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METHODOLOGIES AND APPLICATION

Local multigranulation decision-theoretic rough set in ordered information systems

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Abstract

As a generalized extension of Pawlak's rough set model, the multigranulation decision-theoretic rough set model in ordered information systems utilizes the basic set assignment function to construct probability measure spaces through dominance relations. It is an effective tool to deal with uncertain problems and widely used in practical decision problems. However, when the scale of dataset is large, it takes a lot of time to characterize the approximations of the target concept, as well as complicated calculation processes. In this paper, we develop a novel model called local multigranulation decision-theoretic rough set in an ordered information system to overcome the above-mentioned limitation. Firstly, to reduce the computing time of the information granule independent of the target concept, we only use the characterization of the elements in the target concept to approximate this target concept. Moreover, the corresponding local multigranulation decision-theoretic rough set in an ordered information system is addressed according to the established local model, and the comparisons are made between the proposed local algorithm and the algorithm of original multigranulation decision-theoretic rough set in ordered information systems. Finally, the validity of the local approximation operators is verified through the experimental evaluation using six datasets coming from the University of California-Irvine (UCI) repository.

Keywords Multigranulation decision-theoric rough set · Probabilistic rough set · Local rough set · Ordered information systems

1 Introduction

Rough set theory (RST) (Pawlak 1982), pioneered by the Polish scientist Pawlak Z. in 1982, is a valid paradigm in mathematics to tackle the imprecise, uncertain and tremendous information in intelligent systems (Pawlak 1992). It analyzes the data and extracts hidden knowledge from it, revealing its potential rules. It is also an objective and effective method of data mining. Rough set theory is based on the classification mechanism. It links knowledge to classification and considers knowledge to be the ability to classify objects. The main idea is to approximate inaccurate or uncertain knowledge by using knowledge known in the knowledge base. The most striking difference between this theory and other theories dealing with uncertain and imprecise problems is that it does not need to provide any preliminary or additional information beyond the dataset, so it can be said that the description or treatment of the uncertainty of the problem can be said to be relatively objective. Ever since the inception of RST, it has been successfully applied in many categories such as machine learning, data mining, knowledge discovery in databases, pattern recognition, granular computing and expert systems (Jeon et al. 2016; Duntsh and Gediga 1998; Hu and Cercone 1995; Li et al. 2018; Qian et al. 2014; Pedrycz 2013). However, the original rough set theory is not able to find and deal with inconsistencies from consideration of criteria, that is, attributes with preference-ordered domains, such as test score, product quality, market share and debt ratio. In order to solve this problem, Greco and others put forward that the rough set method is used to sort the attributes under the dominance relation, that is, the extension of the classical rough set theory. It is referred as

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the dominance-based rough set approach (DRSA) (Du and Hu 2017; Greco et al. 2002). Moreover, the DRSA takes into consideration monotonic relationships between descriptions of objects on condition criteria and their class labels (Du and Hu 2016; Li and Li 2015; Shao and Zhang 2005; Susmaga 2014; Zhang et al. 2013). Since its inception, DRSA has been extended to cope with knowledge acquisition in various types of ordered information systems.

After more than 30 years of development, the rough set theory has been applied to many fields and have achieved remarkable results (Xu and Yu 2017; Li et al. 2018; Yu and Xu 2017; Xu and Li 2016; Xu 2013). Pawlak and Wong Pawlak et al. (1988) proposed a probabilistic rough set of conditional probability based on the precision of 0.5. Ziarko discusses the variable precision rough sets based on the error classification rate (Ziarko 1993). Then, Yao introduces rough membership function based on conditional probability into rough set and gives a unified framework for probabilistic rough sets (Yao and Wong 1992; Yao 2008) and discusses generalized probabilistic rough set model (Yao 1998). On the basis of the decision theory rough set model, Ma and Sun (2012) give a probabilistic rough model under the generalized relation in the dual universes. Based on the rough set and two classification problem, Yao proposed three-way decisions theory (Yao 2009, 2007) and gave three domains of classification: the acceptance domain, the rejection region and the disclaimer region (delay decision). Three-way decisions have more one option than the two-way decisions, which is that delay decision. The idea of three-way decisions was proposed based on rough sets and probabilistic rough sets. Greco et al. (2007) discussed a Bayesian decision theory for dominance-based rough set model in 2007. Yao and Zhou (2010) in 2010 put forward Naive Bayesian decision theory rough set model. Li and Xu (2015) studied three multigranulation decision-theoretic rough set models in ordered information systems. They constructed probability measure spaces based on dominance relations by the basic set assignment function (Xu et al. 2010). Yu et al. (2018) combined absolute and relative quantization to construct a double quantization decision theory rough set model in multigranulation approximation space. Qian et al. proposed a generalized multigranular sequential three-way decision model based on multiple different thresholds, which overcomes the situation that the traditional model cannot adapt to multiview granular structure with multiple thresholds (Qian et al. 2019). Recently, the three-way decision based on rough sets has been widely studied and extended and has been applied in many fields, such as investment decision (Liu et al. 2011), government decision (Liu et al. 2012), text classification (Li et al. 2010), cluster analysis (Yu et al. 2014) information recognition (Li et al. 2016) and other aspects (Fang and Min 2019; Zhang and Miao 2017; Chen et al. 2016; Liang et al. 2015, 2016; Sun et al. 2016). Not only in the field of rough sets, but also in recent years, many academic studies have been well developed. For solving NP-complete problems with multiple objectives, Bansal et al. proposed a nature-inspired-based multiobjective optimization algorithm to find the Optimal Golomb rulers in a reasonable time frame (Bansal and Sharma 2018; Bansal 2018). Moreover, Bansal also presented an approach to find near-optimal Golomb ruler sequences based on nature-inspired algorithms (Bansal et al. 2017a, b).

From the point of view of granular computing, the existing approximation concepts of the decision-theoretic rough set model are induced by a single relation on the universe, such as equivalence relations, tolerance relations and dominance relations. However, due to various needs of different people, the concept of approximation is described by using multiple binary relationships. Based on the above, Qian combined the Bayesian decision theory with multigranulation rough set (Qian and Liang 2006) and put forward multigranulation decision-theoretic rough set (Qian et al. 2014). In the original method, all objects need to be calculated to obtain knowledge granules, which is computationally intensive and time-consuming. Thus, Qian et al. put forward a local rough set based on inclusion degree, which only needs to compute the information granules of the target set to obtain the approximation of the target set. It can effectively reduce the interference of invalid information and reduce the time loss caused by calculating all objects. Furthermore, Qian et al. (2017) made full use of the three-way decisions idea and the granular computing idea, and further proposed a local decision-theoretic rough set model based on multigranulation, which provided a fast and convenient method for decision analysis. However, the above research contents are all based on equivalence relations, which are too strict and have great limitations. Therefore, this paper chooses dominance relation instead of indiscernibility relation to build the local rough set model of ordered information systems. Next, the decision making problem is considered using the theory of three-way decisions from a quantitative perspective. Combined with the probability rough set, we would establish a local multigranulation decision-theoretic rough set based on dominance relation. Compared with global (original) multigranulation decision-theoretic rough set model, both need to calculate all dominance classes. But after constructing the approximation space, the local model only need categorize the elements in the target set without considering the classification of all objects. Hence, the local model is far less than the time loss of the global model in terms of time consumption.

The rest of this article is arranged as follows. In order to facilitate discussion, Sect. 2 briefly reviews decision-theoretic rough sets (DTRS) and multigranulation rough sets in ordered information systems. In Sect. 3, the probability approximation space is constructed by using the basic set assignment function, and the local rough set based on dominance relation is established. Furthermore, a fast and time-saving approach to approximate concept of target set is proposed, which is local multigranulation decision-theoretic rough set model. At the same time, two algorithm models are established in Sect. 4, which are the global multigranulation decision-theoretic rough set model and the local multigranulation decision-theoretic rough set model. In the above two models, experimental analysis is carried out to compare the time loss of the concept approximation under different datasets. Finally, Sect. 5 summarizes the work of this paper and puts forward some suggestions for further study.

2 Preliminary

In this section, we will first review some basic concepts and notions in the theory of Pawlak rough set, probabilistic approaches to rough set theory, the decision-theoretic rough set based on Bayesian decision theory and the multigranulation rough set in an ordered information system.

2.1 The rough set in ordered information systems

Definition 2.1 An information system is a triple $I = (U, AT, F)$, where

- $U = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite universe.
- $AT = \{a_1, a_2, \ldots, a_m\}$ is a finite nonempty set of attributes.
- $F = \{f_j | j \leq m\}$ is a set of relationship between U and AT, in which $f_j : U \rightarrow V_j (j \leq m)$ is a total function such that $f_j(x) \in V_j$ for each $a_j \in AT, x \in U$.

In an information system, an attribute is a criterion if the domain of an attribute is partial ordered by a decreasing or increasing preference. An information system is an ordered information system if all attributes are criteria and we denote it as $I^{\geq} = (U, AT, F)$. Each nonempty subset $B \subseteq AT$ determines a preordered relation, which is defined as $R_{\mathrm{B}}^{\geq} = \{(x, y) \in U \times U | f_a(y) \geq f_a(x), \forall a \in B\}$.

According to the relation $R_{\mathrm{B}}^{\geq}$, the universe U is divided into some dominant classes given by $U/B^{\geq} = \{[x]_{\mathrm{B}}^{\geq}, x \in U\}$, where $[x]_{\mathrm{B}}^{\geq} = \{y \in U | (x, y) \in [x]_{\mathrm{B}}^{\geq}\}$.

For an arbitrary subset X of U, $B \subseteq AT$, the lower and upper approximations of X in the ordered information system$I^{\geq}$ are, respectively, defined as follows (Greco et al. 2002).

$$\underline{R_{\mathrm{B}}^{\geq}}(X) = \{x \in U | [x]_{R_{\mathrm{B}}}^{\geq} \subseteq X\}$$
$$\overline{R_{\mathrm{B}}^{\geq}}(X) = \{x \in U | [x]_{R_{\mathrm{B}}}^{\geq} \cap X \neq \emptyset\}$$

If $\underline{R_{\mathrm{B}}^{\geq}}(X) \neq \overline{R_{\mathrm{B}}^{\geq}}(X)$, then we call X as a rough set in this ordered information system. And $\mathrm{pos}(X) = \underline{R_{\mathrm{B}}^{\geq}}(X)$, $\mathrm{neg}(X) =\sim \overline{R_{\mathrm{B}}^{\geq}}(X)$, $\mathrm{bnd}(X) = \overline{R_{\mathrm{B}}^{\geq}}(X) - \underline{R_{\mathrm{B}}^{\geq}}(X)$ are called the positive region, negative region, and boundary region of X, respectively.

2.2 Decision-theoretic rough set based on ordered information systems

In the classical rough set, the positive region is based on the algebraic inclusion relationship, so it can not reflect the tolerance of the concept. To overcome this problem, we knew the probabilistic rough set. And then combining the probabilistic rough sets and decision making, we had decision-theoretic rough sets model proposed by Yao and Wong (1992). That is a way to make decisions under minimum Bayesian expectation risk. Based on the idea of three-way decisions, decision-theoretic rough sets use a state set Ω and an action set A to describe the decision making process. For any subset X of U, we have two states given $\Omega = \{X, X^C\}$ and three actions $A = \{a_{\mathrm{P}}, a_{\mathrm{N}}, a_{\mathrm{B}}\}$, which, respectively, represent the three actions about deciding $x \in \mathrm{pos}(X)$, deciding $x \in \mathrm{neg}(X)$, deciding $x \in \mathrm{bnd}(X)$. The loss function regarding the risk or cost of actions in different states is given by $\lambda_{\mathrm{PP}}, \lambda_{\mathrm{BP}}, \lambda_{\mathrm{NP}}, \lambda_{\mathrm{PN}}, \lambda_{\mathrm{BN}}, \lambda_{\mathrm{NN}}$, where $\lambda_{\mathrm{PP}}, \lambda_{\mathrm{BP}}, \lambda_{\mathrm{NP}}$ represent the losses taking three actions, respectively, when an object belongs to X, and λ_{PN}, λ_{BN}, λ_{NN} indicate the losses when an object is not in X. Given the loss function with equivalence relation R, the expected loss associated with taking the individual actions for the objects in $[x]_R$ can be expressed as:

$$R(a_{\mathrm{P}}|[x]_R) = \lambda_{\mathrm{PP}} P(X|[x]_R) + \lambda_{\mathrm{PN}} P(X^C|[x]_R)$$
$$R(a_{\mathrm{B}}|[x]_R) = \lambda_{\mathrm{BP}} P(X|[x]_R) + \lambda_{\mathrm{BN}} P(X^C|[x]_R)$$
$$R(a_{\mathrm{N}}|[x]_R) = \lambda_{\mathrm{NP}} P(X|[x]_R) + \lambda_{\mathrm{NN}} P(X^C|[x]_R)$$

where $P(X|[x]_R) = |X \cap [x]_R| / |[x]_R|$ represents condition probability of x with respect to X and $P(X^C|[x]_R) = 1 - P(X|[x]_R)$, $|\bullet|$ denotes the cardinality of a set.

By means of Bayesian decision procedure, when it satisfies conditions $\lambda_{\mathrm{PP}} \leq \lambda_{\mathrm{BP}} < \lambda_{\mathrm{NP}}$, $\lambda_{\mathrm{NN}} \leq \lambda_{\mathrm{BN}} < \lambda_{\mathrm{PN}}$, minimum-risk decision rules are followed as:

(P) If $P(X|[x]_R) \geq \alpha$ and $P(X|[x]_R) \geq \gamma$, decide $x \in \mathrm{pos}(X)$;
(B) If $P(X|[x]_R) \leq \alpha$ and $P(X|[x]_R) \geq \beta$, decide $x \in \mathrm{bnd}(X)$;
(N) If $P(X|[x]_R) \geq \beta$ and $P(X|[x]_R) \leq \gamma$, decide $x \in \mathrm{neg}(X)$.

where parameters α, β and γ are defined as:

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}$$
$$\beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}$$
$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}$$

When a loss function with $\lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}, \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$ also satisfies the condition:

$$(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) \geq (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$$

then $0 \leq \beta < \gamma < \alpha \leq 1$. And the DTRS has the following decision rules:

(P) If $P(X|[x]_R) \geq \alpha$, decide $x \in \text{pos}(X)$
(B) If $\beta < P(X|[x]_R) < \alpha$, decide $x \in \text{bnd}(X)$
(N) If $P(X|[x]_R) \leq \beta$, decide $x \in \text{neg}(X)$

Meanwhile, we can get the upper and lower approximations of X based on the DTRS model:

$$\overline{\text{apr}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\}$$
$$\underline{\text{apr}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\}$$

If $\underline{\text{apr}}_{(\alpha,\beta)}(X) = \overline{\text{apr}}_{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is rough. If $\alpha = 1, \beta = 0$, then $\overline{\text{apr}}_{(\alpha,\beta)}(X) = \overline{\text{apr}}(X)$, $\underline{\text{apr}}_{(\alpha,\beta)}(X) = \underline{\text{apr}}(X)$. Therefore, the DTRS model is a generalization of Pawlak's model.

Finally, $\text{pos}_{(\alpha,\beta)}(X) = \underline{\text{apr}}_{(\alpha,\beta)}(X)$, $\text{neg}_{(\alpha,\beta)}(X) =\sim \overline{\text{apr}}_{(\alpha,\beta)}(X)$, $\text{bnd}_{(\alpha,\beta)}(X) = \overline{\text{apr}}_{(\alpha,\beta)}(X) - \underline{\text{apr}}_{(\alpha,\beta)}(X)$ are, respectively, the positive region, negative region and boundary region of X.

In Pawlak's rough set theory, the lower and upper approximation operators can partition the universe U into three disjoint sets, namely positive region, negative region and boundary region. Using the conditional probability $P(X|[x]_R)$, the Bayesian decision procedure can choose how to assign x into these three regions. But in the ordered information system $I^{\geq} = (U, AT, F)$, preordered relations are not same with equivalence relations, which cannot be constructed the probability measure space. To solve this problem, we use an operator to handle the dominance classes and construct a probability measure space.

Definition 2.2 (Xu et al. 2010) Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$. $R_A^{\geq}$ is a dominance relation in $I^{\geq}$. The basic set assignment function h is defined as

$$h(X) = \{x \in U | [x]_{R_A}^{\geq} = X\}, X \in 2^U$$

Apparently, $x \in h(X) \Leftrightarrow [x]_{R_A}^{\geq} = X$.

Meanwhile, $h([x]_{R_A}^{\geq})$ satisfies the following properties:

1. $\bigcup_{X \subseteq U} h(X) = U$;
2. If $X \neq Y$, then $h(X) \bigcap h(Y) = \emptyset$.

Obviously, the function $h([x]_{R_A}^{\geq})$ is the universe partition into equivalence classes. Thus, in the ordered information system, this way can induce probability measure approximation space.

Definition 2.3 (Li and Xu 2015) Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$, $R_A^{\geq}$ is a dominance relation in $I^{\geq}$. And $0 \leq \beta \leq \alpha \leq 1$. For every $X \subseteq U$, the lower and upper approximation based on parameters α, β with respect to relation $R_A^{\geq}$ are defined as follows.

$$\underline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|h([x]_{R_A}^{\geq})) \geq \alpha\}.$$
$$\overline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|h([x]_{R_A}^{\geq})) > \beta\}.$$

If $\underline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \overline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a rough set.

Therefore, the probabilistic positive, negative and boundary regions are

$$\text{POS}(X) = \underline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|h([x]_{R_A}^{\geq})) \geq \alpha\}$$
$$\text{NEG}(X) = U - \overline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|h([x]_{R_A}^{\geq})) \leq \beta\}$$
$$\text{BND}(X) = \overline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) - \underline{\text{hpr}}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U |\ \beta < P(X|h([x]_{R_A}^{\geq})) < \alpha\}$$

2.3 Multigranulation rough set in ordered information systems

Definition 2.4 Let $I^{\geq} = (U, AT, F)$ is an ordered information system, $R_i (i = 1, 2, \ldots, m)$ is a dominance relation in U, and $X \subseteq U$. The optimistic lower and upper approximations of the set X with respect to R_i are defined, respectively,

$$\underline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X) = \left\{x \in U \mid \bigvee_{i=1}^{m} ([x]_{R_i}^{\geq} \subseteq X)\right\}$$
$$\overline{OM_{\sum_{i=1}^m R_i^{\geq}}}(X) = \left\{x \in U \mid \bigwedge_{i=1}^{m} ([x]_{R_i}^{\geq} \cap X \neq \emptyset)\right\}$$

where "$\bigvee$", "$\bigwedge$" represented, respectively, "and", "or". And $[x]_{R_i}^{\geq} = \{y | (x, y) \in R_i^{\geq}\}$, $R_i^{\geq}$ is a dominance relation with respect to the attribute set $R_i^{\geq}$.

If $\underline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \neq \overline{OM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, then we call that X is the optimistic multigranulation rough set with respect to dominance relations.

Definition 2.5 Let $I^{\geq} = (U, AT, F)$ is an ordered information system, $R_i (i = 1, 2, \ldots, m)$ is a dominance relation in U, and $X \subseteq U$. The pessimistic lower and upper approximations of the set X with respect to R_i are

$$\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{ x \in U \mid \bigwedge_{i=1}^{m} ([x]_{R_i}^{\geq} \subseteq X) \right\}$$
$$\overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) = \left\{ x \in U \mid \bigvee_{i=1}^{m} ([x]_{R_i}^{\geq} \cap X \neq \emptyset) \right\}$$

If $\underline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X) \neq \overline{PM_{\sum_{i=1}^{m} R_i^{\geq}}}(X)$, then we call that X is the pessimistic multigranulation rough set with respect to dominance relations.

In ordered information systems, combining multigranulation with decision-theoretic rough set model is to become multigranulation decision-theoretic rough set model based on dominance relation, which is referred to as global multigranulation decision-theoretic rough set (G-DTRS). In the following, we introduce global optimistic multigranulation decision-theoretic rough set and global pessimistic multigranulation decision-theoretic rough set.

Definition 2.6 (Li and Xu 2015) Let $I^{\geq} = (U, AT, F)$ is an ordered information system, $R_1, R_2, \ldots, R_m \subseteq AT$ are dominance relations in U and m granularity structures. For any $X \subseteq U$, the global optimistic multigranulation lower and upper approximations of the set X with respect to R_i are

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X) = \{x \in U | \bigvee_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) \geq \alpha\}$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X) = U - \{x \in U | \bigwedge_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) \leq \beta\}$$

Definition 2.7 (Li and Xu 2015) Let $I^{\geq} = (U, AT, F)$ is an ordered information system, $R_1, R_2, \ldots, R_m \subseteq AT$ are dominance relations in U and m granularity structures. For any $X \subseteq U$, the global pessimistic multigranulation lower and upper approximations of the set X with respect to R_i are

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X) = \{x \in U | \bigwedge_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) \geq \alpha\}$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X) = \{x \in U | \bigvee_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) > \beta\}$$

By the lower approximation $\underline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X)$, $\underline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X)$ and the upper $\overline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X)$, $\overline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X)$, the optimistic multigranulation positive, negative and boundary regions are defined by

$$\mathrm{pos}^{O}(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X)$$
$$\mathrm{neg}^{O}(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X)$$
$$\mathrm{bnd}^{O}(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}^{O}_{(\mathrm{G-DTRS})}(X)$$

Similarly, the pessimistic multigranulation positive, negative and boundary regions are defined by

$$\mathrm{pos}^{P}(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X)$$
$$\mathrm{neg}^{P}(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X)$$
$$\mathrm{bnd}^{P}(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(\mathrm{G-DTRS})}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(G-DTRS)}(X)$$

3 Local multigranulation decision-theoretic rough set in ordered information systems

3.1 Local multigranulation decision-theoretic rough set based on dominance relations

In real life, obtaining the required information $X \in U$ in huge data need handle a number of multifarious screening. A target concept X is often described by upper and lower approximations. The global rough set requires the relationship between each information granule and the target set X, and the amount of computation is large and time-consuming. The local rough set does not consider the information granules $\{ [x] \mid [x] \cap X = \emptyset\}$ that are independent of the target set X, and only filters the information particles related to the target set. Therefore, it is very useful for rough data analysis based on large-scale dataset.

Definition 3.1 Let $I^{\geq} = (U, AT, F)$ be an ordered information system and $\mathcal{D}$ be an inclusion degree defined on $P(U) \times P(U)$. For any $X \subseteq U$, the local α-lower and β-upper approximations are defined as

$$\underline{R^{\geq}_{(L,\alpha)}}(X) = \{ x \mid \mathcal{D}(X/[x]^{\geq}_{R}) \geq \alpha, x \in X\}$$
$$\overline{R^{\geq}_{(L,\beta)}}(X) = \cup \{ [x]^{\geq}_{R} \mid \mathcal{D}(X/[x]^{\geq}_{R}) > \beta, x \in X\}$$

where $0 \leq \beta < \alpha \leq 1$, and

$$\mathcal{D}(X/[x]^{\geq}_{R}) = \frac{|X \cap [x]^{\geq}_{R}|}{|[x]^{\geq}_{R}|}$$

The pair is $\langle \underline{R^{\geq}_{(L,\alpha)}}(X), \overline{R^{\geq}_{(L,\beta)}}(X)\rangle$ called local rough set based on ordered information system.

In particular, if $\alpha = 1$ and $\beta = 0$, then local rough set is transformed into a classic rough set based on preordered relation. So, we can take classic rough set based on preordered relation as a type of global rough set in ordered information systems. Compared with the global rough set in ordered information system, the computation of the upper / lower approximation of the local rough set is based only on the information granules determined by the objects in the target concept, rather than all of objects in a given universe. Therefore, it can greatly reduce the time consumption in the process of calculation approximation. So, $\underline{R^{\geq}_{(L,\alpha)}}(X)$ and $\overline{R^{\geq}_{(L,\beta)}}(X)$ are, respectively, called the local lower approximation and local upper approximation of X based on ordered information.

Remark The local upper approximation in the above is not defined according to the set of points that satisfy the previous conditions, but is defined by the union of all the dominance classes $[x]^{\geq}_{R}$ satisfying the conditions. We know that the local lower and upper approximation elements are taken from the target set $x \in X$. If it is defined in the previous way, the lower and upper approximations are included in the target set X, but the lower and upper approximations of rough sets are used to characterize target sets through approximating. Therefore, the lower approximation is contained in the target set, while the upper approximation should cover the target set. Thus, based on an ordered information we take advantage of the union the dominance classes to define local upper approximation in order to approaching the target set as much as possible. Then, the following simple example to illustrate.

Example 3.1 An ordered information system $I^{\geq} = \{U, AT, F\}$ is shown in Table 1. Suppose that $\alpha = 0.8$, $\beta = 0.45$, $X = \{x_1, x_2, x_4, x_5\}$, and $AT = \{a_1, a_2, a_3, a_4\}$.

According to the definition of local approximation, we first calculate dominance classes of elements of the target set X.

$$[x]^{\geq}_{1} = \{x_1, x_3, x_6\},$$
$$[x]^{\geq}_{2} = \{x_1, x_2, x_3, x_5, x_6\},$$
$$[x]^{\geq}_{4} = \{x_4\},$$
$$[x]^{\geq}_{5} = \{x_5, x_6\}.$$

Table 1 An ordered information system

U	a_1	a_2	a_3	a_4
x_1	1	2	1	3
x_2	1	1	1	3
x_3	1	2	2	3
x_4	2	1	3	1
x_5	3	1	2	3
x_6	3	3	2	3

Then, we adopt the way that the local upper approximation is defined by traditional point set, which means $\overline{R^{\geq}_{(L,\beta)}}(X) = \{x| \ \mathcal{D}(X/[x]^{\geq}_{R}) > \beta, x \in X\}$. Therefore, we have

$$\mathcal{D}(X/[x_1]^{\geq}_{R}) = \frac{1}{3},$$
$$\mathcal{D}(X/[x_2]^{\geq}_{R}) = \frac{3}{5},$$
$$\mathcal{D}(X/[x_4]^{\geq}_{R}) = 1,$$
$$\mathcal{D}(X/[x_5]^{\geq}_{R}) = \frac{1}{2}.$$

Finally, the local upper approximation $\overline{R^{\geq}_{(L,\beta)}}(X) = \{x_2, x_4, x_5\}$ does not cover the target set X. Thus, the definition way of point set is not suitable for the local upper approximation in this paper.

Based on the probabilistic rough set model and DTRS model in ordered information systems, the local DTRS lower and upper approximation under the dominance relation is defined by combining the basic set assignment function $h(X)$.

Definition 3.2 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$, $R^{\geq}_{A}$ is a dominance relation in $I^{\geq}$. And $0 \leq \beta \leq \alpha \leq 1$. For every $X \subseteq U$, the local lower and upper approximation with respect to preordered relation $R^{\geq}_{A}$ are defined as followed

$$\underline{\text{hpr}}^{(\text{L}-\text{DTRS})}_{R^{\geq}_{A}}(X) = \{x \in X | P(X|h([x]^{\geq}_{R_A})) \geq \alpha\}$$
$$\overline{\text{hpr}}^{(\text{L}-\text{DTRS})}_{R^{\geq}_{A}}(X) = \cup\{[x]^{\geq}_{R_A} | P(X|h([x]^{\geq}_{R_A})) > \beta, x \in X\}$$

If $\underline{\text{hpr}}^{(L,\alpha,\beta)}_{R^{\geq}_{A}}(X) \neq \overline{\text{hpr}}^{(L,\alpha,\beta)}_{R^{\geq}_{A}}(X)$, X is called a local rough set based on ordered information systems.

Therefore, the local positive region, local negative region and local boundary region based on preordered relation $R^{\geq}_{A}$ are

$$\mathrm{POS}(X) = \underline{\mathrm{hpr}}_{R_A^{\geq}}^{(\mathrm{L-DTRS})}(X) = \{x \in X \mid P(X|h([x]_{R_A}^{\geq})) \geq \alpha\}$$
$$\mathrm{NEG}(X) = U - \overline{\mathrm{hpr}}_{R_A^{\geq}}^{(\mathrm{L-DTRS})}(X) = \{x \in U \mid P(X|h([x]_{R_A}^{\geq})) \leq \beta\}$$
$$\mathrm{BND}(X) = \overline{\mathrm{hpr}}_{R_A^{\geq}}^{(\mathrm{L-DTRS})}(X) - \underline{\mathrm{hpr}}_{R_A^{\geq}}^{(\mathrm{L-DTRS})}(X)$$

Multigranulation decision-theoretic rough set based on pre-ordered relation is constructed based on a family of probability measure. In an ordered information system, the dominance relation can induce the cover rather than the partition of the universe. The dominant class does not produce a probability measure, and the equivalence class is actually doing this. So, we can use the basic set assignment function $h(X)$ proposed in the second section to construct the probability measure space.

Let $R_i^{\geq}(i = 1, 2, \ldots, m)$ be m coverings of universe$U$ in an ordered information system. For $X \subseteq U$, the lower and upper approximations in a multigranulation rough set approach to ordered information system can be represented as two fusion functions, respectively,

$$\underline{\sum_{i=1}^{m} R_i^{\geq}} = f_l(R_1^{\geq}, R_2^{\geq}, \ldots, R_m^{\geq})$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}} = f_u(R_1^{\geq}, R_2^{\geq}, \ldots, R_m^{\geq})$$

where f_l, f_u are, respectively, called a lower fusion function and a upper fusion function.

According to the Bayesian decision produce, let $\lambda_k(a_i|\omega_j)$ denote the loss or the cost and it is represented taking action a_i when the state ω_j by the dominance relation $R_k^{\geq}$. Let $P(\omega_j|x_k)$ be the conditional probability of an object x_k being in state ω_j, which the object x_k is based on the dominance relation $R_k^{\geq}$. Use the conditional probability $P(\omega_j|x_k)$ and the basic set assignment function $h(X)$, the expected loss associated with taking action a_i is given by

$$R(a_i|x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m}\sum_{j=1}^{s} \lambda_k(a_i|\omega_j) P(\omega_j|x_k)$$

Definition 3.3 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}(i = 1, 2, \ldots, m)$ is a dominance relation in $I^{\geq}$. The basic set assignment function h induced by $R_i^{\geq}$ is defined as

$$h(X) = \{x \in U \mid [x]_{R_i}^{\geq} = X\}, X \in 2^U$$

It is easy to know that $x \in h(X) \Leftrightarrow [x]_{R_i}^{\geq} = X$. Meanwhile, $h([x]_{R_i}^{\geq})$ can form a classification of universe U and transform the nonprobability measure into probability measure space.

In the following, the lower and upper approximation of local optimistic (pessimistic) multigranulation set is introduced, and the decision rules are given under different circumstances.

Definition 3.4 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}(i = 1, 2, \ldots, m)$ is a dominance relation in $I^{\geq}$. $[x]_{Ri}^{\geq}$ is a dominance class induced by $R_i^{\geq}$. For any $X \subseteq U$, parameters α, β satisfied numerical relation $0 \leq \beta < \alpha \leq 1$. The local optimistic multigranulation lower and upper approximations based on a dominance relation $R_i^{\geq}$ are denoted by

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{O}(X) = \left\{x \mid \bigvee_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) \geq \alpha, x \in X\right\}$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{O}(X) = \bigcup_{i=1}^{m} \left\{[x]_{R_i}^{\geq} \mid \bigwedge_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) > \beta, x \in X\right\}$$

where $P(X|h([x]_{R_i}^{\geq}))$ is the conditional probabilistic of the equivalence class $j([x]_{R_i^{\geq}})$ with respect to X.

Then, we have local optimistic multigranulation positive, negative and boundary region of X in the following.

$$\mathrm{pos}^O(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{O}(X)$$
$$\mathrm{neg}^O(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{O}(X)$$
$$\mathrm{bnd}^O(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{O}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{O}(X)$$

Definition 3.5 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_i^{\geq}(i = 1, 2, \ldots, m)$ is a dominance relation in $I^{\geq}$. $[x]_{Ri}^{\geq}$ is a dominance class induced by $R_i^{\geq}$. For any $X \subseteq U$, parameters α, β satisfied numerical relation $0 \leq \beta < \alpha \leq 1$. The local pessimistic multigranulation lower and upper approximations based on a dominance relation $R_i^{\geq}$ are denoted by

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{P}(X) = \left\{x \in X \mid \bigwedge_{i=1}^{m} (P(X|h([x]_{R_i}^{\geq})) \geq \alpha\right\}$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\mathrm{L-DTRS})}^{P}(X) = \bigcup_{i=1}^{m} \overline{\mathrm{hpr}}_{R_i}^{(\mathrm{L-DTRS})}$$

where $P(X|h([x]^{\geq}_{R_i}))$ is the conditional probabilistic of the equivalence class $j([x]_{R_i^{\geq}})$ with respect to X.

Similarly, based on the local pessimistic multigranulation lower and upper approximations, the local pessimistic multigranulation positive, negative and boundary regions of X are defined by

$$\text{pos}^P(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(L-DTRS)}(X)$$

$$\text{neg}^P(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(L-DTRS)}(X)$$

$$\text{bnd}^P(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(L-DTRS)}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}^{P}_{(L-DTRS)}(X)$$

Example 3.2 An ordered information system about the influenza is shown in Table 2, where the universe $U = \{x_1, x_2, \ldots, x_{10}\}$ consists of 10 patients who have 6 clinical features: hyperpyrexia, cough, rhinorrhoea, myodynia, diarrhea, nausea. Therefore we set these six features as six criteria, which are represented by $AT = \{a_1, a_2, \ldots, a_6\}$. The degree of clinical characteristics is represented as None, Slight, Middling, Serious. Suppose $X = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}$ be the set of patients who get suspected the bird flu. And assumes that $\alpha = 0.75$, $\beta = 0.5$.

We set $m = 2$ and granularity structures (dominance relations) are $R_1^{\geq}$, $R_2^{\geq}$, where $R_1^{\geq}$ is a dominance relation formed by three attributes: Hyperpyrexia, Cough, Rhinorrhoea and $R_2^{\geq}$ are made up of three attributes: Myodynia, Diarrhea, Nausea. Later, we need to know which people really get the bird flu, who are possible patients and then exclude unrelated patients. Hence, according to the given Table 1, the local lower and upper multigranulation decision-theoretic rough sets are calculated based on dominance relations $R_1^{\geq}$, $R_2^{\geq}$, and three decisions are obtained.

For the local multigranulation decision-theoretic rough sets based on dominance relations, we only need to obtain dominance classes for these objects from X.

After a computation, we have

$$\begin{aligned}
[x_1]^{\geq}_{R_1} &= [x_6]^{\geq}_{R_1} = [x_7]^{\geq}_{R_1} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_9\},\\
[x_2]^{\geq}_{R_1} &= [x_5]^{\geq}_{R_1} = \{x_2, x_5\},\\
[x_3]^{\geq}_{R_1} &= [x_9]^{\geq}_{R_1} = \{x_3, x_9\},\\
[x_4]^{\geq}_{R_1} &= [x_8]^{\geq}_{R_1} = [x_{10}]^{\geq}_{R_1} = \{x_4, x_8, x_{10}\}.\\
[x_1]^{\geq}_{R_2} &= [x_9]^{\geq}_{R_2} = \{x_1, x_4, x_5, x_8 x_9\},\\
[x_2]^{\geq}_{R_2} &= [x_7]^{\geq}_{R_2} = [x_{10}]^{\geq}_{R_2} = \{x_2, x_7, x_{10}\},\\
[x_3]^{\geq}_{R_2} &= [x_6]^{\geq}_{R_2} = \{x_3, x_6\},\\
[x_4]^{\geq}_{R_2} &= [x_5]^{\geq}_{R_2} = [x_8]^{\geq}_{R_2} = \{x_4, x_5, x_8\}.
\end{aligned}$$

One knows $[x]^{\geq}_{R_i}$ $(i = 1, 2)$ that is formed by coverings of U rather than partitions. According to the basic set assignment function $h([x]^{\geq}_{R_i})$ $(i = 1, 2)$, we can get the equivalence classes of these objects from X.

$$\begin{aligned}
h([x_2]^{\geq}_{R_1}) &= h([x_5]^{\geq}_{R_1}) = \{x_2, x_5\},\\
h([x_4]^{\geq}_{R_1}) &= h([x_8]^{\geq}_{R_1}) = \{x_4, x_8, x_{10}\},\\
h([x_7]^{\geq}_{R_1}) &= \{x_1, x_6, x_7\}, \quad h([x_9]^{\geq}_{R_2}) = \{x_3, x_9\}.\\
h([x_2]^{\geq}_{R_2}) &= h([x_7]^{\geq}_{R_2}) = \{x_2, x_7, x_{10}\},\\
h([x_4]^{\geq}_{R_2}) &= h([x_5]^{\geq}_{R_2}) = h([x_8]^{\geq}_{R_2}) = \{x_4, x_5, x_8\},\\
h([x_9]^{\geq}_{R_2}) &= \{x_1, x_9\}.
\end{aligned}$$

Then the conditional probabilities are shown as following

$$\begin{aligned}
P(X|h([x_2]^{\geq}_{R_1})) &= P(X|h([x_5]^{\geq}_{R_1})) = 1,\\
P(X|h([x_4]^{\geq}_{R_1})) &= P(X|h([x_8]^{\geq}_{R_1})) = \frac{2}{3},\\
P(X|h([x_7]^{\geq}_{R_1})) &= \frac{1}{3}N,\\
P(X|h([x_9]^{\geq}_{R_1})) &= \frac{1}{2}.\\
P(X|h([x_2]^{\geq}_{R_2})) &= P(X|h([x_7]^{\geq}_{R_2})) = \frac{2}{3},\\
P(X|h([x_4]^{\geq}_{R_2})) &= P(X|h([x_5]^{\geq}_{R_2})) = P(X|h([x_8]^{\geq}_{R_2})) = 1,\\
P(X|h([x_9]^{\geq}_{R_2})) &= \frac{1}{2}
\end{aligned}$$

Thus

$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{O}_{(L-DTRS)}(X) = \{x_2, x_4, x_5, x_8\},$$

$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{O}_{(L-DTRS)}(X) = [x_2]^{\geq}_{R_1} \cup [x_4]^{\geq}_{R_1} \cup [x_5]^{\geq}_{R_1} \cup [x_8]^{\geq}_{R_1} \cup [x_2]^{\geq}_{R_2} \cup [x_4]^{\geq}_{R_2} \cup [x_5]^{\geq}_{R_2} \cup [x_8]^{\geq}_{R_2} = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}$$

$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{P}_{(L-DTRS)}(X) = \{x_5\}$$

$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{P}_{(L-DTRS)}(X) = [x_2]^{\geq}_{R_1} \cup [x_4]^{\geq}_{R_1} \cup [x_5]^{\geq}_{R_1} \cup [x_8]^{\geq}_{R_1} \cup [x_2]^{\geq}_{R_2} \cup [x_4]^{\geq}_{R_2}$$

Table 2 An ordered information system

U	a_1	a_2	a_3	a_4	a_5	a_6
x_1	None	Slight	None	Slight	Slight	Middling
x_2	Middling	Serious	Slight	Middling	Middling	Slight
x_3	Slight	Slight	Middling	Middling	None	Serious
x_4	Serious	None	Serious	Slight	Serious	Middling
x_5	Middling	Serious	Slight	Slight	Serious	Middling
x_6	None	Slight	None	Middling	None	Serious
x_7	None	Slight	None	Middling	Middling	Slight
x_8	Serious	None	Serious	Slight	Serious	Middling
x_9	Slight	Slight	Middling	Slight	Slight	Middling
x_{10}	Serious	None	Serious	Middling	Middling	Slight

$$\cup[x_5]^{\geq}_{R_2} \cup [x_7]^{\geq}_{R_2} \cup [x_8]^{\geq}_{R_2}$$
$$= \{x_2, x_4, x_5, x_7, x_8, x_{10}\}$$

Therefore, we can get three decision regions with respect to optimistic and pessimistic cases, respectively:

$$(P^O)\text{pos}^O(X) = \{x_2, x_4, x_5, x_8\}$$
$$(N^O)\text{neg}^O(X) = \{x_1, x_3, x_6, x_9\}$$
$$(B^O)\text{bnd}^O(X) = \{x_7, x_{10}\}$$
$$(P^P)\text{pos}^P(X) = \{x_5\}$$
$$(N^P)\text{neg}^P(X) = \{x_1, x_3, x_6, x_9\}$$
$$(B^P)\text{bnd}^P(X) = \{x_2, x_4, x_7, x_8, x_{10}\}$$

Next, we will explain and analyze the results in two cases:

(1) In accordance with three decision regions of local optimistic multigranulation decision-theoretic rough set in an ordered information system, patients x_2, x_4, x_5, x_8 that belong to positive region do suffer from the bird flu. Then they need to be isolated and start a comprehensive treatment. Patients x_7, x_{10} who belong to boundary region may be infected with the bird flu and need further observation. Patients x_1, x_3, x_6, x_9 that have only common cold and do not infect the bird flu, and no further observation was needed.

(2) In accordance with three decision regions of local pessimistic multigranulation decision-theoretic rough set in an ordered information system, patients x_5 that belong to positive region do suffer from the bird flu. Then they need to be isolated and start a comprehensive treatment. Patients $x_2, x_4, x_7, x_8, x_{10}$ who belong to boundary region may be infected with the bird flu and need further observation. Patients x_1, x_3, x_6, x_9 that have only common cold and do not infect the bird flu, and no further observation was needed.

In the global multigranulation decision-theoretic rough set, we need to calculate the equivalence classes of all the objects from U. By a computation, we have

$$[x_1]^{\geq}_{R_1} = [x_6]^{\geq}_{R_1} = [x_7]^{\geq}_{R_1} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_9\},$$
$$[x_2]^{\geq}_{R_1} = [x_5]^{\geq}_{R_1} = \{x_2, x_5\},$$
$$[x_3]^{\geq}_{R_1} = [x_9]^{\geq}_{R_1} = \{x_2, x_3, x_5, x_9\},$$
$$[x_4]^{\geq}_{R_1} = [x_8]^{\geq}_{R_1} = [x_{10}]^{\geq}_{R_1} = \{x_4, x_8, x_{10}\}.$$
$$[x_1]^{\geq}_{R_2} = [x_9]^{\geq}_{R_2} = \{x_1, x_4, x_5, x_8 x_9\},$$
$$[x_2]^{\geq}_{R_2} = [x_7]^{\geq}_{R_2} = [x_{10}]^{\geq}_{R_2} = \{x_2, x_7, x_{10}\},$$
$$[x_3]^{\geq}_{R_2} = [x_6]^{\geq}_{R_2} = \{x_3, x_6\},$$
$$[x_4]^{\geq}_{R_2} = [x_5]^{\geq}_{R_2} = [x_8]^{\geq}_{R_2} = \{x_4, x_5, x_8\}.$$

Then

$$h([x_1]^{\geq}_{R_1}) = h([x_6]^{\geq}_{R_1}) = h([x_7]^{\geq}_{R_1}) = \{x_1, x_6, x_7\},$$
$$h([x_2]^{\geq}_{R_1}) = h([x_5]^{\geq}_{R_1}) = \{x_2, x_5\},$$
$$h([x_3]^{\geq}_{R_1}) = h([x_9]^{\geq}_{R_2}) = \{x_3, x_9\},$$
$$h([x_4]^{\geq}_{R_1}) = h([x_8]^{\geq}_{R_1}) = h([x_{10}]^{\geq}_{R_1}) = \{x_4, x_8, x_{10}\}.$$
$$h([x_1]^{\geq}_{R_2}) = h([x_9]^{\geq}_{R_2}) = \{x_1, x_9\},$$
$$h([x_2]^{\geq}_{R_2}) = h([x_7]^{\geq}_{R_2}) = h([x_{10}]^{\geq}_{R_2}) = \{x_2, x_7, x_{10}\},$$
$$h([x_3]^{\geq}_{R_2}) = h([x_6]^{\geq}_{R_2}) = \{x_3, x_6\},$$
$$h([x_4]^{\geq}_{R_2}) = h([x_5]^{\geq}_{R_2}) = h([x_8]^{\geq}_{R_2}) = \{x_4, x_5, x_8\}.$$

And

$$P(X|h([x_1]^{\geq}_{R_1})) = P(X|h([x_6]^{\geq}_{R_1})) = P(X|h([x_7]^{\geq}_{R_1})) = \frac{1}{3},$$
$$P(X|h([x_2]^{\geq}_{R_1})) = P(X|h([x_5]^{\geq}_{R_1})) = 1,$$
$$P(X|h([x_3]^{\geq}_{R_1})) = P(X|h([x_9]^{\geq}_{R_1})) = \frac{1}{2},$$
$$P(X|h([x_4]^{\geq}_{R_1})) = P(X|h([x_8]^{\geq}_{R_1})) = P(X|h([x_{10}]^{\geq}_{R_1})) = \frac{2}{3}.$$

$$P(X|h([x_1]^{\geq}_{R_2})) = P(X|h([x_9]^{\geq}_{R_2})) = \frac{1}{2},$$
$$P(X|h([x_2]^{\geq}_{R_2})) = P(X|h([x_7]^{\geq}_{R_2})) = P(X|h([x_{10}]^{\geq}_{R_2})) = \frac{2}{3},$$
$$P(X|h([x_3]^{\geq}_{R_2})) = P(X|h([x_6]^{\geq}_{R_2})) = 0,$$
$$P(X|h([x_4]^{\geq}_{R_2})) = P(X|h([x_5]^{\geq}_{R_2})) = P(X|h([x_8]^{\geq}_{R_2})) = 1.$$

Thus

$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{O}_{(G-DTRS)}(X) = \{x_2, x_4, x_5, x_8\},$$
$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{O}_{(G-DTRS)}(X) = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}$$
$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{P}_{(G-DTRS)}(X) = \{x_5\}$$
$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{P}_{(G-DTRS)}(X) = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}$$

In order to obtain the lower and upper approximations of X for the local multigranulation decision-theoretic rough sets, we just need calculate 12 the basic sets $h(X_i)$ and 12 probability values. However, the corresponding twenty contents need to be computed for the global multigranulation decision-theoretic rough sets. So, we can reduce the computational time with respect to the number of dominance classes. Thus, it is a motivation for our paper.

3.2 The algorithm for computing approximations of MGDTRS in an ordered information system

In the following, the approximate algorithm of the global optimistic multigranulation rough set and the local optimistic multigranulation approximation algorithm will be given. The pessimistic approximation algorithm is similar and can be obtained by analogy, and it is not explained in detail here.

The given Algorithm 1 is a global algorithm for computing the optimistic lower and upper approximations in an ordered information system. First, we set initial values, in which GU represents the global optimistic upper approximation and GL represents the global optimistic lower approximation. In the second step, we calculate the dominance classes of all objects at each granulation, in order to prepare for the construction of approximation space. In the third step, the dominance class is converted into an equivalence class by the basic set assignment function $h(X)$, and all the objects are classified at each granulation by dominance classes of the previous step. Finally, according to the global multigranulation approximation model, the conditional probabilities of all objects are calculated, and the objects satisfying the conditions are put into the lower and upper approximations.

Algorithm 1: The global lower and upper approximations of MGDTRS in an ordered information system.

Input : Given $R_1, R_2, \ldots, R_n \subseteq AT$ n dominance relations on U, α, β, and a target set $X \subseteq U$
Output : The global lower and upper approximation of a target set X.

begin
  1: set $GU \leftarrow \emptyset$, $GL \leftarrow \emptyset$;
  2:**for** $i = 1 : n$ **do**
    for $j = 1 : |U|$ **do**
      compute $[x_j]^{\geq}_{R_i}$; /* the dominance classes of x_j with respect to R_i */
    end
  end
  3:**for** $i = 1 : n$ **do**
    for $j = 1 : |U|$ **do**
      compute $h([x_j]^{\geq}_{R_i})$; /* all of the equivalence classes of $h[x_j]$ based on R_i */
    end
  end
  4:**for** $i = 1 : n$ **do**
    for $j = 1 : |U|$ **do**
      compute $P(X|h([x_j]^{\geq}_{R_i}))$; /* the global optimistic lower and upper approximations */
      if $P(X|h([x_j]^{\geq}_{R_i})) \geq \alpha$ **then**
        $GL \leftarrow GL \cup \{x_j\}$;
      end
      if $P(X|h([x_j]^{\geq}_{R_i})) > \beta$ **then**
        $GU \leftarrow GU \cup \{x_j\}$;
      end
    end
  end
  return : GU, GL;
end

The given Algorithm 2 is a local algorithm for computing the optimistic lower and upper approximations in an ordered information system. First, we set initial values, in which LU represents the local optimistic upper approximation and LL represents the local optimistic lower approximation. Next, as in the second step of the global model, the dominance classes for all objects at each granulation are calculated. Then, at each granulation, only the $h([x_i])$ of elements in the target set needs to be calculated. Finally, according to the local multigranulation approximation model, only the conditional probability of objects in the target set is calculated, and the objects satisfying the conditions are put into the lower and upper approximations.

Since the first step of the local algorithm and the global algorithm is to assign an initial value, its time complexity can be ignored. In the second step, the local algorithm, like the global algorithm, needs to compute all the dominance classes $[x_j]_{R_i^{\geq}}$ at each granularity $R_i (i = 1, 2, \ldots, n)$. Therefore,

Algorithm 2: The local lower and upper approximations of MGDTRS in an ordered information system.

Input : Given $R_1, R_2, \ldots, R_n \subseteq AT$ n dominance relations on U, α, β, and a target set $X \subseteq U$
Output : The local lower and upper approximation of a target set X.

begin
  1: set $LU \leftarrow \emptyset$, $LL \leftarrow \emptyset$;
  2:**for** $i = 1 : n$ **do**
    for $j = 1 : |U|$ **do**
      if $x_j \in X$ **then**
        compute $[x_j]^{\geq}_{R_i}$; /* the dominance classes of x_j with respect to R_i */
      end
    end
  end
  3:**for** $i = 1 : n$ **do**
    for $j = 1 : |U|$ **do**
      if $x_j \in X$ **then**
        compute $h([x_j]^{\geq}_{R_i})$; /* the equivalence classes of $h[x_j]$ based on R_i */
      end
    end
  end
  4:**for** $i = 1 : n$ **do**
    for $j = 1 : |U|$ **do**
      if $x_j \in X$ **then**
        compute $P(X|h([x_j]^{\geq}_{R_i}))$; /* the local optimistic lower and upper approximations */
        if $P(X|h([x_j]^{\geq}_{R_i})) \geq \alpha$ **then**
          $LL \leftarrow LL \cup \{x_j\}$;
        end
        if $P(X|h([x_j]^{\geq}_{R_i})) > \beta$ **then**
          $LU \leftarrow LU \cup \{[x_j]_{R_i^{\geq}}\}$;
        end
      end
    end
  end
  return : LU, LL;
end

the time complexity of the second step on n binary relations is $O(n \times |U|^2)$. In the third step, the local algorithm only needs to calculate $h([x_j]^{\geq}_{R_i})$ of the elements in the target set X, so the complexity of the third step in the local algorithm is $O(n \times |X||U|)$. However, since the global algorithm needs to calculate $h([x_j]^{\geq}_{R_i})$ of all objects on U, the complexity of the third step in the global algorithm is $O(n \times |U|^2)$. Due to $|x| \ll |U|$, the complexity of the local algorithm is much smaller than the complexity of the global algorithm ($O(n \times |X||U|) \ll O(n \times |U|^2)$). Last step, the complexity of computing lower and upper approximations in local algorithm is $O(n \times |X|)$. Nevertheless, the global algorithm requires each object to be compared, so the complexity of the global algorithm in the fourth step is $O(n \times |U|)$. Similarly, $O(n \times |X|) \ll O(n \times |U|)$ in the last step. In conclusion, the time complexity of local algorithm is much less than that of global algorithm after constructing probabilistic approximation space.

4 Experimental analysis

In this section, in order to further illustrate the advantages of the local algorithm in an ordered information system, we carry out a series of experiments to compare time consumption between global algorithm and local algorithm for getting lower and upper approximations by using the three datasets where from the UC Irvine Machine Learning Database Repository (http://archive.ics.uci.edu/ml/datasets.html). Detailed information is shown in Table 3. These experiments are implemented by using MATLAB R2014b and performed on a personal computer with an Intel Core i7-6500, 2.50 GHz CPU, 4.0 GB of memory, and 64-bit Windows 10.

In the experiment, let the original data in each set are divided into ten categories, and fix a target set for each dataset. The first ten percent of each dataset is regarded as the first category of each set, that is first universe, the first 20% of the objects of each dataset is treated as the second universe, and so on. Finally, we get ten universes of each dataset. The horizontal coordinate pertains to ten universes for every set, while the vertical coordinate concerns the computational time. For both global and local algorithms, the first step is to calculate dominance classes of all objects under each relation of granularity, the starting point of the time consumption is as after the first step is finished in order to reflect the difference about algorithms. Therefore, all the charts and data below are recorded and analyzed after the dominance classes have been obtained. In this way, the change of time consumption is compared after making the global into the local. For the completeness of the experiment, we take a pair of parameters ($\alpha = 0.75$, $\beta = 0.5$), and the optimistic and pessimistic approximation are compared under local and global algorithms, respectively. Meanwhile, we preprocessed the data and selected two granular structures in this experiment.

As shown below, Table 4 shows the time results of the six datasets about optimistic approximations when α equals 0.75 and β equals 0.5. All the experimental results of optimistic approximations obtained using the six datasets are shown in Fig. 1. As shown in Fig. 1, the growth trend of the curve of the local algorithm is relatively slow. The time consumption of the global algorithm obviously increases with the increase in the data. By comparison, we can get a much smaller time loss than the global algorithm in the local algorithm.

According to Table 5, the time results about computing pessimistic approximations is shown that local algorithm consumes much less time than the execution time consumed by global algorithm when α equals 0.75 and β equals 0.5.

Table 3 The basic information of datasets

No.	Dataset name	Abbreviation	Objects	Attributes
1	*Statlog (Image Segmentation)*	S(IS)	2310	19
2	*Wine Quality − white*	WQ-w	4898	12
3	*EMG PhysicalAction*	EPA	10,000	8
4	*EEG EyeState*	EES	14,980	15
5	*HTRU2*	HTRU2	17,898	9
6	*Letter Recognition*	LR	20,000	16

Table 4 A comparison of optimistic approximations computational time between Algorithms 1 and 2 (unit is second)

Percentage (%)	S(IS)		WQ-w		EPA		EES		HTRU2		LR	
	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local
10	0.313	0.046	1.529	0.727	5.932	4.350	6.795	2.181	18.105	6.719	6.102	2.732
20	0.855	0.067	2.920	0.881	16.961	6.275	23.683	3.829	53.497	7.611	15.181	2.732
30	1.622	1.045	6.320	1.271	32.810	7.683	49.734	5.532	105.125	9.852	29.051	7.628
40	2.560	1.405	9.945	1.319	55.183	8.713	93.185	7.292	181.085	12.254	42.396	11.247
50	3.732	1.839	14.452	1.520	77.271	10.349	161.030	8.943	261.616	14.158	53.359	13.216
60	5.167	2.089	21.039	1.757	103.683	11.706	213.737	10.917	381.081	16.798	67.474	15.666
70	6.654	2.349	26.736	1.894	141.056	13.384	261.076	12.819	478.790	19.126	81.576	21.555
80	8.345	2.729	34.335	2.125	274.658	16.617	341.038	17.709	604.041	21.226	127.178	23.018
90	10.405	2.833	42.946	2.405	312.085	18.415	441.504	19.734	755.343	26.769	149.302	25.642
100	12.344	3.218	51.182	2.549	417.689	19.528	555.485	21.452	949.194	28.921	240.971	26.871

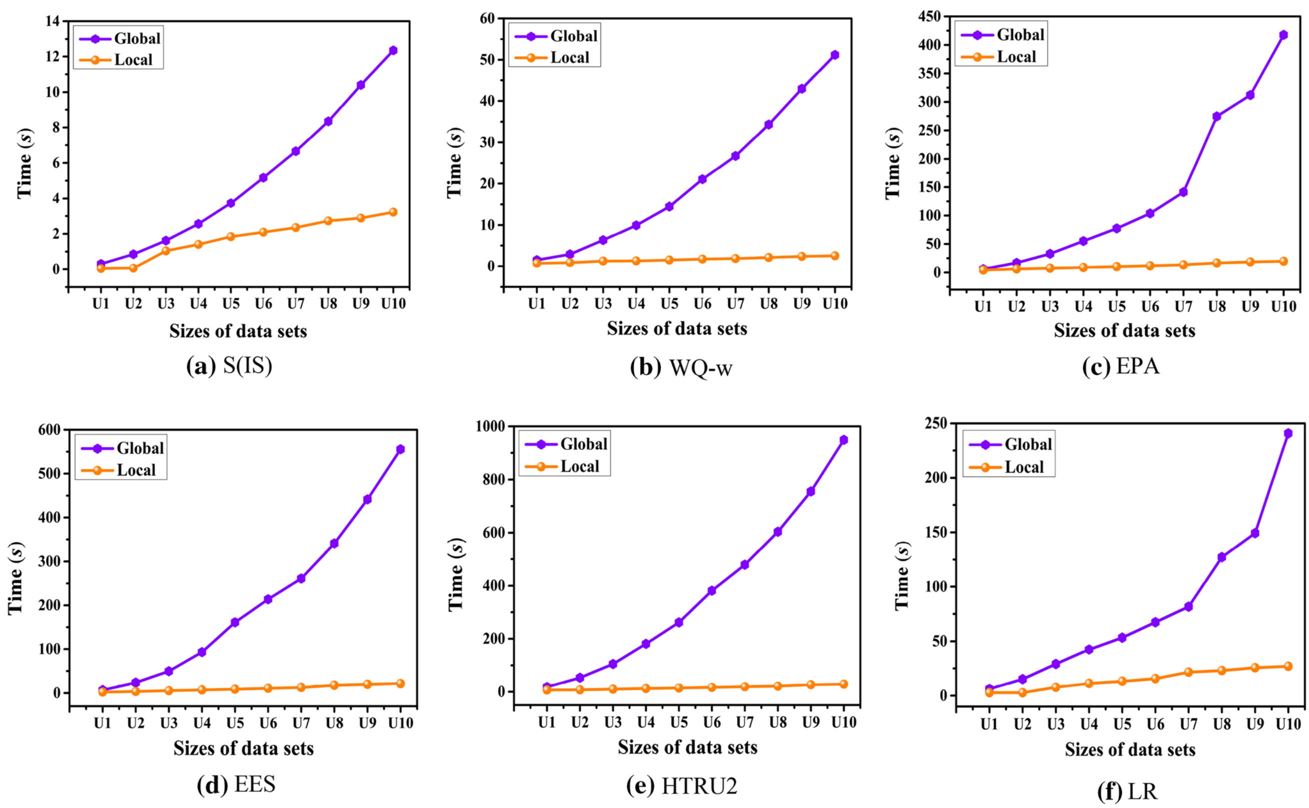

Fig. 1 The computation time comparison of optimistic approximations between local algorithm and global algorithm (α=0.75, β=0.5)

Table 5 A comparison of pessimistic approximations computational time between Algorithms 1 and 2 (unit is second)

Percentage (%)	S(IS)		WQ-w		EPA		EES		HTRU2		LR	
	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local	Global	Local
10	0.321	0.042	1.514	0.727	5.981	4.350	6.795	2.197	18.121	6.719	6.086	2.732
20	0.848	0.061	2.920	0.881	16.946	6.275	23.667	3.829	53.559	7.626	15.118	5.191
30	1.613	0.912	6.332	1.271	32.795	7.668	49.781	5.532	105.172	9.852	29.113	7.628
40	2.498	1.392	9.954	1.319	55.229	8.698	93.169	7.292	181.148	12.254	42.443	11.247
50	3.712	1.804	14.436	1.520	77.239	10.349	161.045	8.747	261.694	14.174	53.389	13.216
60	5.093	2.054	21.070	1.757	103.714	11.751	213.831	10.917	381.222	16.798	67.563	15.666
70	6.598	2.294	26.751	1.894	141.072	13.399	261.139	12.846	478.868	19.127	81.744	21.555
80	8.335	2.712	34.467	2.125	274.642	16.633	340.753	17.726	604.135	21.227	127.241	23.003
90	10.308	2.7930	430.169	2.405	311.944	18.415	441.942	19.734	755.531	26.801	149.390	25.627
100	12.304	3.201	51.303	2.549	417.674	19.544	555.469	21.450	950.147	28.936	241.096	26.871

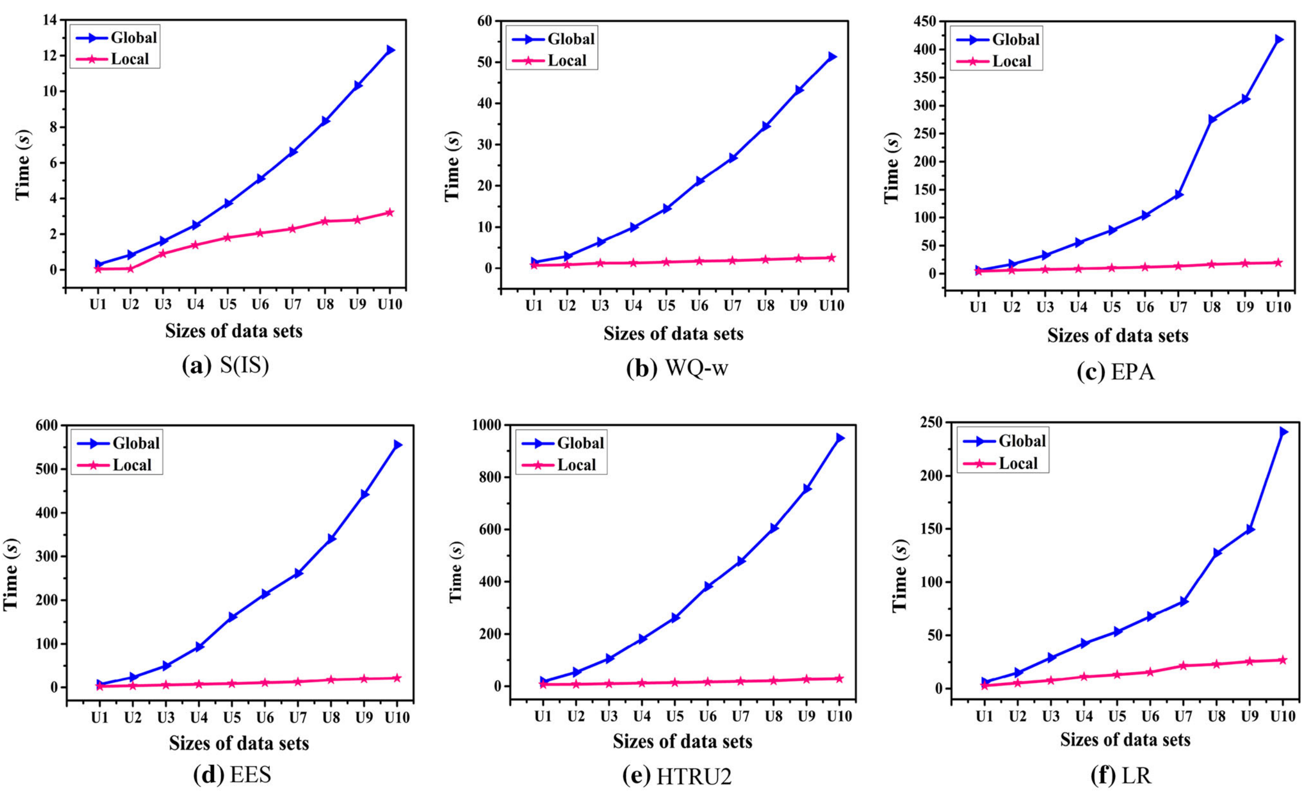

Fig. 2 The computation time comparison of pessimistic approximations between local algorithm and global algorithm (α=0.75, β=0.5)

Similarly, all the experimental results of pessimistic approximations obtained using the six data sets are shown in Fig. 2. From the graph, the local algorithm in all datasets is better than the global algorithm, and the time loss is far less than the corresponding global algorithm. As the scale of the dataset increased, the difference between the time-consuming of the global and the local algorithms is increasing. Therefore, it can be concluded that the local algorithm is better than the global algorithm.

5 Conclusions

In ordered information systems, decision-theoretic rough set divides the information into three decision regions by the minimum Bayesian decision method, which provides a theoretical basis for information screening and classification. But when the scale of dataset is large, it takes a lot of time to characterize the target set and make analysis decisions. Therefore, this paper constructs a local rough set in the ordered information system to reduce time loss. First of all, based on inclusion degree, a local rough set under the dominance relation is defined. Furthermore, combined with the decision-theoretic rough set (DTRS), the probability approximation space is constructed and the local DTRS model is provided. In addition, optimistic and pessimistic local multigranulation decision-theoretic rough set models are, respectively, presented in the viewpoint of granular computing. At the same time, the local algorithm and the global algorithm are given for comparison experiments. The experimental results show that after constructing the approximation space, the local algorithm greatly reduces the time loss compared with the global algorithm and fully proves the advantages of the local model.

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Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

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