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Double threshold construction method for attribute-induced three-way concept lattice in incomplete fuzzy formal context

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Abstract: Concept lattice theory is the basis of formal concept analysis. It is an effective tool for knowledge representation and discovery. Three-way decision theory is a new decision theory based on rough sets and decision rough sets. In this study, a method is proposed for constructing attribute-induced three-way concept lattices in an incomplete fuzzy formal context. First, the authors introduce the related knowledge of the classical formal context, the fuzzy formal context and incomplete fuzzy formal context in detail. They put forward a method to fill the incomplete formal context. Secondly, they propose a double threshold operation, which divides the relationship between objects and attributes into three parts: belonging, not belonging and undetermined. Thirdly, they propose three methods to complement incomplete formal context and introduce their advantages and disadvantages. Fourthly, they deal with incomplete formal context by special symbol completion method and get related algorithms of attribute-induced three-way concept lattice. Finally, a case is given to verify the effectiveness of the algorithm for constructing attribute-induced three-way concept lattices.

1 Introduction

German mathematician Wille put forward formal concept analysis in 1982, which is a powerful tool for data analysis and processing [1–3]. Formal concept analysis is a formal expression of objects, attributes and relationships that make up an ontology [4, 5]. Then, the corresponding concept lattice is constructed according to the formal context. Since concept lattice is the core data structure of formal concept analysis theory, many research achievements focus on the construction and reduction of concept lattice [6, 7].

Fuzzy set theory is a method of describing fuzzy phenomena created by American scholar Zadeh. It expresses the relationship between objects and attributes by a membership function, which is different from the classical set. We can analyse the fuzzy objects through the operation and transformation of fuzzy sets.

Three-way decision theory is a new decision theory proposed by Yao [8] on the basis of rough set. A new concept lattice (three-way concept lattice) is obtained by combining three-way decision theory with concept theory. Then, object-oriented three-way operator and attribute-oriented three-way operator are established. Finally, object-oriented three-way concept lattice and attribute-oriented three-way concept lattice are proposed. The three-way concepts contain more rich language, which can make the description of information more clearer than classical formal concepts [9]. Nowadays, three-way decision theory is playing more and more important roles in attribute reduction and rule extraction. The theory has made many theoretical achievements and has been widely applied.

So far, many scholars have made a lot of achievements about the three-way concepts. Hui and Qing [10] studied the characteristics of three-way concept lattices and three-way rough concept lattices. Ren and Wei [11] studied the attribute reductions of three-way concept lattices. Yao [12] studied interval sets and three-way concept analysis in incomplete contexts. Huang *et al*. [13] studied the three-way concept learning based on cognitive operators: an information fusion viewpoint. Qian *et al*. [14] construct the three-way concept lattices based on apposition and subposition of formal contexts. Yu *et al*. [15] analysed its characteristics of three-way concept lattices and three-way rough concept lattices. Qi *et al*. [16] explored the connections between three-way and classical concept lattices. Qi *et al*. [17] summarised three-way formal concept analysis. Li *et al*. [18] proposed three-way cognitive concept learning via multi-granularity. Singh [19] studied three-way fuzzy concept lattice representation using neutrosophic set. Li and Wang [20] proposed approximate concept construction with three-way decisions and attribute reduction in incomplete contexts. Leijun *et al*. [21] explored analysis and comparison of concept lattices from the perspective of three-way decisions. Liya *et al*. [22] studied dynamic strategy regulation model of three-way decisions based on interval concept lattice and its application. Singh [23] proposed three-way *n*-valued neutrosophic concept lattice at different granulations. Wang and Li [24] summarised three-way decisions, concept lattice and granular computing.

However, the current three-way concepts are only constructed in the complete classical form context. In real life, sometimes the relationship between attributes and objects does not completely belong to or is not completely separate. Fuzzy formal context is obtained by combining formal context with fuzzy set theory. So far, no scholars have studied the construction method of the three-way concepts in the fuzzy form context. We propose a double threshold method to deal with the fuzzy formal context by the three-way decision theory and obtain a method to construct the three-way concept lattice. Sometimes, information is lost due to data storage failure, memory corruption, mechanical failure, loss of data and other reasons. It leads to incomplete formal context.

In view of the incomplete formal context, we further study the construction method of three-way concept in the incomplete fuzzy formal context in order to solve the problem. First, we deal with the incomplete fuzzy formal context with the double threshold method and get a special classical formal context. Then, three methods of completing incomplete fuzzy formal context are proposed and their advantages and disadvantages are introduced in detail. Finally, we construct a three-way concept lattice in the formal context of completion.

2 Preliminary

In this section, we review the complete formal context and some basic notions of the three-way concept lattices.

Definition 1: A formal context $F = (G, M, I)$ consists of two sets G and M and a binary relation I between G and M. The

Table 1 Formal context $F = (G, M, I)$

G/M	a_1	a_2	a_3	a_4	a_5	a_6
x_1	1	0	0	1	1	0
x_2	0	0	0	1	1	0
x_3	0	1	1	1	0	0
x_4	1	0	0	0	1	1

elements of G are called objects and the elements of M are called attributes. For an object x and an attribute a, we call the object x has the attribute a if $(x, a) \in I$ and the object x does not have the attribute a if $(x, a) \notin I$. A detailed description of them can be found in references [1, 2].

A complete formal context can be represented by a two-dimensional table. '1' indicates that the object x has the attribute a and '0' indicates that the object x does not have the attribute a. In this paper, we assume that the object set G and the attribute set M are finite. The complete traditional form context as shown in Table 1.

Definition 2: Let U be a non-empty finite set, $P(U)$ be the power set of the domain U, $\mathscr{DP}(U)$ be the Cartesian product $\mathscr{P}(U) \times \mathscr{P}(U)$. For $(A, B), (C, D) \in \mathscr{DP}(U)$, we have

$$\begin{aligned} &(1) \quad (A, B) \in (C, D) \Leftrightarrow A \subseteq CB \subseteq D \\ &(2) \quad (A, B) \cap (C, D) = (A \cap C, B \cap D) \\ &(3) \quad (A, B) \cup (C, D) = (A \cup C, B \cup D) \end{aligned}$$

Definition 3: Let (G, M, I) be a formal context. For $X \subseteq G$ and $A \subseteq M$, a pair of operators $* : \mathscr{P}(M) \to \mathscr{P}(G)$ and $* : \mathscr{P}(G) \to \mathscr{P}(M)$ are defined by

$$X^* = \{a \in M \mid \forall x \in X, (x, a) \in I\}$$

$$A^* = \{x \in X \mid \forall a \in M, (x, a) \in I\}$$

X^* is the maximal subset of the attributes shared by the objects in X, A^* is the maximal subset of objects that possess all the attributes in A.

Definition 4: Let (G, M, I) be a formal context. For $X, Y \subseteq G$, $A \subseteq M$, a pair of attribute-induced three-way operators $< \cdot : \mathscr{P}(M) \to \mathscr{DP}(G)$ and $\cdot > : \mathscr{DP}(G) \to \mathscr{P}(M)$ are defined by

$$\begin{aligned} A^{<\cdot} &= \left(X^*, X^{\bar{*}}\right) \\ (X, Y)^{\cdot >} &= \{a \in M \mid a \in X^*, a \in Y^{\bar{*}}\} \\ &= X^* \cap Y^{\bar{*}} \end{aligned}$$

Proposition 1: A pair of attribute-induced three-way operators $< \cdot$ and $\cdot >$ has the following properties:

i. $A \subseteq A^{<\cdot\cdot>}, (X, Y) \subseteq (X, Y)^{\cdot><\cdot}$;
ii. $A \subseteq B \Rightarrow A^{<\cdot} \supseteq B^{<\cdot}, (X, Y) \subseteq (Z, W) \Rightarrow (X, Y)^{\cdot>} \supseteq (Z, W)^{\cdot>}$;
iii. $A^{<\cdot\cdot><\cdot} = A^{<\cdot}, (X, Y)^{\cdot><\cdot\cdot>} = (X, Y)^{\cdot>}$;
iv. $A \subseteq (X, Y)^{\cdot>} \Leftrightarrow (X, Y) \subseteq A^{<\cdot}$;
v. $(A \cup B)^{<\cdot} = A^{<\cdot} \cap B^{<\cdot}, (A \cap B)^{<\cdot} \supseteq A^{<\cdot} \cup B^{<\cdot}$;
vi. $((X, Y) \cup (Z, W))^{\cdot>} = (X, Y)^{\cdot>} \cap (Z, W)^{\cdot>}$;
vii. $((X, Y) \cap (Z, W))^{\cdot>} = (X, Y)^{\cdot>} \cup (Z, W)^{\cdot>}$;

where $X, Y, Z, W \subseteq G$ and $A, B \subseteq M$.

Definition 5: Let (G, M, I) be a formal context. For $X, Y \subseteq G$, $A \subseteq M$, $((X, Y), A)$ is called an attribute-induced three-way concept if and only if $A^{<\cdot} = (X, Y)$ and $(X, Y)^{\cdot>} = A$. We abbreviate it as AE-concept. (X, Y) is called the extension of AE-concept, A is called the intension of AE-concept.

$AEL(G, M, I)$ represents the set of all AE-concepts which generated by the formal context (G, M, I). For $((X, Y), A)$, $((W, Z), B) \in AEL(G, M, I)$, the partial order relationship is defined as follows:

$$((X, Y), A) \leq ((W, Z), B) \Leftrightarrow (X, Y) \subseteq (W, Z) \Leftrightarrow A \subseteq B.$$

$((X, Y), A)$ is called the subconcept of $((W, Z), B)$, $((W, Z), B)$ is called the superconcept of $((X, Y), A)$. $AEL(G, M, I)$ is a complete lattice under the partial order relation $\leq$ defined above. It is called AE-concept lattice. The infimum and supremum are given by:

$$((X, Y), A) \vee ((W, Z), B) = \left((X, Y) \cap (W, Z), (A \cup B)^{<\cdot\cdot>}\right);$$

$$((X, Y), A) \wedge ((W, Z), B) = \left((X, Y) \cup (W, Z), (A \cup B)^{<\cdot\cdot>}\right).$$

Definition 6: Let (G, M, I) be a formal context, $< \cdot :$ and $\cdot > :$ are a pair of three-way operators. A pair $((X, (A, B))$ of an object subset $X \subseteq G$ and two attribute subsets $A, B \subseteq M$ is called an object-induced three-way concept if and only if $X^{<\cdot} = (A, B)$ and $X^{<\cdot\cdot>} = (A, B)^{\cdot>} = X$. We abbreviate it as object induced three-way concept (OE-concept). X is called the extent and (A, B) is called the intent of the OE-concept $(X, (A, B))$.

If $(X, (A, B))$ and $(Y, (C, D))$ are OE-concepts, then they can be ordered by

$$(X, (A, B)) \leq (Y, (C, D)) \Leftrightarrow X \subseteq Y \Leftrightarrow (C, D) \subseteq (A, B).$$

All the OE-concepts form a complete lattice, which is called the object-induced three-way concept lattice of (G, M, I) and written as $OEL(G, M, I)$. The infimum and supremum are given by

$$(X, (A, B)) \wedge (Y, (C, D)) = \left(X \cap Y, ((A, B) \cup (C, D))^{\cdot><\cdot}\right);$$
$$(X, (A, B)) \vee (Y, (C, D)) = \left((X \cup Y)^{<\cdot\cdot>}, (A, B) \cap (C, D)\right).$$

Proposition 2: $< \cdot :$ and $\cdot > :$ are a pair of three-way operators. $\mathscr{Q}_1(G, M, I)$ represents the set of all object-induced three-way concept in the formal context G, M, I. $\mathscr{Q}_2(G, M, I)$ represents the set of all attribute-induced three-way concept in the formal context G, M, I. $\forall (X, (A, B)) \in \mathscr{Q}_1(G, M, I)$ and $\forall ((X, Y), A) \in \mathscr{Q}_2(G, M, I)$ have the following properties:

$$(X, (A, B)) = \left(\bigvee_{x \in X} x^{<\cdot\cdot>}, \bigwedge_{x \in X} x^{<\cdot}\right);$$

$$((X, Y), A) = \left(\bigwedge_{a \in A} a^{<\cdot}, \bigvee_{a \in A} a^{<\cdot\cdot>}\right).$$

We can see that the attribute-induced three-way concept and object-induced three-way concept are dual from the definition [5, 6]. Therefore, we only consider the construction for attribute-induced three-way concept lattice in incomplete fuzzy form context.

3 Construction of attribute-induced three-way concept lattice under the incomplete fuzzy form context

In this section, first, we introduce the notion of a complete fuzzy formal context. Secondly, we introduce the notion of an incomplete fuzzy formal context. Thirdly, we define a method to transform

Table 2 Fuzzy form context $F = (G, M, I)$

G/M	a_1	a_2	a_3	a_4
x_1	0.67	0.75	0.74	0.39
x_2	0.65	0.17	0.70	0.03
x_3	0.27	0.04	0.09	0.82
x_4	0.69	0.31	0.95	0.03
x_5	0.43	0.38	0.76	0.79
x_6	0.18	0.48	0.44	0.64

Table 3 Fuzzy form context $F = (G, M, I)$ of Table 2 by Algorithm 1

G/M	a_1	a_2	a_3	a_4
x_1	1	1	1	0
x_2	1	0	1	0
x_3	0	0	0	1
x_4	1	0	1	0
x_4	*	0	1	1
x_4	0	*	*	1

incomplete fuzzy formal context into incomplete classical formal context. Fourthly, we use the maximum probability method to complete the incomplete formal context. Finally, we construct a attribute-induced three-way concept lattice under the incomplete fuzzy form context.

3.1 Fuzzy formal context

In this section, we mainly introduce the definition of fuzzy set and membership function. A detailed description of them can be found in references [3, 4].

Definition 7: Let the set U be the universe. μ_A is a mapping from X to [0, 1]:

$$U \to [0, 1].$$

μ_A is a fuzzy set on U. For $\forall x \in U$, μ_{A_x} is called the membership degree of x to A.

According to the definition, the fuzzy set is different from the classical set. We can only recognise and master it through the membership function. Therefore, the definition of fuzzy sets is often written as follows:

A fuzzy set A is a map from X to [0, 1] on universe U. $A: X \to [0, 1]$.

In this paper, we equate μ_{A_x} with $A(x)$ and μ_A with A.

Definition 8: A formal context $F = (G, M, I)$ consists of two sets G and M and a binary relation I between G and M. G is a collection of all objects and M is a collection of all attributes. I is a fuzzy set defined on domain $G \times M$. Each element $\langle x, m \rangle$ has a membership degree $\mu(x, m), 0 \le \mu(x, m) \le 1$ in the relationship I.

A complete fuzzy form context as shown in Table 2.

3.2 Attribute-induced three-way concept lattices in fuzzy formal context

Attribute-induced three-way concept is described from two aspects: formal context and its complementary context. In the classical form context, the relationship between objects and attributes is only 0 (not belong to) and 1 (belong to). However, the relationship between objects and attributes is expressed by membership degree in the fuzzy form context. Usually, we choose a threshold α. If $\mu_a(x_1) >= \alpha$, then x_1 is related to a. If $\mu_a(x_1) < \alpha$, then x_2 is not related to a. However, this method is not reasonable at some time. For example, we choose the threshold $\alpha = 0.5$. If $\mu_a(x_1) = 0.51$ and $\mu_a(x_1) = 0.49$, then we think x_1 is related to a and x_2 is not related to a. This is clearly unreasonable. So, a method of constructing attribute-induced three-way concept lattice with double thresholds in the fuzzy formal context is proposed based on the idea of three-way decisions in this paper.

Definition 9: Let $F_1 = (G, M, I_1)$ be an original fuzzy formal context and $F_2 = (G, M, I_2)$ be a new fuzzy formal context. F is a mapping from 1 to 2: $F = \{f \mid \mu_a(x) \to I(x, a)\}$, α, β are given two thresholds and $0 < \alpha < ta < 1$. For any $x \in G$, $a \in M$, then

$$I(x, a) = f(\mu_a(x)) = \begin{cases} 1, & \text{if } \mu_a(x) \ge \beta \\ 0, & \text{if } \mu_a(x) \le \alpha \\ *, & \text{if } \alpha \le \mu_a(x) \le \beta \end{cases}$$

If $\mu_a(x) \ge \beta$, we think x is related to a. So, let us make $I(x, a) = 1$ in the new form context. If $\mu_a(x) \le \alpha$, we think x is not related to a. So, let us make $I(x, a) = 0$ in the new form context. If $\alpha \le \mu_a(x) \le \beta$, we think x is not related to a. So, let us make $I(x, a) = 0$ in the new form context. If $\alpha \le \mu_a(x) \le \beta$, then we do not have enough reason to think that x is related to a, nor can we think that x is not related to a. So, we use special symbols * in the new formal context to express the situation that we cannot determine whether x is related to a, $I(x, a) = *$. x is in the boundaries of attribute-induced three-way concepts containing a when forming concepts by the three-way decision theory.

We choose the threshold $a = 0.4$, $B = 0.6$. Table 3 is the new formal context of Table 2 by Algorithm 1 (see Fig. 1).

The following is the related algorithm about getting the new formal context by Definition 8.

3.3 Incomplete fuzzy formal context

In the previous section, we introduced fuzzy formal context. In this section, we will discuss the incomplete fuzzy formal context in depth. A detailed description of them can be found in references [5].

Definition 10: Let $F = (G, M, \{1, ?, 0\}, I)$ be an incomplete formal context. The elements of $G = \{x_1, x_2, \ldots, x_n\}$ are called the objects and the elements of $M = \{a_1, a_2, \ldots, a_m\}$ are called the attributes of the context. $\{1, ?, 0\}$ is set value and $I \subseteq U \times A \times \{1, ?, 0\}$ is a Ternary Relation of $U, A, \{1, ?, 0\}$. $(x, a, 1)$ denotes that object x has attribute a, $(x, a, 0)$ denotes that object x does not have attribute a and $(x, a, ?)$ denotes you are not sure whether the object x has attribute a. Usually we record $\{1, ?, 0\}$ as V. So incomplete formal context $(U, A, \{1, ?, 0\}, I)$ can be recorded as (U, A, V, I).

Definition 11: Let $(G, M, \{A(x), ?\}, I)$ be an incomplete formal context. The elements of $G = \{x_1, x_2, \ldots, x_n\}$ are called the objects and the elements of $M = \{a_1, a_2, \ldots, a_m\}$ are called the attributes of the context. $A(x)$ is called the membership degree of x to A and $I \subseteq U \times A \times \{A(x), ?\}$ is a Ternary Relation of $G, M, \{A(x), ?\}$. $(x, a, A(x))$ denotes that the membership degree of x to A is $A(x)$ and $(x, a, ?)$ denotes that the membership degree of x to A is uncertain. We record $\{A(x), ?\}$ as FV in this paper. So incomplete fuzzy formal context $(U, A, \{A(x), ?\}, I)$ can be recorded as (U, A, FV, I).

An incomplete formal context as shown in Table 4.

A incomplete fuzzy formal context as shown in Table 5.

3.4 Formal context completion method

There are three problems for constructing attribute-induced three-way concept lattice from incomplete fuzzy formal context. (i) We must transform the incomplete fuzzy formal context into the complete fuzzy formal context. (ii) We need to transform the complete fuzzy formal context into the complete classical formal context. (iii) We construct attribute-induced three-way concept lattices for pretreated formal context. Algorithm 2 (see Fig. 2) is proposed for getting the new formal context by Definitions 8 and 13.

In this paper, three methods of completing formal context are introduced. We introduce their merits and demerits in detail and take the mean substitution method as an example to complete the incomplete formal context. A double threshold method is defined to transform the fuzzy formal context into the classical formal context.

```
Input :
(1) Origin fuzzy formal context (G, M, I_1).
(2) The threshold value α and β.
Output :
(1) New formal context (G, M, I_2).
begin
    Origincontext ← load(Fuzzy formal context (G, M, I_1));
    [m, n] = size(Origincontext);
    Newcontext = zero(m, n); % Establishing an empty matrix
    with the same size as the original formal context.
    for all x ∈ G do
        for any a ∈ M do
            if u_a(x) ≥ β then
                Newcontext(i, j) = 1;
            end
            if u_a(x) ≤ α then
                Newcontext(i, j) = 0;
            end
            if α ≤ u_a(x) ≤ β then
                Newcontext(i, j) = ∗;
            end
        end
    end
end
```

Fig. 1 *Algorithm 1: The algorithm for obtaining new formal context*

Definition 12: (Delete Object): Let (G, M, FV, I) be a incomplete fuzzy formal context. For any $x \in G$, we delete object x in the formal context if $\mu_a(x)$ is unknown and $a \in M$. We call it object deletion method.

The object deletion method is to reduce the number of objects in exchange for the integrity of the formal context. It will cause a lot of waste of resources and discard a lot of information which is hidden in these objects. However, it is the most common and easiest way to deal with missing data. This method is very effective if the missing values account for a small proportion.

Definition 13: (Mean/Mode Completer): Let (G, M, FV, I) be an incomplete fuzzy formal context. For $x_i \in G$, we take the average of $\mu_a(x), x \in G$ which is known as the value of $\mu_a(x_i)$ if $\mu_a(x_i), a \in M$ is unknown.

It is a method proposed to solve the shortcomings of object deletion method, which will not affect the mean value. This method has the best effect on the formal context processing with completely random deletion. However, this method produces biased estimates and reduces variances and standard deviations of variables.

Definition 14: (Treating missing attribute values as special values). Let (G, M, FV, I) be an incomplete fuzzy formal context. For $x_i \in G$, we take the * as the value of $\mu_a(x_i)$ if $\mu_a(x_i), a \in M$ is unknown.

This method does not compromise the integrity of a formal context. Inspired by the three decision-making ideas, we use the special value filling method to complete the incomplete formal

```
Input :
(1) Origin fuzzy formal context (G, M, I_1).
(2) The threshold value α and β.
Output :
(1) New formal context (G, M, I_2).
begin
    Origincontext ← load(Fuzzy formal context (G, M, I_1));
    [m, n] = size(Origincontext);
    Newcontext = zero(m, n); % Establishing an empty matrix
    with the same size as the original formal context.
    for all x ∈ G do
        for any a ∈ M do
            if u_a(x) isnt unknow then
                if u_a(x) ≥ β then
                    Newcontext(i, j) = 1;
                end
                if u_a(x) ≤ α then
                    Newcontext(i, j) = 0;
                end
                if α ≤ u_a(x) ≤ β then
                    Newcontext(i, j) = ∗;
                end
            end
            else
                Newcontext(i, j) = ∗
            end
        end
    end
end
```

Fig. 2 *Algorithm 2: The algorithm for obtaining new formal context*

Table 4 Complete fuzzy form context $F = (G, M, I)$

G/M	a_1	a_2	a_3	a_4	a_5	a_6
x_1	1	0	0	1	0	1
x_2	1	1	?	0	0	0
x_3	1	?	0	1	?	0
x_4	0	0	1	0	1	1

Table 5 Incomplete fuzzy form context $F = (G, M, I)$

G/M	a_1	a_2	a_3	a_4	a_5	a_6
x_1	0.67	?	0.27	0.69	0.43	0.18
x_2	0.75	0.17	?	0.31	0.38	0.48
x_3	?	0.70	0.09	0.95	0.76	0.44
x_4	0.39	0.03	0.82	?	0.79	0.64

```
Input :
(1) Formal context from Algorithms 1 or 3.
Output :
(1) All attribute-induced three-way concepts
begin
    [m,n]=size(context);
    Attribute←the set of all attribute in context;
    J←cell(1,n-1); %Establish a cell matrix of one row, n-1 column
    for i1 = 1 : n − 1 do
        J1, i1 = combntns(Attribute, i1);
    end
    %J is all combinations of attributes in attribute-induced three-way concepts
    concepts ← ϕ
    for all Ji ⊆ J do
        for all Aj ⊆ Ji do
            OB11←G;
            OB00←G;
            for all ai1 ⊆ Aj do
                OB1=[];
                OB0=[];%OB1 is used to store objects that are related to the attribute and OB0 is used to store objects that aren't
                  related to the attribute.
                for all x ∈ G do
                    if ua1(x) = 1 then
                        OB1=[OB1;x];
                    end
                    if ua1(x) = 0 then
                        OB0=[OB0;x];
                    end
                end
            end
            OB11 = intersect(OB11, OB1);
            OB11 = intersect(OB11, OB1);
            AT11←G;
            AT00←G;
            for all x ⊆ OB11 do
                AT1← ϕ ;
                for all a ∈ M do
                    if ua(x)=1 then
                        AT1=[AT1;x];
                    end
                end
                AT11= intersect (AT11,AT1);
            end
            for all x ⊆ OB00 do
                AT0← ϕ;
                for all a ∈ M do
                    if ua(x) = 0 then
                        AT01=[AT1;x];
                    end
                end
                AT00=intersect (AT00,AT0);
            end
            AT= intersect (AT11,AT00);
            if AT = Aj then
                < (OB11, OB00), AT >is a attribute-induced three-way concepts concepts = [concepts; < (OB11, OB00), AT >];
            end
        end
    end
end
```

Fig. 3 *Algorithm 3: The algorithm for obtaining all attribute-induced three-way concepts*

context and we order the unknown data as *. We do not know if x is related to a or how much they are related to each other. So, x is in the boundary region of attribute-induced three-way concepts containing a when forming concepts.

Theorem 1: Let $F = (G, M, I_2)$ be a new formal context of incomplete fuzzy formal context by Definitions 8 and 13. $C_i = \langle (X, Y), A \rangle$ is an attribute-induced three-way concept in F. If $x \in G$, $a \in A$ and $\mu_a(x) = *$. Then, $x \notin X$ and $x \notin Y$. So, $x \in G - X - Y$.

Proof: Because $\mu_a(x) = *$, $\mu_a(x) \neq 1$, x is not in the positive region of C_i, $x \notin X$. Because $\mu_a(x) \neq 0$, x is not in the positive region of C_i, $x \notin Y$. So, x is in the boundary region of C_i,

Table 6 Incomplete fuzzy form context $F = (G, M, I)$

G/M	a_1	a_2	a_3	a_4
x_1	1	1	*	0
x_2	*	0	1	0
x_3	0	*	0	1
x_4	1	0	1	*
x_5	*	0	1	1
x_6	0	*	*	1

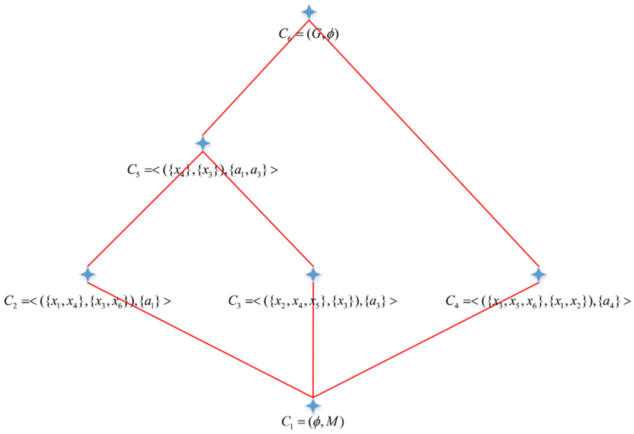

Fig. 4 *Attribute-induced three-way concept lattice*

$x \in G - X - Y$. Algorithm 3 (see Fig. 3) is proposed for obtaining all attribute-induced three-way concepts.□

All attribute-induced three-way concepts are obtained from Table 6 through Algorithm 3.

$$C_1 = (\phi, M); \quad C_2 = \langle (\{x_1, x_4\}, \{x_3, x_6\}), \{a_1\} \rangle;$$
$$C_3 = \langle (\{x_2, x_4, x_5\}, \{x_3\}), \{a_3\} \rangle; \quad C_4 = \langle (\{x_3, x_5, x_6\}, \{x_1, x_2\}), \{a_4\} \rangle;$$
$$C_5 = \langle (\{x_4\}, \{x_3\}), \{a_1, a_3\} \rangle; \quad C_6 = (G, \phi);$$

The following is a figure of attribute-induced three-way concept lattice consisting of attribute-induced three-way concepts (Fig. 4).

4 Conclusions

In this paper, three-way concept lattices are constructed for the first time in an incomplete fuzzy formal context. First, we propose a method to transform the fuzzy formal context into a special classical context based on double thresholds and three-way decision theory. Secondly, we propose three methods to fill in incomplete context. Finally, we propose an algorithm to construct three-way concept lattice in an incomplete fuzzy formal context.

We find that the algorithm of constructing three concept lattices in incomplete fuzzy formal context has a good effect and only *a* few information in the original formal context is lost through the analysis of examples. Next, our research directions include fast generation algorithm of the three-way concept lattice and association rules extraction algorithm in an incomplete fuzzy formal context, distributed storage algorithm, pruning and compression algorithm of large lattices, the extended algorithm of lattice model and so on.

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