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Attribute reducts of multi-granulation information system

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Abstract
In recent years, more and more methods and theories of multi-granulation information systems have been explored. However, there is very limited investigation on the attribute reducts of multi-granulation rough sets. Therefore, the main objective of this paper is to draw attention to the attribute reducts of multi-granulation information system. For any subset of information system, we usually characterize it by its upper and lower approximations. In order to calculate the upper and lower approximations faster, we must reduce the redundant information of the information system. According to the preceding analysis, we first introduce three types of attribute reduct, which are called arbitrary union reduct, neighborhood union reduct and neighborhood intersection reduct, respectively. Then many basic and important results of these reducts are deeply explored. In order to apply the theories of attribute reducts to deal with practical issues, we develop three algorithms so as to compute multi-granulation upper and lower approximations. Next, we further study the interrelationships among these attribute reducts. Finally, we present a multi-granulation information system with respect to thirty students' exam scores and calculate the corresponding attribute reducts by using the algorithms listed in the paper.

Keywords Rough sets · Multi-granulation · Reduct · Lower and upper approximations

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1 Introduction

Rough set theory, proposed by Pawlak Pawlak (1982), provides a useful method for dealing with incomplete and inconsistent knowledge. The theory is widely used in various fields, such as knowledge recognition, data mining, image processing and so on.

We know that classical rough set theory is based on an equivalence relation. Objects in the same equivalence class are indiscernible. Pawlak's rough sets are usually used to deal with data sets described with nominal features. However, in many practical situations, the data sets are not suitable for being handled by Pawlak's rough sets (Bonikowski et al. 1998; Baszczyński et al. 2011; Cattaneo 1998; Chen et al. 2006, 2007, 2011; Diker and Ugur 2012). As a result, many generalized rough set models have been developed in terms of different requirements. Then, the neighborhood-based rough sets (Wang et al. 2016, 2017; Yao 2011), similarity relation rough sets (Slowinski and Vanderpooten 2000; Yao 1998), tolerance relation rough sets (Skowron and Stepaniuk 1996; Xu et al. 2013), and binary relation rough sets (Wang et al. 2017) were constructed, respectively. In particular, some objets have multiple attribute values for a multi-valued attribute. So, we need a covering of the universe instead of a partition. In 1983, Zakowski firstly proposed the covering rough set model Zakowski (1983). After Zakowski, many authors studied the properties of covering rough sets (Bonikowski et al. 1998; Chen et al. 2007; Ge and Li 2011; Kong and Xu 2018a, b; Kong and Wei 2015; Liu and Wang 2011; Liu et al. 2014; Liu and Zhu 2008; Liu and Sai 2009; Shi and Gong 2010). Especially, more and more scholars are working on the attribute reducts of covering rough sets or covering information systems. For example, a pioneering work related to the reducts of covering rough sets was constructed, where the concept of reducts of covering was introduced and the procedure to find a reduct for a covering was shown Zhu and Wang (2003). Meanwhile, the approach to attribute reducts of consistent and inconsistent covering decision systems are firstly introduced by Chen et al. (2007). At the same time, Chen et al. (2007) originally proposed the discernibility to design algorithms that compute all the reducts of consistent and inconsistent covering decision systems. Compared with the attribute reduct method presented in Chen et al. (2007), Wang et al. constructed a new discernibility matrix, which greatly reduced the computational complexity (Wang et al. 2014). In Wang et al. (2015), Wang et al. also provided a new method for constructing simpler discernibility matrix with covering rough sets, and improved some characterizations of attribute reduct provided by Tsang et al. (2008). Moreover, Tan et al. introduced matrix-based methods for computing set approximations and reducts of a covering information system (Tan et al. 2015).

From the perspective of granular computing, an equivalence relation on the universe can be regarded as a granularity, and the corresponding partition can be regarded as a granular structure. Hence, Pawlak's rough set theory is based on a single granularity. In Qian et al. (2010), Qian and Liang firstly extended the single granulation rough sets to the multiple granulation rough sets, where the set approximations were defined by using multiple equivalence relations on the universe. At present, more and more attention has been paid to extending the multi-granulation rough set theory (Li et al. 2016, 2017; Lin et al. 2013; Liu and Wang 2011; Zhang and Kong 2016). Xu et al. developed the multi-granulation rough set model in ordered information systems (Xu et al. 2012). At the same time, based on the multi-granulation rough set theory and fuzzy set theory, Xu et al. proposed a multi-granulation fuzzy rough set model and a multi-granulation fuzzy rough sets in a fuzzy tolerance approximation space (Xu et al. 2014). Meanwhile, Yang also generalized the multi-granulation rough sets into fuzzy rough sets, and discussed the corresponding properties in incomplete information systems Yang et al. (2011). Kong et al. studied the operation and algebraic properties of multi-granulation rough sets and multi-granulation covering rough sets, respectively (Kong et al. 2018; Kong and Wei 2017). She et al. explored the topological structures and obtained many excellent conclusions (She and he 2003). Recently, Xu et al. proposed a generalized multi-granulation rough set model by introducing a support characteristic function and an information level (Xu et al. 2011). However, it is still an open problem regarding the attribute reducts of multi-granulation information systems. Therefore, the main objective of this paper is to study the attribute reduct theory of multi-granulation information system.

The rest of this paper is organized as follows. In Sect. 2, we briefly review some basic concepts of Pawlak's rough sets, multi-granulation rough sets and covering. In Sect. 3, we firstly introduce the concept of the arbitrary union reduct, and discuss some basic properties of this reduct. Furthermore, we develop an algorithm to compute the arbitrary union reduct in an information system. In Sect. 4, we propose the concept of neighborhood union reduct, and study some important properties of neighborhood union reduct. In order to deal with real-life cases, we develop an algorithm to compute the neighborhood union reduct in an information system. In Sect. 5, we define the neighborhood intersection reduct. Many meaningful results of this reduct are explored. In Sect. 6, the interrelationships among the several types of reducts are discussed in detail. In Sect. 7, an illustrate example is given to show how to select the optimal reduct to compute the multi-granulation lower and upper approximations more efficiently. Finally, Sect. 8 concludes this study.

2 Preliminaries

In this section, we review some basic concepts and notions of Pawlak's rough sets, multi-granulation rough sets and covering. More details can be seen in references Chen et al. (2007), Qian et al. (2010), Qian and Liang (2006), Zhu and Wang (2003).

Let $\mathcal{I} = (U, A, V, f)$ be an information system, where U is a nonempty finite set, called a universe; A is a nonempty attribute set; $V = \cup_{a\in A} V_a$, V_a is a set of its values; $f : U \times A \to V$ is an information function with $f(x, a) \in V_a$ for each $a \in A$ and $x \in U$. The family of attribute subsets is denoted by $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, where $A_i \subseteq A,\ i = 1, 2, \ldots, m$. The equivalence class of an object x with respect to $A_i \in \mathcal{A}$ is defined by: $[x]_{A_i} = \{y \in U | f(x, a) = f(y, a), a \in A_i\}$. Let $U/A_i = \{[x_{1_i}]_{A_i}, [x_{2_i}]_{A_i}, \ldots, [x_{n_i}]_{A_i}\}$ is a partition of U and $\cup(\mathcal{A}) = \{[x_{1_1}]_{A_1}, [x_{2_1}]_{A_1}, \ldots, [x_{n_1}]_{A_1}, [x_{1_2}]_{A_2}, [x_{2_2}]_{A_2}, \ldots, [x_{n_2}]_{A_2}, \ldots, [x_{1_m}]_{A_m}, [x_{2_m}]_{A_m}, \ldots, [x_{n_m}]_{A_m}\}$. Then, for each $X \subseteq U$, the lower and upper approximations of X with respect to A_i are defined as follows:

$$\underline{A_i}(X) = \{x \in U | [x]_{A_i} \subseteq X\}, \quad \overline{A_i}(X) = \{x \in U | [x]_{A_i} \cap X \neq \emptyset\}.$$

Definition 2.1 Qian et al. (2010) Let $\mathcal{I} = (U, A, V, f)$ be an information system, $X \subseteq U$, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. The optimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$ are defined as follows:

$$\underline{OM_{\sum_{i=1}^m A_i}}(X) = \{x \mid \vee_{i=1}^m([x]_{A_i} \subseteq X)\}, \quad \overline{OM_{\sum_{i=1}^m A_i}}(X) = \{x \mid \wedge_{i=1}^m([x]_{A_i} \cap X \neq \emptyset)\}.$$

where "$\vee$" means the logical operator "or", which represents that the alternative conditions are satisfied, and "$\wedge$" means the logical operator "and", which represents that all of the conditions are satisfied. Here, the word "optimistic" means that just one granular structure is needed to satisfy with the inclusion between an equivalence class and a target concept when multiple independent granular structures are available in problem processing.

Definition 2.2 Qian and Liang (2006) Let $\mathcal{I} = (U, A, V, f)$ be an information system, $X \subseteq U$, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. The pessimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$ are defined as follows:

$$\underline{PM_{\sum_{i=1}^{m} A_i}}(X) = \{x \mid \wedge_{i=1}^{m}([x]_{A_i} \subseteq X)\}, \quad \overline{PM_{\sum_{i=1}^{m} A_i}}(X) = \{x \mid \vee_{i=1}^{m}([x]_{A_i} \cap X \neq \emptyset)\}.$$

Here, the word "pessimistic" means that all granular structures are needed to satisfy with the inclusion between an equivalence class and a target concept when multiple independent granular structures are available.

Definition 2.3 Zakowski (1983) Let C be a family of nonempty subsets of U. C is called a covering of U if $\cup_{K \in C} K = U$. The ordered pair (U, C) is called a covering approximation space.

Definition 2.4 Zhu and Wang (2003) Let (U, C) be a covering approximation space and $K \in C$. If K is a union of some sets in $C/\{K\}$, we say that K is a union reducible element of C; Otherwise, we say that K is a union irreducible element of C. Meanwhile, for a covering C of U, the new union irreducible covering through above reduction is called a union reduct of C, and denoted by $reduct(C)_U$.

Notice that the union reduct of a covering can be computed by deleting all the union reducible elements. Therefore, the union reduct is the minimum covering by deleting redundancy. Meanwhile, for each subset of the universe, the union reduct of a covering can induce the same lower and upper approximations (Zhu and Wang 2003). However, for the union reduct of a covering, it may contain other redundant elements which are not the union reducible elements. To address this issue, the intersection reduct of a covering is proposed by Chen et al. (2015).

Definition 2.5 Chen et al. (2015) Let (U, C) be a covering approximation space and $K \in C$. If K is an intersection of some sets in $C/\{K\}$, we say that K is an intersection reducible element of C; Otherwise, we say that K is an intersection irreducible element of C. Meanwhile, for a covering C of U, the new intersection irreducible covering through above reduction is called an intersection reduct of C, and denoted by $reduct(C)_I$.

In Chen et al. (2015), the properties of the intersect reduct of a covering are examined. Particularly, the intersection reduct is investigated from the viewpoint of concept lattice theory.

3 Attribute reduction with respect to arbitrary union

In this section, we will introduce the concept of the arbitrary union reduct, and then discuss some interesting properties of this reduct. Meanwhile, we develop an algorithm to compute the arbitrary union reduct.

Definition 3.1 Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ the family of attribute subsets. $A_i \in \mathcal{A}$ is called an arbitrary union reducible element of $\mathcal{A}$, if for each $x \in U$, there exist $\Gamma_x \subseteq \{1, 2, \ldots, i-1, i+1, \ldots, m\}$ and $V_x \subseteq U$ such that $[x]_{A_i} = \cup_{j \in \Gamma_x} \cup_{y \in V_x} [y]_{A_j}$; Otherwise, A_i is called an arbitrary union irreducible element of $\mathcal{A}$. If every element in $\mathcal{A}$ is irreducible, we say that $\mathcal{A}$ is irreducible; Otherwise, $\mathcal{A}$ is reducible.

Definition 3.2 Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ the family of attribute subsets. The new family of attribute subsets $\mathcal{A}^{'} \subseteq \mathcal{A}$ through the above reduct is called the arbitrary union reduct of $\mathcal{A}$, and denoted by $reduct(\mathcal{A})_{AU}$.

Notice that the covering C of a universe U has only one reduct. i.e., $reduct(C)_U$ is unique (Zhu and Wang 2003). Here, we raise a question: is $reduct(\mathcal{A})_{AU}$ unique? In the following, we will employ an example to answer the question.

Example 3.1 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_{20}\}$, $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$.

$$\begin{aligned}
U/A_1 &= \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}, x_{12}\},\\
&\qquad \{x_{13}, x_{14}, x_{15}, x_{16}\}, \{x_{17}, x_{18}\}, \{x_{19}, x_{20}\}\};\\
U/A_2 &= \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7\}, \{x_8\}, \{x_9\}, \{x_{10}\}, \{x_{11}\}, \{x_{12}\},\\
&\qquad \{x_{13}, x_{14}, x_{17}\}, \{x_{15}\}, \{x_{16}, x_{19}, x_{20}\}, \{x_{18}\}\};\\
U/A_3 &= \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}, \{x_9, x_{10}, x_{11}, x_{12}\},\\
&\qquad \{x_{13}, x_{14}, x_{15}\}, \{x_{16}, x_{17}, x_{18}\}, \{x_{19}\}, \{x_{20}\}\};\\
U/A_4 &= \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7\}, \{x_8, x_9, x_{10}\}, \{x_{11}, x_{12}\},\\
&\qquad \{x_{13}, x_{14}, x_{15}\}, \{x_{16}, x_{17}, x_{18}\}, \{x_{19}, x_{20}\}\}.
\end{aligned}$$

It is clear that $reduct(\mathcal{A})_{AU} = \{A_1, A_2, A_3\}$ or $\{A_1, A_2, A_4\}$. Therefore, $reduct(\mathcal{A})_{AU}$ is not unique.

If $K \in C$ is a union reducible element of C and $K_1 \in C/\{K\}$, then K_1 is a union reducible element of C if and only if K_1 is a union reducible element of $C/\{K\}$ Zhu and Wang (2003). Similarly, we have the following result.

Proposition 3.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. Suppose $A_i \in \mathcal{A}$ is an arbitrary union reducible element of $\mathcal{A}$, and $A_j \in \mathcal{A}/\{A_i\}$. If for each $x \in U$, we have $[x]_{A_i} \neq [x]_{A_j}$, then A_j is an arbitrary union reducible element of $\mathcal{A}$ if and only if A_j is an arbitrary union reducible element of $\mathcal{A}/\{A_i\}$.*

Proof ($\Leftarrow$) It is immediate.

($\Rightarrow$) Suppose A_j is an arbitrary union reducible element of $\mathcal{A}$. For each $x \in U$, there exist $\Gamma_x \subseteq \{1, 2, \ldots, j-1, j+1, \ldots, m\}$ and $V_x \subseteq U$ such that $[x]_{A_j} = \cup_{t\in\Gamma_x} \cup_{y\in V_x} [y]_{A_t}$. Case1: If $i \overline{\in} \Gamma_x$, it is clear that A_j is an arbitrary union reducible element of $\mathcal{A}/\{A_i\}$; Case2: If $i \in \Gamma_x$, without lost of generality, let $[x]_{A_j} = [z]_{A_i} \cup (\cup_{t\in\Gamma_x/\{i\}} \cup_{y\in V_x/\{z\}} [y]_{A_t})$, where $z \in V_x$. On the one hand, if $[z]_{A_i} = [x]_{A_i}$, by the assumption in this proposition, we have $[z]_{A_i} \subset [x]_{A_j}$. Since A_i is an arbitrary union reducible element of $\mathcal{A}$, there exist $\Gamma_x^{'} \subseteq \{1, 2, \ldots, i-1, i+1, \ldots, m\}$ and $V_x^{'} \subseteq U$ such that $[z]_{A_i} = \cup_{s\in\Gamma_x^{'}} \cup_{w\in V_x^{'}} [w]_{A_s}$. Because U/A_i is a partition of U, then $j \overline{\in} \Gamma_x^{'}$. Therefore, $[x]_{A_j} = (\cup_{s\in\Gamma_x^{'}} \cup_{w\in V_x^{'}} [w]_{A_s}) \cup (\cup_{t\in\Gamma_x/\{i\}} \cup_{y\in V_x/\{z\}} [y]_{A_t})$. Denote $\widetilde{\Gamma_x} = (\Gamma_x^{'} \cup \Gamma_x)/\{i\}$, $\widetilde{V_x} = (V_x^{'} \cup V_x)/\{z\}$, then $[x]_{A_j} = \cup_{k\in\widetilde{\Gamma_x}} \cup_{u\in\widetilde{V_x}} [u]_{A_k}$. So, A_j is an arbitrary union reducible element of $\mathcal{A}/\{A_i\}$. On the other hand, if $[z]_{A_i} \neq [x]_{A_i}$, then we have $[z]_{A_i} \subset [x]_{A_j}$. According to the above discussion, A_j is an arbitrary union reducible element of $\mathcal{A}/\{A_i\}$. □

In the following, we can develop an algorithm to compute the arbitrary union reduct $reduct(\mathcal{A})_{AU}$.

Algorithm 1: An algorithm for computing $reduct(\mathcal{A})_{AU}$

Input : An information system $\mathcal{I} = (U, A, V, f)$, and $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$;
Output : $reduct(\mathcal{A})_{AU}$.

begin
  $\mathcal{A} \leftarrow reduct(\mathcal{A})_{AU}$;
  for $i = 1 : m; i <= m; i ++$ **do**
    $U \leftarrow \overline{U}$;
    for *each* $x \in \overline{U}$ **do**
      $\emptyset \leftarrow V$;
      for $j = 1 : m; j <= m; j \neq i; j ++$ **do**
        $U \leftarrow \widetilde{U}$;
        for *each* $y \in \widetilde{U}$ **do**
          if $[y]_{A_j} \subseteq [x]_{A_i}$ **then**
            $V \leftarrow V \cup [y]_{A_j}$;
            if $[x]_{A_i} \neq V$ **then**
              $\widetilde{U} \leftarrow \widetilde{U}/[y]_{A_j}$;
            end
          end
        end
      end
      $\overline{U} \leftarrow \overline{U}/[x]_{A_i}$;
    end
    $reduct(\mathcal{A})_{AU} \leftarrow reduct(\mathcal{A})_{AU}/A_i$;
  end
end

In Zhu and Wang (2003), Zhu and Wang indicated that if K is a union reducible element of C and $X \subseteq U$, then the covering lower approximations of X with respect to C and $C/\{K\}$, respectively, are same. Here, we also raise a similar question: if $A_i \in \mathcal{A}$ is an arbitrary union reducible element of $\mathcal{A}$, and $X \subseteq U$, are the optimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, same? To solve this problem, we have the following result.

Proposition 3.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is an arbitrary union reducible element of $\mathcal{A}$, then the optimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

Proof By Definition 2.1, it is obvious that $\underline{OM_{\sum_{i=1,i\neq j}^m A_i}}(X) \subseteq \underline{OM_{\sum_{i=1}^m A_i}}(X)$. Then, for each $x \in \underline{OM_{\sum_{i=1}^m A_i}}(X)$, there exists $A_k \in \mathcal{A}$ such that $[x]_{A_k} \subseteq X$. If $A_k \neq A_j$, then $x \in \underline{OM_{\sum_{i=1,i\neq j}^m A_i}}(X)$. If $A_k = A_j$, since A_j is a reducible element of $\mathcal{A}$, there exists $A_s \in \mathcal{A}$ such that $[x]_{A_s} \subseteq [x]_{A_k} \subseteq X$. According to Definition 2.1, we have $x \in \underline{OM_{\sum_{i=1,i\neq j}^m A_i}}(X)$. So, $\underline{OM_{\sum_{i=1}^m A_i}}(X) \subseteq \underline{OM_{\sum_{i=1,i\neq j}^m A_i}}(X)$. Therefore, $\underline{OM_{\sum_{i=1,i\neq j}^m A_i}}(X) = \underline{OM_{\sum_{i=1}^m A_i}}(X)$. $\square$

Similarly, the result with respect to the optimistic multi-granulation upper approximation is shown as follows.

Proposition 3.3 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is an arbitrary union reducible element of $\mathcal{A}$, then the optimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

According to Propositions 3.2 and 3.3, we have the following result.

Theorem 3.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. Then the optimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$ and $reduct(\mathcal{A})_{AU}$, respectively, are same.*

If $A_i \in \mathcal{A}$ is an arbitrary union reducible element of $\mathcal{A}$, and $X \subseteq U$, then, are the pessimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, same? In order to answer the question, a counterexample is given as follows:

Example 3.2 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_9\}$, $\mathcal{A} = \{A_1, A_2, A_3\}$.

$$\begin{aligned} U/A_1 &= \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}, \{x_7, x_8\}, \{x_9\}\}; \\ U/A_2 &= \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6\}, \{x_7\}, \{x_8, x_9\}\}; \\ U/A_3 &= \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_6, x_7\}, \{x_8, x_9\}\}; \end{aligned}$$

Clearly, A_3 is an arbitrary union reducible element of $\mathcal{A}$. For $X = \{x_7, x_8, x_9\}$, we have $\underline{PM_{\sum_{i=1}^{3} A_i}}(X) = \{x_8, x_9\}$. However, $\underline{PM_{\sum_{i=1}^{2} A_i}}(X) = \{x_7, x_8, x_9\}$. Therefore, $\underline{PM_{\sum_{i=1}^{3} A_i}}(X) \neq \underline{PM_{\sum_{i=1}^{2} A_i}}(X)$.

Similarly, it is not difficult to find that the pessimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, are different.

Proposition 3.4 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A}_1, \mathcal{A}_2$ two families of attribute subsets. If for each $X \subseteq U$, the optimistic multi-granulation lower approximations of X with respect to $reduct(\mathcal{A}_1)_{AU}$ and $reduct(\mathcal{A}_2)_{AU}$, respectively, are same. Then we have that $\cup(reduct(\mathcal{A}_1)_{AU}) = \cup(reduct(\mathcal{A}_2)_{AU})$.*

Proof Suppose that $\mathcal{A}_1 = \{A_{11}, A_{12}, \ldots, A_{1m_1}\}, \mathcal{A}_2 = \{A_{21}, A_{22}, \ldots, A_{2m_2}\}$. For each $K \in \cup(reduct(\mathcal{A}_1)_{AU})$, we have $\underline{OM_{\sum_{k=1}^{m_1} A_{1k}}}(K) = K$. Since the optimistic multi-granulation lower or upper approximations of K with respect to $reduct(\mathcal{A}_1)_{AU}$ and $reduct(\mathcal{A}_2)_{AU}$, respectively, are same. Then we have $\underline{OM_{\sum_{l=1}^{m_2} A_{2l}}}(K) = K$. Therefore, there exist $K_1, K_2, \ldots, K_s \in \cup(reduct(\mathcal{A}_2)_{AU})$ such that $K = \cup_{i=1}^{s} K_i$. Similar to the above proof, there exist $K_{i1}, K_{i2}, \ldots, K_{in_i} \in \cup(reduct(\mathcal{A}_1)_{AU})$ such that $K_i = \cup_{j=1}^{n_i} K_{ij}$. So, $K = \cup_{i=1}^{s} \cup_{j=1}^{n_i} K_{ij}$. By the definition of the arbitrary union reduct, we have $K_{ij} = K$ for all i, j. Therefore, for all i, $K_i = K$. Hence, K is an element of $\cup(reduct(\mathcal{A}_2)_{AU})$.

On the other hand, we can similarly prove that any element of $\cup(reduct(\mathcal{A}_2)_{AU})$ is an element of $\cup(reduct(\mathcal{A}_1)_{AU})$. Therefore, we have $\cup(reduct(\mathcal{A}_1)_{AU}) = \cup(reduct(\mathcal{A}_2)_{AU})$. $\square$

Theorem 3.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A}_1$, $\mathcal{A}_2$ two families of attribute subsets. For each $X \subseteq U$, the optimistic multi-granulation lower approximations of X with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, are same if and only if $\cup(reduct(\mathcal{A}_1)_{AU}) = \cup(reduct(\mathcal{A}_2)_{AU})$.*

Proof ($\Rightarrow$) It is immediate by Proposition 3.4.

($\Leftarrow$) Since $\cup(reduct(\mathcal{A}_1)_{AU}) = \cup(reduct(\mathcal{A}_2)_{AU})$. For each $X \subseteq U$, we have that the optimistic multi-granulation lower approximations of X with respect to $reduct(\mathcal{A}_1)_{AU}$ and $reduct(\mathcal{A}_2)_{AU}$, respectively, are same. By Theorem 3.1, we have that the optimistic multi-granulation lower approximations of X with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, are same. □

Similar to Proposition 3.4 and Theorem 3.2, we have the following results.

Proposition 3.5 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A}_1$, $\mathcal{A}_2$ two families of attribute subsets. For each $X \subseteq U$, the optimistic multi-granulation upper approximations of X with respect to $reduct(\mathcal{A}_1)_{AU}$ and $reduct(\mathcal{A}_2)_{AU}$, respectively, are same. Then we have that $\cup(reduct(\mathcal{A}_1)_{AU}) = \cup(reduct(\mathcal{A}_2)_{AU})$.*

Theorem 3.3 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A}_1$, $\mathcal{A}_2$ two families of attribute subsets. For each $X \subseteq U$, the optimistic multi-granulation upper approximations of X with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, are same if and only if $\cup(reduct(\mathcal{A}_1)_{AU}) = \cup(reduct(\mathcal{A}_2)_{AU})$.*

4 Attribute reduction with respect to neighborhood union

In this section, we firstly propose the concept of neighborhood union reduct. Then we study some important properties of neighborhood union reduct. Finally, we develop an algorithm to compute the neighborhood union reduct in an information system.

Definition 4.1 Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ the family of attribute subsets. $A_i \in \mathcal{A}$ is called a neighborhood union reducible element of $\mathcal{A}$, if each $x \in U$, there exists $\Gamma_x \subseteq \{1, 2, \ldots, i-1, i+1, \ldots, m\}$ such that $[x]_{A_i} = \cup_{j\in\Gamma_x}[x]_{A_j}$; Otherwise, A_i is called a neighborhood union irreducible element of $\mathcal{A}$. If every element in $\mathcal{A}$ is irreducible, we say that $\mathcal{A}$ is irreducible; Otherwise, $\mathcal{A}$ is reducible.

Definition 4.2 Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ the family of attribute subsets. The new family of attribute subsets $\mathcal{A}' \subseteq \mathcal{A}$ through the above reduction is called the neighborhood union reduct of $\mathcal{A}$, and denoted by $reduct(\mathcal{A})_{NU}$.

Example 3.1 shows that the arbitrary union reduct $reduct(\mathcal{A})_{AU}$ is not unique. Now, we investigate that whether the neighborhood union reduct $reduct(\mathcal{A})_{NU}$ is unique. In the following, an example is employed to answer the question.

Example 4.1 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_9\}$, $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$.

$$U/A_1 = \{\{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_6, x_7\}, \{x_8, x_9\}\};$$
$$U/A_2 = \{\{x_1, x_2, x_4, x_5\}, \{x_3\}, \{x_6, x_7, x_9\}, \{x_8\}\};$$
$$U/A_3 = \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_6, x_7, x_8\}, \{x_9\}\};$$

$$U/A_4 = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6, x_7, x_8, x_9\}\};$$
$$U/A_5 = \{\{x_1, x_2, x_4, x_5\}, \{x_3\}, \{x_6, x_7, x_8\}, \{x_9\}\}.$$

Then, we can find that $reduct(\mathcal{A})_{NU} = \{A_1, A_2, A_3, A_4\}$ or $\{A_1, A_2, A_4, A_5\}$. So $reduct(\mathcal{A})_{NU}$ is not unique.

Compared with Proposition 3.1, the similar result with respect to the neighborhood union reduct is shown as follows:

Proposition 4.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. Suppose $A_i \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, and $A_j \in \mathcal{A}/\{A_i\}$. If for each $x \in U$, we have $[x]_{A_i} \neq [x]_{A_j}$, then A_j is a neighborhood union reducible element of $\mathcal{A}$ if and only if A_j is a neighborhood union reducible element of $\mathcal{A}/\{A_i\}$.*

In the following, we give an algorithm to find the neighborhood union reduct $reduct(\mathcal{A})_{NU}$.

Algorithm 2: An algorithm for computing $reduct(\mathcal{A})_{NU}$

Input : An information system $\mathcal{I} = (U, A, V, f)$, and $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$;
Output : $reduct(\mathcal{A})_{NU}$.

begin
  $\mathcal{A} \leftarrow reduct(\mathcal{A})_{NU}$;
  for $i = 1 : m; i <= m; i++$ **do**
    $U \leftarrow \overline{U}$;
    for *each* $x \in \overline{U}$ **do**
      $\emptyset \leftarrow V$;
      for $j = 1 : m; j <= m; j \neq i; j++$ **do**
        if $[x]_{A_j} \subseteq [x]_{A_i}$ **then**
          $V \leftarrow V \cup [y]_{A_j}$;
          if $[x]_{A_i} = V$ **then**
            $\overline{U} \leftarrow \overline{U}/[y]_{A_j}$;
          end
        end
      end
    end
    $reduct(\mathcal{A})_{NU} \leftarrow reduct(\mathcal{A})_{NU}/A_i$;
  end
end

Similarly, we have the following results.

Proposition 4.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, then the optimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

Proof The proof is similar to that of Proposition 3.2. □

Proposition 4.3 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, then the optimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

Proof The proof is similar to that of Proposition 3.3. □

According to Propositions 4.2 and 4.3, we have the following result.

Theorem 4.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. Then the optimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$ and $reduct(\mathcal{A})_{NU}$, respectively, are same.*

If $A_i \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, and $X \subseteq U$, then, are the pessimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, same?

Proposition 4.4 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, then the pessimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

Proof Based on Definition 2.2, it is obvious that $\underline{PM_{\sum_{i=1}^{m} A_i}(X)} \subseteq \underline{PM_{\sum_{i=1, i\neq j}^{m} A_i}(X)}$. On the other hand, for each $x \in \underline{PM_{\sum_{i=1, i\neq j}^{m} A_i}(X)}$, we have that $x \in [x]_{A_i}$, $i = 1, 2, \ldots, j-1, j+1, \ldots, m$. Because $A_j \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, then there exists $\Gamma_x \subseteq \{1, 2, \ldots, j-1, j+1, \ldots, m\}$ such that $[x]_{A_j} = \cup_{t\in\Gamma_x}[x]_{A_t}$. So $x \in [x]_{A_j}$. i.e., $x \in [x]_{A_i}$, $i = 1, 2, \ldots, m$. Therefore, $\underline{PM_{\sum_{i=1, i\neq j}^{m} A_i}(X)} \subseteq \underline{PM_{\sum_{i=1}^{m} A_i}(X)}$. In a word, we have that $\underline{PM_{\sum_{i=1}^{m} A_i}(X)} = \underline{PM_{\sum_{i=1, i\neq j}^{m} A_i}(X)}$. □

If $A_i \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, and $X \subseteq U$, then, are the pessimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, still same?

Proposition 4.5 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, then the pessimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

Proof The proof is similar to that of Proposition 4.4. □

From Propositions 4.4 and 4.5, we have the following result.

Theorem 4.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. Then the pessimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$ and $reduct(\mathcal{A})_{NU}$, respectively, are same.*

5 Attribute reduction with respect to neighborhood intersection

In this section, inspired by the concept of intersection reduct proposed by Chen et al. (2015), the concept of neighborhood intersection reduct are introduced. Meanwhile, some meaningful properties of neighborhood intersection reduct are studied.

Definition 5.1 Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ the family of attribute subsets. $A_i \in \mathcal{A}$ is called a neighborhood intersection reducible element of $\mathcal{A}$, if for each $x \in U$, there exists $\Gamma_x \subseteq \{1, 2, \ldots, i-1, i+1, \ldots, m\}$ such that $[x]_{A_i} = \cap_{j \in \Gamma_x} [x]_{A_j}$; Otherwise, A_i is called a neighborhood intersection irreducible element of $\mathcal{A}$. If every element in $\mathcal{A}$ is irreducible, we say that $\mathcal{A}$ is irreducible; Otherwise, $\mathcal{A}$ is reducible.

Definition 5.2 Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ the family of attribute subsets. The new family of attribute subsets $\mathcal{A}' \subseteq \mathcal{A}$ through the above reduction is called the neighborhood intersection reduct of $\mathcal{A}$, and denoted by $reduct(\mathcal{A})_{NI}$.

Examples 3.1 and 4.1 show that the arbitrary union reduct $reduct(\mathcal{A})_{AU}$ and the neighborhood union reduct $reduct(\mathcal{A})_{NU}$ are both not unique. Next, we wonder if the neighborhood intersection reduct $reduct(\mathcal{A})_{NI}$ is unique. To address this issue, an example is shown as follows.

Example 5.1 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_9\}$, $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$.

$$U/A_1 = \{\{x_1, x_2, x_4\}, \{x_3, x_5, x_6\}, \{x_7, x_8\}, \{x_9\}\};$$
$$U/A_2 = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}\};$$
$$U/A_3 = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}, \{x_9\}\};$$
$$U/A_4 = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7\}, \{x_8, x_9\}\};$$
$$U/A_5 = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}\}.$$

Then, we can find that $reduct(\mathcal{A})_{NI} = \{A_1, A_2, A_3, A_4\}$ or $\{A_1, A_3, A_4, A_5\}$. Therefore, $reduct(\mathcal{A})_{NI}$ is not unique.

In the following, we will construct an algorithm for finding the neighborhood intersection reduct $reduct(\mathcal{A})_{NI}$.

Algorithm 3: An algorithm for computing $reduct(\mathcal{A})_{NI}$

Input : An information system $\mathcal{I} = (U, A, V, f)$, and $\mathcal{A} = \{A_1, A_2, \cdots, A_m\}$;
Output : $reduct(\mathcal{A})_{NI}$.

begin
  $\mathcal{A} \leftarrow reduct(\mathcal{A})_{NI}$;
  for $i = 1 : m;\ i <= m;\ i++$ **do**
    $U \leftarrow \overline{U}$;
    for *each* $x \in \overline{U}$ **do**
      $U \leftarrow V$;
      for $j = 1 : m;\ j <= m;\ j \neq i;\ j++$ **do**
        if $[x]_{A_i} \subseteq [x]_{A_j}$ **then**
          $V \leftarrow V \cap [y]_{A_j}$;
          if $[x]_{A_i} = V$ **then**
            $\overline{U} \leftarrow \overline{U}/[y]_{A_j}$;
          end
        end
      end
    end
    $reduct(\mathcal{A})_{NI} \leftarrow reduct(\mathcal{A})_{NI}/A_i$;
  end
end

Proposition 5.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is a neighborhood intersection reducible element of $\mathcal{A}$, then the pessimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

Proof It is clear that $\underline{PM_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{PM_{\sum_{i=1,i\neq j}^{m} A_i}}(X)$. Meanwhile, for each $x \in \underline{PM_{\sum_{i=1,i\neq j}^{m} A_i}}(X)$, we have $[x]_{A_s} \subseteq X,\ s \in \{1, 2, \ldots, j-1, j+1, \ldots, m\}$. Because A_j is a reducible element of $\mathcal{A}$, then there exists $\Gamma_x \subseteq \{1, 2, \ldots, j-1, j+1, \ldots, m\}$ such that $[x]_{A_j} = \cap_{t\in\Gamma_x}[x]_{A_t}$. Then, $[x]_{A_j} \subseteq [x]_{A_t},\ t \in \Gamma_x$. i.e., $[x]_{A_j} \subseteq X$. So, $x \in \underline{PM_{\sum_{i=1}^{m} A_i}}(X)$. Therefore, $\underline{PM_{\sum_{i=1}^{m} A_i}}(X) = \underline{PM_{\sum_{i=1,i\neq j}^{m} A_i}}(X)$. □

Proposition 5.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. If $A_j \in \mathcal{A}$ is a neighborhood intersection reducible element of $\mathcal{A}$, then the pessimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_j$, respectively, are same.*

According to Propositions 5.1 and 5.2, the following result holds.

Theorem 5.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$. Then the pessimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$ and $reduct(\mathcal{A})_{NI}$, respectively, are same.*

If $A_i \in \mathcal{A}$ is a neighborhood intersection reducible element of $\mathcal{A}$, and $X \subseteq U$, are the optimistic multi-granulation lower approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, same? Then, a counterexample is given as follows:

Example 5.2 (Continued from Example 5.1) According to Example 5.1, we know that A_5 is a neighborhood intersection reducible element of $\mathcal{A}$. For $X = \{x_2, x_4, x_6, x_7\}$, we have that $\underline{OM_{\sum_{i=1}^{5} A_i}}(X) = \{x_2, x_4, x_7\}$, $\underline{OM_{\sum_{i=1}^{4} A_i}}(X) = \{x_4, x_7\}$. Therefore, $\underline{OM_{\sum_{i=1}^{5} A_i}}(X) \neq \underline{OM_{\sum_{i=1}^{4} A_i}}(X)$.

Similarly, if A_i is a neighborhood intersection reducible element of $\mathcal{A}$, then the optimistic multi-granulation upper approximations of X with respect to $\mathcal{A}$ and $\mathcal{A}/A_i$, respectively, are different.

6 Interrelationships among the three types of attribute reductions

In this section, we will deeply explore the interrelationships among several types of attribute reducts.

Proposition 6.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. If $A_j \in \mathcal{A}$ is a neighborhood union reducible element of $\mathcal{A}$, then A_j is an arbitrary union reducible element of $\mathcal{A}$.*

Proof It is clear by Definitions 3.1 and 4.1.

Conversely, if $A_j \in \mathcal{A}$ is an arbitrary union reducible element of $\mathcal{A}$, is A_j a neighborhood union reducible element of $\mathcal{A}$? A counterexample is given as follows: □

Example 6.1 (Continued from Example 3.2) It can be found that A_3 is an arbitrary union reducible element of $\mathcal{A}$. However, A_3 is not a neighborhood union reducible element of $\mathcal{A}$.

Proposition 6.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, then $reduct(reduct(\mathcal{A})_{AU})_{NU}$ and $reduct(reduct(\mathcal{A})_{NU})_{AU}$ are both the arbitrary union reducts of $\mathcal{A}$.*

Proof It is clear by Definitions 3.1 and 4.1. □

Theorem 6.1 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$, then the optimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$, $reduct(reduct(\mathcal{A})_{AU})_{NU}$ and $reduct(reduct(\mathcal{A})_{NU})_{AU}$, respectively, are same.*

Theorem 6.2 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, and $X \subseteq U$, then the pessimistic multi-granulation lower and upper approximations of X with respect to $\mathcal{A}$, $reduct(reduct(\mathcal{A})_{NU})_{NI}$ and $reduct(reduct(\mathcal{A})_{NI})_{NU}$, respectively, are same.*

Example 6.2 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_{10}\}$, $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Let

$$\begin{aligned}
U/A_1 &= \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}, \{x_9, x_{10}\}\};\\
U/A_2 &= \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7\}, \{x_8\}, \{x_9, x_{10}\}\};\\
U/A_3 &= \{\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_7\}, \{x_8\}, \{x_9, x_{10}\}\};\\
U/A_4 &= \{\{x_1, x_2, x_7, x_8, x_9, x_{10}\}, \{x_3, x_4, x_5, x_6\}\};\\
U/A_5 &= \{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4, x_7\}, \{x_8, x_9, x_{10}\}\};\\
U/A_6 &= \{\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_7\}, \{x_8, x_9, x_{10}\}\}.
\end{aligned}$$

Then, It can be found that $reduct(\mathcal{A})_{AU} = \{A_1, A_2\}$ or $\{A_1, A_3\}$, $reduct(\mathcal{A})_{NU} = \{A_1, A_2, A_4, A_5\}$. Furthermore, we have that $reduct(reduct(\mathcal{A})_{AU})_{NU} = reduct(\mathcal{A})_{AU}$.

Proposition 6.3 *Let $\mathcal{I} = (U, A, V, f)$ be an information system, and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$. Then we have that $reduct(reduct(\mathcal{A})_{AU})_{NU} = reduct(\mathcal{A})_{AU}$.*

Proof It is clear by Definitions 3.1 and 4.1. □

According to Proposition 6.3, we have that $reduct(reduct(\mathcal{A})_{AU})_{NU} = reduct(\mathcal{A})_{AU}$. Then, does the equation $reduct(reduct(\mathcal{A})_{NU})_{AU} = reduct(\mathcal{A})_{AU}$ hold? A counterexample is given as follows:

Example 6.3 (Continued from Example 6.2) Let $reduct(\mathcal{A})_{AU} = \{A_1, A_3\}$. However, $reduct(reduct(\mathcal{A})_{NU})_{AU} = \{A_1, A_2\} \neq \{A_1, A_3\}$. Therefore, $reduct(reduct(\mathcal{A})_{NU})_{AU} \neq reduct(\mathcal{A})_{AU}$. Furthermore, by Proposition 6.4, it can be easily found that $reduct(reduct(\mathcal{A})_{NU})_{AU} \neq reduct(reduct(\mathcal{A})_{AU})_{NU}$.

Example 6.4 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_{10}\}$, $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Let

$$\begin{aligned}
U/A_1 &= \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8\}, \{x_9, x_{10}\}\};\\
U/A_2 &= \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_6, x_7, x_8, x_9, x_{10}\}\};\\
U/A_3 &= \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7, x_8\}, \{x_9, x_{10}\}\};\\
U/A_4 &= \{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4\}, \{x_7, x_8, x_9, x_{10}\}\};\\
U/A_5 &= \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6, x_7, x_8, x_9, x_{10}\}\};\\
U/A_6 &= \{\{x_1, x_2, x_4, x_5\}, \{x_3\}, \{x_6, x_7, x_8, x_9, x_{10}\}\}.
\end{aligned}$$

It can be found that $reduct(\mathcal{A})_{NU} = \{A_1, A_3, A_4, A_5, A_6\}$, $reduct(\mathcal{A})_{NI} = \{A_2, A_3, A_4, A_5, A_6\}$, $reduct(reduct(\mathcal{A})_{NU})_{NI} = \{A_3, A_4, A_5, A_6\}$, and $reduct(reduct(\mathcal{A})_{NI})_{NU} = \{A_3, A_4, A_5, A_6\}$.

Then, we have $reduct(reduct(\mathcal{A})_{NI})_{NU} \neq reduct(\mathcal{A})_{NU}$, and $reduct(reduct(\mathcal{A})_{NI})_{NU} \neq reduct(\mathcal{A})_{NI}$. Similarly, it can be found that $reduct(reduct(\mathcal{A})_{NU})_{NI} \neq reduct(\mathcal{A})_{NU}$, and $reduct(reduct(\mathcal{A})_{NU})_{NI} \neq reduct(\mathcal{A})_{NI}$.

Here, we will raise a question: does the equation $reduct(reduct(\mathcal{A})_{NU})_{NI} = reduct(reduct(\mathcal{A})_{NI})_{NU}$ hold? In the following, a counterexample is employed to answer the question.

Example 6.5 Let $\mathcal{I} = (U, A, V, f)$ be an information system, where $U = \{x_1, x_2, \ldots, x_{12}\}$, and $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$. Let

$$\begin{aligned}
U/A_1 &= \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}, x_{12}\}\};\\
U/A_2 &= \{\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}\};\\
U/A_3 &= \{\{x_1, x_2, x_3, x_7, x_8, x_9\}, \{x_4, x_5, x_6, x_{10}, x_{11}, x_{12}\}\};\\
U/A_4 &= \{\{x_1, x_2, x_7, x_9\}, \{x_3, x_6, x_8, x_{11}\}, \{x_4, x_5, x_{10}, x_{12}\}\};\\
U/A_5 &= \{\{x_1, x_3, x_7, x_9\}, \{x_2, x_5, x_9, x_{12}\}, \{x_4, x_6, x_{10}, x_{11}\}\};\\
U/A_6 &= \{\{x_1, x_3, x_8, x_9\}, \{x_2, x_5, x_7, x_{10}\}, \{x_4, x_6, x_{11}, x_{12}\}\};\\
U/A_7 &= \{\{x_1, x_4, x_7, x_{10}\}, \{x_2, x_3, x_8, x_9\}, \{x_5, x_6, x_{11}, x_{12}\}\}.
\end{aligned}$$

Then we have $reduct(reduct(\mathcal{A})_{NU})_{NI} = \{A_1, A_2, A_4, A_5, A_6, A_7\}$, and $reduct(reduct(\mathcal{A})_{NI})_{NU} = \{A_2, A_4, A_5, A_6, A_7\}$. Therefore, $reduct(reduct(\mathcal{A})_{NU})_{NI} \neq reduct(reduct(\mathcal{A})_{NI})_{NU}$.

Next, the main results of this paper are shown in the following tables.

The main results with respect to optimistic multi-granulation lower and upper approximations

Number	Optimistic multi-granulation lower approximations	Optimistic multi-granulation upper approximations
1	$\underline{OM_{\sum\mathcal{A}}}(X) = \underline{OM_{\sum\mathcal{A}_{AU}}}(X)$	$\overline{OM_{\sum\mathcal{A}}}(X) = \overline{OM_{\sum\mathcal{A}_{AU}}}(X)$
2	$\underline{OM_{\sum\mathcal{A}}}(X) = \underline{OM_{\sum\mathcal{A}_{NU}}}(X)$	$\overline{OM_{\sum\mathcal{A}}}(X) = \overline{OM_{\sum\mathcal{A}_{NU}}}(X)$
3	$\underline{OM_{\sum\mathcal{A}}}(X) \neq \underline{OM_{\sum\mathcal{A}_{NI}}}(X)$	$\overline{OM_{\sum\mathcal{A}}}(X) \neq \overline{OM_{\sum\mathcal{A}_{NI}}}(X)$
4	$\underline{OM_{\sum\mathcal{A}}}(X) = \underline{OM_{\sum\mathcal{A}_{AUNU}}}(X) = \underline{OM_{\sum\mathcal{A}_{NUAU}}}(X)$	$\overline{OM_{\sum\mathcal{A}}}(X) = \overline{OM_{\sum\mathcal{A}_{AUNU}}}(X) = \overline{OM_{\sum\mathcal{A}_{NUAU}}}(X)$

The main results with respect to pessimistic multi-granulation lower and upper approximations

Number	Pessimistic multi-granulation lower approximations	Pessimistic multi-granulation upper approximations
1	$\underline{PM_{\sum\mathcal{A}}}(X) \neq \underline{PM_{\sum\mathcal{A}_{AU}}}(X)$	$\overline{PM_{\sum\mathcal{A}}}(X) \neq \overline{PM_{\sum\mathcal{A}_{AU}}}(X)$
2	$\underline{PM_{\sum\mathcal{A}}}(X) = \underline{PM_{\sum\mathcal{A}_{NU}}}(X)$	$\overline{PM_{\sum\mathcal{A}}}(X) = \overline{PM_{\sum\mathcal{A}_{NU}}}(X)$
3	$\underline{PM_{\sum\mathcal{A}}}(X) = \underline{PM_{\sum\mathcal{A}_{NI}}}(X)$	$\overline{PM_{\sum\mathcal{A}}}(X) = \overline{PM_{\sum\mathcal{A}_{NI}}}(X)$
4	$\underline{PM_{\sum\mathcal{A}}}(X) = \underline{PM_{\sum\mathcal{A}_{NUNI}}}(X) = \underline{PM_{\sum\mathcal{A}_{NINU}}}(X)$	$\overline{PM_{\sum\mathcal{A}}}(X) = \overline{PM_{\sum\mathcal{A}_{NUNI}}}(X) = \overline{PM_{\sum\mathcal{A}_{NINU}}}(X)$

Remark In the above two tables, symbols $\underline{OM_{\sum\mathcal{A}}}(X)$, $\underline{OM_{\sum\mathcal{A}_{AU}}}(X)$, $\underline{PM_{\sum\mathcal{A}_{NUNI}}}(X)$ mean the optimistic multi-granulation lower approximations of $X \subseteq U$ with respect to $\mathcal{A}$, $reduct(\mathcal{A})_{AU}$, $reduct(reduct(\mathcal{A})_{NU})_{NI}$, respectively. Other symbols in tables indicate similar meanings.

7 An example

As an application of several types of attribute reducts proposed in the paper, a real-world example of a multi-granulation information system with respect to exam scores is employed. Then, as many researchers are working on various optimization theories Xu et al. (2017) and algorithms (Abualigah et al. 2018a, b, c, d, 2017a, b, c; Abualigah and Khader 2017d; Abualigah et al. 2017e; Abualigah and Hanandeh 2015; Al-Betar and Abualigah 2017), we will explain how to choose the optimal attribute reduct so as to compute the multi-granulation lower and upper approximations more efficiently.

Example 7.1 The following table indicates a multi-granulation information system $\mathcal{I} = (U, A, V, f)$ with respect to exam scores, $U = \{x_1, x_2, \ldots, x_{30}\}$ is a universe including thirty students of mechanical engineering in Jimei University, $\mathcal{A} = \{A_1, A_2, \ldots, A_7\}$ is a family of attribute sets, where $A_1 = \{a_1^1, a_2^1, a_3^1\}$, and a_1^1, a_2^1, a_3^1 stands for three courses: Advanced Mathematics, Linear Algebra, Probability and Statistics, respectively; $A_2 = \{a_1^2, a_2^2\}$, and a_1^2, a_2^2 stands for two courses: College English 1, College English 2, respectively; $A_3 = \{a_1^3, a_2^3, a_3^3\}$, and a_1^3, a_2^3, a_3^3 stands for three courses: Engineering Drawing, Engineering Mechanics, Feedback Control of Dynamic Systems, respectively; $A_4 = \{a_1^4, a_2^4\}$, and a_1^4, a_2^4 stands for two courses: Programming in C, Computer Practice, respectively; $A_5 = \{a_1^5\}$, and a_1^5 stands for College Physics; $A_6 = \{a_1^6, a_2^6\}$, and a_1^6, a_2^6 stands for two courses: College PE 1, College PE 2, respectively; $A_7 = \{a_1^7, a_2^7\}$, and a_1^7, a_2^7 stands for two courses: Experiment 1, Experiment 2, respectively. At the same time, the exam scores are divided into four grades: Good, Medium, Pass and Fail, denoted by 1, 2, 3 and 4, respectively.

An information system with respect to exam scores

U	a_1^1	a_2^1	a_3^1	a_1^2	a_2^2	a_1^3	a_2^3	a_3^3	a_1^4	a_2^4	a_1^5	a_1^6	a_2^6	a_1^7	a_2^7
x_1	2	2	2	2	2	2	1	2	2	1	1	1	2	2	2
x_2	2	2	3	2	2	2	2	3	2	2	1	2	3	2	2
x_3	3	2	2	2	3	2	2	3	2	2	2	3	3	2	1
x_4	3	3	3	3	3	3	3	3	3	2	4	3	4	3	2
x_5	1	1	2	1	1	1	1	2	1	1	1	2	2	1	1
x_6	2	1	2	1	2	2	1	2	1	2	1	1	2	1	2
x_7	4	3	4	3	2	3	2	3	3	3	3	3	3	3	3
x_8	3	3	3	3	3	3	3	3	3	2	4	3	4	3	2
x_9	2	2	3	2	2	2	2	3	2	2	1	3	3	2	2
x_{10}	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1
x_{11}	2	2	3	2	2	2	2	3	2	2	1	2	3	2	2
x_{12}	2	3	3	2	3	3	2	3	2	3	2	3	3	2	3
x_{13}	2	1	1	2	2	1	1	1	1	2	1	2	3	2	2
x_{14}	2	3	4	3	2	3	3	4	3	3	3	2	2	3	3
x_{15}	2	3	3	2	3	3	2	3	2	3	2	3	4	2	3
x_{16}	2	3	2	3	3	2	3	3	2	3	3	3	3	2	3
x_{17}	1	1	2	1	2	1	1	2	1	1	1	2	2	1	1
x_{18}	2	1	2	1	2	2	1	2	1	2	1	1	2	1	2
x_{19}	3	2	2	2	3	2	2	3	2	2	2	3	3	2	1
x_{20}	2	2	2	2	2	2	1	2	2	1	1	2	3	2	2
x_{21}	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1
x_{22}	2	3	2	3	3	2	3	3	2	3	3	3	4	2	3
x_{23}	1	1	2	1	2	1	1	2	1	1	1	2	2	1	1
x_{24}	2	2	3	2	2	2	2	3	2	2	1	2	3	2	2
x_{25}	2	3	4	3	2	3	3	4	3	3	3	1	2	3	3
x_{26}	3	2	2	2	3	2	2	3	2	2	2	3	3	2	1
x_{27}	2	1	2	1	2	2	1	2	1	2	1	2	3	1	2
x_{28}	2	3	3	2	3	3	2	3	2	3	2	3	4	2	3
x_{29}	2	2	2	2	2	2	1	2	2	1	1	1	2	2	2
x_{30}	2	2	2	2	2	2	1	2	2	1	1	2	3	2	2

From the above information system, we have that $U/A_1 = \{\{x_1, x_{20}, x_{29}, x_{30}\}, \{x_2, x_9, x_{11}, x_{24}\}, \{x_3, x_{19}, x_{26}\}, \{x_4, x_8\}, \{x_5, x_{17}, x_{23}\}, \{x_6, x_{18}, x_{27}\}, \{x_7\}, \{x_{10}, x_{21}\}, \{x_{12}, x_{15}, x_{28}\}, \{x_{13}\}, \{x_{14}, x_{25}\}, \{x_{16}, x_{22}\}\}$. Similarly, $U/A_i, i = 2, 3, 4, 5, 6, 7$ can be easily presented.

According to Algorithms 1, 2 and 3, we have $reduct(\mathcal{A})_{AU} = \{A_1, A_2, A_6\}$, $reduct(\mathcal{A})_{NU} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$, $reduct(\mathcal{A})_{NI} = \{A_2, A_3, A_4, A_5, A_6, A_7\}$. Meanwhile, it is clear that $reduct(reduct(\mathcal{A})_{AU})_{NU} = reduct(reduct(\mathcal{A})_{NU})_{AU} = \{A_1, A_2, A_6\}$, $reduct(reduct(\mathcal{A})_{NI})_{NU} = \{A_2, A_3, A_4, A_5, A_6, A_7\}$, and $reduct(reduct(\mathcal{A})_{NU})_{NI} = \{A_2, A_3, A_4, A_5, A_6\}$. Therefore, we have seven attribute reducts with respect to $\mathcal{A} = \{A_1, A_2, \ldots, A_7\}$. i.e., $reduct(\mathcal{A})_{AU}$, $reduct(\mathcal{A})_{NU}$, $reduct(\mathcal{A})_{NI}$, $reduct(reduct(\mathcal{A})_{AU})_{NU}$, $reduct(reduct(\mathcal{A})_{NU})_{AU}$, $reduct(reduct(\mathcal{A})_{NI})_{NU}$, and $reduct(reduct(\mathcal{A})_{NU})_{NI}$.

From the view of granular computing, we will raise a question: which reduct is the best selection to compute the multi-granulation lower and upper approximations? On the one hand, because the optimistic multi-granulation lower and upper approximations with respect to $\mathcal{A}$, $reduct(\mathcal{A})_{AU}$, $reduct(reduct(\mathcal{A})_{AU})_{NU}$, and $reduct(reduct(\mathcal{A})_{NU})_{AU}$, respectively, are same, and from above discussion, it can be found that $reduct(\mathcal{A})_{AU} = reduct(reduct(\mathcal{A})_{AU})_{NU} = reduct(reduct(\mathcal{A})_{NU})_{AU}$. Therefore, $reduct(\mathcal{A})_{AU}$, $reduct(reduct(\mathcal{A})_{AU})_{NU}$, and $reduct(reduct(\mathcal{A})_{NU})_{AU}$ are all the optimal selections to compute the optimistic multi-granulation lower and upper approximations.

On the other hand, we can obtain the following relations: $reduct(reduct(\mathcal{A})_{NU})_{NI} \subset reduct(reduct(\mathcal{A})_{NI})_{NU}$ and $reduct(reduct(\mathcal{A})_{NU})_{NI} \subset reduct(\mathcal{A})_{NI}$, although the pessimistic multi-granulation lower and upper approximations with respect to $\mathcal{A}$, $reduct(\mathcal{A})_{NI}$, $reduct(reduct(\mathcal{A})_{NU})_{NI}$ and $reduct(reduct(\mathcal{A})_{NI})_{NU}$, respectively, are same. Thus $reduct(reduct(\mathcal{A})_{NU})_{NI}$ is the optimal attribute reduct to compute the pessimistic multi-granulation lower and upper approximations.

8 Conclusion

In this part, we first introduce the main conclusions obtained in the paper. Then, we make further prospects for future research work.

(i) Main conclusions of our paper It is clear that reduct theory plays an important role in pattern recognition and machine learning. Although multi-granulation rough set theory has been widely studied, attribute reduct of multi-granulation rough sets has not yet been explored. Therefore, the main content of this paper is to investigate the attribute reduct theory of multi-granulation information system. Specifically, the following problems are solved in this paper. Based on the definitions of multi-granulation upper and lower approximations, three types of attribute reducts are proposed, which are called arbitrary union reduct, neighborhood union reduct and neighborhood intersection reduct, respectively. Then, many important and interesting properties of these attribute reducts are researched. In order to solve practical problems, three effective algorithms are designed. At the same time, the relationships among these three attribute reducts are deeply explored. At last, a real-world example of multi-granulation information system is employed. The multi-granulation information system consists of exam scores of thirty students of mechanical engineering in Jimei University. According to the given algorithms, all the reducts in the information system are calculated one by one. Meanwhile, how to select the optimal reduct is also discussed in detail.

(ii) Future research work The theoretical results in this paper establish a basis for studying attribute reducts of the multi-granulation information systems. On this basis, it is possible to further study the attribute reducts with respect to consistent and inconsistent multi-granulation information systems. Furthermore, we need to develop better algorithms to deal with practical issues in the future.

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