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# Relative Reduction of Incomplete Interval-valued Decision Information Systems Associated with Evidence Theory

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Relative reduction is regarded as a significant problem in rough set theory, which needs to eliminate some attributes that are not required in information system. Dempster-Shafer evidence theory is a serviceable means to explore uncertain information. This article establishes rough set model in incomplete interval-valued decision information system (*IIDIS*). Belief (plausibility) function is introduced for studying relative belief (plausibility) reduction in *IIDIS*. We aim to study several relative reductions based on evidence theory and explore relations among different relative reductions in the consistent/inconsistent *IIDIS* via four importance degrees. Relative reduction is not only equivalent to relative belief reduction but also equivalent to relative plausibility reduction in the consistent *IIDIS*. In the inconsistent *IIDIS*, relative plausibility consistent set can conclude it be deemed as relative belief consistent set, not vice versa. Furthermore, the feasibility about presented theorems are verified by several experiments from six UCI data set.

***Keywords:*** evidence theory, incomplete interval-valued decision information system, granular computing, knowledge discovery, relative reduction, rough theory

## 1. INTRODUCTION

Evidence theory, presented by Dempster in 1967 [1] and extended by Shafer in 1976 [2], is an uncertain reasoning theory. It is referred to as D-S theory belongs to artificial intelligence field and is a novel paradigm for handling uncertain information. D-S theory satisfies conditions that are weaker than Bayesian probability theory and has the ability to express "uncertainty" and "do not know" directly. As fundamental numeric measure, belief/plausibility function is obtained from the sum of basic probability assignment for measuring the values about lower/upper bound of the probability. A belief structure [3, 4] is constructed by an ordered pair that covers the family of all focal elements and basic probability assignment. D-S evidence theory has been expanded into areas of risk assessment, target recognition, comprehensive diagnosis, uncertainty reasoning and so on [5-7].

Pawlak introduced rough set theory (RST) [8] that be considered as ponderable means so as to depict information and knowledge. After decades of development, the

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theory has formed a correspondingly perfect theoretical system. The basic structure of RST is an approximation space that consists of the universe and the binary relation. As two exact concepts, lower and upper approximation can depict imprecise concept. The RST has been applied rapidly and generally to science and technology fields in recent years, which has lots of studying results [9-12]. The generalization of the RST has been researched extensively in machine learning, data analysis and granular computing [13-15] *etc.* The RST has some natural connections with D-S evidence theory in the article [16-18], which pointed at the corresponding relations between lower/upper approximation in approximation space and belief/plausibility function in belief structure.

Usually, potential knowledge and information can be reflected in a table in RST. The table is a basic notion and called an information system (IS). Rows and columns represent object set and attribute set, respectively. With the increase of large amounts of data and the actual needs, the attribute values have changed from single values to continuous values, interval numbers *etc.* Correspondingly, information systems [19, 20] using these values as attribute values are produced and the extension of classical information system. Nevertheless, the phenomena of incompleteness of information, some attribute values of objects are missing but really subsistent, exist diffusely in realistic life. In general, we call that this system is an incomplete information system (IIS) [21-23]. The equivalence relation is applicable to the classical rough set but is not necessarily valid in the incomplete information systems, which limits the application of RST in practical problems. Therefore, some users define a number of binary relations are different from the equivalence relation to continue researching incomplete information systems for mining hidden knowledge and rules in data. Zhang [24] researched a way of extracting rule in incomplete decision tables. The writing [25] improved tolerance relationship and counted the core attributes by extensive tolerance relationship. Xu proposed [26] an approach for fusing multiple fuzzy incomplete information sources by utilizing a binary tolerance relation.

As we all know, attribute reduction [27-30] is a significant studying issue in RST, which is required to delete some attributes that are not relevant or not important under the condition that the knowledge base is classified but the decision-making ability is unchanged. Relative reduction is one of the attribute reductions and performed in decision information system. It is known that many scholars and experts have made valuable and useful results in this area. Sun [31] put forward a range of attribute reduction methods based on the transitive binary relationship in incomplete information system. Wu introduced several attribute reductions [32] in incomplete decision tables via D-S theory. Zhang [33] explored knowledge reduction in the light of inclusion degree and evidence theory. Taking into accounts the ordering of attribute values, Xu [34] and Du [35] utilized dominance relation in order information systems to explore attribute reductions, and discussed the relationships among proposed reductions. But there are few studies in incomplete interval-valued decision information via evidence theory. The motivation of this article is that make attempt to explore relative reductions in consistent/inconsistent *IIDIS* through associating with D-S theory.

Next, Section 2 introduces some concepts concerning *IIDIS* and D-S theory. Based on the interval similarity degree, the paper gives a new definition of tolerance relation that is applied to the incomplete interval-valued decision table in Section 3. Section 4 defines belief/plausibility function to compute several relative reductions in consistent

*IIDIS* and inconsistent *IIDIS* associated with D-S theory, and explores certain relations among raised relative reductions. And an algorithm is displayed for researching relative reductions by means of four importance degrees. Furthermore, the fifth section exhibits relative reductions of six data sets in consistent and in consistent *IIDIS*. Finally, conclusion of this article and plans for next work locates in Section 6.

## 2. BASIC CONCEPTS

We chiefly retrospect indispensable fundamental notions for the next research about relative reductions in *IIDIS* and relevant knowledge of D-S theory.

### 2.1 Preliminaries About IIDIS

Detailed basic concepts and important properties will be displayed in the *IIDIS* [36-38].

An information system can be labeled as $S = (OS, CD, DO, F)$. Assume $CD = CA \cup DA$, $CA \cap DA = \emptyset$, then $DIS = (OS, CA \cup DA, DO, F)$ is referred to as a decision information system. *OS* is seen as universe of discourse. Condition attribute set *CA*, decision attribute set *DA* are subsets of attribute set *CD* that shows features contained by objects in universe. $DO = \prod_{c \in cd} V_c$, where $V_c$ is the domain for $c \in CD$. Total function $F$: $OS \times CD \to DO$ s.t. $F(x, c) \in V_c$ while $x \in OS$, $c \in CD$.

Interval number $F(x, c) = [\mu^-, \mu^+]$, if it is an unknown value but exists in actual, then $F(x, c)$ is treated as *. At this time, $IIS = (OS, CD, DO, F)$($IIDIS = (OS, CA \cup DA, DO, F)$) is an incomplete interval-valued(decision) information system. Next, whole article only takes account of $DA = \{D_c\}$. In the following, we introduce an example for better understand *IIDIS*.

**Example 1:** The Table 1 is the risk investment item evaluation. There have five investment projects (*OS*) and six risk factors (*CA*). The table includes some unknown values, which is an $IIV\ DIS = (OS, CA \cup \{D_c\}, DO, F)$. Where $OS = \{x_1, x_2, \ldots, x_5\}$, $x_j$ show the $j$th investment project ($j = 1, 2, \ldots, 5$). $CA = \{c_1, c_2, \cdots, c_6\}$, $c_i$($i = 1, 2, \ldots, 6$) represent Market, Technique, Management, Environment, Prospect and Finance respectively. $D_c$ exhibits investment risk. $F(x, D_c) \in \{1, 2\}$. 1 denotes the investment risk is high and 2 indicates the investment risk is low.

**Table 1. An *IIDIS*.**

| *OS* | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $D_c$ |
|---|---|---|---|---|---|---|---|
| $x_1$ | [3, 4] | * | [3, 4] | [3, 4] | [2, 3] | [4, 5] | 1 |
| $x_2$ | [2, 4] | [4, 5] | [1, 3] | [1, 3] | * | [1, 3] | 2 |
| $x_3$ | [1, 3] | [1, 2] | * | [3, 4] | [2, 3] | [4, 5] | 1 |
| $x_4$ | * | [4, 5] | [3, 5] | [2, 4] | [3, 5] | [3, 5] | 1 |
| $x_5$ | [2, 3] | [4, 5] | [2, 3] | [1, 2] | [2, 3] | [1, 3] | 2 |

**Definition 1:** [39] Let interval number $I = [\mu^-, \mu^+]$, then its length is denoted by $\rho(I)$

$$\rho(I) = \mu^+ - \mu^-. \tag{1}$$

When $\mu^- = \mu^+$, the interval $I$ represents a single value and $\rho(I) = 0$. $\rho(I) = 0$ while $I$ is an empty. Obviously, for $\forall P$, $Q$ be interval numbers, if $P \cap Q = E$ is nonempty, then $\rho(E) \neq 0$; otherwise, $\rho(E) = 0$.

Jaccard coefficient is a probability and used to compare the similarity and dispersion of the sample set, which could be indicated as intersection of sample set divided by their union (details below).

**Definition 2:** Let an $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$ and $\forall c_k \in CA$, $x_i, x_j \in OS$. $F(x_i, c_k) = P = [\mu^-, \mu^+]$, $F(x_j, c_k) = Q = [v^-, v^+]$, where $P \neq *$, $Q \neq *$, thus interval similarity degree in [40] concerning $x_i$, $x_j$ under the attribute $c_k$ is

$$S_{ij}^k(P,Q) = \frac{\rho(P \cap Q)}{\rho(P \cup Q)}. \tag{2}$$

Since $\rho(P \cup Q) = \rho(P) + \rho(Q) - \rho(P \cap Q)$, then interval similarity degree is written as

$$S_{ij}^k(P,Q) = \frac{\rho(P \cap Q)}{\rho(P) + \rho(Q) - \rho(P \cup Q)} \tag{3}$$

where $\cup(\cap)$ is union(intersection) operation. Obviously, $S_{ij}^k(P, Q) \in [0, 1]$. If $P = Q$, then $S_{ij}^k \cup (P, Q) = S_{ji}^k(P, Q) = 1$. That is the similarity degree satisfies reflexivity. Furthermore, $S_{ij}^k(P, Q) = S_{ji}^k(P, Q)$, so the similarity degree satisfies symmetry.

If there exists a $x_i \in OS$ s.t. $F(x_i, c_k) = *$ for $c_k \in CA$, then the attribute value is viewed as missing value but it actually exists. However, the interval similarity degree is not applicable for this object $x_i$. So we set $S_{ij}^k(P, Q) = 1$ for any $x_j \in OS$ while $F(x_i, c_k) = *$. In addition, the minimal interval similarity degree is defined by the following for any $x_i$, $x_j \in OS$:

$$S_{ij}^{CA} = \min_{c_k \in CA}\{S_{ij}^k(P,Q)\}. \tag{4}$$

### 2.2 Preliminaries About D-S Theory

As a mean to dispose uncertainty in RST, D-S theory is a popularization of probability theory and has a sound theoretical basis.

**Definition 3:** [2, 41] For a universe $OS$, if mapping $m$: $\mathscr{P}(OS) \to [0, 1]$ s.t. two formulas $m(\emptyset) = 0$ and $\sum_{Y \subseteq OS} m(Y) = 1$ hold, then $m$ could be called a basic probability assignment where $m(Y)$ indicates belief degree. $Y \subseteq OS$ is focal element while $m(Y) > 0$. Core of universe $\mathscr{M}$ consists of whole focal elements. $(\mathscr{M}, m)$ be generally termed as belief structure.

Related to basic probability assignment, belief measure and plausibility measure are derived.

**Definition 4:** [2, 41] For a belief structure $(\mathscr{M}, m)$. Set functions: $\mathscr{P}(OS) \to [0, 1]$. Belief and plausibility measure be denoted by, respectively: for any $X \in \mathscr{P}(OS)$, $Y \in M$, then

$$BEL(X) = \sum_{Y \subseteq X} m(Y), \tag{5}$$

$$PL(X) = \sum_{X \cap Y \neq \emptyset} m(Y). \tag{6}$$

These two functions are termed as belief and plausibility function, which indicate sum of probability of set and request the set is definitely and possibly support to $X$, respectively. On the basis of same belief structure, belief/plausibility function corresponds to lower/upper bound of probability and their connection can be represented by $PL(X) = 1 - BEL(X^c)$, Besides, $BEL(X) \leq PL(X)$, $\forall X \in \mathscr{P}(OS)$.

## 3. ROUGH SET IN *IIDIS*

In Pawlak rough set theory, a binary relation $R$ is referred to as an equivalence relation on the universe, which satisfies reflexivity, symmetry as well as transitivity. A universe is partitioned into disjoint sets formed by an equivalence relation, that is equivalence classes. A quotient set is the set of all equivalence classes based on the equivalence relation $R$. Where the equivalence class is $[x]_R = \{y \in OS|xRy\}$ for $x \in OS$. Each uncertainty concept $X \subseteq OS$ can be represented by a pair of exact concepts [8]: lower approximation and upper approximation.

$$\underline{R}(X) = \{x \in OS|[x]_R \subseteq X\} = \cup\{[x]_R|[x]_R \subseteq X\}; \tag{7}$$

$$\overline{R}(X) = \{x \in OS|[x]_R \cap X \neq \emptyset\} = \cup\{[x]_R|[x]_R \cap X \neq \emptyset\}. \tag{8}$$

In this section, a novel tolerance relation can be generated in light of interval similarity degree and rough set model is constructed via this relation in *IIDIS*.

**Definition 5:** Given an $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$, for any $B \subseteq CA$, $x_i, x_j \in OS$. Set $\lambda \in (0.5, 1]$. A tolerance relation concerning attribute set be known as

$$R_B^\lambda = \{(x_i, x_j) \in OS^2 \mid S_{ij}^B \geq \lambda\}. \tag{9}$$

It can be shown that $R_B^\lambda$ is reflexive and symmetrical in the light of the minimal similarity degree, which is a tolerance relation. The tolerance class is:

$$[x_i]_{R_B^\lambda} = \{x_j \in OS \mid (x_i, x_j) \in R_B^\lambda\}, \tag{10}$$

$$OS / R_B^\lambda = \{[x_1]_{R_B^\lambda}, [x_2]_{R_B^\lambda}, \ldots, [x_{|OS|}]_{R_B^\lambda}\}. \tag{11}$$

$[x_i]_{R_B^\lambda}$ represents a cluster objects that the minimal similarity degree with reference to $x_i$, $x_j$ are not less than a given threshold $\lambda$. $OS/R_B^\lambda$ is a cover on $OS$.

As for decision attribute $D_c$, equivalence class $[x_i]_{D_c} = \{x_j \in OS|F(x_i, D_c) = F(x_j, D_c)\}$. Quotient set is called $OS/D_c = \{[x]_{D_c}|\forall x \in OS\} = \{D_c^1, D_c^2, \ldots, D_c^t\}(D_c^i \subseteq OS, i = 1, 2, \ldots, t)$. Distinctly, relation $R_{D_c}$ is an equivalence relation. Certain properties in regard to tolerance relation are discussed below.

**Proposition 1:** Given an $IIDIS$ = ($OS$, $CA\cup\{D_c\}$, $DO$, $F$), if each $B$, $D\subseteq CA$, $x\in OS$. Then:

(1) $R_B^\lambda(x)=\cap_{c_k\in B}R_{c_k}^\lambda(x)$;

(2) If $B\subseteq D$, then $R_D^\lambda(x)\subseteq R_B^\lambda(x)$.

According to definitions of tolerance relation and corresponding tolerance classes, an uncertainty concept can be depicted by two exact notions.

**Definition 6:** Given an $IIDIS$ = ($OS$, $CA\cup\{D_c\}$, $DO$, $F$), for $\forall B\subseteq CA$, $X\subseteq OS$. Lower, upper approximation about tolerance relation $R_B^\lambda$ are denoted by, respectively:

$$\underline{R_B^\lambda}(X)=\{x\in OS\mid [x]_{R_B^\lambda}\subseteq X\}; \tag{12}$$

$$\overline{R_B^\lambda}(X)=\{x\in OS\mid [x]_{R_B^\lambda}\subseteq X\neq\emptyset\}. \tag{13}$$

Furthermore, positive region is $Pos_{R_B^\lambda}(X)=\underline{R_B^\lambda}(X)$, negative region is $Neg_{R_B^\lambda}(X)=OS-\overline{R_B^\lambda}(X)$, boundary region is $Bn_{R_B^\lambda}(X)=\overline{R_B^\lambda}(X)-\underline{R_B^\lambda}(X)$. Next, some properties about $\underline{R_B^\lambda}(X)$ and $\overline{R_B^\lambda}(X)$ are discussed and similar with Pawlak approximation space.

**Theorem 1:** Given an $IIDIS$ = ($OS$, $CA\cup\{D_c\}$, $DO$, $F$), for $\forall B$, $D\subseteq CA$, $Y$, $Z\subseteq OS$.

(1) $\underline{R_B^\lambda}(Y)\subseteq Y\subseteq\overline{R_B^\lambda}(Y)$. (Boundedness)

(2) $\underline{R_B^\lambda}(\sim Y)=\sim\overline{R_B^\lambda}(Y)$; $\overline{R_B^\lambda}(\sim Y)=\sim\underline{R_B^\lambda}(Y)$. (Duality)

(3) $\underline{R_B^\lambda}(\emptyset)=\overline{R_B^\lambda}(\emptyset)=\emptyset$; $\underline{R_B^\lambda}(OS)=\overline{R_B^\lambda}(OS)=(OS)$. (Normality)

(4) $\underline{R_B^\lambda}(Y\cap Z)=\underline{R_B^\lambda}(Y)\cap\underline{R_B^\lambda}(Z)$; $\overline{R_B^\lambda}(Y\cup Z)=\overline{R_B^\lambda}(Y)\cup\overline{R_B^\lambda}(Z)$. (Multiplicativity and Additivity)

(5) If $Y\subseteq Z$, then $\underline{R_B^\lambda}(Y)\subseteq\underline{R_B^\lambda}(Z)$; and $\overline{R_B^\lambda}(Y)\subseteq\overline{R_B^\lambda}(Y)$. (Monotonicity 1)

(6) If $B\subseteq D$, then $\underline{R_B^\lambda}(Y)\subseteq\underline{R_B^\lambda}(Y)$; and $\overline{R_D^\lambda}(Y)\subseteq\overline{R_B^\lambda}(Y)$. (Monotonicity 2)

(7) $\underline{R_B^\lambda}(Y\cup Z)\supseteq\underline{R_B^\lambda}(Y)\cup\underline{R_B^\lambda}(Z)$; $\overline{R_B^\lambda}(Y\cap Z)\subseteq\overline{R_B^\lambda}(Z)\cap\overline{R_B^\lambda}(Z)$. (Inclusion)

## 4. RELATIVE ATTRIBUTE REDUCTIONS IN *IIDIS*

Attribute reduction is to delete some redundancy or not important attributes but remain certain classes unchanged. In the following, we discuss several relative reductions in the case of consistent *IIDIS* and inconsistent *IIDIS*, respectively.

### 4.1 The Belief and Plausibility Functions in *IIDIS*

Since a tolerance relation satisfies reflexivity, inspired by the definition of the mass function in the paper [2, 36], we introduce a mass function, belief/plausibility function in *IIDIS*.

**Definition 7:** Given an *IIDIS* = (*OS*, $CA \cup \{D_c\}$, *DO*, *F*). A basic probability assignment with reference to attribute set $B(B \subseteq CA)$ is depicted by a mapping $m_B$: $\mathscr{P}(OS) \to [0, 1]$. It can be denoted by

$$m_B(X) = \frac{|f_B(X)|}{|OS|} \tag{14}$$

where $f_B(X) = \{x | R_B^\lambda(x) = X\}$ for $x \in OS$. $X$ is a focal element of $m_B$ on condition that $f_B(X) \neq \emptyset$. Naturally, $\mathscr{M}_\mathscr{B} = \{X \in \mathscr{P}(OS) | f_B(X) \neq \emptyset\} = \{R_B^\lambda(x) \in \mathscr{P}(OS) | x \in OS\}$ is the core of *OS* in *IIDIS*. $(\mathscr{M}_\mathscr{B}, m_B)$ be termed as +belief structure in *IIDIS*. Because mass function $m_B$ also makes $m_B(\emptyset) = 0$ and $\sum_{X \subseteq OS} m_B(X) = 1$ hold. The proof is as follows.

***Proof*:**
(1) When $X = \emptyset$. Obviously, $m_B(\emptyset) = 0$.
(2) When $X = \emptyset$, let $OS/R_B^\lambda = \{[x_1]_{R_B^\lambda}, [x_1]_{R_B^\lambda}, \ldots, [x_1]_{R_B^\lambda}\}(k \leq |OS|)$. There are two situations:
(i) For any $i \in \{1, 2, \ldots, k\}$, if $X \not\subseteq OS/R_B^\lambda$, then $X \neq [x_i]_{R_B^\lambda}$, namely, $X \neq R_B^\lambda(x_i)$. So $m_B(X) = \frac{|f_B(X)|}{|OS|} = \frac{|\{x | R_B^\lambda(x) = X, \forall x \in OS\}|}{|OS|} = \frac{|\emptyset|}{|OS|} = 0$.
(ii) If $X \subseteq OS/R_B^\lambda$, then there must exists $i \in \{1, 2, \ldots, k\}$, s.t. $X = [x_i]_{R_B^\lambda}$. Hence $m_B(X) = \frac{|f_B(X)|}{|OS|} = \frac{|\{x | R_B^\lambda(x) = X, \forall x \in OS\}|}{|OS|} = \frac{|\{x | R_B^\lambda(x) = [x_i]_{R_B^\lambda}, \forall x \in OS\}|}{|OS|}$.
If for any $x_l \in [x_i]_{R_B^\lambda}$ such that $[x_l]_{R_B^\lambda} = [x_i]_{R_B^\lambda}$, then $m_B(X) = \frac{|[x_i]_{R_B^\lambda}|}{|OS|}$; Otherwise, $m_B(X) = \frac{|f_B(X)|}{|OS|} = \frac{1}{|OS|}$.
In a word, $\sum_{x \subseteq OS} m_B(X) = 1$.
In conclusion, $m_B$ is a basic probability assignment.

**Definition 8:** Given an *IIDIS* = (*OS*, $CA \cup \{D_c\}$, *DO*, *F*). $m_B$ be treated as basic probability assignment concerning attribute set $B(B \subseteq CA)$. Belief, plausibility functions are denoted by, respectively: for $X \in \mathscr{P}(OS)$, then

$$BLE_B(X) = \sum_{Y \subseteq X} m_B(Y), Y \in OS / R_B^\lambda, \tag{15}$$

$$PL_B(X) = \sum_{Y \cap X \neq \emptyset} m_B(Y), Y \in OS / R_B^\lambda, \tag{16}$$

in the above formulas, set functions $BEL_B$ and $PL_B$ are mappings that convert power set $\mathscr{P}(OS)$ to real numbers in closed interval [0, 1].

Moreover, belief, plausibility function can also be labeled as $BEL_B(X) = \sum_{Y \subseteq X} m_B(Y)$, $PL_B(X) = \sum_{Y \cap X \neq \emptyset} m_B(Y)$ on account of $m_B(Y) = 0$ while $Y \notin OS/R_B^\lambda$.

The functions are connected by a property: $PL_B(X) = 1 - BE\ L_B(X^c)$. And distinctly, $BEL_B(X) \leq PL_B(X)$ holds $X \in \mathscr{P}(OS)$.

**Theorem 2:** Given an *IIDIS* = (*OS*, $CA \cup \{D_c\}$, *DO*, *F*), if each $B \subseteq CA$, then

(1) $BEL_B(X) = P(\underline{R_B^\lambda}(X))$. (2) $PL_B(X) = P(\overline{R_B^\lambda}(X))$,

where $P(X) = \frac{|X|}{|OS|}$, $|X|$ shows the cardinal number of set $X$. $\frac{|X|}{|OS|}$ indicates the probability of concept $X$.

***Proof*:** We can obtain $BEL_B(X) = \sum_{Y\subseteq X} m_B(Y) = \sum_{Y\subseteq X} \frac{|f_B(Y)|}{|OS|} = \sum_{Y\subseteq X} \frac{|\{x\in OS|R_B^\lambda(x)=Y\}|}{|OS|} = \frac{|\{x\in OS|R_B^\lambda(x)\subseteq X\}|}{|OS|} = \frac{|\underline{R_B^\lambda}(X)|}{|OS|} = P(\underline{R_B^\lambda}(X))$ for $Y\in OS/R_B^\lambda$, according to Definition 8. Similarly, $PL_B(X) = P(\overline{R_B^\lambda}(X))$ also holds.

Besides, monotonicity is the property of belief/plausibility function. That is to say when $B\subseteq D\subseteq CA$, one has

$$BEL_B(X) \leq Bel_D(X) \leq P(X) \leq Pl_D(X) \leq PL_B(X)(X\subseteq OS). \tag{17}$$

### 4.2 The Relative Reductions in *IIDIS*

In this section, we give some notions about relative reduction, relative belief/plausibility reduction in the *IIDIS*.

Let $IIDIS = (OS, CA\cup\{D_c\}, DO, F)$, quotient set with regard to decision at tribute could be written as $OS/D_c = \{[x]_{D_c}|\forall x \in OS\} = \{D_c^1, D_c^2, \ldots, D_c^t\}(D_c^i\subseteq OS, i = 1, 2, \ldots, t)$. An *IIDIS* is consistent if $R_{CA}^\lambda\subseteq R_{D_c}$. In other words, let $\Delta_{CA}(x) = \{F(y, D_c)|(x, y) \in R_B^\lambda\}$, if $|\Delta_{CA}(x)| = 1$ holds for every $x \in OS$, then *IIDIS* is consistent. If not, *IIDIS* is inconsistent. This paper studies the consistent and inconsistent *IIDIS*.

**Definition 9:** Given an $IIDIS = (OS, CA\cup\{D_c\}, DO, F)$, for each $B, B' \subseteq CA$, where $B'\subset B$. Then

(1) Suppose $\Delta_B(x) = \Delta_{CA}(x)$ for $\forall x \in OS$, thus $B$ could be termed as relative consistent set of *IIDIS*. Meanwhile, assume every $B'$ is not relative consistent set, hence $B$ could be known as relative reduction in *IIDIS*.
(2) Suppose $BEL_B(D_c^i) = BEL_{CA}(D_c^i)$ for $\forall D_c^i \in OS/D_c$, thus $B$ could be termed as relative belief consistent set in *IIDIS*. At the same time, assume every $B'$ is not relative belief consistent set, hence $B$ could be known as relative belief reduction in *IIDIS*.
(3) Suppose $PL_B(D_c^i) = PL_{CA}(D_c^i)$ for $\forall D_c^i \in OS/D_c$, thus $B$ could be termed as relative plausibility consistent set in *IIDIS*. At the same time, assume every $B'$ is not relative plausibility consistent set, hence $B$ could be known as relative plausibility reduction in *IIDIS*.

From definitions above, we can observe that relative reductions are minimal subsets of attribute set CA, which preserves the consistency of *IIDIS*. Relative belief and plausibility reduction are minimal subsets that promise belief and plausibility degree unchanged, respectively.

Besides, $R_B^\lambda\subseteq R_{D_c}(B \subseteq CA)$ holds iff $B$ can be known as relative consistent set. In their words, as minimal subset promises the researching system being consistent, $B$ can be regard as relative reduction and vice versa.

#### 4.2.1 The relative reductions in consistent *IIDIS*

First of all we explore several properties with reference to relative reductions in consistent *IIDIS*. Here $\Delta_{CA}(x) = \{F(x, D_c)\}$ for every object in universe. To better characterize relative belief, plausibility reduction and for convenience of following statements,

$\sum_{D_c^i \in OS/D_c} BEL_B(X)$ and $\sum_{D_c^i \in OS/D_c} PL_B(X)$ denote the belief, plausibility sum in *IIDIS*, respectively.

**Theorem 3:** Given a consistent *IIDIS* = (*OS*, $CA \cup \{D_c\}$, *DO*, *F*) and $B \subseteq CA$, $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$. Three equivalent statements are acquired:

(1) $\underline{R_B^\lambda}(D_c^i) = D_c^i, \forall 1 \le i \le t$;
(2) $\overline{R_B^\lambda}(D_c^i) = D_c^i, \forall 1 \le i \le t$;
(3) *B* can be regarded as relative consistent set.

***Proof***: (3)⇒(1) Evidently, $\underline{R_B^\lambda}(D_c^i) \subseteq D_c^i$. $\forall x \in D_c^i$, $R_{D_c}(x) = D_c^i$ holds. For *B* is relative consistent set, $R_B^\lambda(x) \subseteq R_{D_c}(x) = D_c^i$, thus $x \in \underline{R_B^\lambda}(D_c^i)$, that is $D_c^i \subseteq \underline{R_B^\lambda}(D_c^i)$. Hence $\underline{R_B^\lambda}(D_c^i) = D_c^i$.

(1)⇒(3) Suppose there exists $x \in OS$ and $R_B^\lambda(x) \not\subseteq R_{D_c}(x)$ holds, then one has $y \in R_B^\lambda(x)$ but $y \notin R_{D_c}(x)$. That is $x \in D_c^i$ and $y \notin D_c^i$ when $F(x, D_c) = D_c^i$. And $\underline{R_B^\lambda}(D_c^i) = D_c^i$ holds, we have $x \in \underline{R_B^\lambda}(D_c^i)$. Thus, $R_B^\lambda(x) \subseteq D_c^i$, $y \in D_c^i$, there exists a contradiction. *B* can be regarded as relative consistent set.

(3)⇒(2) Visibly, $D_c^i \subseteq \overline{R_B^\lambda}(D_c^i)$. Moreover, $\forall x \in \overline{R_B^\lambda}(D_c^i)$, so $[x]_{R_B^\lambda} \cap D_c^i = \emptyset$. There exists $y \in D_c^i$, s.t. $y \in [x]_{R_B^\lambda}$. Because $R_B^\lambda$ satisfies symmetry, so $x \in [y]_{R_B^\lambda}$, $x \in R_B^\lambda(y)$. For *B* can be regarded as relative consistent set, then $R_B^\lambda(y) \subseteq R_{D_c}(y) = D_c^i$. Thus, $x \in D_c^i$. Hence $\overline{R_B^\lambda}(D_c^i) \subseteq D_c^i$. Therefore, $\overline{R_B^\lambda}(D_c^i) = D_c^i$.

(2)⇒(3) Suppose one has $x \in OS$ s.t. $R_B^\lambda(x) \not\subseteq R_{D_c}(x)$, hence one has $y \in R_B^\lambda(x)$ but $y \notin R_{D_c}(x)$. That is $x \in D_c^i$ and $y \notin D_c^i$ while $F(x, D_c) = D_c^i$. And $\overline{R_B^\lambda}(D_c^i) = D_c^i$ holds, so $x \in \overline{R_B^\lambda}(D_c^i)$. Thus, $[x]_{R_B^\lambda} \cap D_c^i = \emptyset$. Furthermore, it can be obtained $x \in [y]_{R_B^\lambda}$ for $R_B^\lambda$ satisfies symmetry and $y \in [x]_{R_B^\lambda}$, thus $[y]_{R_B^\lambda} \cap D_c^i \ne \emptyset$. Namely, $y \in \overline{R_B^\lambda}(D_c^i) = D_c^i$, there exists a contradiction. Hence *B* can be regarded as relative consistent set.

**Theorem 4:** Let a consistent *IIDIS* = (*OS*, $CA \cup \{D_c\}$, *DO*, *F*), $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$, for $\forall B \subseteq CA$. Three equivalent conditions are obtained:

(1) $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 1$;
(2) $\sum_{D_c^i \in OS/D_c} PL_B(D_c^i) = 1$;
(3) *B* can be deemed as relative consistent set.

***Proof***: (3)⇒(1) For any $D_c^i$, we can get $\underline{R_B^\lambda}(D_c^i) = D_c^i$ according to the Theorem 3. Then $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} \frac{|\underline{R_B^\lambda}(D_c^i)|}{|OS|} = \sum_{D_c^i \in OS/D_c} \frac{|D_c^i|}{|OS|} = 1$.

(1)⇔(3) For *CA* is relative consistent set. Then $\sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) = 1$. By Eq. (17), $BEL_B(D_c^i) \le BEL_{CA}(D_c^i)$ holds for any $D_c^i$. So there have $1 = \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) \ge \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 1$. Hence $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) = 1$, which is equivalent to $|\underline{R_B^\lambda}(D_c^i)| = |\underline{R_{CA}^\lambda}(D_c^i)| = |D_c^i|$. Furthermore, $\underline{R_B^\lambda}(D_c^i) \subseteq \underline{R_{CA}^\lambda}(D_c^i) \subseteq D_c^i$ according to Theorem 1. Therefore, $\underline{R_B^\lambda}(D_c^i) = D_c^i$. *B* can be regarded as relative consistent set.

(3)⇔(2) Its proof is similar to (3)⇔(1).

**Theorem 5:** Let a consistent *IIDIS* = (*OS*, $CA\cup\{D_c\}$, *DO*, *F*), $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$, for $\forall B, B' \subseteq CA$, where $B' \subseteq B$. Three equivalent assertions are gained:

(1) $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 1$. Meanwhile, for every $B'$, $\sum_{D_c^i \in OS/D_c} BEL_{B'}(D_c^i) < 1$ holds;
(2) $\sum_{D_c^i \in OS/D_c} PL_B(D_c^i) = 1$. Meanwhile, for every $B_i$, $\sum_{D_c^i \in OS/D_c} PL_{B'}(D_c^i) < 1$ holds;
(3) $B$ is a relative reduction.

***Proof*:** This Theorem can be directly and easily obtained according to Theorem 4 and Definition 9.

**Theorem 6:** Let a consistent *IIDIS* = (*OS*, $CA\cup\{D_c\}$, *DO*, *F*), $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$, for $\forall B \subseteq CA$. Three equivalent assertions are gained:

(1) $B$ can be regarded as relative reduction.
(2) $B$ can be regarded as relative belief reduction.
(3) $B$ can be regarded as relative plausibility reduction.

***Proof*:** Firstly, for $x \in OS$, $D_c^i \in OS/D_c$, according to Theorem 4 we find

$$\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 1 \Leftrightarrow BEL_B(D_c^i) = BEL_{CA}(D_c^i) \Leftrightarrow R_B^\lambda(x) \subseteq R_{D_c}^\lambda(x).$$

As a result, $B' \subset B$ can not be viewed as relative consistent set iff $\sum_{D_c^i \in OS/D_c} BEL_{B'}(D_c^i)$ < 1 iff $B'$ can't be deemed as relative belief consistent set. So (1) is equivalent to (2).

Analogously, (1) is equivalent to (3).

In order to compute relative belief reduction and relative plausibility reduction in this paper, a general algorithm is designed based on four important degrees. The general algorithm is shown in Algorithm 1.

Let *IIDIS* = (*OS*, $CA\cup\{D_c\}$, *DO*, *F*), for $C \subseteq CA$, $\forall c_j \in C$, $OS/D_c = \{[x]_{Dc} \mid \forall x \in OS\} = \{D_c^1, D_c^2, \ldots, D_c^t\}(D_c^i \subseteq OS, i = 1, 2, \ldots, t)$. The inner importance degree of attribute $c_j$ in $C$ is defined to research relative belief reduction:

$$IM_{in}^{BEL}(c_j, C, D_c) = \sum_{D_c^i \in OS/D_c} BEL_C(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_{C-\{c_j\}}(D_c^i). \tag{18}$$

If $IM_{in}^{BEL}(c_j, C, D_c) > 0$, then the attribute $c_j$ is indispensable in $C$. $B = \{c_j \mid IM_{in}^{BEL}(c_j, C, D_c) > 0\}$ is called relative belief core. When $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_C(D_c^i)$, $B$ is a relative belief reduction, which be viewed as a Judgment Rule. Otherwise, some attributes should be added to the relative belief core. Hence another idea should be introduced for computing relative belief reduction.

For any $c_j \in C - B$, there defines the outer importance degree of attribute $c_j$ in $B$:

$$IM_{out}^{BEL}(c_j, B, D_c) = \sum_{D_c^i \in OS/D_c} BEL_{B\cup\{c_j\}}(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i). \tag{19}$$

If $c_j = \arg\max_{c_k \in C-B}(IM_{out}^{BEL}(c_k, B, D_c))$, then $c_j$ should be added to the relative belief core $B$. Until attribute set $C = B\cup\{c_j\}$ satisfies the Judgment Rule, $C$ is a relative belief reduction.

Similarly, the relative plausibility reduction can be acquired by the inner/outer importance degree:

$$IM_{out}^{BEL}(c_j, B, D_c) = \sum_{D_c^i \in OS/D_c} BEL_{B\cup\{c_j\}}(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i), \tag{20}$$

$$IM_{out}^{PL}(c_j, B, D_c) = \sum_{D_c^i \in OS/D_c} PL_B(D_c^i) - \sum_{D_c^i \in OS/D_c} PL_{B\cup\{c_j\}}(D_c^i). \tag{21}$$

**Algorithm 1:** The algorithm for obtaining general reduction in *IIDIS*

**Input:** a testing system $IIDIS = (OS, CA\cup\{D_c\}, DO, F)$, where $C \subseteq CA$.
**Output:** a general core $B$ and reduction $C$ in *IIDIS*.

```
begin
    let B ← ∅, C ← ∅; /* the initialization of core B and reduction C */
    compute OS/D_c = {D_c^1, D_c^2, …, D_c^t};   /* the decision classes D_c^i */
    for c_j ∈ CA do
        compute IM_in(c_j, CA, D_c); /* the inner importance degree of c_j in CA */
        if IM_in(c_j, CA, D_c) > 0 then
            B ← B∪{c_j};
        end
    end
    C ← B;
    while condition do
        for c_j ∈ CA − C do
            compute IM_out(c_j, C, D_c); /* the outer importance degree of c_j in C */
        end
        Selecting an attribute c_j that satisfies c_j = arg max_{ck∈CA-C}(IM_out(c_k, B, D_c));
        C ← C∪{c_j};
    end
    for c_j ∈ C − B do
        if IM_in(c_j, C, D_c) = 0 then
            C ← C − {c_j};
        end
    end
    return: B, C.
end
```

The *condition* means that $\sum_{D_c^i \in OS/D_c} BEL_C(D_c^i) \neq \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$ for obtaining relative belief reduction and $\sum_{D_c^i \in OS/D_c} PL_C(D_c^i) \neq \sum_{D_c^i \in OS/D_c} PL_{CA}(D_c^i)$ to acquire elative plausibility reduction in the Algorithm 1. For sake of description, the inner and outer importance degrees are simply written as $IM_{in}(c_j, E, D_c)$ and $IM_{out}(c_j, E, D_c)(E \subseteq CA)$ in the Algorithm 1 whether relative belief reduction or relative plausibility reduction. The time complexity of Algorithm 1 is $O(|CA|^3|OS|^2)$. And it would be reduced to $O(|CA|^2|OS|^2)$ if *condition* = 0.

**Example 2:** The Table 2 comprises kernels belonging to three different varieties of wheat, which is selected from the data set "seeds" in Section 5. Let *IIVDIS* = ($OS$, $CA \cup \{D_c\}$, $DO$, $F$). Where object set $OS = \{x_1, x_2, \ldots, x_{20}\}$, $x_i$ represents every wheat ($i$ = 1, 2, …, 20), condition attribute set $CA = \{c_1, c_2, \ldots, c_7\}$, $c_j$($j$ = 1, 2, …, 7) show area, perimeter, compactness, length of kernel, width of kernel, asymmetry co-efficient and length of kernel groove, respectively. $D_c$ exhibits the varieties of wheat. Decision values 1, 2 and 3 represent Kama, Rosa and Canadian, respectively. In order to construct an inconsistent *IIDIS* in Example 3, extra decision attribute $D_c'$ is also outlined in Table 2.

**Table 2. An incomplete interval-valued decision information system.**

| $OS$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ | $D_c$ | $D_c'$ |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | [10.40, 12.71] | [11.79, 14.41] | [0.76, 0.93] | [4.65, 5.68] | [2.56, 3.13] | [6.04, 7.39] | [4.46, 5.45] | 3 | 2 |
| $x_2$ | [10.33, 12.63] | [11.75, 14.36] | [0.76, 0.93] | [4.66, 5.70] | [2.48, 3.03] | [5.29, 6.46] | [4.50, 5.50] | 3 | 1 |
| $x_3$ | [16.85, 20.59] | [14.57, 17.81] | [0.81, 0.99] | [5.41, 6.61] | [3.47, 4.24] | [4.79, 5.86] | [5.29, 6.47] | 2 | 1 |
| $x_4$ | [10.11, 12.35] | [11.37, 13.89] | [0.80, 0.97] | [4.41, 5.39] | [2.59, 3.17] | [2.04, 2.50] | [4.23, 5.17] | 1 | 3 |
| $x_5$ | [12.63, 15.43] | [12.74, 15.58] | [0.79, 0.97] | [4.89, 5.98] | [2.88, 3.52] | [1.55, 1.89] | [4.50, 5.50] | 1 | 3 |
| $x_6$ | [15.16, 18.52] | [14.10, 17.24] | [0.78, 0.95] | [5.40, 6.60] | [3.14, 3.83] | [4.21, 5.14] | [5.29, 6.46] | 2 | 1 |
| $x_7$ | [16.59, 20.27] | [14.37, 17.57] | [0.82, 1.00] | [5.38, 6.58] | [3.39, 4.15] | [2.69, 3.28] | [5.31, 6.50] | 2 | 3 |
| $x_8$ | [10.21, 12.47] | [11.58, 14.16] | [0.77, 0.95] | [4.55, 5.56] | [2.56, 3.13] | * | [4.50, 5.50] | 3 | 2 |
| $x_9$ | [13.01, 15.91] | [12.92, 15.79] | [0.79, 0.97] | [4.85, 5.93] | [3.04, 3.71] | [2.52, 3.08] | * | 1 | 2 |
| $x_{10}$ | [10.64, 13.00] | [12.06, 14.74] | [0.74, 0.91] | [4.78, 5.85] | [2.50, 3.05] | [4.02, 4.92] | [4.66, 5.70] | 3 | 3 |
| $x_{11}$ | [14.04, 17.16] | [13.60, 16.62] | [0.77, 0.94] | [5.25, 6.42] | [2.96, 3.61] | [2.45, 3.00] | [5.18, 6.33] | 2 | 3 |
| $x_{12}$ | [15.80, 19.31] | [14.09, 17.23] | [0.81, 0.99] | [5.21, 6.37] | [3.32, 4.06] | [4.83, 5.90] | [5.09, 6.23] | 2 | 3 |
| $x_{13}$ | [11.99, 14.65] | [12.55, 15.33] | [0.78, 0.95] | [4.99, 6.10] | [2.77, 3.38] | [6.33, 7.74] | [4.90, 5.98] | 3 | 2 |
| $x_{14}$ | [16.35, 19.99] | [14.63, 17.89] | [0.78, 0.95] | [5.64, 6.90] | [3.16, 3.86] | [2.57, 3.14] | [5.65, 6.90] | 2 | 2 |
| $x_{15}$ | [13.60, 16.62] | [13.09, 15.99] | [0.81, 0.99] | [5.02, 6.14] | [3.12, 3.81] | [2.82, 3.44] | [4.66, 5.70] | 1 | 1 |
| $x_{16}$ | * | [13.40, 16.38] | [0.79, 0.97] | [5.20, 6.35] | [3.07, 3.75] | [4.47, 5.47] | * | 2 | 3 |
| $x_{17}$ | [11.30, 13.81] | [12.21, 14.93] | [0.77, 0.94] | [4.80, 5.87] | [2.67, 3.26] | [3.98, 4.86] | [4.66, 5.69] | 3 | 1 |
| $x_{18}$ | [11.84, 14.48] | [12.44, 15.20] | [0.78, 0.95] | [4.91, 6.00] | [2.68, 3.27] | [0.77, 0.94] | [4.55, 5.56] | 1 | 1 |
| $x_{19}$ | [12.55, 15.33] | [12.75, 15.59] | [0.79, 0.96] | [5.03, 6.14] | [2.84, 3.47] | * | [4.51, 5.51] | 1 | 1 |
| $x_{20}$ | [13.84, 16.92] | [13.41, 16.39] | [0.78, 0.96] | [5.30, 6.47] | [2.94, 3.59] | [4.02, 4.91] | [5.22, 6.37] | 2 | 3 |

After calculation, $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^3\}$. Where $D_c^1 = \{x_4, x_5, x_9, x_{15}, x_{18}, x_{19}\}$ and $D_c^2 = \{x_3, x_6, x_7, x_{11}, x_{12}, x_{14}, x_{16}, x_{20}\}$, $D_c^3 = \{x_1, x_2, x_8, x_{10}, x_{13}, x_{17}\}$, respectively.

Tolerance class for every object under the condition attribute set $CA$ with $\lambda = 0.6$ is $[x_i]_{R_{CA}^{\lambda}} = \{x_i\}$ apart from $x_1, x_2, x_5, x_8, x_{19}$ in the universe $OS$. $[x_1]_{R_{CA}^{\lambda}} = \{x_1, x_8\}$, $[x_2]_{R_{CA}^{\lambda}} = \{x_2, x_8\}$, $[x_8]_{R_{CA}^{\lambda}} = \{x_1, x_2, x_8\}$, $[x_5]_{R_{CA}^{\lambda}} = [x_{19}]_{R_{CA}^{\lambda}} = \{x_5, x_{19}\}$.

After calculation, $\sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) = 1$. Hence *IIDIS* is consistent.

Let $A = CA - \{c_j\}$, ($j$ = 1, 2, 3, 4), then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 1$.

Therefore, $IM_{in}^{BEL}(c_j, CA, D_c) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0$.

Let $A = CA - \{c_5\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.9$.

Therefore, $IM_{in}^{BEL}(c_5, CA, D_c) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.1 > 0$.

Let $A = CA - \{c_6\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.8$.

Therefore, $IM_{in}^{BEL}(c_6, CA, D_c) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.2 > 0$.

Let $A = CA - \{c_7\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.8$.

Therefore, $IM_{in}^{BEL}(c_7, CA, D_c) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.2 > 0$.
So relative belief core is $B = \{c_5, c_6, c_7\}$. However $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 0.75 \neq 1$.
Let $A = B \cup \{c_1\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 1$.
Therefore, $IM_{out}^{BEL}(c_1, B, D_c) = \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 0.25$.
Let $A = B \cup \{c_2\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 1$.
Therefore, $IM_{out}^{BEL}(c_2, B, D_c) = \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 0.25$.
Let $A = B \cup \{c_3\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 0.75$.
Therefore, $IM_{out}^{BEL}(c_3, B, D_c) = \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 0$.
Let $A = B \cup \{c_4\}$, then $\sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) = 1$.
Therefore, $IM_{out}^{BEL}(c_4, B, D_c) = \sum_{D_c^i \in OS/D_c} BEL_A(D_c^i) - \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = 0.25$.

Since $\sum_{D_c^i \in OS/D_c} BEL_{B \cup \{c_1\}}(D_c^i) = 1$ while $A = B \cup \{c_1\}$ and $IM_{out}^{BEL}(c_1, B, D_c) = \max_{c_k \in CA-B} IM_{out}^{BEL}(c_k, B, D_c)$, so $\{c_1, c_5, c_6, c_7\}$ is a relative belief reduction. Relative plausibility reduction $\{c_1, c_5, c_6, c_7\}$ can be also acquired according to the inner/outer importance degree $IM_{in}^{PL}(c_j, CA, D_c) / IM_{out}^{PL}(c_j, B, D_c)$.

### 4.2.2 The relative reductions in inconsistent *IIDIS*

In an inconsistent $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$, there has an $x$ s.t. $|\Delta_{CA}(x)| \geq 2$. And some objects of the same tolerance class exist distinct decision values. Let $B \subseteq CA$, as a minimal attribute subset of $CA$ to remain the consistency for universe, $B$ can be considered as relative reduction, vice versa. In short, $|\Delta_B(x)| = 1 (\forall x \in OS)$.

**Theorem 7:** Given an inconsistent $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$, $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$, $\forall B \subseteq CA$. There have

(1) $B$ can be regarded as relative belief consistent set $\Leftrightarrow \underline{R_B^\lambda}(D_c^i) = \underline{R_{CA}^\lambda}(D_c^i)$ for any $D_c^i (i \in \{1, 2, \ldots, t\})$.
(2) $B$ can be regarded as relative plausibility consistent set $\Leftrightarrow \overline{R_B^\lambda}(D_c^i) = \overline{R_{CA}^\lambda}(D_c^i)$ for any $D_c^i (i \in \{1, 2, \ldots, t\})$.

***Proof*:** (1) $\Rightarrow BEL_B(D_i) = BEL_{CA}(D_c^i)$ for any $D_c^i \in OS/D_c$ in the light of the Definition 9. That is to say $\frac{|\underline{R_B^\lambda}(D_c^i)|}{|OS|} = \frac{|\underline{R_{CA}^\lambda}(D_c^i)|}{|OS|}$ $(i \in \{1, 2, \ldots, t\})$, $|\underline{R_B^\lambda}(D_c^i)| = |\underline{R_{CA}^\lambda}(D_c^i)|$. And $B \subseteq CA$, thus $\underline{R_B^\lambda}(D_c^i) = \underline{R_{CA}^\lambda}(D_c^i)$. Therefore, $\underline{R_B^\lambda}(D_c^i) = \underline{R_{CA}^\lambda}(D_c^i)$ for any $D_c^i \in OS/D_c$.

$\Leftarrow$, when $\underline{R_B^\lambda}(D_c^i) = \underline{R_{CA}^\lambda}(D_c^i)$ for any $D_c^i \in OS/D_c$. $BEL_B(D_c^i) = \frac{|\underline{R_B^\lambda}(D_c^i)|}{|OS|} = \frac{|\underline{R_B^\lambda}(D_c^i)|}{|OS|} = BEL_{CA}(D_c^i)$. So $B$ can be deemed as relative belief consistent set.

(2) Analogously, its demonstration can be acquired by the Definition 9.

**Theorem 8:** Given an inconsistent $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$, $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$, $\forall B \subseteq CA$. Thus one has

(1) $B$ can be considered as relative belief consistent set $\Leftrightarrow \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$.

(2) $B$ can be considered as relative belief reduction $\Leftrightarrow \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$, meanwhile, for any $B' \subset B$, $\Leftrightarrow \sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$.

***Proof***: (1) $\Rightarrow BEL_B(D_c^i) = BEL_{CA}(D_c^i)$ holds for $D_c^i (i \in \{1, 2, \ldots, t\})$ on account can be seen as relative belief consistent set. Hence $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$.

$\Leftarrow$ There have $\sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$. $\sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$. for any $D_c^i \in OS/D_c$. And for $\forall B' \subseteq B$, $BEL_{B'}(D_c^i) \leq BEL_B(D_c^i)$. So $\sum_{D_c^i \in OS/D_c} BEL_{B'}(D_c^i) \leq \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$. Therefore, $\sum_{D_c^i \in OS/D_c} BEL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} BEL_{CA}(D_c^i)$ for any $D_c^i \in OS/D_c$ forces $BEL_B(D_c^i) = BEL_{CA}(D_c^i)$ holds. $B$ is a relative belief consistent set.

(2) Its demonstration could be directly obtained by (1) and Definition 9.

**Theorem 9:** Given an inconsistent $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$, $OS/D_c = \{D_c^1, D_c^2, \ldots, D_c^t\}$, $\forall B \subseteq CA$. So there have

(1) $B$ can be considered as relative plausibility consistent set $\Leftrightarrow \sum_{D_c^i \in OS/D_c} PL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} PL_{CA}(D_c^i)$.

(2) $B$ can be considered as relative plausibility reduction $\Leftrightarrow \sum_{D_c^i \in OS/D_c} PL_B(D_c^i) = \sum_{D_c^i \in OS/D_c} PL_{CA}(D_c^i)$, meanwhile, for any $B' \subset B$, $\sum_{D_c^i \in OS/D_c} PL_{B'}(D_c^i) > \sum_{D_c^i \in OS/D_c} PL_{CA}(D_c^i)$.

***Proof*:** Its proof has resemblance with Theorem 8.

**Theorem 10:** Given an inconsistent $IIDIS = (OS, CA \cup \{D_c\}, DO, F)$, $B \subseteq CA$. $B$ is a relative plausibility consistent set can conclude that $B$ is a relative belief consistent set.

***Proof***: We can gain that if $[x]_{R_B^\lambda} \subseteq D_c^i$, then $[x]_{R_{CA}^\lambda} \subseteq D_c^i$ from the proof of the Theorem 1 (6). Now, we should prove that $[x]_{R_{CA}^\lambda} \subseteq D_c^i$, can derive $[x]_{R_{CA}^\lambda} \subseteq D_c^i$. If $[x]_{R_{CA}^\lambda} \subseteq D_c^i$, there have $[x]_{R_{CA}^\lambda} \cap D_c^i \neq \emptyset$, and $[x]_{R_{CA}^\lambda} \cap D_c^l \neq \emptyset$ for $OS/D_c$ forms a partition on universe $OS$ when $D_c^l \neq D_c^i$ ($\forall D_c^l \in OS/D_c$). For $B$ be a relative plausibility consistent set, one has $\overline{R_B^\lambda}(D_c^i) = \overline{R_{CA}^\lambda}(D_c^i)$ for any $D_c^i \in OS/D_c$ according to the Theorem 7. Thus, $[x]_{R_B^\lambda} \cap D_c^i \neq \emptyset$ is equivalent to $[x]_{R_{CA}^\lambda} \cap D_c^i \neq \emptyset$ for $x \in OS$. And we can acquire $[x]_{R_B^\lambda} \cap D_c^i \neq \emptyset$ and $[x]_{R_B^\lambda} \cap D_c^l \neq \emptyset$ for $\forall D_c^l \in OS/D_c$ when $D_c^l \neq D_c^i$. Then $[x]_{R_B^\lambda} \subseteq D_c^i$, namely, $[x]_{R_{CA}^\lambda} \subseteq D_c^i$ can elicit that $[x]_{R_B^\lambda} \subseteq D_c^i$ holds. To sum up, $[x]_{R_B^\lambda} \subseteq D_c^i$, then $[x]_{R_{CA}^\lambda} \subseteq D_c^i$ are equivalent for $\forall D_c^i \in OS/D_c$. According to the Theorem 7, $B$ can be termed as relative belief consistent set.

**Example 3:** (Continued from the Example 2) Let $IIDIS = (OS, CA \cup \{D_c'\}, DO, F)$ based on the Example 2. Then decision classes are changed into $D_c^{1\prime} = \{x_2, x_3, x_6, x_{15}, x_{17}, x_{18}, x_{19}\}$, $D_c^{2\prime} = \{x_1, x_8, x_9, x_{13}, x_{14}\}$, and $D_c^{3\prime} = \{x_4, x_5, x_7, x_{10}, x_{11}, x_{12}, x_{16}, x_{20}\}$, which constitutes set $OS/D_c' = \{D_c^{1\prime}, D_c^{2\prime}, D_c^{3\prime}\}$. At this point, there exists an object $x_2$ such that $[x_2]_{R_{CA}^\lambda} \nsubseteq$

$[x_3]_{D_c'} = D_c^{1'}$ when $\lambda = 0.6$. So this *IIDIS* is inconsistent.

Tolerance class for every object under the condition attribute set $CA$ remains unchanged. After calculation, $\sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 1.2$.

Let $A = CA - \{c_j\}$, ($j$ = 2, 3), then $\sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) = 1.2$.

Therefore, $IM_{in}^{PL}(c_j, CA, D_c') = \sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) - \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 0$.

Let $A = CA - \{c_1\}$, then $\sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) = 1.3$.

Therefore, $IM_{in}^{PL}(c_1, CA, D_c') = \sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) - \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 0.1 > 0$.

Let $A = CA - \{c_4\}$, then $\sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) = 1.3$.

Therefore, $IM_{in}^{PL}(c_4, CA, D_c') = \sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) - \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 0.1 > 0$.

Let $A = CA - \{c_5\}$, then $\sum_{D_c^{i'} \in OS/D_c} PL_A(D_c^{i'}) = 1.35$.

Therefore, $IM_{in}^{PL}(c_5, CA, D_c') = \sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) - \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 0.15 > 0$.

Let $A = CA - \{c_6\}$, then $\sum_{D_c^{i'} \in OS/D_c} PL_A(D_c^{i'}) = 1.75$.

Therefore, $IM_{in}^{PL}(c_6, CA, D_c') = \sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) - \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 0.55 > 0$.

Let $A = CA - \{c_7\}$, then $\sum_{D_c^{i'} \in OS/D_c} PL_A(D_c^{i'}) = 1.4$.

Therefore, $IM_{in}^{PL}(c_7, CA, D_c') = \sum_{D_c^{i'} \in OS/D_c'} PL_A(D_c^{i'}) - \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'}) = 0.2 > 0$.

So relative plausibility core is $B = \{c_1, c_4, c_5, c_6, c_7\}$.

Furthermore, $\sum_{D_c^{i'} \in OS/D_c'} PL_B(D_c^{i'}) = 1.2 = \sum_{D_c^{i'} \in OS/D_c'} PL_{CA}(D_c^{i'})$. Therefore, $B$ is a relative plausibility reduction.

## 5. EXPERIMENT ANALYSIS

In this section, some experiments are performed to prove the efficiency of the proposed theorems by six data sets from UCI database. That is "Blood Transfusion Service Center", "Immunotherapy", "seeds", "Page Blocks Classification", "Wine Quality-Red" and "Wine Quality-White", which are shown in Table 3. The testing results are running on personal computer with processor (2.7 GHz Intel Core i5) and memory (8 GB 1867 MHz DDR3). The platform of algorithm is Matlab2016B.

**Table 3. The testing data sets.**

| *Datasets* | *Abbreviation* | *Object* | *Condition Attribute* | *Decision Class* |
|---|---|---|---|---|
| *Blood Transfusion Service Center* | *BTSC* | 748 | 4 | 2 |
| *Immunotherapy* | *IPY* | 90 | 7 | 2 |
| *Seeds* | *SDS* | 210 | 7 | 3 |
| *Page Blocks Classification* | *PBC* | 5473 | 10 | 5 |
| *Wine Quality − Red* | *WQR* | 1599 | 11 | 6 |
| *Wine Quality − White* | *WQW* | 4898 | 11 | 7 |

In fact, the attribute values of six data sets are real numbers. But what we are investigating is *IIDIS*. So we need utilizing multiply error precision $\xi$ and missing rate $\pi$ ($\pi \in$

(0, 1)) to process the data and change the data from real numbers to interval numbers. Let $DIS = (OS, CA \cup \{D_c\}, DO, F)$ be a decision information system. All attribute values are single-valued. For any $x_i \in OS$, $c_j \in CA$, the attribute value of $x_i$ under the attribute $c_j$ can be written as $P = F(x_i, c_j)$. The attribute value of every data set remains unchanged.

- Firstly, we randomly choose $\lfloor \pi \times |OS| \times |CA| \rfloor$($\lfloor \cdot \rfloor$ is the meaning of taking an integer down) attribute values and turn them into missing values in order to construct an incomplete information system. These missing values are written as *;
- Secondly, the interval number can be obtained by formula $P' = [(1-\xi)\times P, (1+\xi)\times P]$. In summary, an *IIDIS* is obtained.

Where $\pi = 0.04$, $\xi = 0.1$ in consistent *IIDIS* and $\pi = 0.004$, $\xi = 0.05$ in inconsistent *IIDIS*. Let $\lambda = 0.6$. Tables 4 and 5 show relative belief and plausibility reduction of consistent/inconsistent *IIDIS* in light of the Algorithm 1. In the following, $C \subseteq CA$ and $B \subseteq C$ indicate relative belief/plausibility reduction and relative belief/plausibility core, respectively.

**Table 4. Relative belief reduction in consistent/inconsistent *IIDIS*.**

| *Data sets* | *Consistent*(*yes/no*) | *Relative belief core* | *Relative belief reduction* |
|---|---|---|---|
| *BT SC* | *no* | {1, 3, 4} | {1, 3, 4} |
| *IPY* | *yes* | {4} | {2, 3, 4, 6} |
| *SDS* | *no* | {6} | {2, 5, 6, 7} |
| *PBC* | *no* | {6} | {4, 6, 7, 8, 9} |
| *WQR* | *yes* | {3, 5, 6, 7} | {2, 3, 4, 5, 6, 7, 10} |
| *WQW* | *no* | {2, 5, 6, 7} | {2, 3, 4, 5, 6, 7} |

**Table 5. Relative plausibility reduction in consistent/inconsistent *IIDIS*.**

| *Data sets* | *Consistent*(*yes/no*) | *Relative plausibility core* | *Relative plausibility reduction* |
|---|---|---|---|
| *BT SC* | *no* | {1, 3, 4} | {1, 3, 4} |
| *IPY* | *yes* | {4} | {2, 3, 4, 6} |
| *SDS* | *no* | {6} | {1, 5, 6, 7} |
| *PBC* | *no* | {6} | {3, 4, 6, 7, 8} |
| *WQR* | *yes* | {3, 5, 6, 7} | {2, 3, 4, 5, 6, 7, 10} |
| *WQW* | *no* | {2, 5, 6, 7} | {2, 4, 5, 6, 7, 10} |

Observed from the Tables 4 and 5, we find a relative belief/plausibility core and a relative belief/plausibility reduction in each data set. Obviously, an attribute subset $C \subseteq CA$ is a relative belief reduction, which is equivalent to $C$ is a relative plausibility reduction in consistent *IIDIS*. Such as data set *IPY* and *WQR*. In addition, relative belief/plausibility reduction $C$ is also equivalent to a relative reduction according to definition 9 and Theorem 5. Therefore, relative reduction is not only equivalent to relative belief reduction but also equivalent to relative plausibility reduction in the consistent *IIDIS*. In inconsistent *IIDIS*, on the one hand, relative belief/plausibility core $B$ of data set *BT SC* is relative belief/plausibility reduction $C$, which indicates $IM_{out}^{BEL}(c_j, B, D_c) = IM_{out}^{PL}(c_j, B, D_c) = 0$ for every $c_j \in CA - B$. On the other hand, relative belief/plausibility core of other three data sets is a subset of relative belief/plausibility reduction. Besides, relative belief reduction is not equivalent to relative plausibility reduction in inconsistent *IIDIS*.

## 6. CONCLUSIONS

D-S evidence theory has close connection with RST. And they are useful and significant tools to handle with uncertainty problem. *IIDIS* is viewed as the extension of classic information system. In the paper, we primarily discuss the properties with reference to approximations by using a novel tolerance relation in *IIDIS*. Relative reduction, relative belief/plausibility reduction are defined associated with D-S theory. After research, the belief (plausibility) function in belief structure is closely related to the lower (upper) approximation in approximate space that is formed by universe and the novel tolerance relation. We find that belief (plausibility) function is cardinal number of the lower (upper) approximation about the concept divided by the cardinality of the discourse. In the consistent *IIDIS*, relative reduction is not only equivalent to relative belief reduction but also equivalent to relative plausibility reduction. In the inconsistent *IIDIS*, relative plausibility consistent set can conclude it be deemed as relative belief consistent set.

We only study relative reductions via evidence theory in *IIDIS*. Later we will further explore other reductions (such as lower/upper approximation reduction, distribute reduction, partially consistent reduction and so on) approaches to deal with issues and look for relationships among these reductions in *IIDIS*.

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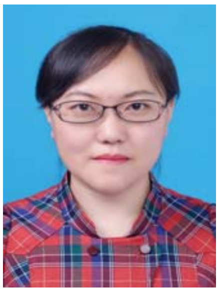

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