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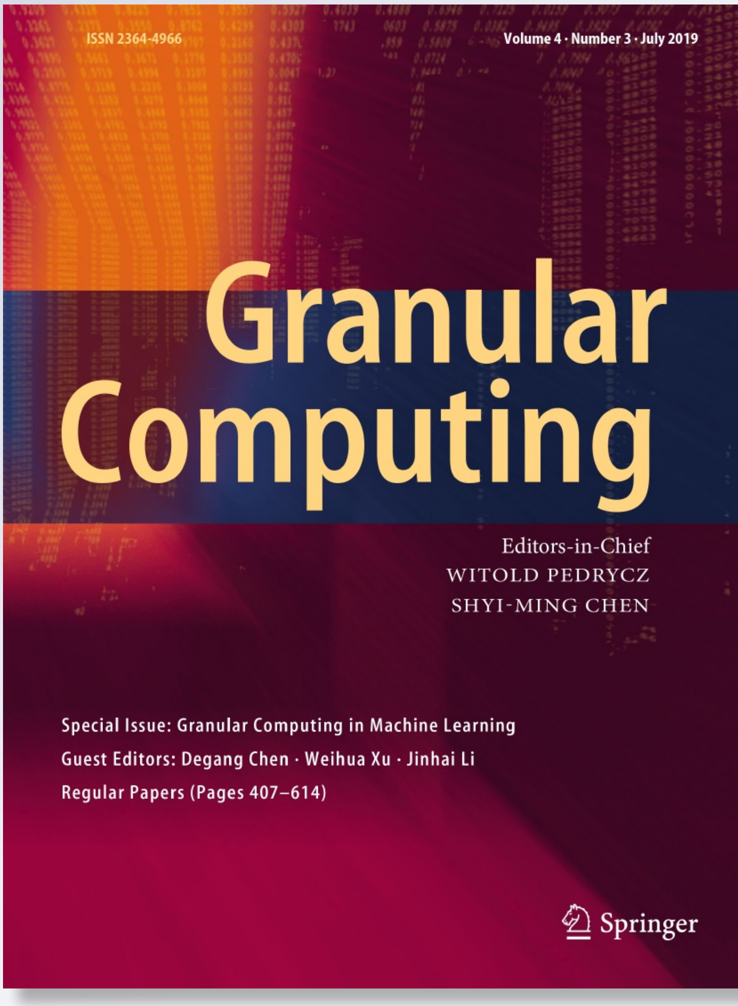

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ORIGINAL PAPER

Operation properties and algebraic properties of multi-covering rough sets

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Abstract

The multi-covering rough sets (MCRSs) are a popular aspect of rough sets. It is easy to see that classical rough sets, covering rough sets (CRSs) and multi-granulation rough sets (MGRSs) are all the special cases of the MCRSs. Recently, the algebraic theory of these rough set models mentioned above have been researched in detail. However, the algebraic theory of MCRSs has not been studied until now. It is necessary for researchers to explore the algebraic theory of MCRSs. In this paper, we focus on the operation and algebraic theories of two types of MCRS models. First, the properties of the two types of multi-covering set approximations are discussed. Especially, the properties of multi-covering approximation operators based on the unary coverings are deeply researched. Second, the operation properties with respect to intersection and union of MCRSs are researched. Meanwhile, to compute the intersection and union of MCRSs, several algorithms are constructed. Finally, on the basis of the operation properties of MCRSs, many meaningful algebraic properties of MCRSs are deeply studied.

Keywords Rough sets · Covering · Unary · Operation properties · Algebraic properties

1 Introduction

According to an equivalence of the universe, the rough set theory was proposed by Pawlak (1982). At present, rough set is one of the most effective ways to deal with complicated and massive data. Meanwhile, this theory is a very excellent method to solve issues of granular computing (Pedrycz and Chen 2011, 2015a, b). Rough sets have been widely used in lots of fields such as uncertainty management, feature acquisition, data processing. (Polkowski and Skowron 1998a, b, c; Pomkala 1988; Wu and Zhang 2006; Yao and Chen 2005; Zhang et al. 2003; Zhu and Wang 2006, 2011).

Based on the considerations of granular computing, scholars usually regard an equivalence relation as a granularity. As we all know, the upper and lower approximations defined by an equivalence play a key role in rough sets. There is no doubt that the single granular structure of classical rough sets is a very fatal shortcoming. Clearly, the classical rough set theory is unable to solve many problems, which are related to multiple granular structures (Apolloni et al. 2016). Therefore, Qian et al. (2005, 2010) generated classical rough sets to the optimistic and pessimistic MGRSs, where the set approximations are constructed based on more than one equivalence relation. Now, lots of scholars are focusing on the developments of MGRS models (Kong and Wei 2017; Li et al. 2016; Lin et al. 2012; Xu and Guo 2016). For example, based on the incomplete information system, Yang et al. (2012) discussed the incomplete MGRS model. Xu et al. (2012) studied the multi-granulation rough sets using of the tolerance relations. Meanwhile, Xu et al. (2013) also deeply studied the MGRSs according to the ordered relation. From the neighborhood point of view, Lin et al. (2012) investigated neighborhood MGRSs. Yao and She (2016) further studied MGRSs and suggested two types of rough set models using of equivalence relations depended on set

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union and intersection, respectively. What is more, Li et al. (2017) investigated the three-way cognitive concept learning with respect to multiple granularity. In addition, according to MGRSs, many authors (Wang et al. 2017; Xu and Wang 2016; Xu et al. 2017; Liang et al. 2018) researched how to select the optimal one from multiple granularities.

In addition, we know that classical rough sets are developed based on an equivalence relation. However, according to the attribute subset, there is no guarantee that we will get an equivalence relation every time. To improve the drawback, many meaningful relations have been proposed to generalize Pawlak rough set model, such as similarity relations, neighborhood relations, and tolerance relations (Skowron and Stepaniuk 1996; Slowinski and Vanderpooten 2000; Yao and Lin 1996; Yao 1998). Zakowski (1983) has proposed the notion of covering and established the CRS theory. It is very meaningful and necessary to research the covering-based rough sets (Dai et al. 2014; D'eer et al. 2016; Ge et al. 2017; Kong and Xu 2018b; Xu and Zhang 2007; Yang and Zhu 2014). Yao (1998, 2003) first proposed two types of rough set approximation operators based on duality and studied the corresponding properties. Meanwhile, according to the tolerance relations, Pomy Kala and Pomy Kala (1988) further explored additional pairs of dual set approximations. In addition, Zhu (2007) also studied several types of approximation operators and discussed their interrelationships. At the same time, many researchers (Chen et al. 2017; Lang and Miao 2016; Lang et al. 2015; Wang et al. 2015) investigated the attribute reduction of CRSs or covering decision information systems.

According to the MGRSs and the CRSs, it is necessary for us to study multi-covering rough sets (MCRSs). At present, lots of authors are doing research on MCRSs (Liu et al. 2014). For example, Wang et al. (2013) developed five types of optimistic and pessimistic MCRS models, and further discussed the relationships among them. Meanwhile, based on the specific practical backgrounds, Lin et al. (2013) constructed several types of MCRS models using different lower and upper set approximations. Moreover, according to the minimal and maximal descriptions, Liu et al. (2014) constructed several types of MCRS models. In addition, Lang et al. (2017, 2018) studied the knowledge reducts of CRSs in dynamic contexts.

At the same time, we note that the algebraic theory of rough sets theory was first explored by Iwiński (1987). Since then, many scholars have been working on the operation theory and the corresponding algebraic theory of classical rough sets (Pagliani 1996; Yao 1998). For instance, Li (2002) investigated many meaningful algebraic theory of classical rough sets in detail. Then Kong and Xu (2018a, b) studied the algebraic properties of CRSs and MGRSs, respectively. However, until now, no one has been engaged in the exploration of algebraic theory of MCRSs. According to the above discussion, it is necessary and important for us to study the algebraic properties of MCRSs.

Here, we concentrates on the study of the operation and algebraic theory of MCRSs, and is organized as follows. In Sect. 2, many important concepts of MGRSs and CRSs are recalled. In Sect. 3, the properties of the first type of multi-covering approximation operators are discussed. Especially, the intersection and union operations and the corresponding operation properties with respect to minimally unary MCRSs are explored. In Sect. 4, the properties of the second type of multi-covering approximation operators are investigated. Furthermore, based on maximally unary multi-covering, the intersection and union operations and corresponding operation theory of the second type of MCRSs are studied. In Sect. 5, according to the operation properties of MCRSs, the algebraic theory of MCRSs is deeply researched. Finally, in Sect. 6, we conclude this study.

2 Preliminaries

In this section, many important notions of MCRSs and CRSs are recalled. More concepts can be found in references (Chen et al. 2007; Zakowski 1983; Zhu and Wang 2006).

2.1 Multi-granulation rough sets

Suppose that $(U, \mathbb{R})$ is an approximation space, where $U = \{a_1, a_2, \dots, a_n\}$ is the universe; and $\mathbb{R}=\{R_1, R_2, \dots, R_m\}$ is a set of the equivalence relations. Meanwhile, $[a]_R = \{b \mid (a, b) \in R\}$ is the equivalence class of $a \in U$.

Definition 2.1 (Qian et al. 2005) Suppose that $(U, \mathbb{R})$ is an approximation space, $R_1, R_2, \dots, R_m \subseteq \mathbb{R}$, and $A \subseteq U$. Denote

$$\underline{\mathrm{OM}_{\sum_{i=1}^m R_i}}(A) = \left\{a \mid \vee_{i=1}^m([a]_{R_i} \subseteq A)\right\};$$
$$\overline{\mathrm{OM}_{\sum_{i=1}^m R_i}}(A) = \sim \underline{\mathrm{OM}_{\sum_{i=1}^m R_i}}(\sim A)$$

we, respectively, call $\underline{\mathrm{OM}_{\sum_{i=1}^m R_i}}(A)$ and $\overline{\mathrm{OM}_{\sum_{i=1}^m R_i}}(A)$ the optimistic lower and upper approximations of A with respect to $(U, \mathbb{R})$.

Definition 2.2 (Qian et al. 2010) Suppose that $(U, \mathbb{R})$ is an approximation space, $R_1, R_2, \dots, R_m \subseteq \mathbb{R}$, and $A \subseteq U$. Denote

$$\underline{\mathrm{PM}_{\sum_{i=1}^m R_i}}(A) = \left\{x \mid \wedge_{i=1}^m([a]_{R_i} \subseteq A)\right\};$$
$$\overline{\mathrm{PM}_{\sum_{i=1}^m R_i}}(A) = \sim \underline{\mathrm{PM}_{\sum_{i=1}^m R_i}}(\sim A),$$

Table 1 A covering about colors

U	Maroon	Scarlet	Dun	Reddish
x_1	Yes	No	No	No
x_2	Yes	No	Yes	No
x_3	No	Yes	No	Yes
x_4	Yes	No	Yes	No
x_5	No	No	Yes	No
x_6	No	No	No	Yes

Table 2 A covering about auto brands

U	Honda	Peugeot	Cadillac	Buick
x_1	No	Yes	No	No
x_2	Yes	Yes	Yes	No
x_3	No	No	Yes	No
x_4	Yes	Yes	Yes	No
x_5	No	No	Yes	No
x_6	No	Yes	No	Yes

we, respectively, call $\underline{\mathrm{PM}_{\sum_{i=1}^{m} R_i}}(A)$ and $\overline{\mathrm{PM}_{\sum_{i=1}^{m} R_i}}(A)$ the pessimistic lower and upper approximations of A with respect to $(U, \mathbb{R})$.

2.2 Covering

In this part, some necessary concepts of CRSs are recalled.

Definition 2.3 (Zhu and Wang 2006) Suppose that U is a universe of discourse, and C is a family of subsets of U. We call C a covering of U, if no subset in C is empty and $\cup C = U$. Meanwhile, we call (U, C) a covering approximation space.

Definition 2.4 (Yao and Yao 2012) Suppose that C is a covering of U, we call $\mathbb{C}(C, a)$ a neighborhood system of $a \in U$, and $\mathbb{C}(C, a)$ is constructed as follows:

$$\mathbb{C}(C, a) = \{K \in C | a \in K\}.$$

Definition 2.5 (Zhu 2007) Suppose that C is a covering of U and $a \in U$, then we call $md(a)$ the minimal description of a, and $\mathrm{md}(a)$ is constructed as follows:

$$\mathrm{md}(a) = \{K \in \mathbb{C}(C, a) | (\forall S \in \mathbb{C}(C, a))(S \subseteq K \Rightarrow K = S)\}.$$

Definition 2.6 (Zhu 2007) Suppose that C is a covering of U. For each $a \in U$, $|\mathrm{md}(x)| = 1$, then we call C the minimally unary covering of U.

Definition 2.7 (Zhu and Wang 2006) Suppose that C is a covering of U and $a \in U$, then we call $\mathrm{MD}(a)$ the maximal description of a, and $MD(a)$ is constructed as follows:

$$\mathrm{MD}(a) = \{K \in \mathbb{C}(C, a) | (\forall S \in \mathbb{C}(C, a))(K \subseteq S \Rightarrow K = S)\}.$$

Definition 2.8 Suppose that C is a covering of U. For each $a \in U$, $|\mathrm{MD}(a)| = 1$, then we call C the maximally unary covering of U.

2.3 Multi-covering

Let U be a nonempty finite set, $\Omega = \{C_1, C_2, \ldots, C_m\}$ a family of covering of U with $C_i = \{K_{i1}, K_{i2}, \ldots, K_{i|C_i|}\}$, $\mathfrak{C}$ is defined by $\mathfrak{C} = \{K_{11}, K_{12}, \ldots, K_{1|C_1|}, K_{21}, K_{22}, \ldots, K_{2|C_2|}, \ldots, K_{m1}, K_{m2}, \ldots, K_{m|C_m|}\}$.

Definition 2.9 Suppose that U is a nonempty finite set, $\Omega = \{C_1, C_2, \ldots, C_m\}$ is a family of covering of U with $C_i = \{K_{i1}, K_{i2}, \ldots, K_{i|C_i|}\}$, we call (U, Ω) the multi-covering approximation space (MCAS). If $\mathfrak{C}$ is a minimally (maximally) unary covering of U, then we call (U, Ω) the minimally (maximally) unary MCAS.

For each $a \in U$, we denote $(\Omega, a) = \{K_{ij} \in C_i | a \in K_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, |C_i|\}$ for simplicity. If $\mathfrak{C}$ is a minimally unary covering of U, for each $K_{ij} \in (\Omega, a)$, there must exist $K_a^{\min} \in (\Omega, a)$ such that $K_a^{\min} \subseteq K_{ij}$. For $A \subseteq U$, denote $\mathfrak{K}_A^{\min} = \{K_{a_1}^{\min}, K_{a_2}^{\min}, \ldots, K_{a_s}^{\min}\}$, where $\mathfrak{K}_A^{\min}$ satisfies two conditions: (1) For $\forall K_{a_i}^{\min}, K_{a_j}^{\min} \in \mathfrak{K}_A^{\min}$, we have $K_{a_i}^{\min} = K_{a_j}^{\min}$ or $K_{a_i}^{\min} \cap K_{a_j}^{\min} = \emptyset$; (2) $\bigcup_{i=1}^{s} K_{a_i}^{\min} = A$. If $\mathfrak{C}$ is a maximally unary covering of U, for each $K_{ij} \in (\Omega, a)$, there must exist $K_a^{\max} \in (\Omega, a)$ such that $K_{ij} \subseteq K_a^{\max}$.

Example 2.1 The universe $U = \{a_1, a_2, \ldots, a_6\}$ stands for six persons. A covering of U about colors is given in

Table 3 A covering about colors

U	Maroon	Scarlet	Dun	Reddish
x_1	Yes	Yes	No	No
x_2	Yes	Yes	No	No
x_3	Yes	No	No	No
x_4	No	No	Yes	Yes
x_5	No	No	Yes	No
x_6	No	No	No	Yes

Table 4 A covering about auto brands

U	Honda	Peugeot	Cadillac	Buick
x_1	No	Yes	No	No
x_2	Yes	No	No	No
x_3	Yes	No	No	No
x_4	No	No	Yes	No
x_5	No	No	Yes	Yes
x_6	No	No	Yes	Yes

Table 1.“Yes” means that the person likes this color.“No” means that the person does not like this color.

Denote $K_{\mathrm{M}} = \{a_1, a_2, a_4\}$, $K_S = \{a_3\}$, $K_{\mathrm{D}} = \{a_2, a_4, a_5\}$, $K_{\mathrm{R}} = \{a_3, a_6\}$. Clearly, $C_1 = \{K_{\mathrm{M}}, K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{R}}\}$ is a covering of U.

A covering of U about auto brands is given in Table 2.“Yes” means that the person likes this auto brand.“No” means that the person does not like this auto brand.

Denote $K_{\mathrm{H}} = \{a_2, a_4\}$, $K_{\mathrm{P}} = \{a_1, a_2, a_4, a_6\}$, $K_{\mathrm{C}} = \{a_2, a_3, a_4, a_5\}$, $K_{\mathrm{B}} = \{a_6\}$. Clearly, $C_2 = \{K_{\mathrm{H}}, K_{\mathrm{P}}, K_{\mathrm{C}}, K_{\mathrm{B}}\}$ is a covering of U.

Let $\Omega = \{C_1, C_2\}$, it can be found that $\mathfrak{C} = \{K_{\mathrm{M}}, K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{R}}, K_{\mathrm{H}}, K_{\mathrm{P}}, K_{\mathrm{C}}, K_B\}$ is a minimally unary covering of U. Then, (U, Ω) is a minimally unary MCAS. For $a_2 \in U$, we have $(\Omega, a_2) = \{K_{\mathrm{M}}, K_{\mathrm{D}}, K_{\mathrm{H}}, K_{\mathrm{P}}, K_{\mathrm{C}}\}$, it is clear that $K_{a_2}^{\min} = K_{\mathrm{H}}$.

Example 2.2 Here, we will replace Tables 1 and 2 in Example 2.1 with Tables 3 and 4 presented below, respectively.

Denote $K_{\mathrm{M}} = \{a_1, a_2, a_3\}$, $K_{\mathrm{S}} = \{a_1, a_2\}$, $K_D = \{a_4, a_5\}$, $K_R = \{a_4, a_6\}$. Clearly, $C_1 = \{K_{\mathrm{M}}, K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{R}}\}$ is a covering of U.

Denote $K_{\mathrm{H}} = \{a_2, a_3\}$, $K_P = \{a_1\}$, $K_C = \{a_4, a_5, a_6\}$, $K_{\mathrm{B}} = \{a_5, a_6\}$. Clearly, $C_2 = \{K_{\mathrm{H}}, K_{\mathrm{P}}, K_{\mathrm{C}}, K_{\mathrm{B}}\}$ is a covering of U.

Let $\Omega = \{C_1, C_2\}$, it can be found that $\mathfrak{C} = \{K_{\mathrm{M}}, K_{\mathrm{S}}, K_{\mathrm{D}}, K_{\mathrm{R}}, K_{\mathrm{H}}, K_{\mathrm{P}}, K_{\mathrm{C}}, K_B\}$ is a maximally unary covering of U. Then, (U, Ω) is a maximally unary MCAS. For $a_2 \in U$, we have $(\Omega, a_2) = \{K_{\mathrm{M}}, K_{\mathrm{S}}, K_{\mathrm{H}}\}$, it is clear that $K_{a_2}^{\max} = K_{\mathrm{M}}$.

3 The first type of MCRSs

In this part, we will investigate the first type of MCRSs, which was first proposed by Lin et al. (2013). Here, we further discuss the properties of the first type of MCRSs, and then present the definitions of intersection and union on the first type of MCRSs. Finally, we study the corresponding operation theory.

Definition 3.1 (Lin et al. 2013) Let (U, Ω) be a MCAS, and $\Omega = \{C_1, C_2, \ldots, C_m\}$ a family of coverings of U with $C_i = \{K_{i1}, K_{i2}, \ldots, K_{it_i}\}$, and $A \subseteq U$. Denote

$$\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) = \cup\{K_{ij} \in C_i | \vee (K_{ij} \subseteq A),$$

$$i \in \{1, 2, \ldots, m\}, j = 1, 2, \ldots, |C_i|\},$$

$$\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) = \sim \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\sim A)$$

we, respectively, call $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$ and $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$ the first type of multi-covering lower and upper approximations of A with respect to (U, Ω).

For $A \subseteq U$, we call $(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A))$ the first type of multi-covering rough set of A. Thus, $\mathbb{C}^F = \{(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)) | A \subseteq U\}$ is all of the first type of MCRSs with respect to (U, Ω).

3.1 The first type of multi-covering approximation operators

In this part, we will study the properties of the first type of multi-covering approximation operators in a MCAS.

Proposition 3.1 (Lin et al. 2013) *Let (U, Ω) be a MCAS and $A, B \subseteq U$, then we have that*

(1) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(U) = \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(U) = U$, $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\emptyset) = \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\emptyset) = \emptyset$;

(2) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \subseteq A \subseteq \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$;

(3) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)) = \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$, $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\overline{FM_{\sum_{i=1}^m C_i}}(A)) = \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$;

(4) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) =\sim \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\sim A)$, $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) = \sim \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\sim A)$;

(5) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A \cap B) \subseteq \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cap \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$, $\overline{FM_{\sum_{i=1}^m C_i}}(A \cup B) \supseteq \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cup \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$;

(6) $A \subseteq B \Rightarrow \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \subseteq \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$ and $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \subseteq \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$.

Proposition 3.2 *Let (U, Ω) be a MCAS. For each $K \in \mathfrak{C}$,then, we have that*

(1) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(K) = K$;

(2) $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\sim K) =\sim K$.

Proof We can prove the proposition by Definition 3.1 and Proposition 3.1. $\square$

Proposition 3.3 *Let (U, Ω)be a minimally unary MCAS and $A, B \subseteq U$, then*

(1) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A \cap B) = \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cap \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$;

(2) $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A \cup B) = \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cup \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$.

Proof

(1) ($\Rightarrow$): It is clear by Proposition 3.1. ($\Leftarrow$): For each $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$, there is a $K \in \mathfrak{C}$ such that $a \in K \subseteq A$.

Then we have that $a \in K_a^{\min} \subseteq A$. Similarly, for each $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$, it can be found that $a \in K_a^{\min} \subseteq B$.

Thus, $a \in K_a^{\min} \subseteq A \cap B$. By Definition 3.1, we have that $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A \cap B)$.

(2) It is immediate by Definition 3.1 and Proposition 3.1. $\square$

Now, we provide an example to further explain Proposition 3.3.

Example 3.1 (Continued from Example 2.1) For $A = \{a_2, a_4, a_6\}$, $B = \{a_1, a_2, a_4, a_5\}$, we have that $\underline{\mathrm{FM}_{C_1+C_2}}(A) = \{a_2, a_4, a_6\}$, $\underline{\mathrm{FM}_{C_1+C_2}}(B) = \{a_1, a_2, a_4\}$. Meanwhile, $\underline{\mathrm{FM}_{C_1+C_2}}(A \cap B) = \{a_2, a_4\}$. Then, $\underline{\mathrm{FM}_{C_1+C_2}}(A \cap B) = \underline{\mathrm{FM}_{C_1+C_2}}(A) \cap \underline{\mathrm{FM}_{C_1+C_2}}(B)$.

At the same time, we have $\overline{\mathrm{FM}_{C_1+C_2}}(A) = \{a_1, a_2, a_4, a_5, a_6\}$, $\overline{\mathrm{FM}_{C_1+C_2}}(B) = \{a_1, a_2, a_4, a_5\}$. In addition, $\overline{\mathrm{FM}_{C_1+C_2}}(A \cup B) = \{a_1, a_2, a_4, a_5, a_6\}$. Hence, $\overline{\mathrm{FM}_{C_1+C_2}}(A \cup B) = \overline{\mathrm{FM}_{C_1+C_2}}(A) \cup \overline{\mathrm{FM}_{C_1+C_2}}(B)$.

Proposition 3.4 *Let (U, Ω) be a minimally unary MCAS and $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$, then $K_a^{\min} \subseteq \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$.*

Proof For $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$, there is a $K \in \mathfrak{C}$ such that $a \in K \subseteq A$. It follows that $a \in K_a^{\min} \subseteq A$. By Proposition 3.1 and Proposition 3.2, we have that $a \in K_a^{\min} = \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(K_a^{\min}) \subseteq \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$. i.e.,$K_a^{\min} \subseteq \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$. $\square$

Proposition 3.5 *Let (U, Ω) be a minimally unary MCAS and $A, B \subseteq U$, then*

(1) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cup \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)) = \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cup \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$;

(2) $\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cap \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)) = \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cap \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$;

(3) $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cup \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)) = \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A) \cup \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$;

(4) $\overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\overline{FM_{\sum_{i=1}^m C_i}}(A) \cap \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)) = \overline{FM_{\sum_{i=1}^m C_i}}(A) \cap \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B)$.

Proof It is immediate from Definition 3.1, Propositions 3.1 and 3.4. $\square$

Proposition 3.6 *Let (U, Ω) be a minimally unary MCAS, $a \in U$and $A \subseteq U$. If $K_a^{\min} = \{a\}$ and $a \in \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$. Then we have that $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$.*

Proof Since $a \in \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$, we can find that $a \in\sim \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(\sim A)$. Thus, $a \in \{a\} = K_a^{\min} \nsubseteq\sim A$. Therefore, we have that $a \in \{a\} = K_a^{\min} \subseteq A$. i.e., $a \in \underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)$. $\square$

3.2 Operation properties of the first type of MCRSs

In this part, we will research the operations of intersection and union on MCRSs. We first propose the concepts of intersection and union of MCRSs.

Definition 3.2 Let (U, Ω) be a MCAS. For any $(\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A)), (\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)) \in \mathbb{C}^F$, the intersection and union of them are constructed as follows.

(1) $(\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A)) \cap (\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)) = (\underline{FM}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B))$;

(2) $(\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A)) \cup (\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)) = (\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cup \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B), \overline{FM}_{\sum_{i=1}^m C_i}(A) \cup \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B))$.

Is the first type of MCRSs closed under set intersection and union? Now, we will provide an example to answer this question.

Example 3.2 Let $U = \{a_1, a_2, \ldots, a_6\}, C_1 = \{\{a_1, a_2, a_4\}, \{a_2, a_3, a_4, a_5\}, \{a_5, a_6\}\}, C_2 = \{\{a_2, a_4\}, \{a_1, a_3, a_4, a_5\}, \{a_4, a_6\}\}$. For $A = \{a_2, a_4, a_6\}$, $B = \{a_3, a_5, a_6\}$, we have that $\underline{\mathrm{FM}}_{C_1+C_2}(A) = \{a_2, a_4, a_6\}, \underline{\mathrm{FM}}_{C_1+C_2}(B) = \{a_5, a_6\}$, and $\underline{\mathrm{FM}}_{C_1+C_2}(A) \cap \underline{\mathrm{FM}}_{C_1+C_2}(B) = \{a_6\}$. Clearly, there is no way to find a subset $E \subseteq U$ such that $\underline{\mathrm{FM}}_{C_1+C_2}(E) = \underline{\mathrm{FM}}_{C_1+C_2}(A) \cap \underline{\mathrm{FM}}_{C_1+C_2}(B)$.

Example 3.2 shows that the first type of MCRSs is not closed under set intersection. Similarly, the first type of MCRSs is not closed under set union.

Proposition 3.7 *Let (U, Ω) be a minimally unary MCAS, $A, B \subseteq U$ and for $\forall a, b \in U$, we have $K_a^{\min} = K_b^{\min}$ or $K_a^{\min} \cap K_b^{\min} = \emptyset$. Then, the first type of MCRSs is closed under set intersection.*

Proof Denote $M = M_2/M_1$, where $M_1 = \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$, and $M_2 = \overline{FM}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$. Let $\mathcal{M} = \{K_a^{\min} | a \in M, K_a^{\min} \cap M_1 = \emptyset\}$ and $\mathcal{M}' = \{K_{a_i}^{\min} | a_i \in M, i = 1, 2, \ldots, s\}$, where $\mathcal{M}'$ must satisfy the following conditions: (a) $\mathcal{M}' \subseteq \mathcal{M}$; (b) For any two elements of $\mathcal{M}'$, the intersection of them is empty; (c) For each $K_a^{\min} \in \mathcal{M}$, we can find $K_{a_i}^{\min} \in \mathcal{M}'$ such that $K_{a_i}^{\min} \subseteq K_a^{\min}$. Denote $K = \{a_i | K_{a_i}^{\min} \in \mathcal{M}', i = 1, 2, \ldots, s\}, E = M_1 \cup K$.

First, we will prove that $\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) = \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$.

For each $a \in \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E)$, by Definition 3.1, Proposition 3.6 and construction of E, we have that $a \in K_a^{\min} \subseteq \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$. Thus, $\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) \subseteq \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$.

From the construction of E, we have $\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B) \subseteq E$. By Propositions 3.1 and 3.5, it is clear that $\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B) = \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)) \subseteq \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E)$. We have that $\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B) \subseteq \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E)$. Therefore, $\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) = \underline{FM}_{\sum_{i=1}^m C_i}(A) \cap \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$.

Second, we will prove that $\overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) = \overline{FM}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$.

According to the construction of E, we have that $E \subseteq \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$. By Propositions 3.1 and 3.6, it is clear that $\overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) \subseteq \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(\overline{FM}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)) = \overline{FM}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$. Thus, $\overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) \subseteq \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$.

Similarly, based on Definition 3.1 and the construction of E, $\overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B) \subseteq \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E)$ holds. Thus, $\overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(E) = \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cap \overline{FM}_{\sum_{i=1}^m C_i}(B)$. □

Proposition 3.8 *Let (U, Ω) be a minimally unary MCAS, $A, B \subseteq U$ and for $\forall a, b \in U$, we have $K_a^{\min} = K_b^{\min}$ or $K_a^{\min} \cap K_b^{\min} = \emptyset$. Then, the first type of MCRSs is closed under set union.*

Proof Denote $N = N_2/N_1$, where $N_1 = \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cup \underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$, $N_2 = \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A) \cup \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B)$. Let $\mathcal{N} = \{K_a^{\min} | a \in N, K_a^{\min} \cap N_1 = \emptyset\}$ and $\mathcal{N}' = \{K_{a_j}^{\min} | a_i \in N, j = 1, 2, \ldots, n\}$, where $\mathcal{N}'$ must satisfy the following conditions: (a) $\mathcal{N}' \subseteq \mathcal{N}$; (b) For any two elements of $\mathcal{N}'$, the intersection of them is empty; (c) For each $K_a^{\min} \in \mathcal{N}$, we can find $K_{a_j}^{\min} \in \mathcal{N}'$ such that $K_{a_j}^{\min} \subseteq K_a^{\min}$. Denote $L = \{a_j | K_{a_j}^{\min} \in \mathcal{N}', j = 1, 2, \ldots, n\}$ and $F = N_1 \cup L$. Similarly, we can prove that $(\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(F), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(F)) = (\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(A)) \cup (\underline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B), \overline{\mathrm{FM}}_{\sum_{i=1}^m C_i}(B))$. □

Remark 3.1 Based on the constructions of E, F, the proofs of Theorems 3.7 and 3.8 can be completed. Meanwhile, we can find that the first type of MCRSs is closed under set union and intersection. In other words, for $A, B \subseteq U$, there are two subsets $E, F \subseteq U$ such that $(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(E), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(E)) = (\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)) \cap (\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B))$; $(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(F), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(F)) = (\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)) \cup (\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B))$. In addition, according to the constructions of subsets E, F presented in Theorems 3.7 and 3.8, it is easy and important for us to develop two algorithms, which could effectively compute the subsets E, F.

Let (U, Ω) be a minimally unary MCAS and for $\forall a, b \in U$, we have $K_a^{\min} = K_b^{\min}$ or $K_a^{\min} \cap K_b^{\min} = \emptyset$. Then, we will design two algorithms which may compute subsets E and F presented in Propositions 3.7 and 3.8.

Algorithm 1: Computing E

Input : A minimally unary MCAS (U, Ω) and $A, B \subseteq U$;
Output : E.

begin
- Compute $\underline{FM_{\sum_{i=1}^m C_i}}(A) \cap \underline{FM_{\sum_{i=1}^m C_i}}(B), \overline{FM_{\sum_{i=1}^m C_i}}(A) \cap \overline{FM_{\sum_{i=1}^m C_i}}(B))$;
- Compute $\mathfrak{K}^{min}_{(\overline{FM_{\sum_{i=1}^m C_i}}(A) \cap \overline{FM_{\sum_{i=1}^m C_i}}(B))/(\underline{FM_{\sum_{i=1}^m C_i}}(A) \cap \underline{FM_{\sum_{i=1}^m C_i}}(B))} = \{K^{min}_{a_1}, K^{min}_{a_2}, \cdots, K^{min}_{a_s}\}$;
- $\emptyset \leftarrow K$;
- **for** $i = 1 : s; i <= s; i++$ **do**
  - **for** *any* $b_i \in K^{min}_{a_i}$ **do**
    - $K \leftarrow K \cup \{b_i\}$;
  - **end**
- **end**
- Compute $(\underline{FM_{\sum_{i=1}^m C_i}}(A) \cap \underline{FM_{\sum_{i=1}^m C_i}}(B)) \cup K$;
  // $E = (\underline{FM_{\sum_{i=1}^m C_i}}(A) \cap \underline{FM_{\sum_{i=1}^m C_i}}(B)) \cup K$ by the construction of E;

end

Algorithm 2: Computing subset F

Input : A minimally unary MCAS (U, Ω) and $X, Y \subseteq U$;
Output : F.

begin
- Compute $\underline{FM_{\sum_{i=1}^m C_i}}(X) \cup \underline{FM_{\sum_{i=1}^m C_i}}(Y), \overline{FM_{\sum_{i=1}^m C_i}}(X) \cup \overline{FM_{\sum_{i=1}^m C_i}}(Y))$;
- Compute $\mathfrak{K}^{min}_{(\overline{FM_{\sum_{i=1}^m C_i}}(X) \cup \overline{FM_{\sum_{i=1}^m C_i}}(Y))/(\underline{FM_{\sum_{i=1}^m C_i}}(X) \cup \underline{FM_{\sum_{i=1}^m C_i}}(Y))} = \{K^{min}_{x_1}, K^{min}_{x_2}, \cdots, K^{min}_{x_t}\}$;
- $\emptyset \leftarrow L$;
- **for** $j = 1 : t; j <= t; j++$ **do**
  - **for** *any* $y_j \in K^{min}_{x_j}$ **do**
    - $L \leftarrow L \cup \{y_j\}$;
  - **end**
- **end**
- Compute $(\underline{FM_{\sum_{i=1}^m C_i}}(X) \cup \underline{FM_{\sum_{i=1}^m C_i}}(Y)) \cup L$;
  // $F = (\underline{FM_{\sum_{i=1}^m C_i}}(X) \cup \underline{FM_{\sum_{i=1}^m C_i}}(Y)) \cup L$ by the construction of F;

end

Clearly, the computational complexities of Algorithms 1 and 2 are $o(s|U|^3)$ and $o(t|U|^3)$, respectively.

Example 3.3 Let $U = \{a_0, a_1, \dots, a_9\}, C_1 = \{\{a_0, a_1\}, \{a_0, a_1, a_2, a_3, a_4, a_5\}, \{a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}\}$, and $C_2 = \{\{a_0, a_1, a_2, a_3, a_4, a_5\}, \{a_2, a_3, a_4, a_5\}, \{a_6, a_7, a_8, a_9\}\}$. For $A = \{a_0, a_1, a_2, a_6, a_7\}, B = \{a_0, a_3, a_4, a_7, a_8\}$, we have that $\underline{\mathrm{FM}_{C_1+C_2}}(A) = \{a_0, a_1\}$, $\underline{\mathrm{FM}_{C_1+C_2}}(B) = \emptyset$, $\overline{\mathrm{FM}_{C_1+C_2}}(A) = U$, and $\overline{\mathrm{FM}_{C_1+C_2}}(B) = U$.

Let $E = \{a_0, a_2, a_6\}, F = \{a_0, a_1, a_2, a_6\}$, then we have

$$\left(\underline{\mathrm{FM}_{C_1+C_2}}(E), \overline{FM_{C_1+C_2}}(E)\right) = \left(\underline{\mathrm{FM}_{C_1+C_2}}(A), \overline{FM_{C_1+C_2}}(A)\right) \cap \left(\underline{\mathrm{FM}_{C_1+C_2}}(B), \overline{FM_{C_1+C_2}}(B)\right) \cdots\cdots \tag{3.1}$$

$$\left(\underline{\mathrm{FM}_{C_1+C_2}}(F), \overline{FM_{C_1+C_2}}(F)\right) = \left(\underline{\mathrm{FM}_{C_1+C_2}}(A), \overline{\mathrm{FM}_{C_1+C_2}}(A)\right) \cup \left(\underline{\mathrm{FM}_{C_1+C_2}}(B), \overline{\mathrm{FM}_{C_1+C_2}}(B)\right) \cdots\cdots \tag{3.2}$$

Clearly, all the subsets $E, F \subseteq U$ satisfying Eqs. (3.1) and (3.2) are not unique. For $A = \{a_0, a_1, a_2, a_6, a_7\}, B = \{a_0, a_3, a_4, a_7, a_8\}$, all the subsets $E, F \subseteq U$ computed from Algorithms 1 and 2 are given in Table 5.

For $C = \{a_0, a_3, a_6, a_7, a_8, a_9\}$. Let $G = \{a_0, a_2\}, H = \{a_0, a_2, a_6\}$, then

$$\left(\underline{\mathrm{FM}_{C_1+C_2}}(G), \overline{FM_{C_1+C_2}}(G)\right) = \left((\underline{\mathrm{FM}_{C_1+C_2}}(A), \overline{\mathrm{FM}_{C_1+C_2}}(A)\right) \cap \left(\underline{\mathrm{FM}_{C_1+C_2}}(B), \overline{FM_{C_1+C_2}}(B))\right) \cup \left(\underline{FM_{C_1+C_2}}(C), \overline{\mathrm{FM}_{C_1+C_2}}(C)\right) \cdots \tag{3.3}$$

$$\left(\underline{\mathrm{FM}_{C_1+C_2}}(H), \overline{FM_{C_1+C_2}}(H)\right) = \left((\underline{\mathrm{FM}_{C_1+C_2}}(A), \overline{\mathrm{FM}_{C_1+C_2}}(A)\right) \cup \left(\underline{\mathrm{FM}_{C_1+C_2}}(B), \overline{\mathrm{FM}_{C_1+C_2}}(B))\right) \cap \left(\underline{\mathrm{FM}_{C_1+C_2}}(C), \overline{\mathrm{FM}_{C_1+C_2}}(C)\right) \cdots \tag{3.4}$$

For $A = \{a_0, a_1, a_2, a_6, a_7\}, B = \{a_0, a_3, a_4, a_7, a_8\}, C = \{a_0, a_3, a_6, a_7, a_8, a_9\}$, all the subsets $G, H \subseteq U$, which satisfy Eqs. (3.3) and (3.4), computed from Algorithms 1 and 2 are given in Table 6.

4 The second type of MCRSs

Similar to first type of MCRSs, first, we propose the multi-covering upper approximation. Then, using the duality, the multi-covering lower approximation will be presented. Therefore, the second type of MCRSs can be constructed as follows:

Definition 4.1 Let (U, Ω) be a MCAS, and $\Omega = \{C_1, C_2, \dots, C_m\}$ a family of coverings of U with $C_i = \{K_{i1}, K_{i2}, \dots, K_{it_i}\}$, and $A \subseteq U$. Denote

Table 5 Subsets E, F

A, B	E	F
$\{a_0, a_1, a_2, a_6, a_7\}$	$\{a_0, a_2, a_6\}, \{a_1, a_2, a_6\}$	$\{a_0, a_1, a_2, a_6\}$
$\{a_0, a_3, a_4, a_7, a_8\}$	$\{a_0, a_2, a_7\}, \{a_1, a_2, a_7\}$	$\{a_0, a_1, a_2, a_7\}$
	$\{a_0, a_2, a_8\}, \{a_1, a_2, a_8\}$	$\{a_0, a_1, a_2, a_8\}$
	$\{a_0, a_2, a_9\}, \{a_1, a_2, a_9\}$	$\{a_0, a_1, a_2, a_9\}$
	$\{a_0, a_3, a_6\}, \{a_1, a_3, a_6\}$	$\{a_0, a_1, a_3, a_6\}$
	$\{a_0, a_3, a_7\}, \{a_1, a_3, a_7\}$	$\{a_0, a_1, a_3, a_7\}$
	$\{a_0, a_3, a_8\}, \{a_1, a_3, a_8\}$	$\{a_0, a_1, a_3, a_8\}$
	$\{a_0, a_3, a_9\}, \{a_1, a_3, a_9\}$	$\{a_0, a_1, a_3, a_9\}$
	$\{a_0, a_4, a_6\}, \{a_1, a_4, a_6\}$	$\{a_0, a_1, a_4, a_6\}$
	$\{a_0, a_4, a_7\}, \{a_1, a_4, a_7\}$	$\{a_0, a_1, a_4, a_7\}$
	$\{a_0, a_4, a_8\}, \{a_1, a_4, a_8\}$	$\{a_0, a_1, a_4, a_8\}$
	$\{a_0, a_4, a_9\}, \{a_1, a_4, a_9\}$	$\{a_0, a_1, a_4, a_9\}$
	$\{a_0, a_5, a_6\}, \{a_1, a_5, a_6\}$	$\{a_0, a_1, a_5, a_6\}$
	$\{a_0, a_5, a_7\}, \{a_1, a_5, a_7\}$	$\{a_0, a_1, a_5, a_7\}$
	$\{a_0, a_5, a_8\}, \{a_1, a_5, a_8\}$	$\{a_0, a_1, a_5, a_8\}$
	$\{a_0, a_5, a_9\}, \{a_1, a_5, a_9\}$	$\{a_0, a_1, a_5, a_9\}$

Table 6 Subsets G, H

A, B, C	G	H
$\{a_0, a_1, a_2, a_6, a_7\}$	$\{a_0, a_2\}$	$\{a_0, a_2, a_6\}, \{a_1, a_2, a_6\}$
$\{a_0, a_3, a_4, a_7, a_8\}$	$\{a_1, a_2\}$	$\{a_0, a_2, a_7\}, \{a_1, a_2, a_7\}$
$\{a_0, a_3, a_6, a_7, a_8, a_9\}$	$\{a_0, a_3\}$	$\{a_0, a_2, a_8\}, \{a_1, a_2, a_8\}$
	$\{a_1, a_3\}$	$\{a_0, a_2, a_9\}, \{a_1, a_2, a_9\}$
	$\{a_0, a_4\}$	$\{a_0, a_3, a_6\}, \{a_1, a_3, a_6\}$
	$\{a_1, a_4\}$	$\{a_0, a_3, a_7\}, \{a_1, a_3, a_7\}$
	$\{a_0, a_5\}$	$\{a_0, a_3, a_8\}, \{a_1, a_3, a_8\}$
	$\{a_1, a_5\}$	$\{a_0, a_3, a_9\}, \{a_1, a_3, a_9\}$
		$\{a_0, a_4, a_6\}, \{a_1, a_4, a_6\}$
		$\{a_0, a_4, a_7\}, \{a_1, a_4, a_7\}$
		$\{a_0, a_4, a_8\}, \{a_1, a_4, a_8\}$
		$\{a_0, a_4, a_9\}, \{a_1, a_4, a_9\}$
		$\{a_0, a_5, a_6\}, \{a_1, a_5, a_6\}$
		$\{a_0, a_5, a_7\}, \{a_1, a_5, a_7\}$
		$\{a_0, a_5, a_8\}, \{a_1, a_5, a_8\}$
		$\{a_0, a_5, a_9\}, \{a_1, a_5, a_9\}$

$$\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) = \cup\{K_{ij} \in \mathcal{C}_i | \vee (K_{ij} \cap A \neq \emptyset), \; i \in \{1,2,\dots,m\}, j = 1,2,\dots, |\mathcal{C}_i|\}, \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) = \sim \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\sim A)$$

we, respectively, call $\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$ and $\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$ the second type of multi-covering upper and lower approximations of A with respect to (U, Ω).

Example 4.1 Let $U = \{a_1, a_2, \cdots, a_6\}, C_1 = \{\{a_1, a_2\}, \{a_2, a_3, a_4\}, \{a_4, a_5\}, \{a_6\}\}$, and $C_2 = \{\{a_1, a_2, a_4\}, \{a_1, a_3, a_5\}, \{a_5, a_6\}\}$. For $A = \{a_5\}$, then we have that $\overline{\mathrm{SM}_{C_1+C_2}}(A) = \{a_4, a_5\} \cup \{a_1, a_3, a_5\} \cup \{a_5, a_6\} = \{a_1, a_3, a_4, a_5, a_6\}$, $\underline{\mathrm{SM}_{C_1+C_2}}(A) = \emptyset$.

For $A \subseteq U$, we call $(\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A))$ the second type of MCRSs of A. Therefore, $\mathbb{C}^S = \{(\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)) | A \subseteq U\}$ is all of the second type of MCRSs with respect to (U, Ω).

4.1 The second type of multi-covering approximation operators

In this part, we will discuss the properties of the second type of multi-covering approximation operators in a MCAS.

Proposition 4.1 *Let (U, Ω) be a MCAS and $A, B \subseteq U$, then*

(1) $\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(U) = \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(U) = U$, $\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\emptyset) = \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\emptyset) = \emptyset$;

(2) $\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \subseteq A \subseteq \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$;

(3) $\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) = \sim \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\sim A)$, $\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) = \sim \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\sim A)$;

(4) $\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A \cap B) = \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \cap \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(B)$, $\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A \cup B) = \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \cup \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(B)$;

(5) $A \subseteq B \Rightarrow \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \subseteq \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(B)$ and $\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \subseteq \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(B)$.

Proof It is clear by Definition 4.1.

Example 4.2 (Continued from Example 4.1) By Example 4.1, then we have $\overline{C_1 + C_2}(A) = \{a_1, a_3, a_4, a_5, a_6\}$. However, $\overline{\mathrm{SM}_{C_1+C_2}}(\overline{SM_{C_1+C_2}}(A)) = U$. Hence, $\overline{\mathrm{SM}_{C_1+C_2}}(\overline{SM_{C_1+C_2}}(A)) \neq \overline{\mathrm{SM}_{C_1+C_2}}(A)$. At the same time, we have that $\underline{\mathrm{SM}_{C_1+C_2}}(\underline{SM_{C_1+C_2}}(A)) \neq \underline{\mathrm{SM}_{C_1+C_2}}(A)$.

Lemma 4.1 *Let (U, Ω) be a maximally unary MCAS and $a \in U$. For each $b \in K_a^{\max}$, we have that $K_b^{\max} = K_a^{\max}$.*

Proof According to $b \in K_a^{\max}$, we have that $K_a^{\max} \subseteq K_b^{\max}$. Suppose that there is $c \in U$ such that $c \in K_b^{\max}/K_a^{\max}$, then $K_a^{\max} \subset K_b^{\max}$. It contradicts with the definition of $K_a^{\max}$. Therefore, $K_b^{\max} = K_a^{\max}$. □

Lemma 4.2 *Let (U, Ω) be a maximally unary MCAS and $A \subseteq U$, then $\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) = \cup_{a \in A} K_a^{\max}$.*

Proof (⇒): For each $b \in \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$, there is $K_{ij} \in \mathfrak{C}$ such that $b \in K_{ij}$ and $K_{ij} \cap A \neq \emptyset$. So, $K_b^{\max} \cap A \neq \emptyset$. Then, there must exist $a \in A$ such that $a \in K_b^{\max}$. By Lemma 4.1, we have that $K_b^{\max} = K_a^{\max}$. Therefore, $b \in K_a^{\max}$, i.e., $b \in \cup_{a \in A} K_a^{\max}$.

(⇐): For each $a \in A$, we can find $K_a^{\max} \cap A \neq \emptyset$. By the Definition 4.1, it is obvious that $K_a^{\max} \subseteq \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$. Hence, $\cup_{a \in A} K_a^{\max} \subseteq \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$.

Proposition 4.2 *Let (U, Ω) be a maximally unary MCAS and $A \subseteq U$, then*

$$(1)\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)) = \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A);$$
$$(2)\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)) = \underline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A).$$

Proof

(1) (⇐): It is obvious by Proposition 4.1. (⇒): For each $b \in \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A))$, there exists $K_{ij} \in \mathfrak{C}$ such that $b \in K_{ij}$ and $K_{ij} \cap \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \neq \emptyset$. Then, we have that $K_b^{\max} \cap \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A) \neq \emptyset$. By Lemma 4.2, there exists $a \in A$ such that $K_b^{\max} \cap K_a^{\max} \neq \emptyset$. Thus, we can choose $c \in K_b^{\max} \cap K_a^{\max}$. By Lemma 4.1, we have that $K_c^{\max} = K_b^{\max}$ and $K_c^{\max} = K_a^{\max}$. It can be obtained that $K_b^{\max} = K_a^{\max}$. Hence, we have that $b \in K_b^{\max} = K_a^{\max} \subseteq \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$. That is, $\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(\overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)) \subseteq \overline{\mathrm{SM}_{\sum_{i=1}^{m} C_i}}(A)$.

(2) We can prove the item by Proposition 4.1 and item (1).

□

Now, we provide an example to further explain the Proposition 4.2.

Example 4.3 (Continued from Example 2.2) For $A = \{a_5\}$, $B = \{a_1, a_2, a_3, a_4\}$, then $\overline{\mathrm{SM}_{C_1+C_2}}(A) = \{a_4, a_5, a_6\}$, $\overline{\mathrm{SM}_{C_1+C_2}}(\overline{\mathrm{SM}_{C_1+C_2}}(A)) = \{a_4, a_5, a_6\}$. Thus, $\overline{\mathrm{SM}_{C_1+C_2}}(\overline{\mathrm{SM}_{C_1+C_2}}(A)) = \overline{\mathrm{SM}_{C_1+C_2}}(A)$. In addition, $\underline{\mathrm{SM}_{C_1+C_2}}(B) = \{a_1, a_2, a_3\}$,

$\underline{\mathrm{SM}}_{C_1+C_2}(\underline{\mathrm{SM}}_{C_1+C_2}(B)) = \{a_1, a_2, a_3\}$. Hence, we can find that $\underline{\mathrm{SM}}_{C_1+C_2}(\underline{\mathrm{SM}}_{C_1+C_2}(B)) = \underline{\mathrm{SM}}_{C_1+C_2}(B)$.

Proposition 4.3 *Let (U, Ω)be a maximally unary MCAS and $a \in \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$, then $K_a^{\max} \subseteq \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$.*

Proof For each $a \in \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$, there is $K \in \mathfrak{C}$ such that $a \in K$ and $K \cap A \neq \emptyset$. We can find that $K_a^{\max} \cap A \neq \emptyset$. By Definition 4.1, $K_a^{\max} \subseteq \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$.

Proposition 4.4 *Let (U, Ω) be a maximally unary MCAS and $A, B \subseteq U$, then*

(1) $\overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(\overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cup \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)) = \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cup \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)$;
(2) $\overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(\overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cap \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)) = \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cap \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)$;
(3) $\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cup \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)) = \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cup \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)$;
(4) $\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cap \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)) = \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cap \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)$.

Proof It is immediate through Propositions 4.1 and 4.2.

Proposition 4.5 *Let (U, Ω) be a maximally unary MCAS, $a \in U$ and $A \subseteq U$ .If $K_a^{\max} = \{a\}$ and $a \in \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$.Then $a \in \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$.*

Proof Suppose that $\{a\} \cap \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) = \emptyset$, we have that $a \in \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(\sim A)$. It can be obtained that $\{a\} \cap (\sim A) \neq \emptyset$. By the assumption in this proposition, it follows that $K_a^{\max} \cap A = \emptyset$. Thus, $\{a\} \cap \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) = \emptyset$. Hence, $a \in \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)$. $\square$

4.2 Operation properties of the second type of MCRSs

In this section, we research the operations of intersection and union on the second type of MCRSs.

Example 4.4 Let $U = \{a_1, a_2, \ldots, a_6\}, C_1 = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_4\}, \{a_5, a_6\}\}, C_2 = \{\{a_1, a_2, a_3\}, \{a_4\}, \{a_3, a_5\}, \{a_6\}\}$. For $A = \{a_1, a_2, a_3\}$, $B = \{a_3, a_5, a_6\}$, we have $\underline{\mathrm{SM}}_{C_1+C_2}(A) = \{a_2\}, \underline{\mathrm{SM}}_{C_1+C_2}(B) = \{a_5, a_6\}, \overline{\mathrm{SM}}_{C_1+C_2}(A) = \{a_1, a_2, a_3, a_4, a_5\}$, and $\overline{\mathrm{SM}}_{C_1+C_2}(B) = \{a_1, a_2, a_3, a_5, a_6\}$. Obviously, we cannot find a subset $F \subseteq U$ such that $(\underline{\mathrm{SM}}_{C_1+C_2}(F), \overline{\mathrm{SM}}_{C_1+C_2}(F)) = (\underline{\mathrm{SM}}_{C_1+C_2}(A), \overline{\mathrm{SM}}_{C_1+C_2}(A)) \cup (\underline{\mathrm{SM}}_{C_1+C_2}(B), \overline{\mathrm{SM}}_{C_1+C_2}(B))$.

Example 4.4 shows us that the second type of MCRSs is not closed under set union and intersection.

Proposition 4.6 *Let (U, Ω) be a maximally unary MCAS and $A, B \subseteq U$. Then the second type of MCRSs is closed under set intersection.*

Proof Denote $M = M_2/M_1$, where $M_1 = \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cap \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B), M_2 = \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cap \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)$. Let $\mathcal{M} = \{K_a^{\max} | a \in M, K_a^{\max} \cap M_1 = \emptyset\}$ and $\mathcal{M}' = \{K_{a_i}^{\max} | a_i \in M, i = 1, 2, \ldots, l\}$, where $\mathcal{M}'$ must satisfy the following conditions: (a) $\mathcal{M}' \subseteq \mathcal{M}$; (b) For any two elements of $\mathcal{M}'$, the intersection of them is empty; (c) For each $K_a^{\max} \in \mathcal{M}$, there exists $K_{a_i}^{\max} \in \mathcal{M}'$ such that $K_{a_i}^{\max} = K_a^{\max}$. Denote $K = \{a_i | K_{a_i}^{\max} \in \mathcal{M}', i = 1, 2, \ldots, l\}$ and $E = M_1 \cup K$. Similarly, we have that $(\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(E), \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(E)) = (\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A), \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)) \cap (\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B), \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B))$. $\square$

Proposition 4.7 *Let (U, Ω)be a maximally unary MCAS and $A, B \subseteq U$.Then the second type of MCRSs is closed under set union.*

Proof Denote $N = N_2/N_1$, where $N_1 = \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cup \underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B), N_2 = \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A) \cup \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B)$. Let $\mathcal{N} = \{K_a^{\max} | a \in N, K_a^{\max} \cap N_1 = \emptyset\}$ and $\mathcal{N}' = \{K_{a_j}^{\max} | a_j \in N, j = 1, 2, \ldots, k\}$, where $\mathcal{N}'$ must satisfy the following conditions: (a) $\mathcal{N}' \subseteq \mathcal{N}$; (b) For any two elements of $\mathcal{N}'$, the intersection of them is empty; (c) For each $K_a^{\max} \in \mathcal{N}$, there exists $K_{a_j}^{\min} \in \mathcal{N}'$ such that $K_{a_j}^{\max} = K_a^{\max}$. Denote $L = \{a_j | K_{a_j}^{\max} \in \mathcal{N}', j = 1, 2, \ldots, k\}$ and $F = N_1 \cup L$. Similarly, we can prove that $(\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(F), \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(F)) = (\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A), \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(A)) \cup (\underline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B), \overline{\mathrm{SM}}_{\sum_{i=1}^{m} C_i}(B))$.

Remark 4.1 On the one hand, based on the constructions of E and F, the proofs of Theorems 4.6 and 4.7 can be completed. According to the constructions of E, F, it can be obtained that the second type of MCRSs is closed under set intersection and union. In other words, for any subsets $A, B \subseteq U$, there are two subsets E, F such that the following two equations hold:

$$\left(\underline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(E), \overline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(E)\right) = \left(\underline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(A)\right) \cap \left(\underline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(B), \overline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(B)\right),$$

$$\left(\underline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(F), \overline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(F)\right) = \left(\underline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(A)\right) \cup \left(\underline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(B), \overline{\mathrm{SM}_{\sum_{i=1}^m C_i}}(B)\right)$$

On the other hand, according to the constructions of subsets E, F presented in Theorems 4.6 and 4.7, it is easy and important for us to develop two algorithms, which could effectively compute the subsets E, F.

Similarly, we can also design the corresponding algorithms to compute subsets E, F shown in Propositions 4.6 and 4.7, respectively. We will not repeat them here.

Example 4.5 Let $U = \{a_0, a_1, \dots, a_9\}, C_1 = \{\{a_0, a_1\}, \{a_2, a_7, a_8, a_9\}, \{a_3\}, \{a_4, a_6\}, \{a_3, a_4, a_5, a_6\}\}$, and $C_2 = \{\{a_0\}, \{a_0, a_1\}, \{a_2, a_7\}, \{a_3, a_4, a_5\}, \{a_6\}, \{a_7, a_8, a_9\}\}$. For $A = \{a_0, a_2, a_6\}, B = \{a_0, a_1, a_6, a_7\}$, then, we have that $\underline{\mathrm{SM}_{C_1+C_2}}(A) = \emptyset$, $\underline{\mathrm{SM}_{C_1+C_2}}(B) = \{a_0, a_1\}$, $\overline{\mathrm{SM}_{C_1+C_2}}(A) = U$, $\overline{\mathrm{SM}_{C_1+C_2}}(B) = U$.

Let $E = \{a_0, a_2, a_3\}, F = \{a_0, a_1, a_2, a_3\}$, then we have

$$\left(\underline{\mathrm{SM}_{C_1+C_2}}(E), \overline{\mathrm{SM}_{C_1+C_2}}(E)\right) = \left(\underline{SM_{C_1+C_2}}(A), \overline{\mathrm{SM}_{C_1+C_2}}(A)\right) \cap \left(\underline{\mathrm{SM}_{C_1+C_2}}(B), \overline{\mathrm{SM}_{C_1+C_2}}(B)\right) \cdots\cdots \tag{4.1}$$

Table 7 Subsets E, F

A, B	E	F
$\{a_0, a_2, a_6\}$	$\{a_0, a_2, a_3\}, \{a_1, a_2, a_3\}$	$\{a_0, a_1, a_2, a_3\}$
$\{a_0, a_1, a_6, a_7\}$	$\{a_0, a_2, a_4\}, \{a_1, a_2, a_4\}$	$\{a_0, a_1, a_2, a_4\}$
	$\{a_0, a_2, a_5\}, \{a_1, a_2, a_5\}$	$\{a_0, a_1, a_2, a_5\}$
	$\{a_0, a_2, a_6\}, \{a_1, a_2, a_6\}$	$\{a_0, a_1, a_2, a_6\}$
	$\{a_0, a_3, a_7\}, \{a_1, a_3, a_7\}$	$\{a_0, a_1, a_3, a_7\}$
	$\{a_0, a_4, a_7\}, \{a_1, a_4, a_7\}$	$\{a_0, a_1, a_4, a_7\}$
	$\{a_0, a_5, a_7\}, \{a_1, a_5, a_7\}$	$\{a_0, a_1, a_5, a_7\}$
	$\{a_0, a_6, a_7\}, \{a_1, a_6, a_7\}$	$\{a_0, a_1, a_6, a_7\}$
	$\{a_0, a_3, a_8\}, \{a_1, a_3, a_8\}$	$\{a_0, a_1, a_3, a_8\}$
	$\{a_0, a_4, a_8\}, \{a_1, a_4, a_8\}$	$\{a_0, a_1, a_4, a_8\}$
	$\{a_0, a_5, a_8\}, \{a_1, a_5, a_8\}$	$\{a_0, a_1, a_5, a_8\}$
	$\{a_0, a_6, a_8\}, \{a_1, a_6, a_8\}$	$\{a_0, a_1, a_6, a_8\}$
	$\{a_0, a_3, a_9\}, \{a_1, a_3, a_9\}$	$\{a_0, a_1, a_3, a_9\}$
	$\{a_0, a_4, a_9\}, \{a_1, a_4, a_9\}$	$\{a_0, a_1, a_4, a_9\}$
	$\{a_0, a_5, a_9\}, \{a_1, a_5, a_9\}$	$\{a_0, a_1, a_5, a_9\}$
	$\{a_0, a_6, a_9\}, \{a_1, a_6, a_9\}$	$\{a_0, a_1, a_6, a_9\}$

$$\left(\underline{\mathrm{SM}_{C_1+C_2}}(F), \overline{\mathrm{SM}_{C_1+C_2}}(F)\right) = \left(\underline{\mathrm{SM}_{C_1+C_2}}(A), \overline{\mathrm{SM}_{C_1+C_2}}(A)\right) \cup \left(\underline{\mathrm{SM}_{C_1+C_2}}(B), \overline{\mathrm{SM}_{C_1+C_2}}(B)\right) \cdots\cdots \tag{4.2}$$

For $A = \{a_0, a_2, a_6\}, B = \{a_0, a_1, a_6, a_7\}$, all the subsets $E, F \subseteq U$ satisfying Eqs. (4.1) and (4.2) are presented in Table 7.

For $C = \{a_0, a_1, a_2, a_3\}$. Let $G = \{a_0, a_1, a_2, a_3\}, H = \{a_0, a_1, a_2, a_3\}$, then

$$\left(\underline{\mathrm{SM}_{C_1+C_2}}(G), \overline{\mathrm{SM}_{C_1+C_2}}(G)\right) = \left((\underline{\mathrm{SM}_{C_1+C_2}}(A), \overline{\mathrm{SM}_{C_1+C_2}}(A)\right) \cap \left(\underline{\mathrm{SM}_{C_1+C_2}}(B), \overline{\mathrm{SM}_{C_1+C_2}}(B))\right) \cup \left(\underline{\mathrm{SM}_{C_1+C_2}}(C), \overline{\mathrm{SM}_{C_1+C_2}}(C)\right) \cdots \tag{4.3}$$

$$\left(\underline{\mathrm{SM}_{C_1+C_2}}(H), \overline{\mathrm{SM}_{C_1+C_2}}(H)\right) = \left((\underline{\mathrm{SM}_{C_1+C_2}}(A), \overline{\mathrm{SM}_{C_1+C_2}}(A)\right) \cup \left(\underline{\mathrm{SM}_{C_1+C_2}}(B), \overline{\mathrm{SM}_{C_1+C_2}}(B))\right) \cap \left(\underline{\mathrm{SM}_{C_1+C_2}}(C), \overline{\mathrm{SM}_{C_1+C_2}}(C)\right) \cdots \tag{4.4}$$

For $A = \{a_0, a_2, a_6\}, B = \{a_0, a_1, a_6, a_7\}, C = \{a_0, a_1, a_2, a_3\}$, all the subsets $G, H \subseteq U$ satisfying Eqs. (4.3) and (4.4) are given in Table 8.

5 Algebraic theory of MCRSs

In this part, according to the operation results of MCRSs, the algebraic theory of MCRSs will be researched in detail. The relevant concepts of algebra can be consulted in reference (Kong and Xu 2018a).

5.1 Algebraic properties of the first type of MCRSs

In this subsection, according to the operation properties of MCRSs, many basic and important algebraic properties of the first type of MCRSs will be further discussed. Let (U, Ω) be a minimally unary MCAS, and for $\forall a, b \in U$, we have $K_a^{\min} = K_b^{\min}$ or $K_a^{\min} \cap K_b^{\min} = \emptyset$. Then, the following results hold.

Theorem 5.1 $(\mathbb{C}^F, \cup, \cap)$ *is a lattice.*

Theorem 5.2 $(\mathbb{C}^F, \cup, \cap)$*is a distributive lattice.*

Proof For $(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(A)))), (\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(B), \overline{FM_{\sum_{i=1}^m C_i}}(B))$, and $(\underline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(C), \overline{\mathrm{FM}_{\sum_{i=1}^m C_i}}(C)) \in \mathbb{C}^F$, then

Table 8 Subsets G, H

A, B, C	G	H
$\{a_0, a_2, a_6\}$	$\{a_0, a_1, a_2, a_3\}$	$\{a_0, a_1, a_2, a_3\}$
$\{a_0, a_1, a_6, a_7\}$	$\{a_0, a_1, a_2, a_4\}$	$\{a_0, a_1, a_2, a_4\}$
$\{a_0, a_1, a_2, a_3\}$	$\{a_0, a_1, a_2, a_5\}$	$\{a_0, a_1, a_2, a_5\}$
	$\{a_0, a_1, a_2, a_6\}$	$\{a_0, a_1, a_2, a_6\}$
	$\{a_0, a_1, a_3, a_7\}$	$\{a_0, a_1, a_3, a_7\}$
	$\{a_0, a_1, a_4, a_7\}$	$\{a_0, a_1, a_4, a_7\}$
	$\{a_0, a_1, a_5, a_7\}$	$\{a_0, a_1, a_5, a_7\}$
	$\{a_0, a_1, a_6, a_7\}$	$\{a_0, a_1, a_6, a_7\}$
	$\{a_0, a_1, a_3, a_8\}$	$\{a_0, a_1, a_3, a_8\}$
	$\{a_0, a_1, a_4, a_8\}$	$\{a_0, a_1, a_4, a_8\}$
	$\{a_0, a_1, a_5, a_8\}$	$\{a_0, a_1, a_5, a_8\}$
	$\{a_0, a_1, a_6, a_8\}$	$\{a_0, a_1, a_6, a_8\}$
	$\{a_0, a_1, a_3, a_9\}$	$\{a_0, a_1, a_3, a_9\}$
	$\{a_0, a_1, a_4, a_9\}$	$\{a_0, a_1, a_4, a_9\}$
	$\{a_0, a_1, a_5, a_9\}$	$\{a_0, a_1, a_5, a_9\}$
	$\{a_0, a_1, a_6, a_9\}$	$\{a_0, a_1, a_6, a_9\}$

$$
\begin{aligned}
&(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A))\\
&\cap ((\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B)) \cup (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(C), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(C)))\\
&= ((\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \cap (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B)))\\
&\cup ((\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{FM_{\sum_{i=1}^{m} C_i}}(A)) \cap (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(C), \overline{FM_{\sum_{i=1}^{m} C_i}}(C)));\\
&(\underline{FM_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A))\\
&\cup ((\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B)) \cap (\underline{\sum_{i=1}^{m}} C_i(C), \overline{\sum_{i=1}^{m}} C_i(C)))\\
&= ((\underline{FM_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \cup (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B)))\\
&\cap ((\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \cup (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(C), \overline{FM_{\sum_{i=1}^{m} C_i}}(C))).
\end{aligned}
$$

Thus, the proposition holds. □

Theorem 5.3 $(\mathbb{C}^F, \cup, \cap, \sim)$ *is a soft algebra.*

Proof It is immediate by the definition of soft algebra. □

For each $(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \in \mathbb{C}^F$, suppose that

$$(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A))^* = (\sim \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \sim \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)),$$

then the following conclusion holds.

Theorem 5.4 $(\mathbb{C}^F, \cup, \cap, \sim, (\emptyset, \emptyset))$ *is a pseudo-complement lattice.*

Proof For $(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \in \mathbb{C}^F$, then

(1) $(\underline{FM_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \cap (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{FM_{\sum_{i=1}^{m} C_i}}(A))^* = (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A) \cap (\sim \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)), \overline{FM_{\sum_{i=1}^{m} C_i}}(A) \cap (\sim \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A))) = (\emptyset, \emptyset)$.

(2) For $(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)), (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B), \overline{FM_{\sum_{i=1}^{m} C_i}}(B)) \in \mathbb{C}^F$. Let $(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)) \cap (\underline{FM_{\sum_{i=1}^{m} C_i}}(B), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B)) = (\emptyset, \emptyset)$. Then $(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A) \cap \underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A) \cap \overline{FM_{\sum_{i=1}^{m} C_i}}(B)) = (\emptyset, \emptyset)$. We have that $\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A) \cap \underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B) = \emptyset$. In other words, $\overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B) \subseteq \sim \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)$. Since $\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A) \subseteq \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)$, so, $\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B) \subseteq \sim \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)$. Then $(\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(B), \overline{FM_{\sum_{i=1}^{m} C_i}}(B)) \subseteq (\underline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A)), \overline{\mathrm{FM}_{\sum_{i=1}^{m} C_i}}(A))^*$. Thus, the proposition holds.

□

Remark 5.1 In this section, some basic algebraic properties of MCRSs are explored. In fact, there are many other algebraic properties that need to be further studied. For example, is $(\mathbb{C}^F, \cup, \cap, \sim, 0)$ a group? For each element of $\mathbb{C}^F$, what is the inverse? Unfortunately, we are not able to answer these questions.

5.2 Algebraic properties of the second type of MCRSs

In this part, according to the operation results of MCRSs, lots of useful algebraic conclusions of the second type of MCRSs can be investigated. Let (U, Ω) be a maximally unary MCAS. It is easy to see that algebraic theory of the second type of MCRSs is similar to those of the first type of MCRSs. Therefore, algebraic theory of the second type of MCRSs will no longer be repeated here.

6 Conclusion

In this part, we first introduce the main conclusions obtained in our paper. Then, we make further prospects for future research work.

1. *Main conclusions of our paper* The MCRS theory is the meaningful development of classical rough sets. Up to now, many excellent results of MCRSs have been presented. The main conclusions of this paper are to develop the operation theory of MCRSs and then further

explore the algebraic properties of MCRSs. First, to find more excellent results, we researched the properties of the two types of covering-based approximation operators with respect to the minimally (maximally) unary MCAS and got many good properties. In addition, the concepts of intersection and union of MCRSs were initiated. Furthermore, we proved that the two types of MCRSs with respect to minimally and maximally unary coverings are closed under set intersection and union, respectively. At the same time, we also develop two algorithms to compute the intersection and union of MCRSs for its further application. Finally, lots of basic and meaningful algebraic properties of MCRSs are further studied.

2. *Further research work* Clearly, on the basis of algebraic theory of MCRSs, new achievements in further research are needed. For example, only a part of algebraic properties of MCRSs is investigated in this paper. More algebraic properties should be studied. Meanwhile, according to the algebraic properties of MCRSs, we can solve lots of practical problems, such as network security, and neural network. Therefore, these problems need to be solved in the future.

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References

Apolloni B, Bassis S, Rota J, Galliani GL, Gioia M, Ferrari L (2016) A neurofuzzy algorithm for learning from complex granules. Granul Comput 1(4):225–246

Chen D, Wang C, Hu Q (2007) A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets. Inf Sci 177:3500–3518

Chen J, Lin Y, Lin G (2017) Attribute reduction of civering decision systems by hypergraph model. Knowl Based Syst 118:93–104

Dai J, Huang D, Su H (2014) Uncertainty measurement for covering rough sets. J Unc Fuzz Knowl Based Syst 22(2):217–233

D'eer L, Restrepo M, Cornelis C (2016) Neighborhood operators for covering-based rough sets. Inf Sci 336:21–44

Ge X, Wang P, Yun Z (2017) The rough membership functions on four types of covering-based rough sets and their applications. Inf Sci 390:1–14

Iwiński T (1987) Algebraic approach to rough sets. Bull Pol Acad Sci (Math) 35(9–10):673–683

Kong Q, Wei Z (2017) Further study of multi-granulation fuzzy rough sets. J Intell Fuzzy Syst 32:2413–2424

Kong Q, Xu W (2018) The comparative study of covering rough sets and multi-granulation rough sets. Soft Comput. https://doi.org/10.1007/s00500-018-3205-y

Kong Q, Xu W (2018b) Operation properties and algebraic application of covering rough sets. Fundam Inf 160:385–408

Lang G, Li Q, Cai M (2015) Characteristic matrixes-based knowledge reduction in dynamic covering decision systems. Knowl Based Syst 85:1–26

Lang G, Miao D (2016) Knowledge reduction of dynamic covering decision information systems when varying covering cardinalities. Inf Sci 346:236–260

Lang G, Miao D, Cai M, Zhang Z (2017) Incremental approaches for updating reducts in dynamic covering information systems. Knowl Based Syst 134:85–104

Lang G, Cai M, Fujita H, Xiao Q (2018) Related families-based attribute reduction of dynamic covering decision information systems. Knowl Based Syst. https://doi.org/10.1016/j.knosys.2018.05.019

Li D (2002) Algebraic aspects and knowledge reduction in rough set theory, Xi'an Jiaotong University Doctor Paper

Li J, Ren Y, Mei C (2016) A comparative study of multi-granulation rough sets and concept lattices via rule acquisition. Knowl Based Syst 91:152–164

Li J, Huang C, Qi J, Qian Y, Liu W (2017) Three-way cognitive concept learning via multi-granulaity. Inf Sci 378:244–263

Liang M, Mi J, Feng T (2018) Optimal granulation selection for multi-label based on multi-granulation rough sets. Granul Comput. https://doi.org/10.1007/s41066-018-0110-9

Lin G, Qian Y, Li J (2012) NMGRS: neighborhood-based multi-granulation rough sets. Int J Approx Reason 53(7):1080–1093

Lin G, Liang J, Qian Y (2013) Multigranulation rough sets: from partition to covering. Inf Sci 241:101–118

Liu C, Miao D, Qian J (2014) On multi-granulation covering rough sets. Int J Approx Reason 55:1404–1418

Pagliani P (1996) Gough sets and Nelson algebras. Fundam Inf 27:205–219

Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11(5):341–356

Pedrycz W, Chen SM (2011) Granular computing and intelligent systems: design with information granules of higher order and higner type. Springer, Heidelberg

Pedrycz W, Chen SM (2015a) Granular computing and decision-making: interactive and interactive approaches. Springer, Heidelberg

Pedrycz W, Chen SM (2015b) Information granularity, big data, and computational intelligence. Springer, Heidelberg

Polkowski L, Skowron A (1998a) Rough sets and current trends in computing, vol 1424. Springer, Berlin

Polkowski L, Skowron A (1998b) Rough sets in knowledge discovery 1: methodology and applications. Studies in fussiness and soft computing, vol 18. Physica C, Heidelberg (ISBN: 978-3-7908-1884-0)

Polkowski L, Skowron A (1998c) Rough sets in knowledge discovery: applications, case studies and, software systems. Physica C, Heidelberg. https://doi.org/10.1007/978-3-7908-1883-3 (ISBN: 3790811203, 9783790811209)

Pomkala J (1988) On definability in the nondeterministic information system. Bulle Pol Acad Sci Math 36:193–210

Pomy Kala J, Pomy Kala JA (1988) The stone algebra of rough sets. Bulle Pol Acad Sci Math 36(7–8):495–508

Qian Y, Liang J, Yao Y, Dang C (2005) MGRS: a multi-granulation rough set. Inf Sci 180:949–970

Qian Y, Liang J, Wei W (2010) Pessimistic rough decision. In: Second international workshop on rough sets theory, Zhoushan, P.R. China, pp 440-449

Skowron A, Stepaniuk J (1996) Tolerance approximation spaces. Fundam Inf 27:245–253

Slowinski R, Vanderpooten D (2000) A generalized definition of rough approximations based on similarity. IEEE Trans Knowl Data Eng 12:331–336

Wang L, Yang X, Wu C (2013) Multi-covering based rough set model. In: Ciucci D et al (eds) RSFDGrC 2013, LNAI 8170. Springer-Verlag, Berlin, pp 236–244

Wang C, Shao M, Sun B (2015) An improved attribute reduction scheme with covering based rough sets. Appl Soft Comput 26:235–243

Wang G, Yang J, Xu J (2017) Granular computing: from granularity optimization to multi-granularity joint problem solving. Granul Comput 2(3):105–120

Wu W, Zhang W (2006) Rough set approximations vs. measurable spaces. In: IEEE GrC, pp 329-332

Xu W, Zhang W (2007) Measuring roughness of generalized rough sets induced by a covering. Fuzzy Sets Syst 158:2443–2455

Xu W, Sun W, Zhang X, Zhang W (2012) Multile granulation rough set approach to ordered information systems. Int J Gen Syst 41(5):471–501

Xu W, Wang Q, Zhang X (2013) Multi-granulation rough sets based on tolerance relations. Soft Comput 17:1241–1252

Xu W, Guo X (2016) Generalized multigranulation double-quantitative decision-theoretic rough set. Knowl Based Syst 105:190–205

Xu W, Li W, Zhang X (2017) Generalized multigranulation rough sets and optimal granularity selection. Granul Comput. https://doi.org/10.1007/s41066-017-0042-9

Xu Z, Wang H (2016) Managing multi-granularity linguistic information in qualitative group decision making: an overview. Granul Comput 1(1):21–35

Yang X, Song X, Chen Z, Yang J (2012) On multigranulation rough sets in incomplete information system. Int J Mach Learn Cybern 3:223–232

Yang B, Zhu W (2014) A new type of covering-based rough sets, In: 9th International conference on rough sets and knowledge technology, Shanghai, P.R.China, pp 489–499

Yao Y, Lin T (1996) Generalization of rough sets using model logic. Intell Autom Soft Comput Int J 2:103–120

Yao Y (1998) Relational interpretations of neighborhood operators and rough set approximation operators. Inf Sci 101:239–259

Yao Y (2003) On generalizing rough set theory. In: Proceeding of the ninth international conference on rough sets, fuzzy sets, data mining and granular computing LNCS(LNAI) 2639, pp 44-51

Yao Y, Chen Y (2005) Subsystem based generalizations of rough set approximations. LNCS 3488:210–218

Yao Y, Yao B (2012) Covering based rough set approximations. Inf Sci 200(1):91–107

Yao Y, She Y (2016) Rough set models in multigranulation spaces. Inf Sci 327:40–56

Zakowski W (1983) Approximations in the space (u, π). Demonstr Math 16:761–769

Zhang N, Yao Y, Ohshima M (2003) Pecularity oriented multidatabase mining. IEEE Trans Knowl Data Eng 15(4):952–960

Zhu W (2007) Generalized rough sets based on relations. Inf Sci 177(22):4997–5001

Zhu W, Wang F (2006) Covering based granular computing for conflict analysis. In: IEEE ISI vol 3975 of LNCS, pp 566-571

Zhu W, Wang S (2011) Matroidal approaches to generalized rough sets based on relations. Int J Mach Learn Cybern 2(4):273–279