Information Sciences 500 (2019) 87–112

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Local logical disjunction double-quantitative rough sets

Yanting Guo[a], Eric C.C. Tsang[a,*], Weihua Xu[b], Degang Chen[c]

[a] Faculty of Information Technology, Macau University of Science and Technology, Taipa, Macau, PR China
[b] School of Mathematics and Statistics, Southwest University, Chongqing 400715, PR China
[c] Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, PR China

ARTICLE INFO

Article history:
Received 12 September 2018
Revised 1 May 2019
Accepted 10 May 2019
Available online 21 May 2019

Keywords:
Decision-theoretic rough sets
Local rough sets
Graded rough sets
Rough approximation

ABSTRACT

Local rough sets as a generalization of classical rough sets not only inherit the advantages of classical rough sets which can handle imprecise, fuzzy and uncertain data, but also break through the limitation of classical rough sets requiring large amount of labeled data. The existing researches on local rough sets mainly use the relative quantitative information between a target concept and equivalence classes of those objects contained in the target concept to approximate the target concept. This ignores the information differences of equivalence classes concerned containing the relevant concept, namely the absolute quantitative information. We propose Local Logical Disjunction Double-quantitative Rough Sets (LLDDRS) model based on the importance, completeness and complementary nature of the relative and absolute quantitative information to describe an approximation space. This provides an effective tool for discovering knowledge and making decisions in relation to large data sets. In this paper we first study the important properties, optimal computing of rough regions and decision rules of the LLDDRS model. Then we explore the relationships of the proposed LLDDRS model and other representative models. Finally, we present experimental comparisons showing the computational efficiency and approximate accuracy of the LLDDRS model in concept approximation.

1. Introduction

With the development of information technology, mass data with a large number of uncertainties and complex types have emerged. This creates higher requirements for data analysis tools. Rough set theory (RS) [20] is a mathematical tool proposed by Pawlak in 1982 to quantitatively analyze imprecise, inconsistent and incomplete information and knowledge. This theory has been widely used in intelligent information processing field such as pattern recognition [27], knowledge discovery [47], uncertainty analysis [17], and so on. Compared with other uncertainty analysis tools, we have found that the most significant advantage of RS is that it does not require any prior information besides the data to be dealt with itself. Thus the description of the uncertainty of the problem using RS will be more objective [30].

In the RS model, the requirement of the set inclusion relation between equivalence classes and an approximated concept is very strict, so that it lacks fault tolerance capabilities. In order to enhance the practicability of the RS model, researchers have extended Pawlak's rough sets from binary relations to approximation sets and inclusion operators. Many extension models are proposed based on the constructive method, such as tolerance rough sets [28], dominance-based rough sets [5],

* Corresponding author.
E-mail address: cctsang@must.edu.mo (E.C.C. Tsang).

https://doi.org/10.1016/j.ins.2019.05.033

fuzzy rough sets and rough fuzzy sets [2], graded rough sets [35], and probabilistic rough sets [29] and so on. In particular, conditional probability is a suitable inclusion measure [34], which can reflect the relative quantitative information between equivalence classes and the approximated concept. Based on conditional probability, various probabilistic rough set models have been proposed. These include 0.5-probabilistic rough sets [22], decision-theoretic rough sets [37], variable precision rough sets [43], parameterized rough sets [21], Bayesian rough sets [26] and game-theoretic rough sets [7]. Meanwhile, Yao and Lin [35] proposed a graded rough set model basing on a non-probabilistic inclusion measure, which mainly reflects the absolute quantitative information between sets. The relative and the absolute quantitative information are two distinctive objective sides that describe an approximation space, and each has its own virtues and application environments so that neither can be neglected. Many double quantification models have been proposed. Zhang et al. [48–50] proposed different double-quantitative rough set models based on logical and crossed combinations of precision and grade. More generally, Yao and Deng [34] proposed a framework of quantitative rough sets based on different classes of subsethood measures which are generalizations of the set inclusion relation. In order to reduce the loss caused by inappropriate decision-making resulting from insufficient information consideration, this paper studies concept approximation and decision theory in large data sets based on relative and absolute quantitative information.

Decision-theoretic rough sets (DTRS) [33] are an outstanding model for describing an approximation space based on relative quantitative information. There are many remarkable theoretical researches on the DTRS model [3,12,14,15,19,23,38,44], and the DTRS theory has been widely used in many fields such as text classification [36], email filtering [45], oil exploitation [16], policy decisions [13], web-based medical decision support systems [46] and E-learning systems [1]. Considering that the completeness of double quantification can reduce the uncertainty of concept description to some extent in complex environments [48–50], many researchers proposed more general decision rough set models based on the DTRS model and absolute quantitative information. Li and Xu [11] proposed two double-quantitative decision-theoretic rough set models based on the crossed combinations of decision-theoretic and graded rough sets. Xu and Guo [32] proposed generalized multi-granulation double-quantitative decision-theoretic rough sets based on the majority decision principle and double-quantitative decision-theoretic rough sets. Fan et al. [4] studied two kinds of double-quantitative rough fuzzy sets based on the logical combinations of decision-theoretic and graded rough sets. Yu et al. [42] studied double-quantitative decision-theoretic approaches based on optimistic, pessimistic and medium risk preferences.

Nowadays, the amount of available data is rapidly increasing and the scale is getting larger and larger. All the above extension models of rough sets, called global rough sets by some scholars, need to consider equivalence classes of all objects in the universe (the whole data set) when calculating upper and lower approximations of a target concept. Therefore, processing with rough data analysis tools based on global rough sets are very time-consuming or even infeasible in large data sets [24]. Moreover, from the perspective of machine learning, these global rough sets often require a large amount of labeled data for knowledge discovery, and so can be regarded as supervised learning methods. However, in massive data environment, labeling data is an expensive and laborious task and sometimes even infeasible.

In order to enhance the feasibility of rough data analysis in large data sets, some scholars have proposed new data analysis methods based on rough set theory [9,10,18,24,25,31,39–41]. Qian et al. [24] first proposed local rough sets (LRS) for large data sets based on the DTRS and RS models, which only consider equivalence classes of objects in the target concept when calculating upper and lower approximations of a target concept. Later, Wang et al. [31] and Qian et al. [25] proposed local neighborhood rough sets and multi-granulation local rough sets, respectively. The main differences between LRS and the corresponding global rough sets are: (1) the time complexity of computing the upper and lower approximations of a concept in LRS is always linear in large data sets, but the computation of approximations in global rough sets is non-linear and extremely time-consuming [24,25,31]. (2) In large data sets, LRS is a semi-supervised learning method to discover knowledge, while the corresponding global rough set model is a supervised learning method to discover knowledge [24,25,31]. (3) The classifier based on LRS may have certain generalization ability when compared to the classifier based on global rough sets [25,31]. In this paper, we mainly study concept approximation and decision theory in large data sets based on local theory and double-quantitative information.

The existing researches on local rough sets only consider relative quantitative information between equivalence classes and the approximated target concept. However, the relative quantitative information merely reflects the intersection relation between equivalence classes and the approximated concept from the point of view of quantity. It ignores the information differences of different equivalence classes containing the target concept. When information in one aspect is not well described by the equivalence relation in approximation spaces, the description of the relative quantitative information will have a certain deviation from the actual state. The relative and absolute quantitative information are often close, complementary and dialectical. These two kinds of information are indispensable in describing an approximation space.

In real life, there are many problems that need to consider relative and absolute quantitative information. Such problems include talent recruitment, resource allocation and company financing and so on [6,11,42,48–50]. For example, in university teacher recruitment, the university would put forward different requirements regarding the scientific research ability of graduating PhD students concerned based on the quality and quantity of their published articles. Therefore, this paper studies concept approximation and decision theory in large data sets based on local idea and double-quantitative information of the set inclusion relation between equivalence classes and the concept to be approximated. In consideration of the graded rough set (GRS) model [35], a representative model for approximating concepts using absolute quantitative information, we propose a Local Logical Disjunction Double-quantitative Rough Set (LLDDRS) model based on local rough sets (LRS) and graded rough sets (GRS). It is useful to combine local rough sets with graded rough sets based on the advantage

Table 1
A loss function.

	$X(P)$	$X^C(N)$
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

of computational efficiency of local rough sets and the ability of more accurate approximation space description of double quantification model.

The main contributions of this paper are as follows: (1) we propose a LLDDRS model as a generalization of classical rough sets. This model has strong double fault tolerance capabilities which can adapt to increasing complex environments and provide more accurate concept descriptions. (2) We propose an effective rough data analysis method for large data sets, which can make full use of information to discover knowledge and make decisions. (3) We provide the expressions of optimal computation for rough regions, and the detailed and understandable decision rules.

The rest of the paper is organized as follows. In Section 2, some basic concepts of rough sets and DTRS, LRS and GRS are briefly introduced. In Section 3, we describe our proposed Local Logical Disjunction Double-quantitative Rough Sets (LLDDRS) model and study its important properties, rough regions and decision rules. At the same time, we design two algorithms to calculate the upper and lower approximations and rough regions of the LLDDRS model, and study the relationships of the LLDDRS model and other models in detail. In Section 4, the LLDDRS theory and the advantages of the LLDDRS model are illustrated by a medical example. In Section 5, the feasibility, necessity and validity of the proposed LLDDRS model for rough analysis on large data sets are verified by numerical experiments from the perspective of computational efficiency and approximation accuracy of concept approximation. Section 6 concludes the paper and elaborates on future studies.

2. Preliminaries

In this section, some basic concepts of rough sets, decision-theoretic rough sets, local rough sets and graded rough sets are briefly introduced. More details can be found in [20,24,33,35,37]. In this paper, we assume that the universe U is a nonempty set, $P(U)$ is a power set which is made up of all subsets of U and R is an equivalence relation on the universe. And X^C denotes the complementary set of set X and the symbol $|\cdot|$ denotes the cardinality of a set.

2.1. Pawlak rough sets (RS)

Let U be a finite universe and R be an equivalence relation on the U, then (U, R) is called an approximation space. For an arbitrary subset $X \subseteq U$, the upper and lower approximations of X are defined as follows [20]:

$\overline{R}(X) = \cup\{[x]_R : [x]_R \cap X \neq \emptyset\} = \{x \in U : [x]_R \cap X \neq \emptyset\}$,

$\underline{R}(X) = \cup\{[x]_R : [x]_R \subseteq X\} = \{x \in U : [x]_R \subseteq X\}$, where $[x]_R$ is the equivalence class of x under the equivalence relation R, which are made up of objects with the same description as x under the relation R, namely $[x]_R = \{y \in U : (x, y) \in R\}$. The equivalence classes are the basic building blocks to construct rough set approximations. Based on the lower and upper approximations, the positive region $pos(X)$, negative region $neg(X)$, and boundary region $bnd(X)$ of X are defined as: $pos(X) = \underline{R}(X)$, $neg(X) = (\overline{R}(X))^C$, $bnd(X) = \overline{R}(X) - \underline{R}(X)$, respectively. Obviously, the three regions form a partition of the universe.

Let $(U, C \cup D)$ be a decision table, where C is a condition attribute set and D is a decision attribute set, which can generate different equivalence relations. The approximation accuracy of C with respect to D in the RS model is defined as $\rho(C, D) = \frac{\sum\{|\underline{R_C}(Y_i)|:Y_i \in U/R_D\}}{\sum\{|Y_i|:Y_i \in U/R_D\}}$, where $U/R_D = \{Y_1, Y_2, \ldots, Y_s\}$ is a partition of U under D and R, R_D denote the equivalence relations of U under C and D, respectively.

2.2. Decision-theoretic rough sets (DTRS)

Decision-theoretic rough sets [33,37] are the relative quantitative generalization of the qualitative rough sets, which have fault tolerance capabilities and enhance the practicability of the RS model. As a special probabilistic generalization model, it makes decisions based on the minimum Bayesian expectation risk.

Let U be the universe, and X be any subset of U. For any object $x \in U$ about an approximated concept X, there are two states. Let $\Omega = \{X, X^C\}$ denote the set of states, which indicates that an object is in a decision class X and not in X. And under different states, there are three actions that can be taken for any object x. The set of actions can be denoted by $A = \{a_P, a_B, a_N\}$, where a_P, a_B and a_N denote the three actions in classifying an object, deciding $pos(X)$, deciding $bnd(X)$, and deciding $neg(X)$, respectively. Let λ_{iP} and λ_{iN} $(i \in \{P, B, N\})$ represent the loss incurred for taking action a_i when an object belongs to X and does not belong to X, respectively. The loss function is shown in Table 1.

Let $P(X|[x]_R)$ and $P(X^C|[x]_R)$ represent the probabilities that an object x in the equivalence class $[x]_R$ belongs to X and X^C, respectively. Base on the loss function, the expected loss associated with taking the individual actions for the objects in $[x]_R$

can be expressed as:

$$R(a_P|[x]_R) = \lambda_{PP}P(X|[x]_R) + \lambda_{PN}P(X^C|[x]_R);$$
$$R(a_B|[x]_R) = \lambda_{BP}P(X|[x]_R) + \lambda_{BN}P(X^C|[x]_R);$$
$$R(a_N|[x]_R) = \lambda_{NP}P(X|[x]_R) + \lambda_{NN}P(X^C|[x]_R).$$

where $P(X|[x]_R) = |X \cap [x]_R|/|[x]_R|$ and $P(X|[x]_R) + P(X^C|[x]_R) = 1$.

We have the following minimum-risk decision rules by using the Bayesian decision procedure:

(P) If $R(a_P|[x]_R) \le R(a_B|[x]_R)$ and $R(a_P|[x]_R) \le R(a_N|[x]_R)$, decide $pos(X)$;

(B) If $R(a_B|[x]_R) \le R(a_P|[x]_R)$ and $R(a_B|[x]_R) \le R(a_N|[x]_R)$, decide $bnd(X)$;

(N) If $R(a_N|[x]_R) \le R(a_P|[x]_R)$ and $R(a_N|[x]_R) \le R(a_B|[x]_R)$, decide $neg(X)$.

In actual situations, we always assume that the loss of classifying an object x belonging to X into the positive region $pos(X)$ is less than or equal to the loss of classifying x into the boundary region $bnd(X)$, and both of these losses are strictly less than the loss of classifying x into the negative region $neg(X)$. The reverse order of losses is used for classifying an object that does not belong to X. The details are $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$. In this case, the decision rules can be expressed as

(P) If $P(X|[x]_R) \ge \alpha$ and $P(X|[x]_R) \ge \gamma$, decide $pos(X)$,

(B) If $P(X|[x]_R) \le \alpha$ and $P(X|[x]_R) \ge \beta$, decide $bnd(X)$,

(N) If $P(X|[x]_R) \le \beta$ and $P(X|[x]_R) \le \gamma$, decide $neg(X)$, where parameters $\alpha,\ \beta,\ \gamma$ can be calculated by $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$, and $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$.

If a loss function further satisfies the condition: $(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) \ge (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, then we can get $0 \le \beta < \gamma < \alpha \le 1$. The decision rules are further simplified to

(P) If $P(X|[x]_R) \ge \alpha$, decide $pos(X)$;

(B) If $\beta < P(X|[x]_R) < \alpha$, decide $bnd(X)$;

(N) If $P(X|[x]_R) \le \beta$, decide $neg(X)$.

This paper mainly studies the decision model under a loss function satisfying the condition $(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) \ge (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$. And the upper and lower approximations of the DTRS model in such case can be expressed as $\overline{R}_{(\alpha,\beta)}(X) = \{x \in U : P(X|[x]_R) > \beta\}$; $\underline{R}_{(\alpha,\beta)}(X) = \{x \in U : P(X|[x]_R) \ge \alpha\}$. Similarly, rough regions of X in the DTRS model can be defined as $pos_{(\alpha,\beta)}(X) = \underline{R}_{(\alpha,\beta)}(X)$, $neg_{(\alpha,\beta)}(X) = (\overline{R}_{(\alpha,\beta)}(X))^C$, $bnd_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_{(\alpha,\beta)}(X)$.

When $\alpha = 1,\ \beta = 0$, we have $\overline{R}_{(\alpha,\beta)}(X) = \overline{R}(X)$ and $\underline{R}_{(\alpha,\beta)}(X) = \underline{R}(X)$. Therefore, the decision-theoretic rough set model is a generalization of the rough set model.

2.3. Local rough sets (LRS)

Considering the three challenges of global rough sets in big data sets namely limited labeled property of big data, computational inefficiency and over-fitting in attribute reduction, Qian et al. [24] proposed local rough sets with fault tolerance capabilities. Local rough sets (LRS) provide an effective and efficient rough data analysis tool for large data sets, which makes it possible for rough set theory to be widely applied in big data sets.

Let (U, R) be an approximation space, and $\mathcal{D}$ be an inclusion degree defined on the $P(U) \times P(U)$. For any subset $X \subseteq U$, the upper and lower approximations of X are defined as

$\overline{LR}_{(\alpha,\beta)}(X) = \cup\{[x]_R : \mathcal{D}(X/[x]_R) > \beta, x \in X\}$,

$\underline{LR}_{(\alpha,\beta)}(X) = \{x : \mathcal{D}(X/[x]_R) \ge \alpha, x \in X\}$.

Without loss of generality, in this paper we choose conditional probability function as the inclusion degree, namely $\mathcal{D}(X/[x]_R) = P(X|[x]_R) = |[x]_R \cap X|/|[x]_R|$. Other inclusion degrees are mentioned in the literature [24]. When $\alpha = 1,\ \beta = 0$, we have $\overline{LR}_{(\alpha,\beta)}(X) = \overline{R}(X)$ and $\underline{LR}_{(\alpha,\beta)}(X) = \underline{R}(X)$. Therefore, the local rough set model is a generalization of the Pawlak rough set model.

In the following, we give the semantics of and differences between local rough sets and rough sets.

First, in the Pawlak rough set, the upper and lower approximations of X can further be expressed as

$\overline{R}(X) = \cup\{[x]_R | [x]_R \cap X \ne \emptyset, x \in U\} = \cup\{[x]_R | P(X|[x]_R) > 0, x \in U\} = \cup\{[x]_R | P(X|[x]_R) > 0, x \in X\}$,

$\underline{R}(X) = \cup\{[x]_R | [x]_R \subseteq X, x \in U\} = \{x : P(X|[x]_R) = 1, x \in U\} = \{x : P(X|[x]_R) = 1, x \in X\}$.

In the Pawlak rough set model, $\overline{R}(X)$ is the union of the equivalence classes of objects in X whose probability (inclusion degree) with respect to X exceeds 0; $\underline{R}(X)$ is the objects set where the probability (inclusion degree) of each object belonging to X equals 1. In the local rough set model, $\overline{LR}_{(\alpha,\beta)}(X)$ is the union of the equivalence classes of objects in X whose probability (inclusion degree) with respect to X exceeds β; $\underline{LR}_{(\alpha,\beta)}(X)$ is the objects set where the probability (inclusion degree) of each object belonging to X is greater than or equal to α.

The local rough set model excavates the deeper essence of Pawlak rough set, which can be used as a semi-supervised learning method for data mining. Moreover, the introduction of decision-making risk threshold parameters $\alpha,\ \beta$ makes the local rough set model have certain fault tolerance capabilities. Therefore, the LRS model is not only a generalization of the Pawlak rough set model, but also an innovation of the classical model in computational efficiency and fault tolerance capability.

At the same time, we give the relationships between LRS and the corresponding global rough set model.

In the corresponding global rough set model [24], the upper and lower approximations of X are defined as

$\overline{GR}_{(\alpha,\beta)}(X) = \cup\{[x]_R | P(X|[x]_R) > \beta, x \in U\} = \cup\{[x]_R | P(X|[x]_R) > \beta, x \in X\}$,

$\underline{GR}_{(\alpha,\beta)}(X) = \cup\{[x]_R | P(X|[x]_R) \geq \alpha, x \in U\} = \cup\{[x]_R | P(X|[x]_R) \geq \alpha, x \in X\}$.

After further analysis, Qian et al. [25] pointed out that $\overline{GR}_{(\alpha,\beta)}(X) = \overline{LR}_{(\alpha,\beta)}(X)$ and $\underline{LR}_{(\alpha,\beta)}(X) \subseteq X \subseteq \underline{GR}_{(\alpha,\beta)}(X)$. Moreover, $\underline{LR}_{(\alpha,\beta)}(X) = \underline{GR}_{(\alpha,\beta)}(X)$ if and only if $\forall x \in X$, $[x]_R \subseteq X$ namely $\alpha = 1$. Therefore, LRS is not only a generalization of probabilistic models, but also an innovation of probabilistic models in computational efficiency.

In this paper, in order to speed up the computational efficiency of approximations and rough regions in local rough sets, we propose the following simplest expressions after excluding objects from the perspective of set element description, namely

$\overline{LR}_{\beta}(X) = \cup\{[x]_R : P(X|[x]_R) > \beta, x \in X\} = \{x : P(X|[x]_R) > \beta, x \in U\} = \{x : P(X|[x]_R) > \beta, x \in U_0\}$;

$\underline{LR}_{\alpha}(X) = \{x : P(X|[x]_R) \geq \alpha, x \in X\}$, where $U_0 = U - \cup\{[x]_R : [x]_R \cap X = \emptyset\} = \overline{R}(X) = \cup\{[x]_R : x \in X\}$ and parameters α, β are determined by the loss function. Because the upper approximation $\overline{R}(X)$ of the RS model contains all objects with respect to X with non-zero probability, we search the objects with probability greater than β in this scope to further accelerate the calculation speed of $\overline{LR}_{\beta}(X)$.

2.4. Graded rough sets (GRS)

Yao and Lin [35] focused on the absolute quantitative information depicting intersecting degree of the target concept and equivalence classes during the process of approximating concepts, and proposed graded rough sets. The GRS model with fault tolerance capabilities is an absolute quantitative generalization of rough sets. The graded rough set model degenerates into the rough set model when grade parameter equals 0.

Let (U, R) be an approximation space, and grade parameter $k \in \mathbf{N}$ be a nonnegative integer. For any subset $X \subseteq U$, the upper and lower approximations of X under the grade k are defined as follows:

$\overline{R}_k(X) = \{x \in U : |X \cap [x]_R| > k\}$,

$\underline{R}_k(X) = \{x \in U : |[x]_R| - |X \cap [x]_R| \leq k\} = \{x \in U : |[x]_R \cap X^C| \leq k\}$, where $|X \cap [x]_R|$ denotes the internal grade of $[x]_R$ with respect to X and $|[x]_R| - |X \cap [x]_R|$ denotes the external grade of $[x]_R$ with respect to X [49].

3. Local logical disjunction double-quantitative rough sets (LLDDRS)

Local rough sets (LRS) provide a new method for rough data analysis under big data sets. In order to extract more useful information in the boundary region of the LRS model, we introduce the absolute quantitative information characterized by grade index into it. The completeness of the double quantification leads to a more refined description of the uncertainty, and the uncertainty is thereby decreased to some extent.

First, the local graded rough set is proposed by introducing local ideas into the graded rough set model. The details are shown in the following. In order to facilitate descriptions, we provide some explanations to the symbols used in this section. Symbol (U, R) denotes an approximation space, X and $U_0 = \cup\{[x]_R : x \in X\}$ denote two subsets of the universe U, $\mathbf{N}$ is the set of nonnegative integers, $k \in \mathbf{N}$, α, $\beta \in [0, 1]$ and $\beta \leq \alpha$.

3.1. Local logical disjunction double-quantitative rough set model

Definition 3.1. Let (U, R) be an approximation space, the local grade upper and lower approximations of X with respect to R are defined as:

$\overline{LR_k}(X) = \cup\{[x]_R : |[x]_R \cap X| > k, x \in X\} = \{x \in U_0 : |[x]_R| - |[x]_R \cap X| \leq k, U_0 = \cup\{[x]_R : x \in X\}\}$;

$\underline{LR_k}(X) = \{x \in X : |[x]_R| - |[x]_R \cap X| \leq k\}$.

Based on the two approximations $\overline{LR_k}$ and $\underline{LR_k}$, a new local rough set model, namely local grade rough sets (LGRS) can be obtained, which can be denoted by $(U, \overline{LR_k}, \underline{LR_k})$.

By analyzing the above Definition 3.1, we find that for any $k \geq 1 (k \in \mathbf{N})$, we have $\overline{LR_k}(X) \subseteq \overline{LR_0}(X) = \{x \in U_0 : |[x]_R \cap X| > 0\}$ and $\underline{LR_k}(X) \supseteq \underline{LR_0}(X) = \{x \in X : |[x]_R| - |[x]_R \cap X| \leq 0\}$. When $k = 0$, we have $\overline{LR_k}(X) = \overline{R}(X)$ and $\underline{LR_k}(X) = \underline{R}(X)$. Therefore, the local graded rough set (LGRS) model is a generalization of the Pawlak rough set model. More importantly, with the increase of k, the upper approximation will become smaller and the lower approximation set will become larger. In other words, the boundary region of the LGRS model is smaller than that of the Pawlak rough set, which can provide more knowledge from the boundary region for decision making.

In the universe U, there are always such objects that can not be distinguished within the probability threshold (grade threshold), but they can be distinguished by using grade information (probability information). In order to identify such objects to the greatest extent, this paper studies a logical disjunction double-quantitative model in the local idea when people are optimistic about fault tolerance. Of course, people can study different logical combination or crossed combination models according to their own needs. For example, when people can accept certain fault tolerance capabilities, but they want the information of each equivalence class containing concepts to be as valuable as possible in both quantity and quality, they may consider logical and double-quantitative models. Based on the completeness of double-quantitative information on the description of approximation spaces and the computational efficiency of local models in large data sets, other local double-quantitative combination models are a promising research direction. This paper only studies the logical disjunction combination model based on the uniqueness and complementarity of relative and absolute quantitative information.

Based on the simultaneous consideration of double-quantitative information and decision risk in the decision-making process, a Local Logical Disjunction Double-quantitative Rough Set model (LLDDRS) is proposed on the basis of the local rough set (LRS) and the local graded rough set (LGRS), which can describe a concept from two aspects of relative and absolute information. The details are given as follows:

Definition 3.2. Let (U, R) be an approximation space, the local logical disjunction double-quantitative upper and lower approximations of X with respect to R are defined as:

$\overline{LR_{(\alpha,\beta)\vee k}}(X) = \cup\{[x]_R : P(X|[x]_R) > \beta \ \ or \ \ |[x]_R \cap X| > k, x \in X\}$
$= \{x \in U_0 : P(X|[x]_R) > \beta \ \ or \ \ |[x]_R \cap X| > k, U_0 = \cup\{[x]_R : x \in X\}\};$
$\underline{LR_{(\alpha,\beta)\vee k}}(X) = \{x \in X : P(X|[x]_R) \geq \alpha \ \ or \ \ |[x]_R| - |[x]_R \cap X| \leq k\}.$

In LLDDRS model, $\overline{LR_{(\alpha,\beta)\vee k}}(X)$ is the union of the equivalence classes of objects in X whose probability (inclusion degree) with respect to X exceeds β or whose internal grade with respect to X exceeds k; $\underline{LR_{(\alpha,\beta)\vee k}}(X)$ is the object set where the probability (inclusion degree) of each object belonging to X is not less than α or the external grade of the object belonging to X is not more than k.

Based on the two approximations $\overline{LR_{(\alpha,\beta)\vee k}}$ and $\underline{LR_{(\alpha,\beta)\vee k}}$, a new local rough set model can be obtained, namely Local Logical Disjunction Double-quantitative Rough Sets (LLDDRS). The new rough set model can be denoted by $(U, \overline{LR_{(\alpha,\beta)\vee k}}, \underline{LR_{(\alpha,\beta)\vee k}})$. The positive region, negative region, upper boundary region, lower boundary region and boundary region of subset X with respect to R in the $(U, \overline{LR_{(\alpha,\beta)\vee k}}, \underline{LR_{(\alpha,\beta)\vee k}})$ model are defined as:

$posLR_{\vee}(X) = \overline{LR_{(\alpha,\beta)\vee k}}(X) \cap \underline{LR_{(\alpha,\beta)\vee k}}(X);$
$negLR_{\vee}(X) = (\overline{LR_{(\alpha,\beta)\vee k}}(X) \cup \underline{LR_{(\alpha,\beta)\vee k}}(X))^C;$
$UbnLR_{\vee}(X) = \overline{LR_{(\alpha,\beta)\vee k}}(X) - \underline{LR_{(\alpha,\beta)\vee k}}(X);$
$LbnLR_{\vee}(X) = \underline{LR_{(\alpha,\beta)\vee k}}(X) - \overline{LR_{(\alpha,\beta)\vee k}}(X);$
$bnLR_{\vee}(X) = UbnLR_{(\alpha,\beta)\vee k}(X) \cup LbnLR_{(\alpha,\beta)\vee k}(X).$

Obviously, the positive region, negative region, upper boundary region, lower boundary region can form a partition of the universe, namely $U = posLR_{\vee}(X) \cup negLR_{\vee}(X) \cup UbnLR_{\vee}(X) \cup LbnLR_{\vee}(X)$. We can find that the upper approximation $\overline{LR_{(\alpha,\beta)\vee k}}(X)$ is the union of the positive region $posLR_{\vee}(X)$ and the upper boundary region $UbnLR_{\vee}(X)$, and the lower approximation $\underline{LR_{(\alpha,\beta)\vee k}}(X)$ is the union of the positive region $posLR_{\vee}(X)$ and the lower boundary region $LbnLR_{\vee}(X)$.

Next, we define the following measures to evaluate the performance of the LLDDRS model and the importance of the double-quantitative information.

Definition 3.3. Let $(U, C \cup D)$ be a decision table, where C is a condition attribute set and D is a decision attribute set. The approximation accuracy of C with respect to D in the LLDDRS model is defined as

$\rho_{\vee}(C, D) = \frac{\sum\{|\underline{LR_{(\alpha,\beta)\vee k}}(Y_i)| : Y_i \in U/R_D\}}{\sum\{|Y_i| : Y_i \in U/R_D\}}$ where $U/R_D = \{Y_1, Y_2, \ldots, Y_s\}$ is a partition of the universe U under the equivalence relation generated by D and $i, j \in \{1, 2, \ldots, s\}$.

The unique contribution rates of relative, absolute quantitative information to concept approximation and the shared contribution rate of relative and absolute quantitative information to concept approximation are defined as

$C_{\overline{p}} = \frac{\sum\{|\underline{LR_{(\alpha,\beta)\vee k}}(Y_i) - \underline{LR_k}(Y_i)| : Y_i \in U/R_D\}/|U|}{\rho_{\vee}(C,D)}$, $C_{\overline{g}} = \frac{\sum\{|\underline{LR_{(\alpha,\beta)\vee k}}(Y_i) - \underline{LR_{(\alpha,\beta)}}(Y_i)| : Y_i \in U/R_D\}/|U|}{\rho_{\vee}(C,D)}$, $C_{p\vee g} = 1 - C_{\overline{p}} - C_{\overline{g}}$, respectively.

The unique contribution rate of relative quantitative information $C_{\overline{p}}$ is characterized by the percentage of objects that can only be depicted by relative quantitative information in concept approximation; the unique contribution rate of absolute quantitative information $C_{\overline{g}}$ is characterized by the percentage of objects that can only be depicted by absolute quantitative information in concept approximation; and the shared contribution rate of relative and absolute quantitative information $C_{p\vee g}$ is characterized by the percentage of objects that can be depicted either by relative or absolute quantitative information in concept approximation.

According to Definition 3.2 and the concepts of rough regions, we find that the positive, negative, upper boundary and lower boundary regions of the LLDDRS model can be known by calculating the upper and lower approximations, and the upper and lower approximation sets can be calculated by the positive, upper boundary and lower boundary regions. Next, we first design Algorithm 1 to compute the local logical disjunction double-quantitative upper and lower approximations of a target concept X.

In Algorithm 1, steps 1–3 calculate the equivalence class of any object x in the target concept X, the cardinality of sets $[x]_R$ and $[x]_R \cap X$ and the conditional probability of the equivalence class $[x]_R$ with respect to the concept X, and its time complexity is $\mathcal{O}(|X||U|)$. Step 4 initializes the upper and lower approximations of X, and its time complexity is constant. Steps 5–12 calculate the upper and lower approximations $\overline{LR_{(\alpha,\beta)\vee k}}(X)$, $\underline{LR_{(\alpha,\beta)\vee k}}(X)$, and the time complexity is $\mathcal{O}(|X|)$. At

Algorithm 1: Upper and lower approximations of a target concept in the LLDDRS model.

Input: An approximation space (U, R), a target concept X, decision risk parameters α, β and grade parameter k
Output: $\overline{LR_{(\alpha,\beta)\vee k}}(X)$, $\underline{LR_{(\alpha,\beta)\vee k}}(X)$

```
for each x ∈ X do
    Compute [x]_R, |[x]_R|, |[x]_R ∩ X| and P(X|[x]_R) of x    // [x]_R is the equivalence class of x with respect to R
end for
Initialize UpperLR_{(α,β)∨k}(X) ← ∅, LowerLR_{(α,β)∨k}(X) ← ∅
for each x ∈ X do
    if P(X|[x]_R) > β  or  |[x]_R ∩ X| > k then
        UpperLR_{(α,β)∨k}(X) = UpperLR_{(α,β)∨k}(X) ∪ [x]_R
    end if
    if P(X|[x]_R) ≥ α  or  |[x]_R| − |[x]_R ∩ X| ≤ k then
        LowerLR_{(α,β)∨k}(X) = LowerLR_{(α,β)∨k}(X) ∪ x
    end if
end for
return UpperLR_{(α,β)∨k}(X), LowerLR_{(α,β)∨k}(X)
```

last, return the upper and lower approximations of the target concept X. The time complexity of Algorithm 1 is equal to $\mathcal{O}(|X||U|)$.

When calculating the upper and lower approximations, the time complexities of the LLDDRS model and the LRS model are the same. However, the time complexity of upper and lower approximations in the LLDDRS model is less or even far less than that in the corresponding global logical disjunction double-quantitative rough set (GLDDRS) ($\mathcal{O}(|U|^2)$). Therefore, the proposed LLDDRS model is efficient in computing approximation sets in large data sets.

3.2. The important properties of LLDDRS

Theorem 3.1. *Let (U, R) be an approximation space, for any subset $X \subseteq U$, the following conclusions are obtained:*
$\overline{LR_{(\alpha,\beta)\vee k}}(X) = \overline{LR_{(\alpha,\beta)}}(X) \cup \overline{LR_k}(X)$;
$\underline{LR_{(\alpha,\beta)\vee k}}(X) = \underline{LR_{(\alpha,\beta)}}(X) \cup \underline{LR_k}(X)$.

Proof: It can directly be proved by Definitions 3.1 and 3.2.

According to Theorem 3.1, the upper (lower) approximation of the LLDDRS model is the union of the upper (lower) approximations of the LRS and LGRS models.

Theorem 3.2. *Let (U, R) be an approximation space, for any subset $X \subseteq U$, the following conclusions are obtained:*
(1)For $\forall X \subseteq U$, there is $\underline{\mathrm{LR}_{(\alpha,\beta)\vee k}}(X) \subseteq X$;
(2)If $\beta \in [0, \min\{P(X|[x]_R): x \in X\}]$ or $k \leq \min\{|[x]_R \cap X|: x \in X\}$, then $X \subseteq \overline{LR_{(\alpha,\beta)\vee k}}(X)$;
(3)Let $\beta_1, \beta_2, \alpha_1, \alpha_2 \in [0, 1]$ and $\beta_1 < \beta_2$, $\alpha_1 < \alpha_2$, there are
$\overline{LR_{(\alpha,\beta_1)\vee k}}(X) \supseteq \overline{LR_{(\alpha,\beta_2)\vee k}}(X)$, $\underline{LR_{(\alpha_1,\beta)\vee k}}(X) \supseteq \underline{LR_{(\alpha_2,\beta)\vee k}}(X)$;
(4)Let $k_1, k_2 \in \mathbf{N}$ and $k_1 < k_2$, there are $\overline{LR_{(\alpha,\beta)\vee k_1}}(X) \supseteq \overline{LR_{(\alpha,\beta)\vee k_2}}(X)$ and $\underline{LR_{(\alpha,\beta)\vee k_1}}(X) \subseteq \underline{LR_{(\alpha,\beta)\vee k_2}}(X)$;
(5)$\overline{LR_{(\alpha,\beta)\vee k}}(X) \supseteq \overline{LR_{(\alpha,\beta)}}(X)$, $\underline{\mathrm{LR}_{(\alpha,\beta)\vee k}}(X) \supseteq \underline{\mathrm{LR}_{(\alpha,\beta)}}(X)$; $\overline{LR_{(\alpha,\beta)\vee k}}(X) \supseteq \overline{LR_k}(X)$, $\underline{\mathrm{LR}_{(\alpha,\beta)\vee k}}(X) \supseteq \underline{\mathrm{LR}_k}(X)$;
(6)$\overline{LR_{(\alpha,\beta)\vee k}}(X) \subseteq \overline{R}(X)$, $\underline{\mathrm{LR}_{(\alpha,\beta)\vee k}}(X) \supseteq \underline{R}(X)$;
(7)$\overline{LR_{(1,0)\vee k}}(X) = \overline{R}(X)$, $\underline{LR_{(1,0)\vee k}}(X) = \underline{LR_k}(X) \supseteq \underline{R}(X)$; $\overline{LR_{(\alpha,\beta)\vee 0}}(X) = \overline{R}(X)$, $\underline{LR_{(\alpha,\beta)\vee 0}}(X) = \underline{LR_{(\alpha,\beta)}}(X) \supseteq \underline{R}(X)$;
(8)$\overline{LR_{(1,0)\vee 0}}(X) = \overline{R}(X)$, $\underline{LR_{(1,0)\vee 0}}(X) = \underline{R}(X)$.

Proof: (1) It can directly be proved by the definition of lower approximation in Definition 3.2.

(2) When $\beta \in [0, \min\{P(X|[x]_R): x \in X\}]$, for any object x in the concept X, we have $P(X|[x]_R) \geq \beta$. Then we obtain $x \in \overline{LR_{(\alpha,\beta)\vee k}}(X)$. When $k \leq \min\{|[x]_R \cap X|: x \in X\}$, for any object x in the concept X, we have $|[x]_R \cap X| > k$, then there is $x \in \overline{LR_{(\alpha,\beta)\vee k}}(X)$. All in all, in any case, the conclusion $X \subseteq \overline{LR_{(\alpha,\beta)\vee k}}(X)$ is true.

(3) When $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, for any $k \in \mathbf{N}$, we have $\underline{LR_{(\alpha_2,\beta)}}(X) \subseteq \underline{LR_{(\alpha_1,\beta)}}(X)$ and $\overline{LR_{(\alpha,\beta_2)}}(X) \subseteq \overline{LR_{(\alpha,\beta_1)}}(X)$. According to Theorem 3.1, the conclusions $\overline{LR_{(\alpha,\beta_1)\vee k}}(X) \supseteq \overline{LR_{(\alpha,\beta_2)\vee k}}(X)$ and $\underline{LR_{(\alpha_1,\beta)\vee k}}(X) \supseteq \underline{LR_{(\alpha_2,\beta)\vee k}}(X)$ are true.

(4) When $k_1 < k_2$, we have $\overline{LR_{k_2}}(X) \subseteq \overline{LR_{k_1}}(X)$ and $\underline{LR_{k_1}}(X) \subseteq \underline{LR_{k_2}}(X)$. For any $X \subseteq U$ and any $\alpha, \beta \in [0, 1]$, according to Theorem 3.1, we can get $\overline{LR_{(\alpha,\beta)\vee k_1}}(X) \supseteq \overline{LR_{(\alpha,\beta)\vee k_2}}(X)$ and $\underline{LR_{(\alpha,\beta)\vee k_1}}(X) \subseteq \underline{LR_{(\alpha,\beta)\vee k_2}}(X)$.

(5) Equations can directly be proved by Theorem 3.1.

(6) When $k>0$, for any $X\subseteq U$, we have $\overline{LR_k}(X)\subseteq\overline{LR_0}(X)$ and $\underline{LR_k}(X)\supseteq\underline{LR_0}(X)$. And when $k=0$, we have $\overline{LR_0}(X)=\overline{R}(X)$ and $\underline{LR_0}(X)=\underline{R}(X)$. Therefore, according to Theorem 3.1, we have $\overline{LR_{(\alpha,\beta)\vee k}}(X)\subseteq\overline{LR_{(\alpha,\beta)\vee 0}}(X)$ and $\underline{LR_{(\alpha,\beta)\vee k}}(X)\supseteq\underline{LR_{(\alpha,\beta)\vee 0}}(X)$. Because $\overline{LR_{(\alpha,\beta)\vee 0}}(X)=\overline{LR_{(\alpha,\beta)}}(X)\cup\overline{R}(X)$ and $\overline{LR_{(\alpha,\beta)}}(X)\subseteq\overline{LR_{(\alpha,0)}}(X)=\overline{R}(X)$, we have the conclusion $\overline{LR_{(\alpha,\beta)\vee 0}}(X)=\overline{R}(X)$. In addition, because $\underline{LR_{(\alpha,\beta)\vee 0}}(X)=\underline{LR_{(\alpha,\beta)}}(X)\cup\underline{R}(X)$ and $\underline{R}(X)=\underline{LR_{(1,\beta)}}(X)\subseteq\underline{LR_{(\alpha,\beta)}}(X)$, we have $\underline{LR_{(\alpha,\beta)\vee 0}}(X)=\underline{LR_{(\alpha,\beta)}}(X)\supseteq X$. Therefore, the conclusions $\overline{LR_{(\alpha,\beta)\vee k}}(X)\subseteq\overline{R}(X)$ and $\underline{LR_{(\alpha,\beta)\vee k}}(X)\supseteq\underline{R}(X)$ are true.

(7) For any $X\subseteq U$, we have $\overline{LR_{(1,0)\vee k}}(X)=\overline{R}(X)\cup\overline{LR_k}(X)$. For any $k\in\mathbf{N}$, $\overline{LR_k}(X)\subseteq\overline{LR_0}(X)=\overline{R}(X)$. Therefore, the conclusion $\overline{LR_{(1,0)\vee k}}(X)=\overline{R}(X)$ is true. Similarly, for any $X\subseteq U$, we have $\underline{LR_{(1,0)\vee k}}(X)=\underline{R}(X)\cup\underline{LR_k}(X)=\underline{LR_0}(X)\cup\underline{LR_k}(X)=\underline{LR_k}(X)$. Therefore, the conclusion $\underline{LR_{(1,0)\vee k}}(X)=\underline{LR_k}(X)\supseteq\underline{R}(X)$ is true. For any $X\subseteq U$, we have $\overline{LR_{(\alpha,\beta)\vee 0}}(X)=\overline{LR_{(\alpha,\beta)}}(X)\cup\overline{R}(X)$. Because $\overline{LR_{(\alpha,\beta)}}(X)\subseteq\overline{LR_{(\alpha,0)}}(X)=\overline{R}(X)$, so we can get $\overline{LR_{(\alpha,\beta)\vee 0}}(X)=\overline{R}(X)$. For any $X\subseteq U$, we have $\underline{LR_{(\alpha,\beta)\vee 0}}(X)=\underline{LR_{(\alpha,\beta)}}(X)\cup\underline{R}(X)$. Because $\underline{R}(X)=\underline{LR_{(1,\beta)}}(X)\subseteq\underline{LR_{(\alpha,\beta)}}(X)$, so we can get $\underline{LR_{(\alpha,\beta)\vee 0}}(X)=\underline{LR_{(\alpha,\beta)}}(X)\supseteq\underline{R}(X)$.

(8)When $\alpha=1,\beta=0,k=0$, we have $\overline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in U:[x]_R\cap X\neq\emptyset\}$ and $\underline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in U:[x]_R\subseteq X\}$. Therefore, the conclusions $\overline{LR_{(1,0)\vee 0}}(X)=\overline{R}(X)$ and $\underline{LR_{(1,0)\vee 0}}(X)=\underline{R}(X)$ are true.

Based on the above conclusions (6) and (8), we find that the Local Logical Disjunction Double-quantitative Rough Set model is a generalization model of the Pawlak rough sets, and the upper approximation of the LLDDRS model becomes smaller and its lower approximation becomes larger due to the introduction of parameters α, β, k. So the LLDDRS model can provide more valuable information for decision making. In particular, when conditions $\alpha=1,\beta=0,k=0$ are satisfied, the LLDDRS model degenerates to the Pawlak rough set.

3.3. The rough regions of the LLDDRS model

After simplifying Definition 3.2, we get the following expressions of upper and lower approximations

$\overline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in U_0:|[x]_R\cap X|>min(\beta|[x]_R|,k),U_0=\cup\{[x]_R:x\in X\}\}$;

$\underline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in X:|[x]_R\cap X|\geq\min(\alpha|[x]_R|,|[x]_R|-k)\}$.

From these two equations, we can see that different parameter values of α, β $(0\leq\beta<\alpha\leq1)$ will lead to different upper and lower approximation sets. The lower approximation set describes objects within acceptable tolerance that definitely belong to a target concept. Considering the acceptability level of people for fault tolerance in real life, we set the threshold parameter α of the lower approximation to be greater than 1/2. The approximation sets and rough regions of the LLDDRS model are analyzed below.

Firstly, we analyze the first case $1/2\leq\beta<\alpha\leq1$.

When $1/2\leq\beta<\alpha\leq1$, we have $\alpha+\beta>1$. We can get inequality $k/\beta<k/(1-\alpha)$. If $|[x]_R|<k/\beta$, then the upper approximation is $\overline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in U_0:|[x]_R\cap X|>\beta|[x]_R|,U_0=\cup\{[x]_R:x\in X\}\}$. The lower approximation can be obtained by inequality $|[x]_R|<k/\beta<k/(1-\alpha)$, namely $\underline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in X:|[x]_R\cap X|\geq|[x]_R|-k\}$. If $|[x]_R|\geq k/\beta$, the upper approximation is $\overline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in U_0:|[x]_R\cap X|>k,U_0=\cup\{[x]_R:x\in X\}\}$. The lower approximation needs to be further compared with values $\alpha|[x]_R|$ and $|[x]_R|-k$. Under the condition $k/\beta\leq|[x]_R|<k/(1-\alpha)$, the lower approximation is $\underline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in X:|[x]_R\cap X|\geq|[x]_R|-k\}$; under the condition $|[x]_R|\geq k/(1-\alpha)$, the lower approximation is $\underline{LR_{(\alpha,\beta)\vee k}}(X)=\{x\in X:|[x]_R\cap X|\geq\alpha|[x]_R|\}$. Therefore, we can get the following expressions of upper and lower approximations in the case of $1/2\leq\beta<\alpha\leq1$ $(U_0=\cup\{[x]_R:x\in X\})$, which are

- If $|[x]_R|<k/\beta$, then $\overline{LR_\vee}(X)=\{x\in U_0:|[x]_R\cap X|>\beta|[x]_R|\}$, $\underline{LR_\vee}(X)=\{x\in X:|[x]_R\cap X|\geq|[x]_R|-k\}$;
- If $k/\beta\leq|[x]_R|<k/(1-\alpha)$, then $\overline{LR_\vee}(X)=\{x\in U_0:|[x]_R\cap X|>k\}$, $\underline{LR_\vee}(X)=\{x\in X:|[x]_R\cap X|\geq|[x]_R|-k\}$;
- If $|[x]_R|\geq k/(1-\alpha)$, then $\overline{LR_\vee}(X)=\{x\in U_0:|[x]_R\cap X|>k\}$, $\underline{LR_\vee}(X)=\{x\in X:|[x]_R\cap X|\geq\alpha|[x]_R|\}$;

Rough regions in the case of $1/2\leq\beta<\alpha\leq1$ can be obtained based on upper and lower approximation sets and the inclusion relation $X\subseteq U_0\subseteq U$.

If $|[x]_R|\in(0,\,k/\beta)$, then we have $|[x]_R|-k<\beta|[x]_R|$ $(k/\beta<k/(1-\beta))$. We can obtain the following conclusions by the intersection, union and complement operations of sets:

$posLR_\vee(X)=\underline{LR_\vee}(X)\cap\overline{LR_{(\vee}}(X)=\{x\in X:|[x]_R\cap X|>\beta|[x]_R|\}$;

$negLR_\vee(X)=(\underline{LR_\vee}(X)\cup\overline{LR_{(\vee}}(X))^C=\{x\in U_0-X:|[x]_R\cap X|\leq\beta|[x]_R|\}\cup\{x\in X:|[x]_R\cap X|<|[x]_R|-k\}\cup\{U-U_0\}$;

$Ubn_\vee(X)=\overline{LR_{(\vee}}(X)-\underline{LR_\vee}(X)=\{x\in U_0-X:|[x]_R\cap X|>\beta|[x]_R|\}$;

$Lbn_\vee(X)=\underline{LR_\vee}(X)-\overline{LR_{(\vee}}(X)=\{x\in X:|[x]_R|-k\leq|[x]_R\cap X|\leq\beta|[x]_R|\}$.

If $[k/\beta,k/(1-\alpha))$, then we need to compare k and $|[x]_R|-k$. When $[k/\beta,2k)$, we have $|[x]_R|-k<k$. Then rough regions are $posLR_\vee(X)=\{x\in X:|[x]_R\cap X|>k\}$;

$negLR_\vee(X)=\{x\in U_0-X:|[x]_R\cap X|\leq k\}\cup\{x\in X:|[x]_R\cap X|<|[x]_R|-k\}\cup\{U-U_0\}$;

$Ubn_\vee(X)=\{x\in U_0-X:|[x]_R\cap X|>k\}$;

$Lbn_\vee(X)=\{x\in X:|[x]_R|-k\leq|[x]_R\cap X|\leq k\}$.

When $[2k,k/(1-\alpha))$, we have $|[x]_R|-k\geq k$. Then rough regions are

$posLR_\vee(X)=\{x\in X:|[x]_R\cap X|\geq|[x]_R|-k\}$;

$negLR_\vee(X) = \{x \in U_0 : |[x]_R \cap X| \le k\} \cup \{U - U_0\}$;
$Ubn_\vee(X) = \{x \in U_0 - X : |[x]_R \cap X| > k\} \cup \{x \in X : k < |[x]_R \cap X| < |[x]_R| - k\}$;
$Lbn_\vee(X) = \emptyset$.

If $|[x]_R| \in [k/(1-\alpha), +\infty)$, we have $\alpha|[x]_R| > k$ $(k/(1-\alpha) > k/\alpha)$. The rough regions are
$posLR_\vee(X) = \{x \in X : |[x]_R \cap X| \ge \alpha|[x]_R|\}$;
$negLR_\vee(X) = \{x \in U_0 : |[x]_R \cap X| \le k\} \cup \{U - U_0\}$;
$Ubn_\vee(X) = \{x \in U_0 - X : |[x]_R \cap X| > k\} \cup \{x \in X : k < |[x]_R \cap X| < \alpha|[x]_R|\}$;
$Lbn_\vee(X) = \emptyset$.

Consequently, in the case of $1/2 \le \beta < \alpha \le 1$, we have the following conclusions by simplification:
$posLR_\vee(X) = \{x \in X : |[x]_R| \ge k/(1-\alpha), |[x]_R \cap X| \ge \alpha|[x]_R|\} \cup \{x \in X : 2k \le |[x]_R| < k/(1-\alpha), |[x]_R \cap X| \ge |[x]_R| - k\}$
$\cup \{x \in X: k/\beta \le |[x]_R| < 2k, |[x]_R \cap X| > k\} \cup \{x \in X: |[x]_R| < k/\beta, |[x]_R \cap X| > \beta|[x]_R|\}$;
$negLR_\vee(X) = \{x \in U_0 : |[x]_R| \ge 2k, |[x]_R \cap X| \le k\} \cup \{x \in (U_0 - X) : k/\beta \le |[x]_R| < 2k, |[x]_R \cap X| \le k\}$
$\cup \{x \in (U_0 - X) : |[x]_R| < k/\beta, |[x]_R \cap X| \le \beta|[x]_R|\} \cup \{x \in X : |[x]_R| < 2k, |[x]_R \cap X| \le |[x]_R| - k\} \cup \{U - U_0\}$;
$UbnLR_\vee(X) = \{x \in (U_0 - X) : |[x]_R| \ge k/\beta, |[x]_R \cap X| > k\} \cup \{x \in (U_0 - X) : |[x]_R| < k/\beta, |[x]_R \cap X| > \beta|[x]_R|\}$
$\cup \{x \in X : |[x]_R| \ge k/(1-\alpha), k < |[x]_R \cap X| < \alpha|[x]_R|\}$
$\cup \{x \in X : 2k \le |[x]_R| < k/(1-\alpha), k < |[x]_R \cap X| < |[x]_R| - k\}$;
$LbnLR_\vee(X) = \{x \in X : k/\beta \le |[x]_R| < 2k, |[x]_R| - k \le |[x]_R \cap X| \le k\}$
$\cup \{x \in X : |[x]_R| < k/\beta, |[x]_R| - k \le |[x]_R \cap X| \le \beta|[x]_R|\}$.

Secondly, we analyze rough regions in the case of $0 \le \beta < 1/2 < \alpha \le 1$, $\alpha + \beta \ge 1$ and the case of $0 \le \beta < 1/2 < \alpha \le 1$, $\alpha + \beta < 1$ by using the similar method. The details are presented as follows:

In these two cases, we have the same positive region, negative region and lower boundary region expressions, which are
$posLR_\vee(X) = \{x \in X : |[x]_R| \ge k/(1-\alpha), |[x]_R \cap X| \ge \alpha|[x]_R|\}$
$\cup \{x \in X : k/(1-\beta) \le |[x]_R| < k/(1-\alpha), |[x]_R \cap X| \ge |[x]_R| - k\}$
$\cup \{x \in X : |[x]_R| < k/(1-\beta), |[x]_R \cap X| > \beta|[x]_R|\}$;
$negLR_\vee(X) = \{x \in U_0 : |[x]_R| \ge k/\beta, |[x]_R \cap X| \le k\}$
$\cup \{x \in U_0 : k/(1-\beta) \le |[x]_R| < k/\beta, |[x]_R \cap X| \le \beta|[x]_R|\}$
$\cup \{x \in (U_0 - X) : |[x]_R| < k/(1-\beta), |[x]_R \cap X| \le \beta|[x]_R|\}$
$\cup \{x \in X : |[x]_R| < k/(1-\beta), |[x]_R \cap X| < |[x]_R| - k\} \cup \{U - U_0\}$;
$Lbn_\vee(X) = \{x \in X : |[x]_R| < k/(1-\beta), |[x]_R| - k \le |[x]_R \cap X| \le \beta|[x]_R|\}$.

However there are different expressions for the upper boundary regions under these two cases. In case $0 \le \beta < 1/2 < \alpha \le 1$, $\alpha + \beta < 1$, the upper boundary region is
$Ubn_\vee(X) = \{x \in (U_0 - X) : |[x]_R| \ge k/\beta, |[x]_R \cap X| > k\} \cup \{x \in X: |[x]_R| \ge k/\beta, k < |[x]_R \cap X| < \alpha|[x]_R|\}$
$\cup \{x \in (U_0 - X) : |[x]_R| < k/\beta, |[x]_R \cap X| > \beta|[x]_R|\}$
$\cup \{x \in X : k/(1-\alpha) \le |[x]_R| < k/\beta, \beta|[x]_R| < |[x]_R \cap X| < \alpha|[x]_R|\}$
$\cup \{x \in X : k/(1-\beta) \le |[x]_R| < k/(1-\alpha), \beta|[x]_R| < |[x]_R \cap X| < |[x]_R| - k\}$.

In case $0 \le \beta < 1/2 < \alpha \le 1$, $\alpha + \beta \ge 1$, the upper boundary region is
$Ubn_\vee(X) = \{x \in (U_0 - X) : |[x]_R| \ge k/\beta, |[x]_R \cap X| > k\} \cup \{x \in (U_0 - X) : |[x]_R| < k/\beta, |[x]_R \cap X| > \beta|[x]_R|\}$
$\cup \{x \in X : |[x]_R| \ge k/(1-\alpha), k < |[x]_R \cap X| < \alpha|[x]_R|\}$
$\cup \{x \in X : k/\beta \le |[x]_R| < k/(1-\alpha), k < |[x]_R \cap X| < |[x]_R| - k\}$
$\cup \{x \in X : k/(1-\beta) \le |[x]_R| < k/\beta, \beta|[x]_R| < |[x]_R \cap X| < |[x]_R| - k\}$.

In this paper, two methods for calculating rough regions are presented. One method is based on the upper and lower approximations, and the other is to directly use the above rough region expressions. When people focus only on one or some rough regions, the second method is more practical. In order to facilitate the solution, we design Algorithm 2 to calculate the related regions of the LLDDRS model.

In Algorithm 2, steps 1–3 calculate the equivalence class of any object x in the target concept X and the cardinality of sets $[x]_R$ and $[x]_R \cap X$, and its time complexity is $\mathcal{O}(|X||U|)$. Step 4 initializes positive region $posLR_\vee(X)$, negative region $negLR_\vee(X)$, lower boundary region $LbnLR_\vee(X)$ as empty set and U_0 as the union of equivalence classes of all elements in X, and its time complexity is constant. Steps 5–32 calculate positive, negative, lower boundary and upper boundary regions and boundary regions under two cases $0 \le \beta < 1/2 < \alpha \le 1$ and $1/2 \le \beta < \alpha \le 1$ in the LLDDRS model, and the time complexity is $\mathcal{O}(|U_0|)$. At last, return rough regions of the target concept X. The time complexity of Algorithm 2 is equal to $\mathcal{O}(|X||U|)$.

3.4. The decision rules in the LLDDRS model

Based on the above rough regions of the LLDDRS model, we can get the corresponding decision rules. For example, when $\beta \ge 1/2$, we obtain the following positive region and negative region decision rules
$(P^\vee)$ If $x \in X$, $|[x]_R| \ge k/(1-\alpha)$, $|[x]_R \cap X| \ge \alpha|[x]_R|$, then accept
$(P^\vee)$ If $x \in X$, $2k \le |[x]_R| < k/(1-\alpha)$, $|[x]_R \cap X| \ge |[x]_R| - k$, then accept
$(P^\vee)$ If $x \in X$, $k/\beta \le |[x]_R| < 2k$, $|[x]_R \cap X| > k$, then accept
$(P^\vee)$ If $x \in X$, $|[x]_R| < k/\beta$, $|[x]_R \cap X| > \beta|[x]_R|$, then accept
$(N^\vee)$ If $x \in (U - U_0)$, then reject
$(N^\vee)$ If $x \in U_0$, $|[x]_R| \ge 2k$, $|[x]_R \cap X| \le k$, then reject

Algorithm 2: Rough regions of a target concept in the LLDDRS model.

Input: An approximation space (U, R), a target concept X, decision risk parameters α, β and grade parameter k
Output: $posLR_\vee(X), negLR_\vee(X), LbnLR_\vee(X), UbnLR_\vee(X), bnLR_\vee(X)$

for each $x \in X$ **do**
  Compute $[x]_R$, $|[x]_R|$ and $|[x]_R \cap X|$ of x // $[x]_R$ is the equivalence class of x with respect to R
end for
Initialize $posLR_\vee(X) \leftarrow \emptyset$, $negLR_\vee(X) \leftarrow \emptyset$,$LbnLR_\vee(X) \leftarrow \emptyset$, $U_0 \leftarrow \cup\{[x]_R : x \in X\}$
if $beta < 1/2$ **then**
  for each $x \in U_0$ **do**
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : |[x]_R| \geq k/(1-\alpha), |[x]_R \cap X| \geq \alpha|[x]_R|, x \in X\}$
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : |[x]_R| < k/(1-\beta), |[x]_R \cap X| > \beta|[x]_R|, x \in X\}$
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : k/(1-\beta) \leq |[x]_R| < k/(1-\alpha), |[x]_R \cap X| \geq |[x]_R| - k, x \in X\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : |[x]_R| \geq k/\beta, |[x]_R \cap X| \leq k\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : |[x]_R| < k/(1-\beta), |[x]_R \cap X| < |[x]_R| - k, x \in X\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : k/(1-\beta) \leq |[x]_R| < k/\beta, |[x]_R \cap X| \leq \beta|[x]_R|\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : |[x]_R| < k/(1-\beta), |[x]_R \cap X| \leq \beta|[x]_R|, x \in (U_0 - X)\}$
    $LbnLR_\vee(X) \leftarrow LbnLR_\vee(X) \cup \{x : |[x]_R| < k/(1-\beta), |[x]_R| - k \leq |[x]_R \cap X| \leq \beta|[x]_R|, x \in X\}$
  end for
else
  for each $x \in U_0$ **do**
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : |[x]_R| \geq k/(1-\alpha), |[x]_R \cap X| \geq \alpha|[x]_R|, x \in X\}$
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : k/\beta \leq |[x]_R| < 2k, |[x]_R \cap X| > k, x \in X\}$
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : 2k \leq |[x]_R| < k/(1-\alpha), |[x]_R \cap X| \geq |[x]_R| - k, x \in X\}$
    $posLR_\vee(X) \leftarrow posLR_\vee(X) \cup \{x : |[x]_R| < k/\beta, |[x]_R \cap X| > \beta|[x]_R|, x \in X\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : |[x]_R| \geq 2k, |[x]_R \cap X| \leq k\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : k/\beta \leq |[x]_R| < 2k, |[x]_R \cap X| \leq k, x \in (U_0 - X)\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : |[x]_R| < k/\beta, |[x]_R \cap X| \leq \beta|[x]_R|, x \in (U_0 - X)\}$
    $negLR_\vee(X) \leftarrow negLR_\vee(X) \cup \{x : |[x]_R| < 2k, |[x]_R \cap X| \leq |[x]_R| - k, x \in X\}$
    $LbnLR_\vee(X) \leftarrow LbnLR_\vee(X) \cup \{x : k/\beta \leq |[x]_R| < 2k, |[x]_R| - k \leq |[x]_R \cap X| \leq k, x \in X\}$
    $LbnLR_\vee(X) \leftarrow LbnLR_\vee(X) \cup \{x : |[x]_R| < k/\beta, |[x]_R| - k \leq |[x]_R \cap X| \leq \beta|[x]_R|, x \in X\}$
  end for
end if
$negLR_\vee(X) \leftarrow negLR_\vee(X) \cup (U - U_0)$
$UbnLR_\vee(X) \leftarrow U - posLR_\vee(X) - negLR_\vee(X) - LbnLR_\vee(X)$
$bnLR_\vee(X) \leftarrow LbnLR_\vee(X) \cup UbnLR_\vee(X)$
return $posLR_\vee(X), negLR_\vee(X), LbnLR_\vee(X), UbnLR_\vee(X), bnLR_\vee(X)$

$(N^\vee)$ If $x \in (U_0 - X), k/\beta \leq |[x]_R| < 2k, |[x]_R \cap X| \leq k$, then reject
$(N^\vee)$ If $x \in (U_0 - X), |[x]_R| < k/\beta, |[x]_R \cap X| \leq \beta|[x]_R|$, then reject
$(N^\vee)$ If $x \in X, |[x]_R| < 2k, |[x]_R \cap X| < |[x]_R| - k$, then reject
When we only focus on the decision rules of one region, this method is very convenient.

Next, we present all the decision rules in a more understandable way. For objects in the positive and negative regions, we make acceptance and rejection decisions, respectively. For objects in the upper and lower boundary regions, we need more information to make definite decisions. In order to reduce the losses caused by incorrect acceptance and incorrect rejection, we defer decision-making on these objects in boundary regions, namely noncommitment.

In the case $1/2 \leq \beta < \alpha \leq 1$, we have the inequality $k/\beta \leq 2k < k/(1-\alpha)$. Considering that the universe U consists of three parts X, $U_0 - X$ and $U - U_0$, we present the following rule information. Details are shown in Table 2, where $\mathbf{N}^+$ is a set of natural numbers.

In the case $0 \leq \beta < 1/2 < \alpha \leq 1$ $(\alpha + \beta \geq 1)$, we have the inequality $k/(1-\beta) < k/\beta < k/(1-\alpha)$. Decision rules are shown in Table 3.

In the case $0 \leq \beta < 1/2 < \alpha \leq 1$ $(\alpha + \beta < 1)$, we have the inequality $k/(1-\beta) < k/(1-\alpha) < k/\beta$. Decision rules are shown in Table 4.

3.5. The relationships of the LLDDRS model with other models

Firstly, we study the relationships between the LLDDRS model and the four models (RS, DTRS, GRS and LRS) proposed in the preliminaries. Details are shown in Fig. 1.

In Fig. 1, rough sets (RS) lack fault tolerance capabilities because of the strict requirement of the set inclusion relation between equivalence classes and an approximated concept. In order to enhance the practicability of the RS model, decision-

Table 2
The decision rules of LLDDRS model in the case $1/2 \leq \beta < \alpha \leq 1$.

Conditions	$\lvert[x]_R\rvert$	$\lvert[x]_R \cap X\rvert$	Regions	Decisions
$x \in X$	$< k/\beta$	$> \beta\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$[\lvert[x]_R\rvert - k, \beta\lvert[x]_R\rvert]$	$LbnLR_{\vee}$	Noncommitment
		$< \lvert[x]_R\rvert - k$	$negLR_{\vee}$	Rejection
	$[k/\beta, 2k)$	$> k$	$posLR_{\vee}$	Acceptance
		$[\lvert[x]_R\rvert - k, k]$	$LbnLR_{\vee}$	Noncommitment
		$< \lvert[x]_R\rvert - k$	$negLR_{\vee}$	Rejection
	$[2k, k/(1-\alpha))$	$\geq \lvert[x]_R\rvert - k$	$posLR_{\vee}$	Acceptance
		$(k, \lvert[x]_R\rvert - k)$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
	$\geq k/(1-\alpha)$	$\geq \alpha\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$(k, \alpha\lvert[x]_R\rvert)$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
$x \in U_0 - X$	$< k/\beta$	$> \beta\lvert[x]_R\rvert$	$UbnLR_{\vee}$	Noncommitment
		$\leq \beta\lvert[x]_R\rvert$	$negLR_{\vee}$	Rejection
	$(k/\beta, 2k)$	$> k$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
	$\geq 2k$	$> k$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
$x \in U - U_0$	$\mathbf{N}^+$	$\mathbf{N}^+$	$negLR_{\vee}$	Rejection

Table 3
The decision rules of LLDDRS in the case $0 \leq \beta < 1/2 < \alpha \leq 1$, $\alpha + \beta \geq 1$.

Conditions	$\lvert[x]_R\rvert$	$\lvert[x]_R \cap X\rvert$	Regions	Decisions
$x \in X$	$< k/(1-\beta)$	$> \beta\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$[\lvert[x]_R\rvert - k, \beta\lvert[x]_R\rvert]$	$LbnLR_{\vee}$	Noncommitment
		$< \lvert[x]_R\rvert - k$	$negLR_{\vee}$	Rejection
	$[k/(1-\beta), k/\beta)$	$\geq \lvert[x]_R\rvert - k$	$posLR_{\vee}$	Acceptance
		$(\beta\lvert[x]_R\rvert, \lvert[x]_R\rvert - k)$	$UbnLR_{\vee}$	Noncommitment
		$\leq \beta\lvert[x]_R\rvert$	$negLR_{\vee}$	Rejection
	$[k/\beta, k/(1-\alpha))$	$\geq \lvert[x]_R\rvert - k$	$posLR_{\vee}$	Acceptance
		$(k, \lvert[x]_R\rvert - k)$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
	$\geq k/(1-\alpha)$	$\geq \alpha\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$(k, \alpha\lvert[x]_R\rvert)$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
$x \in U_0 - X$	$< k/\beta$	$> \beta\lvert[x]_R\rvert$	$UbnLR_{\vee}$	Noncommitment
		$\leq \beta\lvert[x]_R\rvert$	$negLR_{\vee}$	Rejection
	$\geq k/\beta$	$> k$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
$x \in U - U_0$	$\mathbf{N}^+$	$\mathbf{N}^+$	$negLR_{\vee}$	Rejection

Table 4
The decision rules of LLDDRS in the case $0 \leq \beta < 1/2 < \alpha \leq 1$, $\alpha + \beta < 1$.

Conditions	$\lvert[x]_R\rvert$	$\lvert[x]_R \cap X\rvert$	Regions	Decisions
$x \in X$	$< k/(1-\beta)$	$> \beta\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$[\lvert[x]_R\rvert - k, \beta\lvert[x]_R\rvert]$	$LbnLR_{\vee}$	Noncommitment
		$< \lvert[x]_R\rvert - k$	$negLR_{\vee}$	Rejection
	$[k/(1-\beta), k/(1-\alpha))$	$\geq \lvert[x]_R\rvert - k$	$posLR_{\vee}$	Acceptance
		$(\beta\lvert[x]_R\rvert, \lvert[x]_R\rvert - k)$	$UbnLR_{\vee}$	Noncommitment
		$\leq \beta\lvert[x]_R\rvert$	$negLR_{\vee}$	Rejection
	$[k/(1-\alpha), k/\beta)$	$\alpha\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$(\beta\lvert[x]_R\rvert, \alpha\lvert[x]_R\rvert)$	$UbnLR_{\vee}$	Noncommitment
		$\leq \beta\lvert[x]_R\rvert$	$negLR_{\vee}$	Rejection
	$\geq k/\beta$	$\geq \alpha\lvert[x]_R\rvert$	$posLR_{\vee}$	Acceptance
		$(k, \alpha\lvert[x]_R\rvert)$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
$x \in U_0 - X$	$< k/\beta$	$> \beta\lvert[x]_R\rvert$	$UbnLR_{\vee}$	Noncommitment
		$\leq \beta\lvert[x]_R\rvert$	$negLR_{\vee}$	Rejection
	$\geq k/\beta$	$> k$	$UbnLR_{\vee}$	Noncommitment
		$\leq k$	$negLR_{\vee}$	Rejection
$x \in U - U_0$	$\mathbf{N}^+$	$\mathbf{N}^+$	$negLR_{\vee}$	Rejection

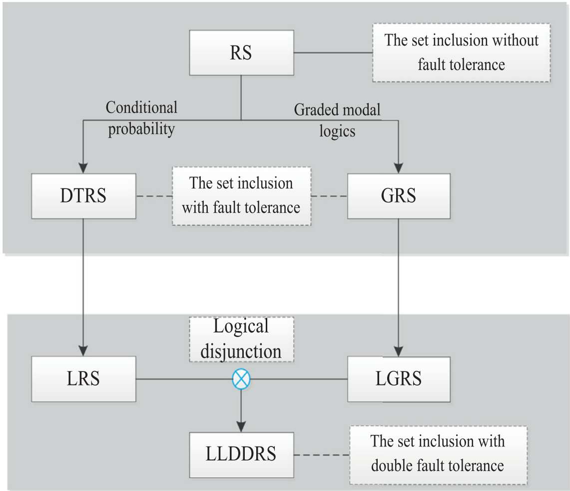


Fig. 1. The relationships of the LLDDRS, RS, DTRS, GRS and LRS models.

theoretic rough sets (DTRS) and graded rough sets (GRS) are two quantitative generalizations of the qualitative RS. The DTRS model quantifies the set inclusion relation based on conditional probability in the approximate operators, which is a representative model considering relative quantitative information between equivalence classes and an approximated concept and has certain fault tolerance capabilities. The GRS model quantifies the set inclusion relation from external grade and internal grade based on the idea of graded modal logics, which is a representative model considering absolute quantitative information between equivalence classes and a concept and has certain fault tolerance capabilities. When decision risk parameters $\alpha = 1, \beta = 0$ and grade parameter $k = 0$, both DTRS and GRS degenerate into RS. These models in first gray box are called global rough sets.

Local rough sets (LRS) can do effectively and efficiently rough data analysis in large data sets by deeply mining the essence of approximation concepts of rough set model. The LRS model is an extension and innovation of RS, which has certain fault tolerance capabilities. When $\alpha = 1, \beta = 0$, LRS degenerates into RS. Considering the relative and absolute quantitative information are two distinctive objective sides that describe approximation spaces and none can be neglected, we first propose local graded rough sets (LGRS), then propose logical disjunction double-quantitative rough sets (LLDDRS) based on relative and absolute quantitative information. The LLDDRS model is computationally efficient in large data sets, which provides an effective tool for rough data analysis in large data sets. Moreover, the LLDDRS model as the extension and innovation of RS has double fault tolerance capabilities in concept approximation, which can provide more accurate descriptions of concepts when compared it with LRS. When $k = 0$, LGRS degenerates into RS, and when $\alpha = 1, \beta = 0, k = 0$, LLDDRS degenerates into RS. These models in second gray box are called local rough sets. Compared with the corresponding global models, each local model has computational advantages in large data sets.

Secondly, from the perspective of computational efficiency, we present the connections of the proposed LLDDRS model and some incremental learning models. The common points of the following models and the LLDDRS model are that the basic rough set models studied need to be improved to adapt to complex data changes such as large-scale data, dynamic data, and these rough data analysis methods cannot satisfy the requirements of efficient computation in large data sets. Therefore, scholars propose efficient rough approximation updating or attribute reduction methods for data mining in large or dynamic data sets.

- Jing et al. studied reduction approaches based on knowledge granularity in large-scale decision systems with the change of objects [9] and with the simultaneous change of objects and attributes [10]. Their focus is to propose an efficient attribute reduction method based on knowledge granularity for dynamic large data sets. We focus more on developing an efficient knowledge discovery tool to provide more accurate approximation of concepts based on double-quantitative information.
- Luo et al. [18] studied systematically the updating processes of condition granules, decision granules and dominance-based rough approximations with the cut refinement and coarsening of attribute value taxonomies in hierarchical multi-criteria decision systems. Their focus is to propose efficient concept approximation methods for hierarchical multicriteria decision systems with dynamic change of attribute values. Our focus is to propose efficient concept approximation and decision methods for large data sets.
- Yang et al. [39] proposed two efficient incremental algorithms for fuzzy rough set based feature selection in large-scale real-valued data sets from the viewpoint of the successive arrival of sample subsets. Their focus is on feature selection methods for efficient processing of large-scale real-valued data sets. The emphasis of this paper is to propose efficient concept description and decision methods in large data sets.

Table 5
Initial medical data.

U	a_1	a_2	d	U	a_1	a_2	d	U	a_1	a_2	d
x_1	0	0	0	x_{13}	0	0	0	x_{25}	0	2	0
x_2	1	1	0	x_{14}	2	1	1	x_{26}	2	2	1
x_3	0	2	1	x_{15}	0	1	1	x_{27}	1	1	0
x_4	2	1	0	x_{16}	1	1	0	x_{28}	2	0	1
x_5	1	0	1	x_{17}	0	2	0	x_{29}	2	1	1
x_6	2	2	1	x_{18}	2	1	1	x_{30}	0	0	0
x_7	0	0	0	x_{19}	0	0	0	x_{31}	1	2	0
x_8	1	2	0	x_{20}	1	2	1	x_{32}	0	1	0
x_9	2	2	1	x_{21}	2	0	1	x_{33}	2	1	1
x_{10}	1	1	1	x_{22}	0	0	0	x_{34}	1	1	1
x_{11}	1	2	1	x_{23}	2	1	0	x_{35}	0	0	0
x_{12}	2	0	0	x_{24}	1	2	1	x_{36}	2	0	0

- Yang et al. proposed the general model and multilevel incremental mechanisms and algorithms of sequential three-way decisions [40] and developed the methods for dynamic updating three-way regions in multilevel variations of data [41]. Their focuses are to provide a cost-effective decision method from the perspective of multiple granular structures for complex problem solving and propose a unified framework for incrementally updating three-way probabilistic regions, respectively. Our focus is to propose effective and efficient concept approximation and decision-making methods based on double-quantitative information and local rough sets for large data sets.

Finally, from the perspective of quantitative completeness extensions, we analyze the relationships between some double-quantitative rough set models.

- Yao and Deng [34] proposed a framework of quantitative rough sets encompassing both probabilistic and non-probabilistic based on subsethood measures, and established some relationships between these models based on different classes of subsethood measures from a theoretical point of view. Their focus is to put forward a more general framework of quantitative global rough sets. We mainly focus on the computational efficiency of the double quantification model in large data sets.
- Zhang et al. studied systematically two specific double-quantitative rough set models based on the crossed combinations of the approximations in the variable precision rough sets and GRS models [49], and deeply studied quantitative semantics, complete system and optimal calculation of double-quantification and rough set models in the double-quantitative approximation space of precision and grade (PG-Approx-Space) [50]. Their focus is to propose powerful tools for approximation description and knowledge discovery in increasingly complex environments based on double-quantitative information. Our focus is also to propose such powerful tools for approximation description and knowledge discovery in large data sets. Meanwhile, we pay special attention to the computational efficiency of the proposed double-quantitative model.

Based on the above analyses, the LLDDRS model with strong double fault tolerance capabilities can not only provide a thorough description of the approximation space but also satisfies the requirement of efficient computation to some extent in large data sets.

4. Case study

Compared with Pawlak rough sets and local rough sets, the feasibility and superiority of the LLDDRS model are illustrated by a medical example from literature [48]. Let $I = \{U, A \cup d, V, f\}$ be a decision information table, where the universe U is composed of 36 patients. The condition attribute set $A = \{a_1, a_2\}$ represents *fever* and *headache*, respectively. And the decision attribute d represents *cold*. The detailed statistics are shown in Table 5, where the attribute values {0, 1, 2} denote *have no such symptom, suffer from the symptom* and *have the severe symptom*, respectively.

There are 2 decision equivalence classes that can be obtained, which can be denoted by $D_1 = \{x_1, x_2, x_4, x_7, x_8, x_{12}, x_{13}, x_{16}, x_{17}, x_{19}, x_{22}, x_{23}, x_{25}, x_{27}, x_{30}, x_{31}, x_{32}, x_{35}, x_{36}\}$ and $D_2 = \{x_3, x_5, x_6, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{34}\}$. D_1 and D_2 denote *not cold* and *cold*, respectively. The conditional equivalence classes and the intersections about decision classes are shown in Table 6, where P_1 and P_2 are the conditional probabilities of equivalence classes about D_1 and D_2, respectively.

According to the information in Table 6, if only the relative quantitative information (namely conditional probability) is considered, the class x_6, x_9, x_{26} and the class x_5 are indiscernible in the LRS model. But the cardinalities of intersection of these two classes about D_2 are completely different. So, x_6, x_9, x_{26} and x_5 are discernible in the LGRS model. Therefore, the LRS model has some disadvantages sometimes. Introducing the grade information into the LRS model is helpful to the approximation and decision of the target concept. In addition, if only the absolute relative quantitative information (namely the degree of information) is considered, the class equivalence x_{11}, x_{20}, x_{24} and the equivalence class x_6, x_9, x_{26} are indiscernible in the LGRS model. However, the class equivalence x_{11}, x_{20}, x_{24} and the equivalence class x_6, x_9, x_{26} are

Table 6
The information of equivalence classes.

(i, j)	$[x]_R$	$\|[x]_R\|$	$[x]_R \cap D_1$	$\|[x]_R \cap D_1\|$	P_1	$[x]_R \cap D_2$	$\|[x]_R \cap D_2\|$	P_2
(0,0)	$x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}, x_{35}$	7	$x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}, x_{35}$	7	1	$\emptyset$	0	0
(0,1)	x_{15}, x_{32}	2	x_{32}	1	1/2	x_{15}	1	1/2
(0,2)	x_3, x_{17}, x_{25}	3	x_{17}, x_{25}	2	2/3	x_3	1	1/3
(1,0)	x_5	1	$\emptyset$	0	0	x_5	1	1
(1,1)	$x_2, x_{10}, x_{16}, x_{27}, x_{34}$	5	x_2, x_{16}, x_{27}	3	3/5	x_{10}, x_{34}	2	2/5
(1,2)	$x_8, x_{11}, x_{20}, x_{24}, x_{31}$	5	x_8, x_{31}	2	2/5	x_{11}, x_{20}, x_{24}	3	3/5
(2,0)	$x_{12}, x_{21}, x_{28}, x_{36}$	4	x_{12}, x_{36}	2	1/2	x_{21}, x_{28}	2	1/2
(2,1)	$x_4, x_{14}, x_{18}, x_{23}, x_{29}, x_{33}$	6	x_4, x_{23}	2	1/3	$x_{14}, x_{18}, x_{29}, x_{33}$	4	2/3
(2,2)	x_6, x_9, x_{26}	3	$\emptyset$	0	0	x_6, x_9, x_{26}	3	1

discernible in the LRS model. That is to say, the LGRS model also has some disadvantages in some circumstances. The LRS model and the LGRS model are complementary to each other. Therefore, by considering the user's requirement for the approximation accuracy of concepts, we propose the LLDDRS model.

4.1. The elaboration of the LLDDRS model

In the following, D_2 is selected as the concept to illustrate the proposed local logical double-quantitative rough set theory. $U_2 = \cup\{[x]_R | x \in D_2\} = U - \{x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}, x_{35}\}$. The decision risk parameters α, β is determined by the decision loss function given by the experts in the relevant field and grade parameter k is determined by user requirements. In this section, the parameter k is set to 1. Since rough regions of the LLDDRS model are related to two variables β and $\alpha + \beta$, the local logical disjunction double-quantitative rough set theory is elaborated from four cases. These cases are mutually exclusive. When the loss function is given, it must belong to one of the four cases.

Case 1: $\beta < 1/2, \alpha + \beta = 1$

When loss parameters $\lambda_{PP} = 0, \lambda_{PN} = 14,\ \lambda_{BP} = 8, \lambda_{BN} = 2,\ \lambda_{NP} = 11, \lambda_{NN} = 0$ in a loss function, the decision risk parameters can be calculated as $\alpha = 0.6,\ \beta = 0.4$.

According to Definition 3.2, the upper and lower approximations in the LLDDRS model are

$\overline{LR_{(0.6,0.4)\vee 1}}(D_2) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{27}, x_{28}, x_{29}, x_{31}, x_{32}, x_{33}, x_{34}, x_{36}\}$,

$\underline{LR_{(0.6,0.4)\vee 1}}(D_2) = \{x_5, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}, x_{33}\}$.

And the rough regions of the LLDDRS model can be obtained as follows:

$posLR_\vee(D_2) = \{x_5, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}, x_{33}\}$;

$negLR_\vee(D_2) = \{x_1, x_3, x_7, x_{13}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{35}\}$;

$UbnLR_\vee(D_2) = \{x_2, x_4, x_8, x_{10}, x_{12}, x_{16}, x_{21}, x_{23}, x_{27}, x_{28}, x_{31}, x_{32}, x_{34}, x_{36}\}$;

$LbnLR_\vee(D_2) = \emptyset$.

Of course, rough regions of the LLDDRS model can be calculated by logical disjunction decision rules. When $\alpha = 0.6$, $\beta = 0.4$, we have $k/\alpha = k/(1-\beta) = 5/3,\ k/\beta = k/(1-\alpha) = 5/2$. Patients x_6, x_9, x_{11}, x_{14}, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}, x_{33} have a cold by the first rule of positive region rules $P^\vee$; patient x_{15} has a cold by the second rule of $P^\vee$; and patient x_5 has a cold by the third rule of $P^\vee$. Therefore, patients x_5, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{24}, x_{26}, x_{29}, x_{33} have a cold according to the positive region rules. In addition, patients x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}, x_{35} have no cold by the first rule of negative region rules $N^\vee$; according to the second and third rules of $N^\vee$, the same conclusion was obtained, that is, patients x_3, x_{17}, x_{25} have no cold; and no patient meets the conditions of the fourth and fifth rules of $N^\vee$. Therefore, the patients who have no cold are x_1, x_3, x_7, x_{13}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{35}. Similarly, patients x_2, x_4, x_8, x_{10}, x_{12}, x_{16}, x_{21}, x_{23}, x_{27}, x_{28}, x_{31}, x_{32}, x_{34}, x_{36} are less likely to have a cold according to the upper boundary region rules, and no patient is more likely to catch a cold by the lower boundary region rules.

Two methods are provided to calculate rough regions in this paper, which are the solution method based on upper and lower approximations and the simplest expression method of rough regions in Section 3.3. The second method is more direct and flexible when we only need to know about one region or several regions.

Case 2: $\beta < 1/2, \alpha + \beta < 1$

When $\lambda_{PP} = 0, \lambda_{PN} = 19,\ \lambda_{BP} = 10, \lambda_{BN} = 9,\ \lambda_{NP} = 21, \lambda_{NN} = 0$, we have $\alpha = 0.5$ and $\beta = 0.45$.

The upper and lower approximations and rough regions of the LLDDRS model are given as follows:

$\overline{LR_{(0.5,0.45)\vee 1}}(D_2) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{27}, x_{28}, x_{29}, x_{31}, x_{32}, x_{33}, x_{34}, x_{36}\}$,

$\underline{LR_{(0.5,0.45)\vee 1}}(D_2) = \{x_5, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}\}$.

$posLR_\vee(D_2) = \{x_5, x_6, x_9, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}\}$;

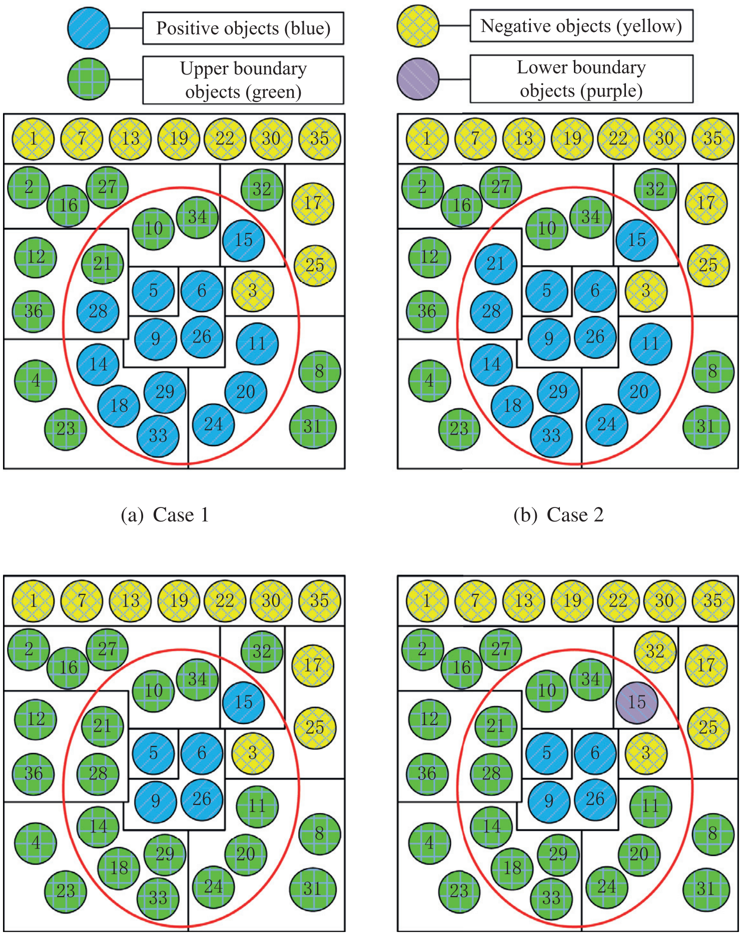
$negLR_\vee(D_2) = \{x_1, x_3, x_7, x_{13}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{35}\}$;

$UbnLR_\vee(D_2) = \{x_2, x_4, x_8, x_{10}, x_{12}, x_{16}, x_{23}, x_{27}, x_{31}, x_{32}, x_{34}, x_{36}\}$;

$LbnLR_\vee(D_2) = \emptyset$.

Case 3: $\beta < 1/2, \alpha + \beta > 1$

When $\lambda_{PP} = 0, \lambda_{PN} = 22,\ \lambda_{BP} = 6, \lambda_{BN} = 8,\ \lambda_{NP} = 18, \lambda_{NN} = 0$, we have $\alpha = 0.7$ and $\beta = 0.4$.



(a) Case 1 (b) Case 2

(c) Case 3 (d) Case 4

Fig. 2. Rough region information of the LLDDRS model under different cases.

The upper and lower approximations and rough regions of the LLDDRS model are presented as follows:
$\overline{LR_{(0.7,0.4)\vee 1}}(D_2) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{15}, x_{16}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{27}, x_{28}, x_{29}, x_{31}, x_{32}, x_{33}, x_{34}, x_{36}\}$,
$\underline{LR_{(0.7,0.4)\vee 1}}(D_2) = \{x_5, x_6, x_9, x_{15}, x_{26}\}$.
$\overline{posLR_{\vee}(D_2)} = \{x_5, x_6, x_9, x_{15}, x_{26}\}$;
$negLR_{\vee}(D_2) = \{x_1, x_3, x_7, x_{13}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{35}\}$;
$UbnLR_{\vee}(D_2) = \{x_2, x_4, x_8, x_{10}, x_{11}, x_{12}, x_{14}, x_{16}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{27}, x_{28}, x_{29}, x_{31}, x_{32}, x_{33}, x_{34}, x_{36}\}$;
$LbnLR_{\vee}(D_2) = \emptyset$. Case 4: $\beta \geq 1/2, \alpha + \beta > 1$
When $\lambda_{PP} = 0, \lambda_{PN} = 42,\ \lambda_{BP} = 6, \lambda_{BN} = 18,\ \lambda_{NP} = 18, \lambda_{NN} = 0$, we have $\alpha = 0.8$ and $\beta = 0.6$.
The upper and lower approximations and rough regions of the LLDDRS model are given as follows:
$\overline{LR_{(0.8,0.6)\vee 1}}(D_2) = \{x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{14}, x_{16}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{26}, x_{27}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$,
$\underline{LR_{(0.8,0.6)\vee 1}}(D_2) = \{x_5, x_6, x_9, x_{15}, x_{26}\}$.
$\overline{posLR_{\vee}(D_2)} = \{x_5, x_6, x_9, x_{26}\}$;
$negLR_{\vee}(D_2) = \{x_1, x_3, x_7, x_{13}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{32}, x_{35}\}$;
$UbnLR_{\vee}(D_2) = \{x_2, x_4, x_8, x_{10}, x_{11}, x_{12}, x_{14}, x_{16}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{27}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}, x_{36}\}$;
$LbnLR_{\vee}(D_2) = \{x_{15}\}$.

The region information of the LLDDRS model under different cases is reflected in Fig. 2. It is necessary to point out that in Fig. 2, polygons represent equivalence classes; the red ellipse represents the decision class D_2; and the blue, yellow, green and purple solid circles denote the elements in the positive, negative, upper, and lower boundary regions, respectively. And $i \in 1, 2, \ldots, 36$ is the abbreviation of x_i, which represents the ith patient.

From Fig. 2, we can find objects x_5, x_6, x_9, x_{26} always belong to the positive region $posLR_{\vee}(D_2)$ in four cases and object x_{15} belongs to the positive region in three cases; objects x_1, x_3, x_7, x_{13}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{32}, x_{35} always belong to

Table 7
The added objects of lower approximation sets of two classes.

Grade k	$(\alpha=0.6, \beta=0.4)$		$(\alpha=0.5, \beta=0.45)$	
	D_1	D_2	D_1	D_2
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	x_{32}	x_{15}	$\emptyset$	$\emptyset$
2	x_{12}, x_{32}, x_{36}	$x_3, x_{15}, x_{21}, x_{28}$	$\emptyset$	x_3
3	$x_8, x_{12}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{15}, x_{21}, x_{28}, x_{34}$	x_8, x_{31}	x_3, x_{10}, x_{34}
4	$x_4, x_8, x_{12}, x_{23}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{15}, x_{21}, x_{28}, x_{34}$	x_4, x_8, x_{23}, x_{31}	x_3, x_{10}, x_{34}
5	$x_4, x_8, x_{12}, x_{23}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{15}, x_{21}, x_{28}, x_{34}$	x_4, x_8, x_{23}, x_{31}	x_3, x_{10}, x_{34}
6	$x_4, x_8, x_{12}, x_{23}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{15}, x_{21}, x_{28}, x_{34}$	x_4, x_8, x_{23}, x_{31}	x_3, x_{10}, x_{34}
7	$x_4, x_8, x_{12}, x_{23}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{15}, x_{21}, x_{28}, x_{34}$	x_4, x_8, x_{23}, x_{31}	x_3, x_{10}, x_{34}
k	$(\alpha=0.7, \beta=0.4)$		$(\alpha=0.8, \beta=0.6)$	
	D_1	D_2	D_1	D_2
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	x_{17}, x_{25}, x_{32}	x_{15}	x_{17}, x_{25}, x_{32}	x_{15}
2	$x_2, x_{12}, x_{16}, x_{17}$ $x_{25}, x_{27}, x_{32}, x_{36}$	$x_3, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}$	$x_2, x_{12}, x_{16}, x_{17}$ $x_{25}, x_{27}, x_{32}, x_{36}$	$x_3, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}$
3	$x_2, x_8, x_{12}, x_{16}, x_{17}$ $x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$	$x_2, x_8, x_{12}, x_{16}, x_{17}$ $x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$
4	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$
5	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$
6	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$
7	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$	$x_2, x_4, x_8, x_{12}, x_{16}, x_{17}$ $x_{23}, x_{25}, x_{27}, x_{31}, x_{32}, x_{36}$	$x_3, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}$ $x_{20}, x_{21}, x_{24}, x_{28}, x_{29}, x_{33}, x_{34}$

Table 8
The approximation accuracies of LLDDRS.

(α, β)	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
(0.6, 0.4)	$\frac{23}{36}$	$\frac{25}{36}$	$\frac{5}{6}$	$\frac{17}{18}$	1	1	1	1
(0.5, 0.45)	$\frac{29}{36}$	$\frac{29}{36}$	$\frac{5}{6}$	$\frac{17}{18}$	1	1	1	1
(0.7, 0.4)	$\frac{11}{36}$	$\frac{15}{36}$	$\frac{5}{6}$	$\frac{17}{18}$	1	1	1	1
(0.8, 0.6)	$\frac{11}{36}$	$\frac{15}{36}$	$\frac{5}{6}$	$\frac{17}{18}$	1	1	1	1

the negative region $negLR_{\vee}(D_2)$ in four cases; objects x_2, x_4, x_8, x_{10}, x_{12}, x_{16}, x_{23}, x_{27}, x_{31}, x_{32}, x_{34}, x_{36} belong to the upper boundary region $UbnLR_{\vee}(D_2)$ in four cases and objects x_{21}, x_{32} belong to the upper boundary region $UbnLR_{\vee}(D_2)$ in three cases; the lower boundary region is almost close to $\emptyset$; and objects x_{11}, x_{14}, x_{18}, x_{20}, x_{24}, x_{28}, x_{29}, x_{33} lie between the positive region and the upper boundary region.

4.2. Comparison of different models

Firstly, according to the definition of the lower approximation in the Pawlak rough set, there are
$\underline{R_P}(D_1)=\{x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}, x_{35}\}$, $\underline{R_P}(D_2)=\{x_5, x_6, x_9, x_{26}\}$.

Then the approximation accuracy of A with respect to d in the RS model is $\rho_p(A,d)=\frac{|\underline{R_P}(D_1)|+|\underline{R_P}(D_2)|}{|D_1|+|D_2|}=\frac{11}{36}$.

Similarly, the approximation accuracies of A with respect to d in the LRS model can be calculated as follows:
$\rho_{(0.6,0.4)}(A,d)=\frac{23}{36}$, $\rho_{(0.5,0.45)}(A,d)=\frac{29}{36}$, $\rho_{(0.7,0.4)}(A,d)=\frac{11}{36}$, $\rho_{(0.8,0.6)}(A,d)=\frac{11}{36}$.

Before solving the approximation accuracy in the LLDDRS model, the objects added to the lower approximation sets of decision classes due to the introduction of grade information are given in Table 7.

From Table 7, when grade k is equal to 0, the lower approximation sets of two decision classes in the LLDDRS model are equivalent to the lower approximation sets in the LRS model. When $k \geq 1$, in most cases, the lower approximation sets of two decision classes in the LLDDRS model contains the lower approximation sets in the LRS model. Then under different grade demands, the approximation accuracies in the LLDDRS model are shown in Table 8.

From Table 8, when $k=0$, the approximation accuracy of A with respect to d in the LLDDRS model is equivalent to the corresponding result of the LRS model, and the approximation accuracies of these two models are greater than or equal to the result of the RS model. With the increase of k, the performance of the LLDDRS model is much better than the RS model and the LRS model. Therefore, it is necessary to introduce grade information into local rough sets (LRS). Parameter k is used to characterize the intersection degree of equivalence classes and the complementary set of target concept in the lower approximation, which is also called external grade. The increase of k means that the requirement for the external

Table 9
Data information.

Data sets	Abbreviation	Samples	Condition attributes	Decision classes
Car evaluation	Car	1728	6	4
Chess (King-Rook vs. King-Pawn)	Chess	3196	36	2
Statlog (landsat satellite)	Landsat	6435	36	6
Nursery	Nursery	12,960	8	5
Online news popularity	Online	39,797	60	2
Statlog (Shuttle)	Shuttle	58,000	8	17

inclusion degree of equivalence classes about a target concept is reduced. In real life, people can set the value of k according to different actual demands.

5. Experimental analysis

In this section, we verify the feasibility and validity of the proposed Local Logical Disjunction Double-quantitative Rough Set (LLDDRS) model for rough analysis on large data sets. Firstly, by comparing the running time of the lower approximation set of LLDDRS with that of the corresponding global logical disjunction double-quantitative rough set model (GLDDRS), the superiority of the LLDDRS model in dealing with large data sets is verified. Secondly, the necessity of considering double quantification indices in local models is illustrated by the contribution rate of double quantification indices in concept approximation. Finally, the advantage of the LLDDRS model in concept approximation is verified by comparing it with classical Pawlak rough sets (RS) [20] and local rough sets (LRS) [24] based on approximation accuracy.

The experimental data and basic settings are as follows. Six data sets derived from UCI Repository of machine learning databases (http://archive.ics.uci.edu/ml/datasets.html) are used in the experimental analysis. Detailed information is shown in Table 9. All the experiments are performed in Matlab 2015b and run in a hardware environment with 2.6 GHz CPU, 8.0 GB of memory and 64-bit Windows 10. For numerical data sets (Landsat, Online and Shuttle), we employ a fuzzy C-means clustering (FCM) technique to discretize numerical data into two nominal values based on attributes according to the comparative results of different discretization methods in [8].

Ten experiments were conducted on each data set and each decision class in each data set is divided into 10 parts. Let the universe U_i of the $i^{th}(i\in\{1, 2, 3, 4, 5, 6\})$ data set S_i be divided into t decision classes based on decision attributes D^i, namely $U/D^i = \{D_1, D_2, \ldots, D_t\}$. In the first experiment, we selected the top ten percent of each decision class as experimental data, namely $D_{11}+D_{21}+\cdots+D_{t1}$. In the second experiment, we selected the top twenty percent of each decision class as experimental data, namely $D_{11}+D_{12}+D_{21}+D_{22}+\cdots+D_{t1}+D_{t2}$. By analogy, the tenth experiment runs on the entire data set S_i.

5.1. The computational efficiency of the LLDDRS model

In this subsection, we compare the time consumption of local and global logical disjunction double-quantitative rough set models (LLDDRS and GLDDRS) in computing approximation sets to illustrate the superiority of the LLDDRS model in processing large data. In each experiment on each data set, we select the first decision class on the current data set as the target concept. Considering the moderate risk preference, we set the decision risk parameters α, β to 0.8 and 0.2, respectively. In order to enhance the reliability of the experiments, we increase k from 1 to 5 step by step, and take the average of five results as the final experimental results. According to the analysis in Section 2.3, the time consumption of upper approximation of these two models is the same in essence by the optimization of global model. We give the computational time of the lower approximation in LLDDRS and GLDDRS models. Detailed results are shown in Table 10, where the computational time is measured in seconds.

As can be seen from Table 10, the time consumption of the LLDDRS model in approximate calculation is always less than that of the global logical disjunction double-quantitative rough set (GLDDRS) model. More intuitive comparisons are shown in Fig. 3. From Fig. 3, we can see intuitively that the LLDDRS model performs better than the GLDDRS model in terms of computing time of concept approximation.

At the same time, in order to understand the fluctuation of running time with the change of grade parameter k, the average running time of each model is calculated on six data sets. Details are shown in Table 11.

From Table 11, we can also see that the LLDDRS model is stable and good. In fact, by using time complexity analysis in Section 3.1, the running time of the GLDDRS model is approximately $|U|/|X|$ times that of the LLDDRS model, where $|U|$ is the number of objects in a data set and $|X|$ is the number of objects in the target concept X. When $|X| \ll |U|$, the superiority of the LLDDRS model in processing large data is obvious.

5.2. The importance of double quantification indices

The LLDDRS model uses conditional probability and internal and external grades to reflect the relative and absolute quantitative information between information granules and the target concept. We illustrate the importance of these two

Table 10
The running time of the lower approximation of the target concept in LLDDRS and GLDDRS models.

Data	Models	Experiments									
		1st	2nd	3th	4th	5th	6th	7th	8th	9th	10th
Car	LLDDRS	0.0371	0.0801	0.1682	0.2781	0.4334	0.5780	0.7563	0.9568	1.2027	1.4744
	GLDDRS	0.0494	0.1091	0.2216	0.4462	0.5963	0.7992	1.0493	1.3532	1.6999	1.9939
Chess	LLDDRS	0.0683	0.1913	0.4622	0.6989	1.0623	1.4688	1.9713	2.5465	3.2955	4.1513
	GLDDRS	0.1190	0.3937	0.7872	1.3312	1.9632	2.7916	3.5119	4.4665	4.9637	5.9914
Landsat	LLDDRS	0.0778	0.2615	0.5659	0.9814	1.5279	2.1945	3.0069	3.7305	4.8846	5.7193
	GLDDRS	0.3479	1.0660	2.3773	3.9871	6.1417	8.7966	12.1174	15.7762	19.7405	24.0861
Nursery	LLDDRS	0.4145	1.6067	3.3070	5.8738	8.5053	10.7569	14.3608	18.6430	23.6335	28.8814
	GLDDRS	1.0796	4.1672	8.8823	14.1527	21.9379	31.4657	42.8750	56.7955	71.2620	88.2769
Online	LLDDRS	1.8262	7.1961	16.8827	28.9959	46.8481	67.6585	92.7806	119.7735	153.3008	182.8564
	GLDDRS	9.8514	40.1275	90.7077	159.1994	259.1084	363.4057	498.6125	649.3364	838.5681	999.1519
Shuttle	LLDDRS	14.5389	56.9905	128.1822	228.8923	352.8161	511.8825	696.5359	914.8937	1156.8072	1401.4428
	GLDDRS	18.7416	79.4400	180.4212	323.6495	496.7949	717.2040	979.6348	1293.0284	1633.6303	1982.0055

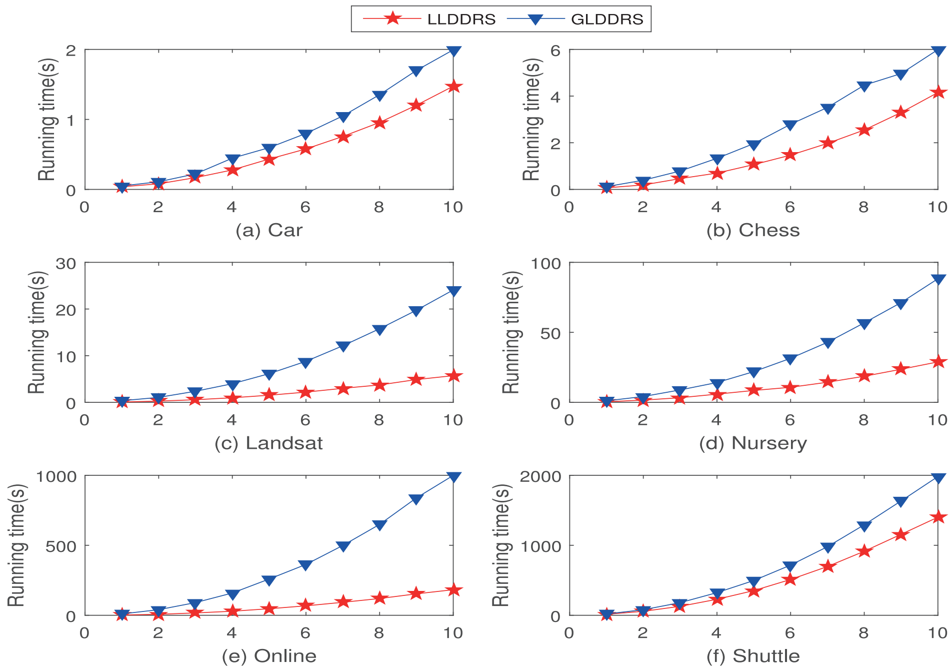


Fig. 3. Comparisons of running time results in the LLDDRS and GLDDRS models.

quantitative indices by their contribution rates in approximate concept. On the above six data sets, the contribution rates of two indicators in each experiment are shown in Tables 12–17. In Tables 12–17, symbol T denotes the ordinal number of experiments, and symbol * denotes that no object is distinguished only by relative (or absolute) information. And $C_{\overline{g}}$ is the unique contribution rate of absolute quantitative information, $C_{\overline{p}}$ is the unique contribution rate of relative quantitative information and $C_{p\vee g}$ is the shared contribution rate of double-quantitative information.

From Tables 12 and 13, on data Car and Chess, many objects cannot be distinguished within the probability threshold, but they can be distinguished by using grade information. From Tables 14, 17, on data Landsat and Shuttle, some objects can not be distinguished within the probability threshold, but they can be distinguished by using grade information, and many objects can not be distinguished within grade threshold, but they can be distinguished by using probability information. From Table 15, on data Nursery, some objects can not be distinguished within the probability threshold, but they can be distinguished by using grade information in the first four experiments. From Table 16, on data Online, many objects can not be distinguished within the probability threshold, but they can be distinguished by using grade information, and some objects can not be distinguished within grade threshold, but they can be distinguished by using probability information. Furthermore, on six data sets, we can see that many objects can only be identified by grade information. This further indicates that only the relative quantitative information considering in concept approximation in local models may lead to the inaccuracy and uncertainty of concept approximation, and the conclusion is the same as that in literature [49,50].

Table 11
The average running time of the lower approximation of the target concept in LLDDRS and GLDDRS models.

Data	Models	Experiments									
		1st	2nd	3th	4th	5th	6th	7th	8th	9th	10th
Car	LLDDRS	0.0364 ± 0.0020	0.0802 ± 0.0035	0.1778 ± 0.0251	0.2891 ± 0.0275	0.4385 ± 0.0559	0.5957 ± 0.0425	0.7572 ± 0.0106	0.9623 ± 0.0268	1.2064 ± 0.0259	1.4883 ± 0.0306
	GLDDRS	0.0492 ± 0.0018	0.1041 ± 0.0144	0.2134 ± 0.0315	0.4308 ± 0.1308	0.6242 ± 0.0635	0.7995 ± 0.0082	1.0482 ± 0.007	1.3637 ± 0.0310	1.7081 ± 0.0293	1.9402 ± 0.1035
Chess	LLDDRS	0.0687 ± 0.0029	0.1919 ± 0.0077	0.4536 ± 0.0602	0.7141 ± 0.0520	1.0814 ± 0.0665	1.4705 ± 0.0316	1.9739 ± 0.0408	2.5761 ± 0.0766	3.3855 ± 0.6797	4.2696 ± 1.1092
	GLDDRS	0.1219 ± 0.0083	0.4125 ± 0.0562	0.7950 ± 0.0800	1.3425 ± 0.0630	1.9704 ± 0.0479	2.7985 ± 0.0394	3.4860 ± 0.2351	4.3476 ± 0.5336	4.9932 ± 0.1661	6.0173 ± 0.1057
Landsat	LLDDRS	0.0773 ± 0.0017	0.2615 ± 0.0076	0.5619 ± 0.0188	0.9736 ± 0.0403	1.5037 ± 0.0464	2.1731 ± 0.0964	3.0099 ± 0.1439	3.7522 ± 0.075	4.8225 ± 0.161	5.7135 ± 0.0255
	GLDDRS	0.4405 ± 0.1617	1.1395 ± 0.1331	2.5343 ± 0.3254	3.9999 ± 0.0834	6.1199 ± 0.1084	8.8208 ± 0.1526	12.2677 ± 0.4359	15.9254 ± 0.5072	19.9567 ± 0.6812	24.3987 ± 0.7449
Nursery	LLDDRS	0.4139 ± 0.0170	1.6320 ± 0.1076	3.2508 ± 0.1374	5.8035 ± 0.2640	8.4906 ± 0.7127	10.8145 ± 0.2867	14.4031 ± 0.0957	18.5986 ± 0.2285	23.6097 ± 0.1573	28.8507 ± 0.2824
	GLDDRS	1.0681 ± 0.1214	4.0951 ± 0.5001	9.0208 ± 0.9924	14.1647 ± 0.1926	21.9224 ± 0.1495	31.3861 ± 0.2536	42.8043 ± 0.4388	57.9888 ± 2.4203	72.3376 ± 2.1061	90.2136 ± 4.4167
Online	LLDDRS	1.8327 ± 0.0783	7.2023 ± 0.1191	17.5414 ± 1.4362	29.1379 ± 0.8499	46.9845 ± 0.7310	67.9139 ± 1.8280	92.4512 ± 0.7750	119.7205 ± 0.7520	153.9327 ± 1.6165	183.2612 ± 2.0977
	GLDDRS	9.9168 ± 0.2447	41.0220 ± 1.9328	91.1574 ± 1.3320	159.1034 ± 1.9668	258.5769 ± 1.8871	363.9353 ± 2.9406	498.4074 ± 1.9284	648.4028 ± 3.8551	846.5226 ± 20.0261	998.0077 ± 4.7834
Shuttle	LLDDRS	14.5178 ± 0.1718	57.2001 ± 0.4694	128.1599 ± 0.7019	227.9144 ± 2.6598	352.8372 ± 0.8394	511.4598 ± 2.1838	696.3785 ± 2.5774	912.4170 ± 7.2827	1153.0499 ± 8.6884	1399.9962 ± 5.4039
	GLDDRS	18.8832 ± 0.8395	79.4746 ± 0.7327	180.3319 ± 0.6729	323.6641 ± 0.7755	496.9539 ± 0.9562	717.2569 ± 0.7930	979.7860 ± 0.8818	1293.0906 ± 0.6848	1633.5896 ± 0.5416	1981.5798 ± 1.3955

Table 12
The unique and shared contribution rates of quantitative information to concept approximation on data Car.

T	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$
1	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*
2	96.50	3.50%	*	95.11	4.89%	*	95.11	4.89%	*	95.11	4.89%	*	95.11	4.89%	*
3	77.31	22.69%	*	70.50	29.50%	*	70.50	29.50%	*	70.50	29.50%	*	70.50	29.50%	*
4	80.83	19.17%	*	75.14	24.86%	*	75.14	24.86%	*	75.14	24.86%	*	75.14	24.86%	*
5	74.49	25.51%	*	67.13	32.87%	*	67.13	32.87%	*	67.13	32.87%	*	67.13	32.87%	*
6	69.06	30.94%	*	60.73	39.27%	*	60.73	39.27%	*	60.73	39.27%	*	60.73	39.27%	*
7	63.36	36.64%	*	54.52	45.48%	*	54.52	45.48%	*	54.52	45.48%	*	54.52	45.48%	*
8	65.59	34.41%	*	52.30	47.70%	*	52.30	47.70%	*	52.30	47.70%	*	52.30	47.70%	*
9	62.87	37.13%	*	49.74	50.26%	*	49.74	50.26%	*	49.74	50.26%	*	49.74	50.26%	*
10	67.07	32.93%	*	48.09	51.91%	*	48.09	51.91%	*	48.09	51.91%	*	48.09	51.91%	*

Table 13
The unique and shared contribution rates of quantitative information to concept approximation on data Chess.

T	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$
1	77.44	22.56	*	73.02	26.98	*	72.33	27.67	*	71.88	28.13	*	71.88	28.13	*
2	75.26	24.74	*	67.77	32.23	*	67.24	32.76	*	67.03	32.97	*	67.03	32.97	*
3	70.15	29.85	*	63.11	36.89	*	62.32	37.68	*	62.19	37.81	*	62.19	37.81	*
4	73.87	26.13	*	64.36	35.64	*	63.39	36.61	*	62.65	37.35	*	62.50	37.50	*
5	80.55	19.45	*	70.15	29.85	*	69.09	30.91	*	68.44	31.56	*	68.31	31.69	*
6	84.26	15.74	*	75.24	24.76	*	74.29	25.71	*	73.71	26.29	*	73.59	26.41	*
7	86.78	13.22	*	78.84	21.16	*	77.99	22.01	*	77.47	22.53	*	77.37	22.63	*
8	86.54	13.46	*	78.83	21.17	*	78.02	21.98	*	77.47	22.53	*	77.38	22.62	*
9	86.19	13.81	*	78.35	21.65	*	77.50	22.50	*	76.99	23.01	*	76.91	23.09	*
10	86.60	13.40	*	78.70	21.30	*	77.90	22.10	*	77.29	22.71	*	77.22	22.78	*

Table 14
The unique and shared contribution rates of quantitative information to concept approximation on data Landsat.

T	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$ (%)	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$ (%)	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$ (%)	$C_{\bar p}$
1	91.61	1.07	7.32	91.28	1.42	7.30	98.58	1.42	*	98.58	1.42	*	98.58	1.42	*
2	88.06	2.55	9.39	88.93	2.99	8.08	88.93	2.99	8.08%	97.01	2.99	*	97.01	2.99	*
3	85.90	3.63	10.47	85.76	4.75	9.49	85.71	4.81	9.49%	85.65	4.86	9.48%	95.02	4.98	*
4	83.33	3.96	12.70	84.50	5.28	10.22	83.99	5.85	10.16%	83.87	5.98	10.14%	83.75	6.12	10.13%
5	82.16	4.18	13.66	83.38	5.97	10.65	82.62	6.83	10.55%	82.37	7.11	10.52%	81.97	7.57	10.47%
6	82.43	4.12	13.44	84.47	5.66	9.87	83.91	6.28	9.81%	83.12	7.17	9.71%	82.70	7.63	9.67%
7	83.41	3.84	12.76	85.56	5.57	8.87	84.89	6.31	8.80%	84.32	6.94	8.74%	83.62	7.71	8.67%
8	83.32	4.05	12.63	84.17	5.78	10.05	84.59	6.72	8.69%	84.05	7.31	8.63%	83.50	7.92	8.58%
9	81.94	4.10	13.96	82.46	6.15	11.39	83.33	6.58	10.10%	82.99	7.64	9.37%	82.55	8.13	9.32%
10	73.83	4.49	21.68	73.87	6.60	19.53	83.13	7.15	9.72%	82.98	8.02	9.00%	82.58	8.47	8.96%

Table 15
The unique and shared contribution rates of quantitative information to concept approximation on data Nursery.

T	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$	$C_{p\vee g}$ (%)	$C_{\bar g}$	$C_{\bar p}$
1	79.20	20.80%	*	73.71	26.29%	*	46.92	53.08%	*	40.73	59.27%	*	35.21	64.79%	*
2	100	*	*	98.03	1.97%	*	95.12	4.88%	*	72.50	27.51%	*	66.67	33.33%	*
3	100	*	*	100	*	*	97.81	2.19%	*	94.76	5.24%	*	85.02	14.98%	*
4	97.30	*	2.70%	100	*	*	99.11	0.90%	*	99.11	0.90%	*	99.11	0.90%	*
5	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*
6	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*
7	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*
8	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*
9	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*
10	100	*	*	100	*	*	100	*	*	100	*	*	100	*	*

Table 16
The unique and shared contribution rates of quantitative information to concept approximation on data Online.

T	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$ (%)	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$
1	74.54	24.28	1.18	65.57	34.43	*	59.52	40.48	*	55.59	44.41	*	52.43	47.57	*
2	74.33	24.93	0.74	64.36	35.64	*	58.54	41.46	*	55.08	44.93	*	52.45	47.55	*
3	73.32	25.20	1.48	63.47	36.53	*	56.74	43.26	*	52.68	47.32%	*	50.02%	49.98	*
4	72.56	25.52	1.92	62.58	36.90	0.52%	56.53	43.47	*	52.30	47.70	*	49.27	50.74	*
5	71.72	26.56	1.72	61.21	38.20	0.59%	54.94	44.89	0.16%	50.79	49.21	*	47.77	52.23	*
6	72.44	26.01	1.55	61.53	38.19	0.28%	54.83	45.17	*	50.58	49.42	*	47.35	52.65	*
7	71.94	26.08	1.97	60.93	38.56	0.51%	54.04	45.96	*	49.69	50.31	*	46.68	53.32	*
8	71.35	26.35	2.30	60.59	38.71	0.70%	53.79	46.09	0.12%	49.10	50.90	*	46.15	53.85	*
9	71.20	25.66	3.14	61.22	37.74	1.04%	54.21	45.21	0.58%	49.44	50.03	0.53%	46.15	53.48	0.37%
10	71.91	24.96	3.13	61.52	37.06	1.42%	54.81	44.64	0.55%	50.09	49.64	0.27%	46.89	52.86	0.25%

Table 17
The unique and shared contribution rates of quantitative information to concept approximation on data Shuttle.

T	$k=1$			$k=2$			$k=3$			$k=4$			$k=5$		
	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$ (%)	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$ (%)	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$ (%)	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$ (%)	$C_{p\vee g}$ (%)	$C_{\tilde{g}}$ (%)	$C_{\tilde{p}}$ (%)
1	59.31	0.15	40.53	59.30	0.18	40.52	59.21	0.33	40.46	85.90	0.68	13.42	87.50	0.98	11.52
2	48.49	0.04	51.46	52.19	0.15	47.66	53.57	0.18	46.25	54.73	0.24	45.03	59.06	0.26	40.68
3	42.26	0.04	57.70	51.69	0.14	48.17	51.98	0.22	47.80	53.40	0.22	46.38	54.54	0.22	45.24
4	41.01	0.04	58.95	46.92	0.07	53.01	51.64	0.07	48.29	51.85	0.15	47.99	51.81	0.24	47.95
5	35.97	0.02	64.01	44.98	0.03	54.99	51.66	0.05	48.29	52.00	0.06	47.94	52.00	0.07	47.94
6	31.11	0.01	68.87	39.59	0.03	60.38	45.28	0.06	54.66	52.11	0.06	47.83	52.11	0.06	47.83
7	21.61	0.02	78.37	39.32	0.04	60.64	45.05	0.06	54.88	50.34	0.06	49.60	51.71	0.07	48.22
8	21.65	0.01	78.34	39.42	0.02	60.57	40.01	0.02	59.97	45.54	0.04	54.42	51.60	0.04	48.36
9	21.41	0.01	78.59	39.32	0.01	60.67	39.91	0.02	60.08	40.54	0.02	59.44	41.94	0.04	58.02
10	21.53	0.01	78.47	39.34	0.01	60.64	39.91	0.06	60.03	40.38	0.06	59.56	40.69	0.08	59.23

Table 18
The approximation accuracy on data Car.

Times	RS	LRS	LLDDRS				
			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0.9511	0.9511	0.9856	1.0000	1.0000	1.0000	1.0000
3	0.7050	0.7050	0.9119	1.0000	1.0000	1.0000	1.0000
4	0.7514	0.7514	0.9296	1.0000	1.0000	1.0000	1.0000
5	0.6713	0.6713	0.9011	1.0000	1.0000	1.0000	1.0000
6	0.6073	0.6073	0.8793	1.0000	1.0000	1.0000	1.0000
7	0.5452	0.5452	0.8604	1.0000	1.0000	1.0000	1.0000
8	0.5230	0.5230	0.7974	1.0000	1.0000	1.0000	1.0000
9	0.4974	0.4974	0.7912	1.0000	1.0000	1.0000	1.0000
10	0.4809	0.4809	0.7170	1.0000	1.0000	1.0000	1.0000

In summary, the relative and absolute quantitative information are two unique objective aspects in describing approximation spaces. Each has its own virtues and application environments, so none can be neglected.

5.3. The superiority of the LLDDRS model in concept approximation

In the following, we verify the advantages of the LLDDRS model in concept approximation by comparing the proposed LLDDRS model with classical Pawlak rough sets (RS) [20] and local rough sets (LRS) [24] based on approximation accuracy. The approximation accuracies of the three rough set models are shown in Tables 18–23, where times denote the ordinal number of experiments.

From Table 18, in the first experiment of data Car, the approximation accuracies of the LLDDRS, RS, LRS models are the same when k changes from 1 to 5. In the second experiment, the approximation accuracies of the LLDDRS model is slightly higher than that of the RS and LRS models when k changes from 1 to 5. The next eight experiments show that the approximation accuracies of the LLDDRS model is much higher than those of the other two models when k changes from 1 to 5. The approximation accuracy of the LLDDRS model is increasing as people's requirements for the external inclusion grade of equivalence classes about concepts are reduced (namely k is increasing). From Tables 19–23, on data Chess, Landsat, Online and Shuttle, the 10 experimental results show that the approximation accuracies of the LLDDRS model are much higher than those of the RS and LRS models when k changes from 1 to 5. From Table 21, on data Nursery, the first four

Table 19
The approximation accuracy on data Chess.

Times	RS	LRS	LLDDRS				
			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	0.6937	0.7188	0.9281	0.9844	0.9938	1.0000	1.0000
2	0.6578	0.6703	0.8906	0.9891	0.9969	1.0000	1.0000
3	0.6135	0.6219	0.8865	0.9854	0.9979	1.0000	1.0000
4	0.6023	0.6250	0.8461	0.9711	0.9859	0.9977	1.0000
5	0.6650	0.6831	0.8481	0.9738	0.9888	0.9981	1.0000
6	0.7208	0.7359	0.8734	0.9781	0.9906	0.9984	1.0000
7	0.7607	0.7737	0.8915	0.9812	0.9920	0.9987	1.0000
8	0.7609	0.7738	0.8941	0.9816	0.9918	0.9988	1.0000
9	0.7590	0.7691	0.8924	0.9816	0.9924	0.9990	1.0000
10	0.7631	0.7722	0.8917	0.9812	0.9912	0.9991	1.0000

Table 20
The approximation accuracy on data Landsat.

Times	RS	LRS	LLDDRS				
			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	0.7941	0.8576	0.8669	0.8700	0.8700	0.8700	0.8700
2	0.7477	0.8274	0.8491	0.8529	0.8529	0.8529	0.8529
3	0.7054	0.8075	0.8380	0.8478	0.8483	0.8488	0.8498
4	0.6552	0.7783	0.8104	0.8216	0.8266	0.8278	0.8289
5	0.6399	0.7601	0.7932	0.8084	0.8158	0.8183	0.8223
6	0.6398	0.7619	0.7946	0.8075	0.8130	0.8207	0.8248
7	0.5871	0.7651	0.7957	0.8103	0.8167	0.8222	0.8291
8	0.5764	0.7603	0.7924	0.8069	0.8150	0.8202	0.8257
9	0.5602	0.7527	0.7848	0.8020	0.8056	0.8149	0.8192
10	0.5610	0.7543	0.7897	0.8076	0.8124	0.8200	0.8241

Table 21
The approximation accuracy on data Nursery.

Times	RS	LRS	LLDDRS				
			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	0.3398	0.3521	0.4445	0.4777	0.7504	0.8644	1.0000
2	0.3421	0.3451	0.3451	0.3521	0.3629	0.4761	0.5177
3	0.3413	0.3439	0.3439	0.3439	0.3516	0.3629	0.4045
4	0.3332	0.3424	0.3424	0.3424	0.3455	0.3455	0.3455
5	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333
6	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333
7	0.3334	0.3334	0.3334	0.3334	0.3334	0.3334	0.3334
8	0.3334	0.3334	0.3334	0.3334	0.3334	0.3334	0.3334
9	0.3334	0.3334	0.3334	0.3334	0.3334	0.3334	0.3334
10	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333	0.3333

Table 22
The approximation accuracy on data Online.

Times	RS	LRS	LLDDRS				
			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	0.4013	0.4215	0.5566	0.6428	0.7081	0.7583	0.8039
2	0.4111	0.4350	0.5794	0.6758	0.7430	0.7898	0.8293
3	0.3557	0.3821	0.5108	0.6020	0.6734	0.7252	0.7639
4	0.3162	0.3443	0.4623	0.5456	0.6090	0.6582	0.6988
5	0.2847	0.3096	0.4216	0.5010	0.5618	0.6096	0.6481
6	0.2646	0.2888	0.3903	0.4673	0.5268	0.5710	0.6099
7	0.2473	0.2725	0.3686	0.4435	0.5042	0.5483	0.5837
8	0.2319	0.2581	0.3504	0.4211	0.4788	0.5257	0.5593
9	0.2241	0.2514	0.3382	0.4038	0.4589	0.5032	0.5405
10	0.2171	0.2439	0.3250	0.3875	0.4405	0.4842	0.5173

Table 23
The approximation accuracy on data Shuttle.

Times	RS	LRS	LLDDRS				
			$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	0.3593	0.7806	0.7818	0.7820	0.7832	0.7860	0.7884
2	0.3356	0.7829	0.7832	0.7841	0.7842	0.7848	0.7849
3	0.2070	0.7829	0.7833	0.7840	0.7847	0.7847	0.7847
4	0.1680	0.7827	0.7830	0.7832	0.7833	0.7839	0.7846
5	0.1584	0.7812	0.7814	0.7814	0.7816	0.7817	0.7817
6	0.1589	0.7811	0.7812	0.7814	0.7815	0.7816	0.7816
7	0.1590	0.7809	0.7811	0.7812	0.7814	0.7814	0.7815
8	0.1593	0.7805	0.7805	0.7806	0.7806	0.7808	0.7808
9	0.1586	0.7804	0.7804	0.7805	0.7805	0.7805	0.7807
10	0.1579	0.7798	0.7799	0.7799	0.7803	0.7803	0.7804

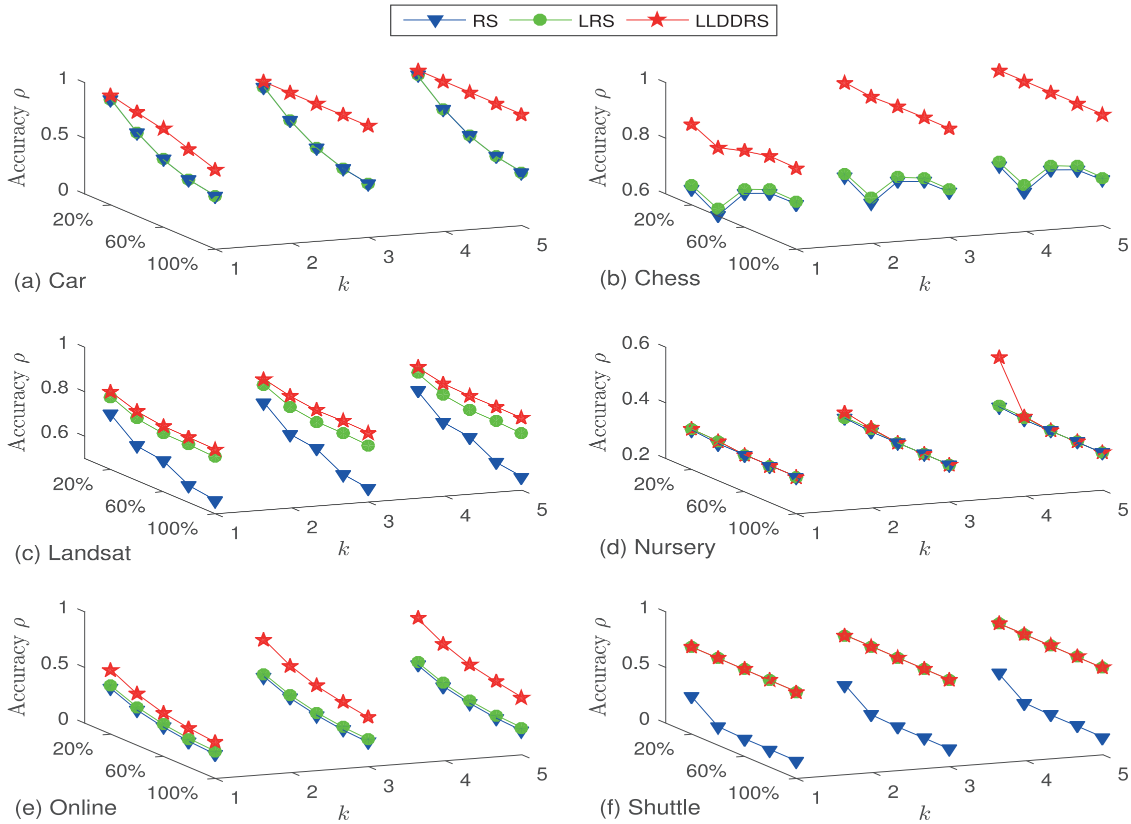


Fig. 4. The approximation accuracy results of the three models of the 5 experiments.

experiments show that the performance of the LLDDRS model is the best, followed by the LRS model, and the RS model is the worst one. In the remaining six experiments, the three models showed the same performance. Through experimental comparisons, the overall performances of the LLDDRS model and the LRS model are always better than that of the RS model, and the LLDDRS model performs best in the three models. Therefore, it is meaningful to study knowledge discovery and rule extraction based on local rough set theory and double-quantitative information.

In order to more directly reflect the advantage of the LLDDRS model, the results of the 2nd, 4th, 6th, 8th and 10th experiments under $k=1,3,5$ are shown in Fig. 4. It can be seen from Fig. 4 that the performance of the proposed LLDDRS model is quite good.

At the same time, in order to show more vividly the advantage of combining relative information with absolute information to approximate concepts more accurately, we show the results of the 2nd, 6th and 10th experiments under $k=1,2,3,4,5$ in Figs. 5–7. In order to make the trend of data change more obvious, we fine-tune the data without affecting the logical size of the data.

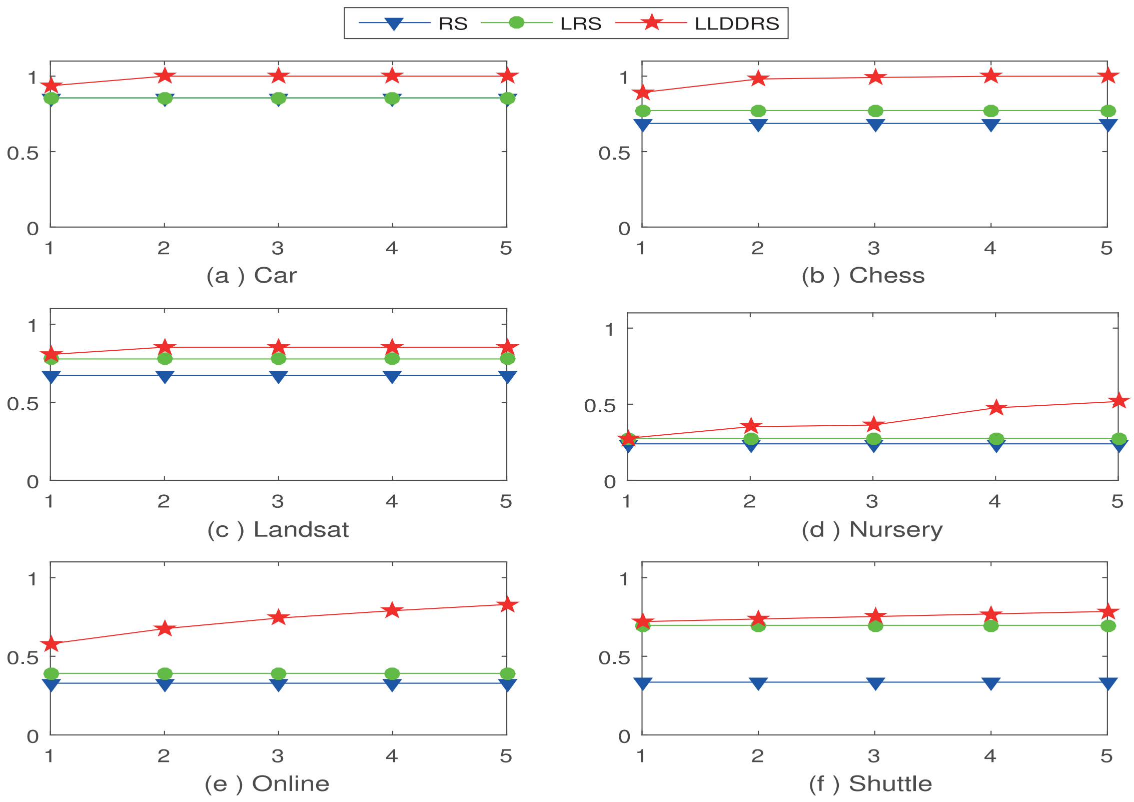


Fig. 5. Comparisons of approximation accuracies of three models under different grades in the 2nd experiment.

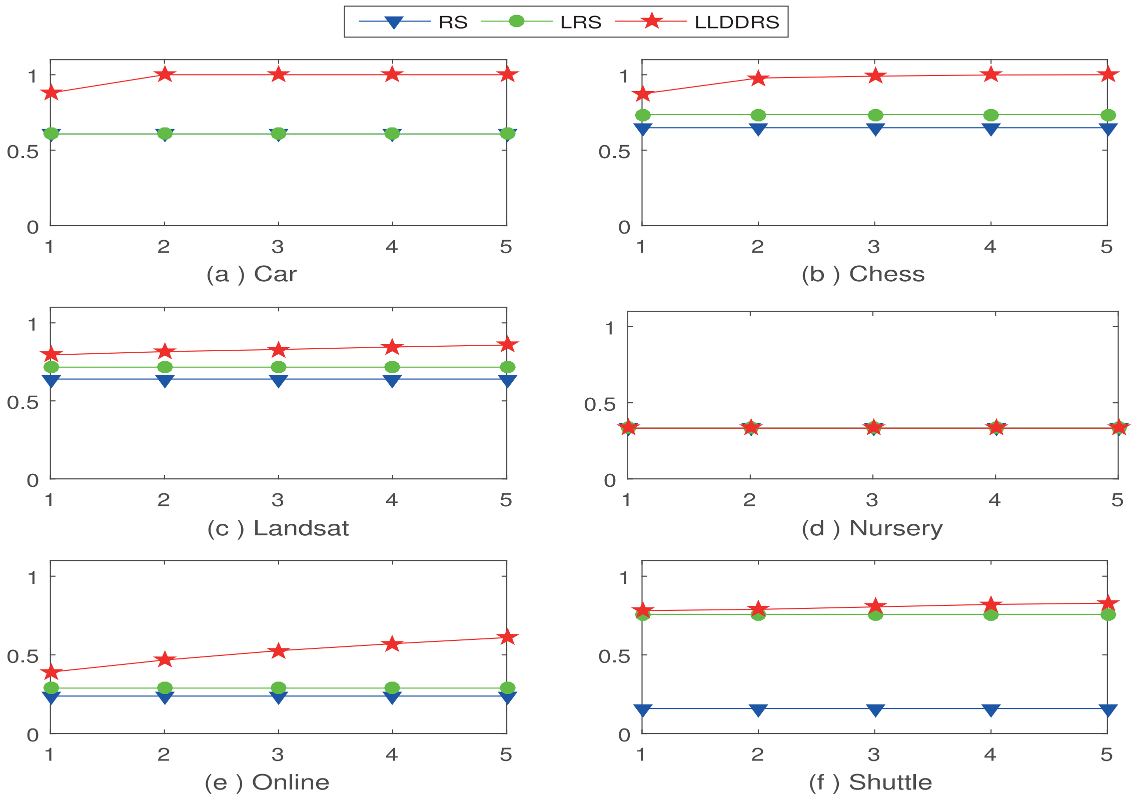


Fig. 6. Comparisons of approximation accuracies of three models under different grades in the 6th experiment.

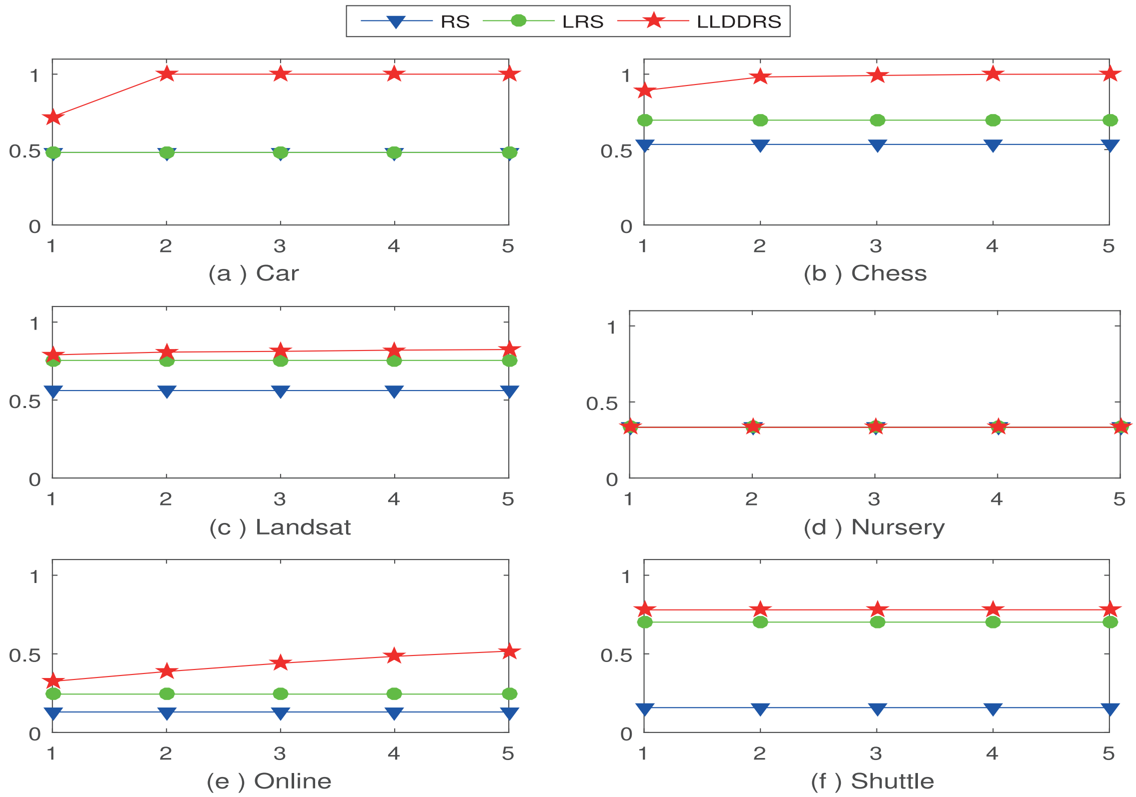


Fig. 7. Comparisons of approximation accuracies of three models under different grades in the 10th experiment.

In Figs. 5–7, the results of the RS and LRS models are not affected by the variation of the grade parameter k, so the corresponding results with the change of k are on a straight broken line. The purpose is to compare the approximation accuracy of the LLDDRS model with that of the RS and LRS models under different grades, respectively. From Figs. 5–7, although the approximation accuracy of the proposed LLDDRS model decreases with the decrease of k, the results of the LLDDRS model are still better than those of the other two models. Therefore, compared with the RS and LRS models, the proposed LLDDRS model has some advantages in the concept approximation even when people require higher grade information.

6. Conclusions

Relative and absolute quantitative information reflect the essence of objective things from different sides and neither can be neglected for effective description. In order to identify objects to the greatest extent, we proposed a Local Logical Disjunction Double-quantitative Rough Set model (LLDDRS) based on the logical disjunction combination of the double quantification information. The important properties, rough regions and decision rules of the LLDDRS model are studied, and the corresponding algorithms are designed. Meanwhile, the relationships of the LLDDRS model and other models are analyzed systematically. Theoretical analyses and experimental results show that: (1) the LLDDRS model based on local idea is efficient to compute approximations of concepts in large data sets when compared with global models. (2) The LLDDRS model based on double-quantitative information provides more accurate approximation space description when compared with single quantitative models. (3) The LLDDRS model provides an efficient method for rough data analysis in large data sets. This paper provides a framework of local logical disjunction double-quantitative rough sets under equivalence relations. In the future, we can further study local logical disjunction double-quantitative rough sets from different binary relations, fuzzy concepts, two universes and multi-granulation to enhance the usability of the local logical disjunction double-quantitative rough set theory. These studies are of great significance and value, which can help people mine data, discover knowledge and extract rules in large-scale data situations.

Conflict of interest

None.

Acknowledgments

This work is supported by the Macau Science and Technology Development Fund (no. 081/2015/A3), the Natural Science Foundation of China (no. 61772002). We would like to thank the editor, associate editor and reviewers for their valuable suggestions. We would also like to thank professor John W.T. Lee for his help in proof-reading this paper.

References

[1] R.A. Ayad, J. Liu, Supporting e-learning system with modified Bayesian rough set model, in: Proceedings of International Symposium on Neural Networks: Advances in Neural Networks, 2009, pp. 192–200.
[2] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (2–3) (1990) 191–209.
[3] X.F. Deng, Y.Y. Yao, A multifaceted analysis of probabilistic three-way decisions, Fundam. Inf. 132 (3) (2014) 291–313.
[4] B.J. Fan, E.C.C. Tsang, W.H. Xu, J.H. Yu, Double-quantitative rough fuzzy set based decisions: a logical operations method, Inf. Sci. 378 (2017) 264–281.
[5] S. Greco, B. Matarazzo, R. Slowinski, Rough sets theory for multicriteria decision analysis, Eur. J. Oper. Res. 129 (1) (2001) 1–47.
[6] Y.T. Guo, E.C.C. Tsang, W.H. Xu, D.G. Chen, Logical disjunction double-quantitative fuzzy rough sets, in: Proceedings of International Conference on Machine Learning and Cybernetics, 2018, pp. 415–421.
[7] J.P. Herbert, J.T. Yao, Game-theoretic risk analysis in decision-theoretic rough sets, in: Proceedings of International Conference on Rough Sets and Knowledge Technology, 2008, pp. 132–139.
[8] Q.H. Hu, D.R. Yu, Z.X. Xie, Information-preserving hybrid data reduction based on fuzzy-rough techniques, Pattern Recogn. Lett. 27 (5) (2006) 414–423.
[9] Y.G. Jing, T.R. Li, H. Fujita, B.L. Wang, et al., An incremental attribute reduction method for dynamic data mining, Inf. Sci. 465 (2018) 202–218.
[10] Y.G. Jing, T.R. Li, H. Fujita, Z. Yu, et al., An incremental attribute reduction approach based on knowledge granularity with a multi-granulation view, Inf. Sci. 411 (2017) 23–38.
[11] W.T. Li, W.H. Xu, Double-quantitative decision-theoretic rough set, Inf. Sci. 316 (2015) 54–67.
[12] P. Lingras, M. Chen, D.Q. Miao, Rough cluster quality index based on decision theory, IEEE Trans. Knowl. Data Eng. 21 (7) (2009) 1014–1026.
[13] D. Liu, T.R. Li, D.C. Liang, Three-way government decision analysis with decision-theoretic rough sets, Int. J. Uncert. Fuzzin. Knowl. Based Syst. 20 (1) (2012) 119–132.
[14] D. Liu, T.R. Li, D.C. Liang, Incorporating logistic regression to decision-theoretic rough sets for classifications, Int. J. Approx. Reason. 55 (1) (2014) 197–210.
[15] D.C. Liang, Z.S. Xu, D. Liu, A new aggregation method-based error analysis for decision-theoretic rough sets and its application in hesitant fuzzy information systems, IEEE Trans. Fuzzy Syst. 25 (6) (2017) 1685–1697.
[16] D. Liu, Y.Y. Yao, T.R. Li, Three-way investment decisions with decision-theoretic rough sets, Int. J. Comput. Intell. Syst. 4 (1) (2011) 66C74.
[17] H.X. Li, M.H. Wang, X.Z. Zhou, J.B. Zhao, An interval set model for learning rules from incomplete information table, Int. J. Approx. Reason. 53 (1) (2012) 24–37.
[18] C. Luo, T.R. Li, H.M. Chen, H. Fujita, et al., Incremental rough set approach for hierarchical multicriteria classification, Inf. Sci. 429 (2018) 72–87.
[19] W.M. Ma, B.Z. Sun, On relationship between probabilistic rough set and bayesian risk decision over two universes, Int. J. Gen. Syst. 41 (3) (2012) 225–245.
[20] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[21] Z. Pawlak, A. Skowron, Rough sets: some extensions, Inf. Sci. 177 (1) (2007) 28–40.
[22] Z. Pawlak, S.K.M. Wong, W. Ziarko, Rough sets: probabilistic versus deterministic approach, Int. J. Man Mach. Stud. 29 (1) (1988) 81–95.
[23] Y.H. Qian, H. Zhang, Y.L. Sang, J.Y. Liang, Multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 55 (1) (2014) 225–237.
[24] Y.H. Qian, X.Y. Liang, Q. Wang, J.Y. Liang, et al., Local rough set: a solution to rough data analysis in big data, Int. J. Approx. Reason. 97 (2018) 38–63.
[25] Y.H. Qian, X.Y. Liang, G.P. Lin, Q. Guo, et al., Local multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 82 (2017) 119–137.
[26] D. Ślęzak, Rough sets and Bayes factor, in: Transactions on Rough Sets III, Springer, 2005, pp. 202–229.
[27] Q. Shen, A. Chouchoulas, A rough-fuzzy approach for generating classification rules, Pattern Recogn. 35 (11) (2002) 2425–2438.
[28] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, Fundam. Inf. 27 (2-3) (1996) 245–253.
[29] S.K.M. Wong, W. Ziarko, Comparison of the probabilistic approximate classification and the fuzzy set model, Fuzzy Set Syst. 21 (3) (1987) 357–362.
[30] G.Y. Wang, Y.Y. Yao, H. Yu, A survey on rough set theory and applications, Chin. J. Comput. 32 (7) (2009) 1229–1246.
[31] Q. Wang, Y.H. Qian, X.Y. Liang, Q. Guo, et al., Local neighborhood rough set, Knowl. Based Syst. 153 (2018) 53–64.
[32] W.H. Xu, Y.T. Guo, Generalized multigranulation double-quantitative decision-theoretic rough set, Knowl.-Based Syst. 105 (2016) 190–205.
[33] Y.Y. Yao, The superiority of three-way decisions in probabilistic rough set models, Inf. Sci. 181 (6) (2011) 1080–1096.
[34] Y.Y. Yao, X.F. Deng, Quantitative rough sets based on subsethood measures, Inf. Sci. 267 (2014) 306–322.
[35] Y.Y. Yao, T.Y. Lin, Generalization of rough sets using modal logics, Intell. Autom. Soft Co. 2 (2) (1996) 103–119.
[36] Y.Y. Yao, Y.H. She, Rough set models in multigranulation spaces, Inf. Sci. 327 (2016) 40–56.
[37] Y.Y. Yao, S.K.M. Wong, P. Lingras, A decision-theoretic rough set model, in: Proceedings of the Fifth International Symposium on Methodologies for Intelligent Systems, 1990, pp. 17–24.
[38] H. Yu, G.Y. Wang, Y.Y. Yao, Current research and future perspectives on decision-theoretic rough sets, Chin. J. Comput. 38 (8) (2015) 1628–1639.
[39] Y.Y. Yang, D.G. Chen, H. Wang, X.Z. Wang, Incremental perspective for feature selection based on fuzzy rough sets, IEEE Trans. Fuzzy Syst. 26 (3) (2018) 1257–1273.
[40] X. Yang, T.R. Li, H. Fujita, D. Liu, et al., A unified model of sequential three-way decisions and multilevel incremental processing, Knowl. Based Syst. 134 (2017) 172–188.
[41] X. Yang, T.R. Li, D. Liu, H.M. Chen, et al., A unified framework of dynamic three-way probabilistic rough sets, Inf. Sci. 420 (2017) 126–147.
[42] J.H. Yu, B. Zhang, M.H. Chen, W.H. Xu, Double-quantitative decision-theoretic approach to multigranulation approximate space, Int. J. Approx. Reason. 98 (2018) 236–258.
[43] W. Ziarko, Variable precision rough set model, J. Comput. Syst. Sci. 46 (1) (1993) 39–59.
[44] X.R. Zhao, B.Q. Hu, Fuzzy and interval-valued fuzzy decision-theoretic rough set approaches based on fuzzy probability measure, Inf. Sci. 298 (2015) 534–554.
[45] Y. Zhang, J.T. Yao, Determining three-way decision regions with Gini coeffcients, in: Proceedings of International Conference on Rough Sets and Current Trends in Computing, 2014, pp. 160–171.
[46] Y. Zhang, J.T. Yao, Rule measures tradeoff using game-theoretic rough sets, in: Proceedings of International Conference on Brain Informatics, 2012, pp. 348–359.
[47] A. Zeng, D. Pan, Q.L. Zheng, H. Peng, Knowledge acquisition based on rough set theory and principal component analysis, IEEE Intell. Syst. 21 (2) (2006) 78–85.
[48] X.Y. Zhang, Z.W. Mo, F. Xiong, W. Cheng, Comparative study of variable precision rough set model and graded rough set model, Int. J. Approx. Reason. 53 (1) (2012) 104–116.
[49] X.Y. Zhang, D.Q. Miao, Two basic double-quantitative rough set models of precision and grade and their investigation using granular computing, Int. J. Approx. Reason. 54 (8) (2013) 1130–1148.
[50] X.Y. Zhang, D.Q. Miao, Quantitative information architecture, granular computing and rough set models in the double-quantitative approximation space of precision and grade, Inf. Sci. 268 (2014) 147–168.