 | 0.2031 | 0.0000 | 1.7969 | 0.0000 | 59.4219 | 0.0000 | 391.2969 | 0.0625 | 296.1563 | 0 |
| $a_5, x_5$ | 13.8125 | 0.0000 | 0.1719 | 0.0000 | 1.6719 | 0.0000 | 60.1563 | 0.0000 | 389.9219 | 0.0000 | 324.6532 | 0 |
| $a_6, x_6$ | 12.2656 | 0.0313 | 0.2031 | 0.0000 | 1.7813 | 0.0000 | 59.5000 | 0.0000 | 389.2344 | 0.0156 | 318.9752 | 0 |
| $a_7, x_7$ | 12.3438 | 0.0000 | 0.1406 | 0.0000 | 1.8438 | 0.0000 | 59.2344 | 0.0000 | 396.5781 | 0.0156 | 355.928 | 0 |
| $a_8, x_8$ | 13.1094 | 0.0000 | 0.1719 | 0.0000 | 1.7813 | 0.0000 | 60.625 | 0.0000 | 394.2344 | 0.0156 | 342.5610 | 0 |
| $a_9, x_9$ | 12.3281 | 0.0156 | 0.1719 | 0.0000 | 1.8125 | 0.0000 | 58.7188 | 0.0000 | 383.6250 | 0.0000 | 335.7236 | 0 |
| $a_{10}, x_{10}$ | 12.6563 | 0.0000 | 0.2188 | 0.0000 | 1.75 | 0.0000 | 59.8594 | 0.0781 | 395.7656 | 0.0625 | 342.5671 | 0 |

concept. Therefore, we can consider that the dynamic updating method of the AE-granular concepts is much better than the static updating method in the case of deleting an object.

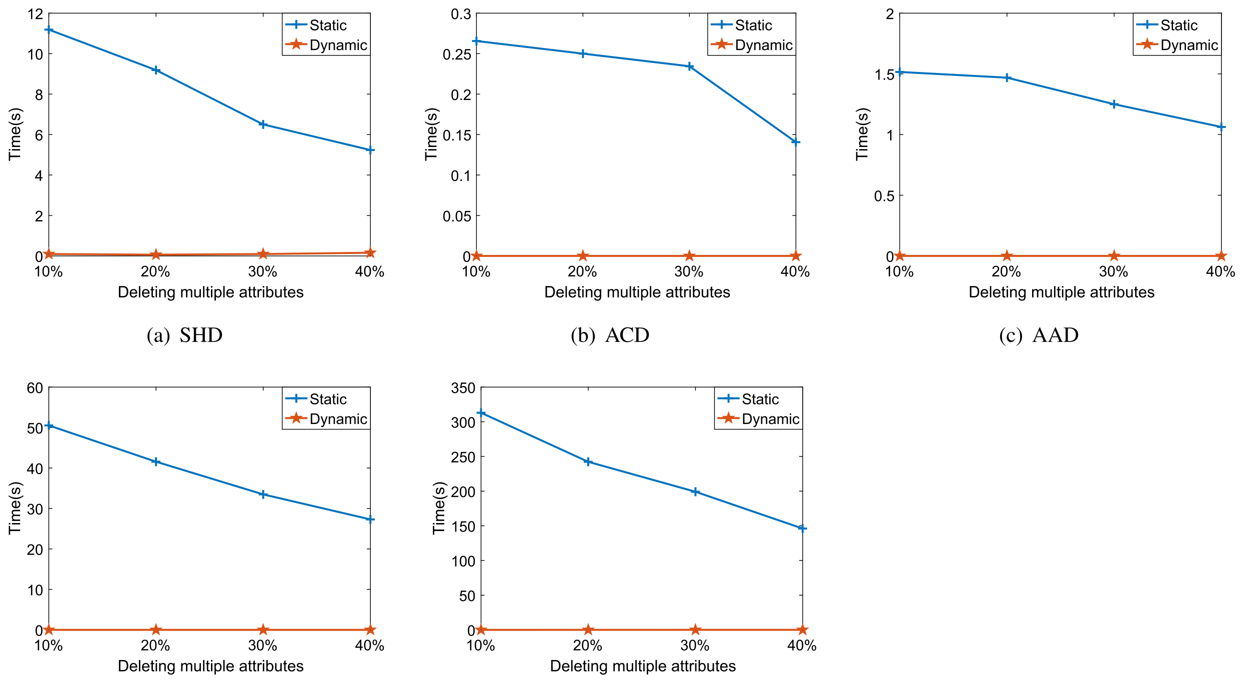
Finally, we randomly delete an attribute and object from the original context and compare the time of static updating method with dynamic updating method for AE-granular concepts. We repeat 10 experiments in order to ensure the accuracy and objectivity of the algorithm. As shown below, Table 11 shows the time to update the AE-granular concept by dynamic and static algorithm for six data sets when an object and object are deleted. The time consumption of dynamic algorithm and static algorithm for deleting an attribute and object can be obtained in Fig. 5. The deleted object and attribute as abscissa and time consumption is used as ordinate. It can be seen from Table 15 and Fig. 5 that the computation time of dynamic algorithm is much less than that of the static algorithm after an object are deleted. In particular, the time of updating AE-concept by dynamic approach is close to zero. So, the dynamic algorithm is better than the static algorithm in formal context.

### *6.2. In the case of deleting multiple attributes or objects, the time consumption of the AE-granular concepts is compared*

In this subsection, some numerical experiments are conducted to evaluate the performance of the proposed dynamic updating algorithm when multiple attributes or objects are deleted.

**Table 16**
Comparison of time consumption between static update method and dynamic update method of AE-granular concept when deleting multiple attributes (unit is second).

| Attributes | SHD | | ACD | | AAD | | DUS | | AASN | |
|---|---|---|---|---|---|---|---|---|---|---|
| | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic |
| 10% | 11.1875 | 0.0938 | 0.2656 | 0.0000 | 1.5156 | 0.0000 | 50.5313 | 0.0000 | 313.0000 | 0.0000 |
| 20% | 9.1875 | 0.0625 | 0.2500 | 0.0000 | 1.4688 | 0.0000 | 41.5625 | 0.0000 | 242.2813 | 0.0156 |
| 30% | 6.5000 | 0.0938 | 0.2344 | 0.0000 | 1.2500 | 0.0000 | 33.4844 | 0.0000 | 199.1406 | 0.0313 |
| 40% | 5.2344 | 0.1563 | 0.1406 | 0.0000 | 1.0625 | 0.0000 | 27.2969 | 0.0000 | 146.0781 | 0.0313 |



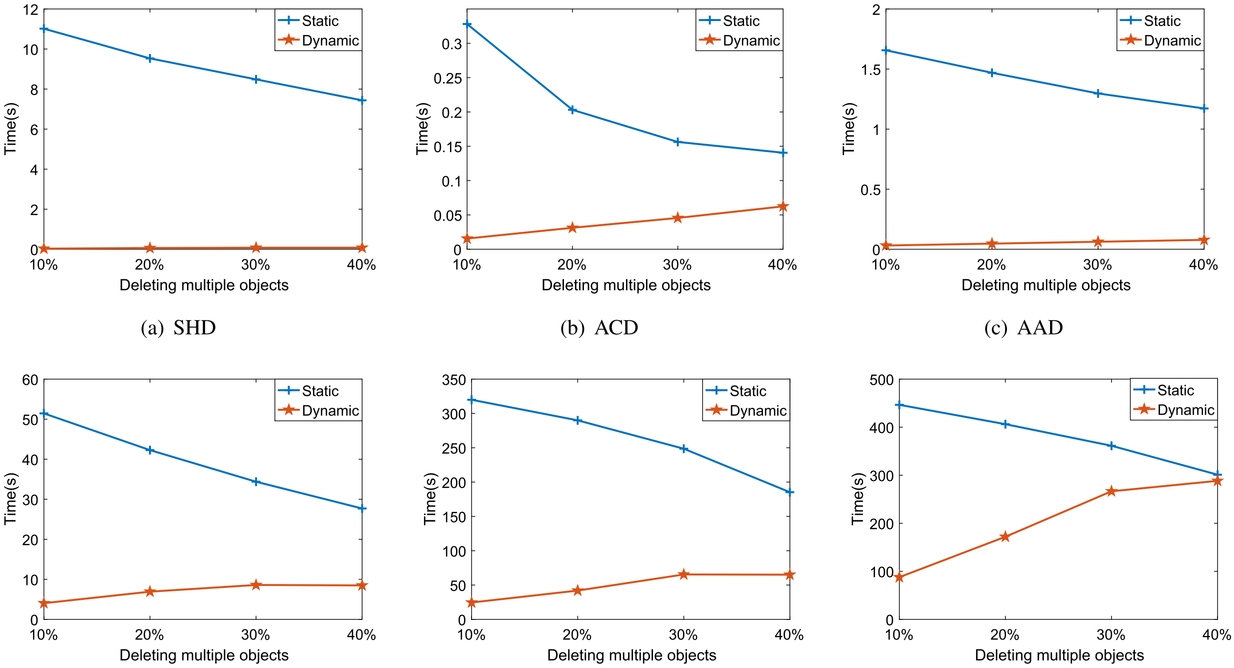

(a) SHD (b) ACD (c) AAD

(d) DUS (e) AASN

**Fig. 6.** The time consumption between static and dynamic update method when deleting multiple attributes.

Firstly, we randomly delete multiple attributes proportionally in the original context and compare the time of updating AE-granular concepts by static and dynamic methods. In order to ensure the accuracy and objectivity of the experiment, we delete 10%, 20%, 30% and 40% attributes, respectively. Because the number (only 4) of attributes in data set sp is too small, we do not consider the update time of dynamic and static methods when deleting multiple attributes. Table 16 shows the time to update the AE-granular concept by dynamic and static algorithm for six data sets when multiple attributes are deleted. The time consumption of dynamic algorithm and static algorithm for deleting multiple attributes can be obtained in Fig. 6. The deleted objects and attributes as abscissa and time consumption is used as ordinate. It can be seen from Table 15 and Fig. 6 that the computation time of dynamic algorithm is much less than that of the static algorithm after an object are deleted. Thus, the dynamic algorithm is much better than the static algorithm in formal context.

Secondly, we randomly delete multiple objects proportionally in the original context and compare the time of updating AE-granular concepts by static and dynamic methods. In order to ensure the accuracy and objectivity of the experiment, we delete 10%, 20%, 30% and 40% objects, respectively. Table 16 shows the time to update the AE-granular concept by dynamic and static algorithm for six data sets when multiple objects are deleted. The time consumption of dynamic algorithm and static algorithm for deleting multiple object can be obtained in Fig. 7. The deleted objects as abscissa and time consumption is used as ordinate. We can see from Table 17 and Fig. 7 that the time of static update algorithm decreases with the increase of deleted objects, and the time of dynamic update algorithm increases with the increase of deleted objects. From the first five data sets (SHD, ACD, AAD, DUS, AASN), we can see that the dynamic update algorithm is still better than the static update algorithm. From the data set sp, we can see that the time of dynamic update algorithm is close to that of static algorithm when the deleted object reaches 40%. This is because the data set sp has only four attributes, and the dynamic algorithm works better in the data set with more attributes than in the data set with fewer attributes. Hence, we think the dynamic algorithm is better than the static algorithm in formal context.




(a) SHD (b) ACD (c) AAD

(d) DUS (e) AASN (f) sp

**Fig. 7.** The time consumption between static and dynamic update method when deleting multiple objects.

**Table 17**
Comparison of time consumption between static update method and dynamic update method of AE-granular concept when deleting multiple objects (unit is second).

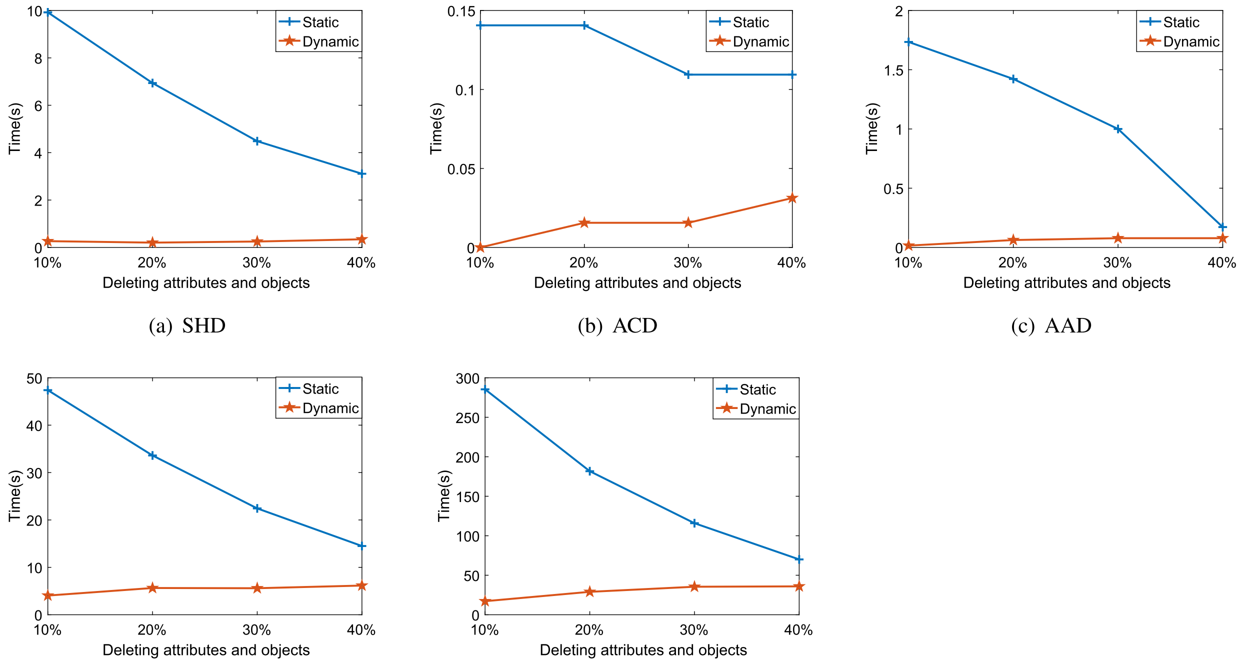
| Objects | SHD | | ACD | | AAD | | DUS | | AASN | | sp | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic |
| 10% | 11.0156 | 0.0313 | 0.3281 | 0.0156 | 1.6563 | 0.0313 | 51.4375 | 4.0469 | 319.8906 | 24.5938 | 446.4844 | 87.7656 |
| 20% | 9.5313 | 0.0625 | 0.2031 | 0.0313 | 1.4688 | 0.0469 | 42.2656 | 6.9219 | 289.9063 | 41.9688 | 406.2031 | 171.9531 |
| 30% | 8.4844 | 0.0781 | 0.1563 | 0.0469 | 1.2969 | 0.0625 | 34.375 | 8.5938 | 248.5781 | 65.4219 | 361.2969 | 266.5469 |
| 40% | 7.4375 | 0.0781 | 0.1406 | 0.0625 | 1.1719 | 0.0781 | 27.6875 | 8.4844 | 185.2969 | 73.0938 | 301.0625 | 288.6816 |

Finally, we randomly delete multiple attributes and multiple objects proportionally in the original context and compare the time of updating AE-granular concepts by static and dynamic methods. In order to ensure the accuracy and objectivity of the experiment, we delete 10%, 20%, 30% and 40% attributes and objects, respectively. Because the number (only 4) of attributes in data set sp is too small, we do not consider the update time of dynamic and static methods when deleting multiple attributes and objects. Table 17 shows the time to update the AE-granular concept by dynamic and static algorithm for six data sets when multiple attributes and multiple objects are deleted. The time consumption of dynamic algorithm and static algorithm for deleting multiple object can be obtained in Fig. 8. The deleted objects and attributes as abscissa and time consumption is used as ordinate. We can see from Table 18 and Fig. 8 that the time of static update algorithm decreases with the increase of deleted attributes and objects, and the time of dynamic update algorithm increases with the increase of deleted attributes and objects. Although the advantage of dynamic update method when 40% attributes and objects are deleted at the same time is not as obvious as that of single deletion of attributes or objects, dynamic update algorithm is still better than static update algorithm. By this token, we think the dynamic algorithm is better than the static algorithm in formal context.

## 7. Conclusions and future works

In this part, we first introduce the main conclusions of this paper. Then, the future research work is prospected.

**(1) Main conclusions of our paper** In real life, the formal context changes constantly as information is updated. If the information needed to be rediscovered through the updated formal context, the time cost will be greatly increased. Therefore, it is very important for knowledge discovery and feature extraction to find a fast way to acquire AE-granular concepts. Then, we study a method for dynamically extracting AE-granular concepts from the original formal context. This paper mainly discusses the dynamic updating rules and calculation methods of AE-granular concepts when multiple attributes or




(a) SHD (b) ACD (c) AAD

(d) DUS (e) AASN

**Fig. 8.** The time consumption between static and dynamic update method when deleting multiple objects and attributes.

**Table 18**
Comparison of time consumption between static update method and dynamic update method of AE-granular concept when deleting multiple attributes and objects (unit is second).

| ATs and OBs | SHD | | ACD | | AAD | | DUS | | AASN | |
|---|---|---|---|---|---|---|---|---|---|---|
| | static | dynamic | static | dynamic | static | dynamic | static | dynamic | static | dynamic |
| 10% | 9.9219 | 0.2656 | 0.1406 | 0.0000 | 1.7344 | 0.0156 | 47.3906 | 4.0469 | 285.3906 | 17.1406 |
| 20% | 6.9375 | 0.2031 | 0.1406 | 0.0156 | 1.4219 | 0.0625 | 33.6094 | 5.6406 | 181.7188 | 29.000 |
| 30% | 4.4844 | 0.2500 | 0.1094 | 0.0156 | 1.0000 | 0.0781 | 22.4375 | 5.5938 | 115.8906 | 35.4375 |
| 40% | 3.1094 | 0.3438 | 0.1094 | 0.0313 | 0.1719 | 0.0781 | 14.5000 | 6.1563 | 70.0313 | 35.9219 |

objects are deleted. Firstly, a dynamic updating method of AE-granular concept is proposed. Secondly, a dynamic updating method of AE-granular concept is proposed to overcome the shortcomings of static updating algorithm, such as high time complexity and large space occupation. The dynamic updating algorithm can get the AE-granular concept in the sub-context directly by utilizing the relationship between the AE-granular concept and the deleted attributes or objects in the original formal context. This updating method avoids the repeated calculation of the AE-granular concept. Finally, the superiority of the dynamic update method is proved through experimental analysis.

**(2) Future research work** Three-way granular concepts play an important role in the field of conceptual cognition. The dynamic updating method proposed in this paper is significant to simplify the temporal and spatial complexity of conceptual cognitive model in dynamic context. On this basis, the dynamic update algorithm also provides a new idea for the rapid construction of concept lattices. In addition, the dynamic updating model of three-way granular concepts has broad application prospects in artificial intelligence, machine learning, data mining and other fields.

## Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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