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Generalized multi-granulation double-quantitative decision-theoretic rough set of multi-source information system

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ABSTRACT

Traditionally, multi-source information system (MsIS) is typically integrated into a single information table for knowledge acquisition. Therefore, discovering knowledge directly from MsIS without information loss is a valuable research direction. In this paper, we propose the generalized multi-granulation double-quantitative decision-theoretic rough set of multi-source information system (MS-GMDQ-DTRS) to handle this issue. First, we propose a generalized multi-granulation rough set model for MsIS (MS-GMRS) as the basis of other models. In this model, each single information system is treated as a granular structure. Next, we combine MS-GMRS with double-quantitative decision-theoretic rough set to obtain two new models. They have better fault tolerance capability compared with MS-GMRS. Furthermore, we propose corresponding algorithms to calculate the approximation accuracy of the proposed models. Experiments are carried out on four datasets downloaded from UCI. Experimental results show that the two new models have better fault tolerance in directly acquiring knowledge from MsIS.

1. Introduction

As the information age continues to evolve, data form becomes more complex. This has promoted the development of some knowledge representation forms for complex data. Among these representation forms, multi-source information system (MsIS) [1] is an important representative, which is a family of homogeneous single source information systems. In real-world applications, the MsIS is widely used in various fields, such as natural language processing [2], extracting trips [3], energy consumption prediction [4], deep learning [5] and so on. In recent years, multi-source fusion [6–11], as a common method for dealing with MsIS, has been widely concerned, which can integrate information from MsIS into a

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single information table, acquire knowledge, and extract rules. This topic has attracted the attention of researchers because of its wide applications.

Integrating MsIS into a single information table is the most common multi-source fusion strategy. Then knowledge is acquired through the obtained information table. The integration method of MsIS is a necessary prerequisite for processing MsIS. In recent years, many multi-source fusion methods have been proposed. Particularly, Xu et al. proposed a multi-source fusion method based on information entropy [6,8] and information source selection principle [7]. In these works, the multi-source fusion method uses a metric to select reliable information from a MsIS for integration into a single information table. In fact, the essence of this method is to integrate the description of objects from all information tables in MsIS into one classic information table. This is called integrating MsIS from the perspective of objects. It is worth noting that in the process of this integration, information loss will inevitably occur. Therefore, how to discover knowledge directly from MsIS without information loss motivates this study. In this paper, we focus on introduce the multi-granulation rough set theory into multi-source fusion.

Multi-granulation rough set is an important theory in granular computing. The concept of granular computing has been widely concerned since it was proposed by Zadeh [12]. Recently, Yao [13] proposed three-way granular computing for processing information by integrating granular computing and three-way decision. From the perspective of granular computing, in the universe, an equivalence relation is a granular structure, multiple equivalence relations are multiple granular structures. In real-world applications, an object set needs to be described by multiple relations according to the user's needs or target of problem handling. To meet the actual needs, Qian et al. [14] proposed the theoretical framework of multi-granulation rough set (MGRS), which was used to process an information table with multiple granular structure. In recent years, the MGRS model has been widely extended and applied [15–21]. Yang et al. [22] proposed a multi-level granular structure-based sequential three-way approach for solving multi-class decision issues. Attribute reduction based on single criterion is useless for complex problems. To address this issue, Li et al. [23] presented a multi-objective attribute reduction method. To solve linguistic information-based multiple attribute group decision making issue, Sun et al. [24] proposed three-way approach in the framework of decision-theoretic rough set.

Multi-granulation rough set theory provides a new way for information fusion. From the perspective of information fusion, the objective of the MGRS is to integrate information from one information table with multiple granular structures into the approximations of a target concept. In such an information table, the granular structure is usually composed of multiple attributes. Thus an information fusion strategy based on the idea of multi-granulation is called fusing information from the perspective of attributes. In fact, each single information system in MsIS can be regarded as a granular structure. Inspired by the above, we argue that it is a feasible method to integrate MsIS based on multi-granulation rough set theory. It is worth pointing out that this method can directly generate approximations of a target concept in MsIS, thus avoiding information loss.

However the MGRS model is too strict or too loose in describing approximations, it does not take into account the situation that the minority is subordinate to the majority. To overcome this deficiency, Xu et al. [25] proposed the theoretical framework of generalized multi-granulation rough set (GMRS), which is the extension of MGRS model. The key constituent of this model is to use an information level $\beta \in (0.5, 1]$ to control objects selection. By adjusting this parameter, the objects can be positively described in most classifications, and the objects that may be described below the corresponding level are deleted. So the GMRS model has better practicality. Based on this model, Qian et al. [26] proposed a multiple thresholds-based generalized multi-granulation sequential three-way decision model for solving the issue of multi-granulation structure. Considering the advantages of the GMRS model, this paper exploits this model to deal with MsIS. Therefore, we build a new model, called generalized multi-granulation rough set model of MsIS (MS-GMRS), to acquire the knowledge of MsIS.

In order to improve the fault tolerance capability of the MS-GMRS model, we combine this model with double-quantitative decision-theoretic rough set [27]. Then we obtain two new models, called two kinds of generalized multi-granulation double-quantitative decision-theoretic rough set model for MsIS (MS-GMDQ-DTRS). Double-quantitative decision-theoretic rough set, as an important extension of Pawlak rough set theory, has been widely studied [28–30]. The Pawlak rough set theory [31] is an important mathematical tool, which has been widely applied to attribute reduction [32–36], uncertainty measurement [37,38], and decision theory [39,40], cost-sensitive learning [41–46], rough data analysis [47] etc. However, the Pawlak rough set model has limitation, which is sensitive to noisy data. Since the degree of intersection between target set and knowledge granules is not considered, the Pawlak rough set model has no fault tolerant effect in processing information. To solve this issue, many meaningful works about the extension of Pawlak rough set model have been investigated. Among these extended models, double-quantitative decision-theoretic rough set model is an important representative, which is built by combining the graded rough set model [48] and the decision-theoretic rough set model [49]. These two rough set models have good fault tolerance, but their quantitative relations are different. In the graded rough set model, the absolute quantitative relation between target set and knowledge granules is considered, but the relative quantitative relation between them is neglected. In contrast, the decision-theoretic rough set model considers relative quantitative relation between target set and knowledge granules, while ignoring the absolute quantitative relation. To complement each other, it is necessary and valuable to combine the two quantitative models. In addition, in the decision-theoretic rough set, Yao et al. [49] offered an appropriate semantic explanation for decision-making process. This shows that the double-quantitative decision-theoretic rough set not only has strong fault tolerance, but has a reasonable decision-making process. This is the main reason why this model is exploited in this paper.

The main contributions of the work are four-folds. First, we build the generalized multi-granulation rough set model for MsIS and its relevant properties are investigated and proved. Second, based on the proposed model, we introduce the double-quantitative decision-theoretic rough set to obtain two new models, called two kinds of generalized multi-granulation double-quantitative decision-theoretic rough set model of MsIS. Meanwhile, the respective decision rules are presented. Third, we discuss the relations between the three models mentioned above and verify them through an illustrative case. Fourth, we respectively define the approximation accuracy of the three models and propose the corresponding algorithm, the objective is to compare the fault tolerance of the proposed models. Experiments on four data sets from UCI show that the fault tolerance capability of the proposed models.

The rest of the paper is organized as follows: Section 2 reviews the related work. Section 3 presents the MS-MRS model and proposes two MS-GMDQ-DTRS models based on it. Meanwhile, relevant properties and relations of the proposed models are investigated and proved. Section 4 presents an illustrative case for verifying the relevant properties and relations. Section 5 sets up the experiment and discusses the results. Section 6 concludes the work and outlines the future research.

2. Preliminaries

In this section, we briefly introduce the some basic concepts of Pawlak rough set [31] and some of its extended models [25,27,48,49].

2.1. Pawlak rough set

Let $IS = (U, AT, V, f)$ be an information system, where $U = \{x_1, x_2, \dots, x_n\}$ is a non-empty and finite set of objects, AT is a non-empty and finite set of attributes, $V = \bigcup_{a\in AT} V_a$, V_a is the domain of attribute a, and $f : U \times AT \to V$ is an information function, $f(x,a) \in V_a$ $(a \in AT)$. For any $A \subseteq AT$, an indiscernibility relation is $R_A = \{(x, y) \in U \times U | \forall a \in A, f_a(x) = f_a(y)\}$. The R_A is called an equivalence relation, which can generate a partition of U, denoted by $U/R_A = \{[x]_A | x \in U\}$. The $[x]_A$ represents the equivalence class of x with respect to R_A. The Pawlak approximation space [31] is (U, R_A), briefly written as (U, R). For any $X \subseteq U$, the lower and upper approximations of X are

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\}, \overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\}.$$

The positive region, negative region, and boundary region of X are $POS(X) = \underline{R}(X)$, $NEG(X) = (\overline{R}(X))^c$, and $BND(X) = \overline{R}(X) - \underline{R}(X)$. The approximation accuracy and roughness of X are $\alpha_R(X) = |\underline{R}(X)|/|\overline{R}(X)|$ and $\rho_R(X) = 1 - \alpha_R(X)$, where $|\bullet|$ represents cardinality of a set.

Let $DS = (U, AT \cup DT, V, f)$ be a decision system, where AT is a set of conditional attributes, DT is a set of decision attributes. The $U/DT = \{D_1, D_2, \dots, D_m\}$ is the partition of the universe U on decision attributes. The lower and upper approximations of the partition U/DT are

$$\underline{R}(U/DT) = \underline{R}(D_1) \cup \underline{R}(D_2) \cup \cdots \cup \underline{R}(D_m), \overline{R}(U/DT) = \overline{R}(D_1) \cup \overline{R}(D_2) \cup \cdots \cup \overline{R}(D_m).$$

For U/DT, the approximation accuracy is

$$\alpha_R(U/DT) = \frac{\sum_{D_i \in U/DT} |\underline{R}(D_i)|}{\sum_{D_i \in U/DT} |\overline{R}(D_i)|}.$$

2.2. Some extended rough sets models

(1) The graded rough set (GRS).

The GRS [48] mainly describes the absolute quantitative relation between knowledge granules and basic concepts. The upper and lower approximations with grade $k \in N$ are

$$\overline{R}_k(X) = \{x \in U | |[x]_R \cap X| > k\}, \underline{R}_k(X) = \{x \in U | |[x]_R| - |[x]_R \cap X| \leq k\}.$$

The positive region, negative region, and boundary region of X are $POS(X) = \underline{R}_k(X)$, $NEG(X) = (\overline{R}_k(X))^c$, and $BND(X) = \overline{R}_k(X) - \underline{R}_k(X)$.

(2) The decision-theoretic rough set (DTRS).

The DTRS proposed a way about how to make decision under minimum Bayesian expectation risk [49]. Based on the idea of three-way decisions, the DTRS describes the decision-making process with a state set Ω and an action set A. The $\Omega = \{X, X^c\}$, where X and X^c denote that $x \in X$ and $x \in X^c$, respectively. The $A = \{a_P, a_B, a_N\}$, where a_P, a_B, and a_N represent three actions about deciding $x \in POS(X)$, $x \in BND(X)$, and $x \in NEG(X)$, respectively. Let λ_{PP}, λ_{BP}, and λ_{NP} denote the losses caused by take actions a_P, a_B, and a_N, respectively, when $x \in X$. Let λ_{PN}, λ_{BN}, and λ_{NN} denote the losses caused by take the same when $x \in X^c$. Given the loss function, for any $x \in [x]_R$, the expected loss for different actions are

$$R(a_P|[x]_R) = \lambda_{PP}P(X|[x]_R) + \lambda_{PN}P(X^c|[x]_R),$$
$$R(a_B|[x]_R) = \lambda_{BP}P(X|[x]_R) + \lambda_{BN}P(X^c|[x]_R),$$
$$R(a_N|[x]_R) = \lambda_{NP}P(X|[x]_R) + \lambda_{NN}P(X^c|[x]_R),$$

where $P(X|[x]_R) = |X \cap [x]_R|/|[x]_R|$ and $P(X^c|[x]_R) = 1 - P(X|[x]_R)$.

According to Bayesian decision procedure, minimum-risk decision rules are

(P) If $R(a_P|[x]_R) \le R(a_B|[x]_R)$ and $R(a_P|[x]_R) \le R(a_N|[x]_R)$, decide $x \in POS(X)$,
(B) If $R(a_B|[x]_R) \le R(a_P|[x]_R)$ and $R(a_B|[x]_R) \le R(a_N|[x]_R)$, decide $x \in BND(X)$,
(N) If $R(a_N|[x]_R) \le R(a_P|[x]_R)$ and $R(a_N|[x]_R) \le R(a_B|[x]_R)$, decide $x \in NEG(X)$.

Taking into account the actual situations, there is an ordered relation between the decision cost values, i.e., $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$. Then the above rules are re-expressed as

(P) If $P(X|[x]_R) \ge \alpha$ and $P(X|[x]_R) \ge \gamma$, decide $x \in POS(X)$,
(B) If $P(X|[x]_R) \le \alpha$ and $P(X|[x]_R) \ge \beta$, decide $x \in BND(X)$,
(N) If $P(X|[x]_R) \ge \beta$ and $P(X|[x]_R) \le \gamma$, decide $x \in NEG(X)$,

where

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}, \beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}, \gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}.$$

If decision costs values meet the condition: $(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) > (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, then we can get $0 \le \beta < \gamma < \alpha \le 1$. Then the rules of DTRS are

(P) If $P(X|[x]_R) \ge \alpha$, decide $x \in POS(X)$,
(B) If $\beta < P(X|[x]_R) < \alpha$, decide $x \in BND(X)$,
(N) If $P(X|[x]_R) \le \beta$, decide $x \in NEG(X)$.

In addition, according to the above rules, the upper and lower approximations of the DTRS model are

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\}, \underline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \ge \alpha\}.$$

The positive region, negative region, and boundary region of X are $POS(X) = \underline{R}_{(\alpha,\beta)}(X)$, $NEG(X) = (\overline{R}_{(\alpha,\beta)}(X))^c$, $BNG(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_{(\alpha,\beta)}(X)$.

(3) The double-quantitative decision-theoretic rough set (Dq-DTRS).

Both DTRS and GRS have strong fault tolerance, so they can not be ignored. By introducing absolute quantitative information in DTRS, two kinds of double-quantitative DTRS (Dq-DTRS) (i.e., DqI-DTRS and DqII-DTRS) are proposed [27], respectively.

i. The DqI-DTRS

The DqI-DTRS is made up of $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k)$. For any $X \subseteq U$, the upper and lower approximations of X are

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\}, \underline{R}_k(X) = \{x \in U || [x]_R| - |[x]_R \cap X| \le k\}.$$

The positive region, negative region, and upper and lower boundary region of X are $POS^I(X) = \overline{R}_{(\alpha,\beta)}(X) \cap \underline{R}_k(X)$, $NEG^I(X) = (\overline{R}_{(\alpha,\beta)}(X) \cup \underline{R}_k(X))^c$, $UBN^I(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_k(X)$, $LBN^I(X) = \underline{R}_k(X) - \overline{R}_{(\alpha,\beta)}(X)$. Then, the following decision rules are

(P^I) If $x \in X$ satisfies $P(X|[x]_R) > \beta$ and $|[x]_R| - |[x]_R \cap X| \le k$, then $x \in POS^I(X)$,
(N^I) If $x \in X$ satisfies $P(X|[x]_R) \le \beta$ and $|[x]_R| - |[x]_R \cap X| > k$, then $x \in NEG^I(X)$,
(Ub^I) If $x \in X$ satisfies $P(X|[x]_R) > \beta$ and $|[x]_R| - |[x]_R \cap X| > k$, then $x \in UBN^I(X)$,
(Lb^I) If $x \in X$ satisfies $P(X|[x]_R) \le \beta$ and $|[x]_R| - |[x]_R \cap X| \le k$, then $x \in LBN^I(X)$.

ii. The DqII-DTRS

The DqII-DTRS is made up of $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)})$. For any $X \subseteq U$, the upper and lower approximations of X are

$$\overline{R}_k(X) = \{x \in U || [x]_R \cap X| > k\}, \underline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \ge \alpha\}.$$

Similarly, the positive region, negative region, and upper and lower boundary region of X are $POS^{II}(X) = \overline{R}_k(X) \cap \underline{R}_{(\alpha,\beta)}(X)$, $NEG^{II}(X) = (\overline{R}_k(X) \cup \underline{R}_{(\alpha,\beta)}(X))^c$, $UBN^{II}(X) = \overline{R}_k(X) - \underline{R}_{(\alpha,\beta)}(X)$, $LBN^{II}(X) = \underline{R}_{(\alpha,\beta)}(X) - \overline{R}_k(X)$. Then, decision rules are

(P^{II}) If $x \in X$ satisfies $P(X|[x]_R) \ge \alpha$ and $|[x]_R \cap X| \le k$, then $x \in POS^{II}(X)$,
(N^{II}) If $x \in X$ satisfies $P(X|[x]_R) < \alpha$ and $|[x]_R \cap X| \le k$, then $x \in NEG^{II}(X)$,
(Ub^{II}) If $x \in X$ satisfies $P(X|[x]_R) < \alpha$ and $|[x]_R \cap X| > k$, then $x \in UBN^{II}(X)$,
(Lb^{II}) If $x \in X$ satisfies $P(X|[x]_R) \ge \alpha$ and $|[x]_R \cap X| \le k$, then $x \in LBN^{II}(X)$.

(4) Generalized multi-granulation rough set (GMRS).

Given an information system $IS = (U, AT, V, f)$, for any $A \subseteq AT$, R_A is an equivalence relation on U with respect to attribute set A, and U/R_A is a partition of U with respect to R_A. In the view of granular computing, the R_A is seen as a granulation, expressed as A. The U/A is a granulation structure with respect to A. In many cases, U is partitioned by multiple equivalence relations R_{A_i} ($A_i \subseteq AT, i = 1, 2, \ldots, s$), which are seen as multiple granulations (i.e., multi-granulation), expressed as $A_i \subseteq AT, i = 1, 2, \ldots, s$ ($s \le 2^{|AT|}$). For any $X \subseteq U$,

$$\mathcal{S}_X^{A_i}(x) = \begin{cases} 1, & \text{if } ([x]_{A_i} \subseteq X, i \le 2^{|AT|}); \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{S}_X^{A_i}$ is called support feature function of x for X, which is used to describe the inclusion relation between equivalence class $[x]_{A_i}$ and X. The lower and upper approximations [25] of X for $\sum_{i=1}^{s} A_i$ are

$$\underline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} \mathcal{S}_X^{A_i}(x)}{s} \ge \beta \right.\right\}, \overline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s}(1 - \mathcal{S}_{X^c}^{A_i}(x))}{s} > 1 - \beta \right.\right\},$$

where β is an adjustable standard of information with respect to $\sum_{i=1}^{s} A_i$. The positive region, negative region, and boundary region of X are $POS(X) = \underline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta$, $NEG(X) = (\overline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta)^c$, $BND(X) = \overline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta - \underline{GM}_{\sum_{i=1}^{s} A_i}(X)_\beta$.

If $\beta = 0$, the GMRS model is degenerated into optimistic MGRS model, the lower and upper approximations [25] of X for $\sum_{i=1}^{s} A_i$ are

$$\underline{OM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} \mathcal{S}_X^{A_i}(x)}{s} \ge 0 \right.\right\}, \overline{OM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s}(1 - \mathcal{S}_{X^c}^{A_i}(x))}{s} > 1 \right.\right\}.$$

If $\beta = 1$, the GMRS model is degenerated into pessimism MGRS model, the lower and upper approximations [25] of X for $\sum_{i=1}^{s} A_i$ are

$$\underline{PM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} \mathcal{S}_X^{A_i}(x)}{s} \ge 1 \right.\right\}, \overline{PM}_{\sum_{i=1}^{s} A_i}(X)_\beta = \left\{ x \in U \left| \frac{\sum_{i=1}^{s}(1 - \mathcal{S}_{X^c}^{A_i}(x))}{s} > 0 \right.\right\}.$$

3. MS-GMDQ-DTRS: generalized multi-granulation double-quantitative decision-theoretic rough set model of multi-source information system

In this section, a MS-GMRS model is firstly proposed. To further improve the fault tolerance of this model, a pair of MS-GMDQ-DTRS models are developed by introducing the double-quantitative decision-theoretic rough set. The decision rules for this pair of models are given, respectively. We present relevant properties and relations of the proposed models.

3.1. MS-GMRS: generalized multi-granulation rough set model for multi-source information system

In this subsection, we propose a generalized multi-granulation rough set model for MsIS (MS-GMRS) and its the relevant properties are studied. In order to evaluate the fault tolerance of this model, the approximation accuracy is defined. First, the definition of MsIS is introduced.

Definition 3.1. [18] A multi-source information system (MsIS) consists of multiple $IS_i = (U, AT, V_i, f_i)$. For any $i \in N^*$, the IS_i represents the ith information system of the MsIS. Therefore, a MsIS can be defined as

$$MS = \{IS_1, IS_2, \ldots, IS_s\}. \tag{1}$$

Similarly, a multi-source decision system (MsDS) consists of multiple $DS_i = (U, AT \cup DT, V_i, f_i)$. For any $i \in N^*$, DS_i represents the ith decision system of the MsDS. Therefore, a MsDS can be defined as

$$MDS = \{DS_1, DS_2, \ldots, DS_s\}. \tag{2}$$

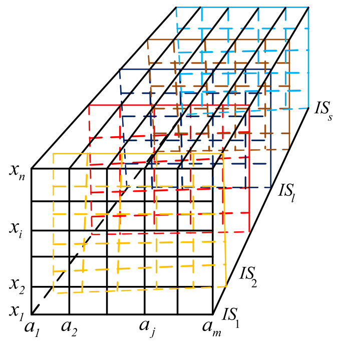

Fig. 1. A multi-source information box.

Multiple single-source information systems (decision system) are grouped together to form a MsIS (MsDS) similar to an information box, as shown in Fig. 1, where $x_1, x_2, \ldots, x_n$ are the objects in the U, $a_1, a_2, \ldots, a_m$ are the attributes in the AT, $IS_1, IS_2, \ldots, IS_s$ are s single information systems that constitute a MsIS. Note: in this paper, the MsIS is isomorphic, i.e., all single information systems have the same set of attributes and objects, but in different single information systems, the value of the same object under the same attribute may be different.

Definition 3.2. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the MS-GMRS, for any $X \subseteq U$, the lower and upper approximations are defined by

$$\underline{MS\text{-}GM}_{MS}(X) = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}S_X^{SI_i}(x)}{s} \geq \varphi \right. \right\}, \tag{3}$$

$$\overline{MS\text{-}GM}_{MS}(X) = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} (1 - MS\text{-}S_{X^c}^{IS_i}(x))}{s} > 1 - \varphi \right. \right\}, \tag{4}$$

where $\varphi \in (0.5, 1]$ is an adjustable information standard with respect to MS, X^c is a complement to X. Under IS_i, the support feature functions of $x \in U$ with respect to X and X^c are

$$MS\text{-}S_X^{IS_i}(x) = \begin{cases} 1, & if\,([x]_{IS_i} \subseteq X); \\ 0, & otherwise. \end{cases} \tag{5}$$

$$MS\text{-}S_{X^c}^{IS_i}(x) = \begin{cases} 1, & if\,([x]_{IS_i} \cap X = \emptyset); \\ 0, & if\,([x]_{IS_i} \cap X \neq \emptyset). \end{cases} \tag{6}$$

Note that the $[x]_{IS_i}$ represents the equivalence class of x with respect to AT in IS_i. If X satisfies $\overline{MS\text{-}GM}_{MS}(X) = \underline{MS\text{-}GM}_{MS}(X)$, the X is a definable target set in MsIS. Conversely, the X is a rough target set. This model is called the generalized multi-granulation rough set model of MsIS (MS-GMRS). Then the positive region, negative region, and boundary region of X are

$$POS(X) = \underline{MS\text{-}GM}_{MS}(X), NEG(X) = (\overline{MS\text{-}GM}_{MS}(X))^c, BND(X) = \overline{MS\text{-}GM}_{MS}(X) - \underline{MS\text{-}GM}_{MS}(X).$$

Here are two extreme forms of MS-GMRS model, namely pessimism multi-granulation rough set model of MsIS (MS-PMRS), and optimism multi-granulation rough set model of MsIS (MS-OMRS).

Definition 3.3. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the MS-PMRS model, for any $X \subseteq U$, the lower and upper approximations are defined by

$$\underline{MS\text{-}PM}_{MS}(X) = \{x \in U \mid \wedge_{i=1}^{s} ([x]_{IS_i} \subseteq X)\} = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}S_X^{IS_i}(x)}{s} \geq 1 \right. \right\}; \tag{7}$$

$$\overline{MS\text{-}PM}_{MS}(X) = \{x \in U \mid \vee_{i=1}^{s} ([x]_{IS_i} \cap X \neq \emptyset)\} = \left\{ x \in U \left| \frac{\sum_{i=1}^{s}(1 - MS\text{-}S_{X^c}^{IS_i}(x))}{s} > 0 \right. \right\}. \tag{8}$$

The expression of the $POS(X)$, $NEG(X)$, and $BND(X)$ of MS-PMRS model are the same as MS-GMRS model.

Definition 3.4. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the MS-OMRS model, for any $X \subseteq U$, the lower and upper approximations are defined by

$$\underline{MS\text{-}OM}_{MS}(X) = \{x \in U \mid \wedge_{i=1}^{s} ([x]_{IS_i} \subseteq X)\} = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}S_{X}^{IS_i}(x)}{s} > 0 \right. \right\}; \tag{9}$$

$$\overline{MS\text{-}OM}_{MS}(X) = \{x \in U \mid \vee_{i=1}^{s} ([x]_{IS_i} \cap X \neq \emptyset)\} = \left\{ x \in U \left| \frac{\sum_{i=1}^{s}(1 - MS\text{-}S_{X^c}^{IS_i}(x))}{s} \geq 1 \right. \right\}. \tag{10}$$

The expression of the $POS(X)$, $NEG(X)$, and $BND(X)$ of MS-OMRS model are the same as MS-GMRS model.

Proposition 3.1. *Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$, for any $X \subseteq U$, $\varphi \in (0.5, 1]$. The following conclusions hold:*

(1) $\underline{MS\text{-}PM}_{MS}(X) \subseteq \underline{MS\text{-}GM}_{MS}(X) \subseteq \underline{MS\text{-}OM}_{MS}(X)$;
(2) $\overline{MS\text{-}OM}_{MS}(X) \subseteq \overline{MS\text{-}GM}_{MS}(X) \subseteq \overline{MS\text{-}PM}_{MS}(X)$.

Proof. (1) For any $x \in U$, one can prove $\underline{MS\text{-}PM}_{MS}(X) \subseteq \underline{MS\text{-}GM}_{MS}(X)$ through $x \in \underline{MS\text{-}PM}_{MS}(X) \Leftrightarrow \frac{\sum_{i=1}^{s} MS\text{-}S_{X}^{IS_i}(x)}{s} \geq 1$. As $\varphi \in (0.5, 1]$, $\frac{\sum_{i=1}^{s} MS\text{-}S_{X}^{IS_i}(x)}{s} \geq 1 \geq \varphi$. Then, $x \in \underline{MS\text{-}GM}_{MS}(X)$. Therefore, $\underline{MS\text{-}PM}_{MS}(X) \subseteq \underline{MS\text{-}GM}_{MS}(X)$. Analogously, $x \in \underline{MS\text{-}GM}_{MS}(X) \Leftrightarrow \frac{\sum_{i=1}^{s} MS\text{-}S_{X}^{IS_i}(x)}{s} \geq \varphi > 0 \Rightarrow x \in \underline{MS\text{-}OM}_{MS}(X)$. Therefore, $\underline{MS\text{-}GM}_{MS}(X) \subseteq \underline{MS\text{-}OM}_{MS}(X)$. This conclusion is proved.

(2) This conclusion can be proved similarly.

Proposition 3.2. *Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$, for any $X \subseteq U$, $\varphi \in (0.5, 1]$, the following properties are true.*

(L_1) $\underline{MS\text{-}GM}_{MS}(X^c) = (\overline{MS\text{-}GM}_{MS}(X))^c$; ($U_1$) $\overline{MS\text{-}GM}_{MS}(X^c) = (\underline{MS\text{-}GM}_{MS}(X))^c$.
(L_2) $\underline{MS\text{-}GM}_{MS}(X) \subseteq X$; ($U_2$) $X \subseteq \overline{MS\text{-}GM}_{MS}(X)$.
(L_3) $\underline{MS\text{-}GM}_{MS}(\emptyset) = \emptyset$; ($U_3$) $\overline{MS\text{-}GM}_{MS}(\emptyset) = \emptyset$.
(L_4) $\underline{MS\text{-}GM}_{MS}(U) = U$; ($U_4$) $\overline{MS\text{-}GM}_{MS}(U) = U$.
(L_5) $X \subseteq Y \Rightarrow \underline{MS\text{-}GM}_{MS}(X) \subseteq \underline{MS\text{-}GM}_{MS}(Y)$; ($U_5$) $X \subseteq Y \Rightarrow \overline{MS\text{-}GM}_{MS}(X) \subseteq \overline{MS\text{-}GM}_{MS}(Y)$.
(L_6) $\underline{MS\text{-}GM}_{MS}(X \cap Y) \subseteq \underline{MS\text{-}GM}_{MS}(X) \cap \underline{MS\text{-}GM}_{MS}(Y)$;
(U_6) $\overline{MS\text{-}GM}_{MS}(X \cup Y) \supseteq \overline{MS\text{-}GM}_{MS}(X) \cup \overline{MS\text{-}GM}_{MS}(Y)$.
(L_7) $\underline{MS\text{-}GM}_{MS}(X \cup Y) \supseteq \underline{MS\text{-}GM}_{MS}(X) \cup \underline{MS\text{-}GM}_{MS}(Y)$;
(U_7) $\overline{MS\text{-}GM}_{MS}(X \cap Y) \subseteq \overline{MS\text{-}GM}_{MS}(X) \cap \overline{MS\text{-}GM}_{MS}(Y)$.

Proof. (L_1) $\forall X \subseteq U$, $x \in \overline{MS\text{-}GM}_{MS}(X) \Leftrightarrow \frac{\sum_{i=1}^{s}(1-MS\text{-}S_{X^c}^{IS_i}(x))}{s} > 1 - \varphi$, we have that $x \in (\overline{MS\text{-}GM}_{MS}(X))^c \Leftrightarrow \frac{\sum_{i=1}^{s}(1-MS\text{-}S_{X^c}^{IS_i}(x))}{s} \leq 1 - \varphi \Leftrightarrow \frac{\sum_{i=1}^{s} MS\text{-}S_{X^c}^{IS_i}(x)}{s} \geq \varphi \Leftrightarrow \underline{MS\text{-}GM}_{MS}(X^c)$. Therefore, the ($L_1$) is proved. The ($U_1$) can be proved in the same way as (L_1).

(L_2) $\forall x \in \underline{MS\text{-}GM}_{MS}(X)$, we have $\frac{\sum_{i=1}^{s} MS\text{-}S_X^{IS_i}(x)}{s} \geq \varphi > 0$. Obviously, $\exists i \leq s$ such that $[x]_{IS_i} \subseteq X$. So, $x \in X$ can be obtained. Therefore, the (L_2) is proved.

(U_2) Based on (L_1) and (L_2), we can get $(\overline{MS\text{-}GM}_{MS}(X))^c = \underline{MS\text{-}GM}_{MS}(X^c) \subseteq X^c$. So $X \subseteq \overline{MS\text{-}GM}_{MS}(X^c)$ can be directly proved.

(L_3), (U_3), (L_4), and (U_4) can all be directly proved by Eqs. (3), (4).

(L_5) $\forall x \in \underline{MS\text{-}GM}_{MS}(X) \Rightarrow \frac{\sum_{i=1}^{s} MS\text{-}S_X^{IS_i}(x)}{s} \geq \varphi$. $X \subseteq Y \Rightarrow \sum_{i=1}^{s} MS\text{-}S_X^{IS_i}(x) \leq \sum_{i=1}^{s} MS\text{-}S_Y^{IS_i}(x) \Rightarrow \frac{\sum_{i=1}^{s} MS\text{-}S_Y^{IS_i}(x)}{s} \geq \varphi$. Then, we can get $x \in \underline{MS\text{-}GM}_{MS}(Y)$. Therefore, the (L_5) is proved. The (U_5) can be proved in the same way as (L_5).

(L_6) Based on the (L_5), we can get $X \cap Y \subseteq X \Rightarrow \underline{MS\text{-}GM}_{MS}(X \cap Y) \subseteq \underline{MS\text{-}GM}_{MS}(X)$ and $X \cap Y \subseteq Y \Rightarrow \underline{MS\text{-}GM}_{MS}(X \cap Y) \subseteq \underline{MS\text{-}GM}_{MS}(Y)$. Thus, $\underline{MS\text{-}GM}_{MS}(X \cap Y) \subseteq \underline{MS\text{-}GM}_{MS}(X) \cap \underline{MS\text{-}GM}_{MS}(Y)$ is proved. Similarly, (U_6) can be proved.

(L_7) can be directly certified according to (L_5). (U_7) can be directly certified according to (U_5).

For survey the classification ability of MS-GMRS model, the approximation accuracy is defined in MsDS. The specific definition is as follows.

Definition 3.5. Let $MDS = \{DS_1, DS_2, \ldots, DS_s\}$ be a MsDS, where $DS_i = (U, AT \cup DT, V_i, f_i)$, $U/DT = \{D_1, D_2, \ldots, D_n\}$ is a set of decision classes. In MS-GMRS model, the approximation accuracy of U/DT is defined by

$$\alpha_{MDS}(U/DT) = \frac{\sum_{j=1}^{n} \left|\underline{MS\text{-}GM}_{MS}(D_j)\right|}{\sum_{j=1}^{n} \left|\overline{MS\text{-}GM}_{MS}(D_j)\right|}. \tag{11}$$

The following we design an algorithm for calculating the approximation accuracy of MS-GMRS model, which is Algorithm 1. The time complexity of this Algorithm 1 is analyzed as : the time complexity of steps 3 - 13 is $O(m \times s)$, the time complexity of steps 15 - 22 is $O(m)$. Therefore, the time complexity of Algorithm 1 is $O((m \times s + m) \times n) = O(mns)$.

3.2. IMS-GMDQ-DTRS: the first kind of generalized multi-granulation double-quantitative decision-theoretic rough set for multi-source information system

In this subsection, we proposed the first kind of generalized multi-granulation double-quantitative decision-theoretic rough set model of MsIS (IMS-GMDQ-DTRS) by introducing the DqI-DTRS model. The corresponding decision rules of this model are investigated.

Definition 3.6. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the IMS-GMDQ-DTRS model, for any $X \subseteq U$, the upper and lower approximations are defined by

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}USI_X^{IS_i}(x)}{s} > 1 - \varphi \right. \right\}, \tag{12}$$

$$\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \left\{ x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}LSI_X^{IS_i}(x)}{s} \geq \varphi \right. \right\}, \tag{13}$$

where $\varphi \in (0.5, 1]$ is an adjustable information standard with respect to MS. Under IS_i, the upper support feature function of $x \in U$ with respect to X is

$$MS\text{-}USI_X^{IS_i}(x) = \begin{cases} 1, & if\, P(X|[x]_{IS_i}) > \beta; \\ 0, & otherwise. \end{cases} \tag{14}$$

Under IS_i, the lower support feature function of $x \in U$ with respect to X is

$$MS\text{-}LSI_X^{IS_i}(x) = \begin{cases} 1, & if\, |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k; \\ 0, & otherwise. \end{cases} \tag{15}$$

Algorithm 1: The approximation accuracy is calculated in MS-GMRS model.

Input : $MDS=\{DS_1, DS_2, \ldots, DS_s\}$, $U/D=\{D_1, D_2, \ldots, D_n\}$, φ.
Output : The approximation accuracy $\alpha_{MDS}(U/DT)$.
begin
  for $g=1:n$ **do**
    for $i=1:m$ **do**
      $MS\text{-}US(i) \leftarrow 0$; `/* It represents` $MS\text{-}S_{D_g}^{DS_i}(x)$ `*/`
      $MS\text{-}LS(i) \leftarrow 0$; `/* It represents` $1-MS\text{-}S_{(D_g)^c}^{DS_i}(x)$ `*/`
      for $j=1:s$ **do**
        if $[x_i]_{DS_j} \subseteq D_g$ **then**
          $MS\text{-}LS(i) \leftarrow MS\text{-}LS(i)+1$;
        end
        if $[x_i]_{DI_j} \cap D_g \neq \emptyset$ **then**
          $MS\text{-}US(i) \leftarrow MS\text{-}US(i)+1$;
        end
      end
    end
    $\underline{MS\text{-}GM}_{MS}(D_g) \leftarrow \emptyset$; $\overline{MS\text{-}GM}_{MS}(D_g) \leftarrow \emptyset$;
    for $i=1:m$ **do**
      if $\frac{MS\text{-}LS(i)}{s} \geq \varphi$ **then**
        $\underline{MS\text{-}GM}_{MS}(D_g) \leftarrow \underline{MS\text{-}GM}_{MS}(D_g) \cup \{x_i\}$;
      end
      if $\frac{MS\text{-}US(i)}{s} > 1-\varphi$ **then**
        $\overline{MS\text{-}GM}_{MS}(D_g) \leftarrow \overline{MS\text{-}GM}_{MS}(D_g) \cup \{x_i\}$;
      end
    end
  end
  return : $\alpha_{MDS}(U/DT) \leftarrow \frac{\sum_{k=1}^{n}\left|\underline{MS\text{-}GM}_{MS}(D_g)\right|}{\sum_{k=1}^{n}\left|\overline{MS\text{-}GM}_{MS}(D_g)\right|}$.
end

If X satisfies $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X)=\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X)$, X is a definable target set in MsIS. Conversely, X is a rough target set. The positive region, negative region, upper and lower boundary region of this model are

$$\begin{aligned} &POS^{\mathrm{I}}(X)=\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \cap \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X);\ NEG^{\mathrm{I}}(X)=(\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \cup \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X))^c;\\ &UBN^{\mathrm{I}}(X)=\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X)-\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X);\ LBN^{\mathrm{I}}(X)=\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X)-\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X). \end{aligned} \tag{16}$$

Here are two extreme forms of IMS-GMDQ-DTRS model, namely the first kind of pessimism multi-granulation double-quantitative decision-theoretic rough set model of MsIS (IMS-PMDQ-DTRS), and the first kind of optimism multi-granulation double-quantitative decision-theoretic rough set model of MsIS (IMS-OMDQ-DTRS).

Definition 3.7. Let $MS=\{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i=(U, AT, V_i, f_i)$. In the IMS-PMDQ-DTRS, for any $X \subseteq U$, the upper and lower approximations are defined by

$$\overline{MS\text{-}PM}^{\mathrm{I}}_{MS}(X)=\{x \in U \mid \vee_{i=1}^{s}(P(X|[x]_{IS_i})>\beta)\}=\left\{x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}USI_X^{IS_i}(x)}{s}>0\right.\right\}; \tag{17}$$

$$\underline{MS\text{-}PM}^{\mathrm{I}}_{MS}(X)=\{x \in U \mid \wedge_{i=1}^{s}(|[x]_{IS_i}|-|[x]_{IS_i} \cap X| \leq k)\}=\left\{x \in U \left| \frac{\sum_{i=1}^{s} MS\text{-}LSI_X^{IS_i}(x)}{s} \geq 1\right.\right\}. \tag{18}$$

The expression of the positive region, negative region, upper and lower boundary region of this model are the same as IMS-GMDQ-DTRS model.

Definition 3.8. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the IMS-OMDQ-DTRS model, for any $X \subseteq U$, the upper and lower approximations are defined by

$$\overline{MS\text{-}OM}^{\mathrm{I}}_{MS}(X) = \{x \in U \mid \wedge_{i=1}^{s} (P(X|[x]_{IS_i}) > \beta)\} = \left\{ x \in U \left| \frac{\sum\limits_{i=1}^{s} MS\text{-}USI_X^{IS_i}(x)}{s} \geq 1 \right. \right\}; \tag{19}$$

$$\underline{MS\text{-}OM}^{\mathrm{I}}_{MS}(X) = \{x \in U \mid \vee_{i=1}^{s} (|[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k)\} = \left\{ x \in U \left| \frac{\sum\limits_{i=1}^{s} MS\text{-}LSI_X^{IS_i}(x)}{s} > 0 \right. \right\}. \tag{20}$$

The expression of the positive region, negative region, upper and lower boundary region of this model are the same as IMS-GMDQ-DTRS model.

Proposition 3.3. *Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$, for any $X \subseteq U$, $\varphi \in (0.5, 1]$. The following conclusions hold:*

(1) $\overline{MS\text{-}PM}^{I}_{MS}(X) \subseteq \overline{MS\text{-}GM}^{I}_{MS}(X) \subseteq \overline{MS\text{-}OM}^{I}_{MS}(X)$;
(2) $\overline{MS\text{-}OM}^{I}_{MS}(X) \subseteq \overline{MS\text{-}GM}^{I}_{MS}(X) \subseteq \overline{MS\text{-}PM}^{I}_{MS}(X)$.

Proof. According to Eqs. (12), (13), (17), (18), (19), (20), the above properties are easily verified.

In what follow, we introduce the decision rules of the proposed models.

Rule 3.1. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. For any $X \subseteq U$, the decision rules of IMS-GMDQ-DTRS are
(P^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| > s(1-\varphi)$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq s\varphi$, then $x \in POS^{\mathrm{I}}(X)$;
(N^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| \leq s(1-\varphi)$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| < s\varphi$, then $x \in NEG^{\mathrm{I}}(X)$;
(UB^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| > s(1-\varphi)$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| < s\varphi$, then $x \in UBN^{\mathrm{I}}(X)$;
(LB^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| \leq s(1-\varphi)$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq s\varphi$, then $x \in LBN^{\mathrm{I}}(X)$.
Considering the idea of pessimism, the decision rules of IMS-PMDQ-DTRS model can be deduced, which are
(P^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| \geq 1$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| = s$, then $x \in POS^{\mathrm{I}}(X)$;
(N^{I}) If $|IS_i : P(X|[x]_{IS_i}) \leq \beta| = s$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq 1$, then $x \in NEG^{\mathrm{I}}(X)$;
(UB^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| \geq 1$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq 1$, then $x \in UBN^{\mathrm{I}}(X)$;
(LB^{I}) If $|IS_i : P(X|[x]_{IS_i}) \leq \beta| = s$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| = s$, then $x \in LBN^{\mathrm{I}}(X)$.
Similarly, according to the idea of optimism, the decision rules of IMS-OMDQ-DTRS model can be obtained, which are
(P^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| = s$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq 1$, then $x \in POS^{\mathrm{I}}(X)$;
(N^{I}) If $|IS_i : P(X|[x]_{IS_i}) \leq \beta| \geq 1$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| > k| = s$, then $x \in NEG^{\mathrm{I}}(X)$;
(UB^{I}) If $|IS_i : P(X|[x]_{IS_i}) > \beta| = s$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| > k| = s$, then $x \in UBN^{\mathrm{I}}(X)$;
(LB^{I}) If $|IS_i : P(X|[x]_{IS_i}) \leq \beta| \geq 1$, $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq 1$, then $x \in LBN^{\mathrm{I}}(X)$.

Definition 3.9. Let $MDS = \{DS_1, DS_2, \ldots, DS_s\}$ be a MsDS, where $DS_i = (U, AT \cup DT, V_i, f_i)$, $U/DT = \{D_1, D_2, \ldots, D_n\}$ is a set of decision classes. In IMS-GMDQ-DTRS model, the approximation accuracy of U/DT is defined by

$$\alpha_{IMDS}(U/DT) = \frac{\sum\limits_{j=1}^{n} \left| \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_j) \right|}{\sum\limits_{j=1}^{n} \left| \overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_j) \right|}. \tag{21}$$

In order to calculate the approximation accuracy of IMS-GMDQ-DTRS model, we design Algorithm 2. The time complexity is similar to that of Algorithm 1.

3.3. IIMS-GMDQ-DTRS: the second kind of generalized multi-granulation double-quantitative decision-theoretic rough set for multi-source information system

This subsection introduces the second kind of generalized multi-granulation double-quantitative decision-theoretic rough set model (IIMS-GMDQ-DTRS). Then the decision rules of this model are investigated.

Algorithm 2: The approximation accuracy is calculated in IMS-GMDQ-DTRS model.

Input : $MDS = \{DS_1, DS_2, \dots, Ds_s\}$, $U/D = \{D_1, D_2, \dots, D_n\}$, φ, β, and k.
Output : The approximation accuracy $\alpha_{IMDS}(U/DT)$.

begin
  for $g = 1 : n$ **do**
    for $i = 1 : m$ **do**
      $MS\text{-}USI(i) \leftarrow 0$; $MS\text{-}LSI(i) \leftarrow 0$;
      for $j = 1 : s$ **do**
        if $|[x]_{DS_j}| - |[x]_{DS_j} \cap D_g| \le k$ **then**
          $MS\text{-}LSI(i) \leftarrow MS\text{-}LSI(i) + 1$;
        end
        if $P(X|[x]_{DS_j}) > \beta$ **then**
          $MS\text{-}USI(i) \leftarrow MS\text{-}USI(i) + 1$;
        end
      end
    end
    $\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g) \leftarrow \emptyset$; $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g) \leftarrow \emptyset$;
    for $i = 1 : m$ **do**
      if $\frac{MS\text{-}LSI(i)}{s} \ge \varphi$ **then**
        $\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g) \leftarrow \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g) \cup \{x_i\}$;
      end
      if $\frac{MS\text{-}USI(i)}{s} > 1 - \varphi$ **then**
        $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g) \leftarrow \overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g) \cup \{x_i\}$;
      end
    end
  end
  return : $\alpha_{IMDS}(U/D) \leftarrow \frac{\sum_{k=1}^{n} \left|\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g)\right|}{\sum_{k=1}^{n} \left|\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(D_g)\right|}$.
end

Definition 3.10. Let $MS = \{IS_1, IS_2, \dots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the IIMS-GMDQ-DTRS model, for any $X \subseteq U$, the upper and lower approximations are defined by

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{s} MS\text{-}USII^{IS_i}_X(x)}{s} > 1 - \varphi \right\}, \tag{22}$$

$$\underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{s} MS\text{-}LSII^{IS_i}_X(x)}{s} \ge \varphi \right\}, \tag{23}$$

where $\varphi \in (0.5, 1]$ is an adjustable information standard with respect to MS. Under IS_i, the upper support feature function of $x \in U$ with respect to X is

$$MS\text{-}USII^{IS_i}_X(x) = \begin{cases} 1, & if\, |[x]_{IS_i} \cap X| > k; \\ 0, & otherwise. \end{cases} \tag{24}$$

Under IS_i, the lower support feature function of $x \in U$ with respect to X is

$$MS\text{-}LSII^{IS_i}_X(x) = \begin{cases} 1, & if\, P(X|[x]_{IS_i}) \ge \alpha; \\ 0, & otherwise. \end{cases} \tag{25}$$

If X satisfies $\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X)$, X is a definable target set in MsIS. Conversely, X is a rough target set. The positive region, negative region, upper and lower boundary region of this model are

$$\begin{aligned} &POS^{\mathrm{II}}(X) = \overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) \cap \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X); \; NEG^{\mathrm{II}}(X) = (\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) \cup \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X))^c; \\ &UBN^{\mathrm{II}}(X) = \overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) - \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X); \; LBN^{\mathrm{II}}(X) = \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) - \overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X). \end{aligned} \tag{26}$$

Here are two extreme forms of IIMS-GMDQ-DTRS model, namely the second kind of pessimism multi-granulation double-quantitative rough set model of MsIS (IIMS-PMDQ-DTRS), and the second kind of optimism multi-granulation double-quantitative rough set model of MsIS (IIMS-OMDQ-DTRS).

Definition 3.11. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the IIMS-PMDQ-DTRS model, for any $X \subseteq U$, the upper and lower approximations are defined by

$$\overline{MS\text{-}PM}^{\mathrm{II}}_{MS}(X) = \{x \in U \mid \vee_{i=1}^{s} (|[x]_{IS_i} \cap X| > k)\} = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{s} MS\text{-}USII^{IS_i}_{X}(x)}{s} > 0 \right\}; \tag{27}$$

$$\underline{MS\text{-}PM}^{\mathrm{II}}_{MS}(X) = \{x \in U \mid \wedge_{i=1}^{s} (P(X|[x]_{IS_i}) \geq \alpha)\} = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{s} MS\text{-}LSII^{IS_i}_{X}(x)}{s} \geq 1 \right\}. \tag{28}$$

The positive region, negative region, upper and lower boundary region of this model are the same as IIMS-GMDQ-DTRS model.

Definition 3.12. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$. In the IIMS-OMDQ-DTRS model, for any $X \subseteq U$, the upper and lower approximations are defined by

$$\overline{MS\text{-}OM}^{\mathrm{II}}_{MS}(X) = \{x \in U \mid \wedge_{i=1}^{s} (|[x]_{IS_i} \cap X| > k)\} = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{s} MS\text{-}USII^{IS_i}_{X}(x)}{s} \geq 1 \right\}; \tag{29}$$

$$\underline{MS\text{-}OM}^{\mathrm{II}}_{MS}(X) = \{x \in U \mid \vee_{i=1}^{s} (P(X|[x]_{IS_i}) \geq \alpha)\} = \left\{ x \in U \,\middle|\, \frac{\sum_{i=1}^{s} MS\text{-}LSII^{IS_i}_{X}(x)}{s} > 0 \right\}. \tag{30}$$

The positive region, negative region, upper and lower boundary region of this model are the same as IIMS-GMDQ-DTRS model.

Proposition 3.4. *Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$, for any $X \subseteq U$, $\varphi \in (0.5, 1]$. The following conclusions hold:*

(1) $\overline{MS\text{-}PM}^{II}_{MS}(X) \subseteq \overline{MS\text{-}GM}^{II}_{MS}(X) \subseteq \overline{MS\text{-}OM}^{II}_{MS}(X)$;
(2) $\overline{MS\text{-}OM}^{II}_{MS}(X) \subseteq \overline{MS\text{-}GM}^{II}_{MS}(X) \subseteq \overline{MS\text{-}PM}^{II}_{MS}(X)$.

Proof. According to Eqs. (22), (23), (27), (28), (29), (30), the above properties are easily verified.

The following the decision rules of the proposed models are presented.

Rule 3.2. Let $MS = \{IS_1, IS_2, \ldots, IS_s\}$ be a MsIS, where $IS_i = (U, AT, V_i, f_i)$, for any $X \subseteq U$. The decision rules of IIMS-GMDQ-DTRS are
(P^{II}) If $|IS_i : |[x]_{IS_i} \cap X| > k| > s(1-\varphi), |IS_i : P(X|[x]_{IS_i}) \geq \alpha| \geq s\varphi$, then $x \in POS^{\mathrm{II}}(X)$;
(N^{II}) If $|IS_i : |[x]_{IS_i} \cap X| > k| \leq s(1-\varphi), |IS_i : P(X|[x]_{IS_i}) \geq \alpha| < s\varphi$, then $x \in NEG^{\mathrm{II}}(X)$;
(UB^{II}) If $|IS_i : |[x]_{IS_i} \cap X| > k| > s(1-\varphi), |IS_i : P(X|[x]_{IS_i}) \geq \alpha| < s\varphi$, then $x \in UBN^{\mathrm{II}}(X)$;
(LB^{II}) If $|IS_i : |[x]_{IS_i} \cap X| > k| \leq s(1-\varphi), |IS_i : P(X|[x]_{IS_i}) \geq \alpha| \geq s\varphi$, then $x \in LBN^{\mathrm{II}}(X)$.

Combining the idea of pessimism, the decision rules of IIMS-PMDQ-DTRS model can be deduced, which are
(P^{II}) If $|IS_i : |[x]_{IS_i} \cap X| > k| \geq 1, |IS_i : P(X|[x]_{IS_i}) \geq \alpha| = s$, then $x \in POS^{\mathrm{II}}(X)$;
(N^{II}) If $|IS_i : |[x]_{IS_i} \cap X| \leq k| = s, |IS_i : P(X|[x]_{IS_i}) \geq \alpha| \geq 1$, then $x \in NEG^{\mathrm{II}}(X)$;
(UB^{II}) If $|IS_i : |[x]_{IS_i} \cap X| > k| \geq 1, |IS_i : P(X|[x]_{IS_i}) \geq \alpha| \geq 1$, then $x \in UBN^{\mathrm{II}}(X)$;
(LB^{II}) If $|IS_i : |[x]_{IS_i} \cap X| \leq k| = s, |IS_i : P(X|[x]_{IS_i}) \geq \alpha| = s$, then $x \in LBN^{\mathrm{II}}(X)$.

Similarly, based on the idea of optimism, the decision rules of IIMS-OMDQ-DTRS model can be obtained, which are
(P^{II}) If $|IS_i : |[x]_{IS_i} \cap X| \leq k| = s, |IS_i : P(X|[x]_{IS_i}) \geq \alpha| \geq 1$, then $x \in POS^{\mathrm{II}}(X)$;

Algorithm 3: The approximation accuracy is calculated in IIMS-GMDQ-DTRS model.

Input : $MDS = \{DS_1, DS_2, \ldots, DS_s\}$, $U/D = \{D_1, D_2, \ldots, D_n\}$, φ, α, and k.
Output : The approximation accuracy $\alpha_{IIMDS}(U/DT)$.

begin
  for $g = 1 : n$ **do**
    for $i = 1 : m$ **do**
      $MS\text{-}US\text{II}(i) \leftarrow 0$; $MS\text{-}LS\text{II}(i) \leftarrow 0$;
      for $j = 1 : s$ **do**
        if $P(X|[x]_{DS_j}) \geq \alpha$ **then**
          $MS\text{-}LS\text{II}(i) \leftarrow MS\text{-}LS\text{II}(i) + 1$;
        end
        if $|[x]_{DS_j} \cap D_g| > k$ **then**
          $MS\text{-}US\text{II}(i) \leftarrow MS\text{-}US\text{II}(i) + 1$;
        end
      end
    end
    $\underline{MS\text{-}GM}^{\text{II}}_{MS}(D_g) \leftarrow \emptyset$; $\overline{MS\text{-}GM}^{\text{II}}_{MS}(D_g) \leftarrow \emptyset$;
    for $i = 1 : m$ **do**
      if $\frac{MS\text{-}LSII(i)}{s} \geq \varphi$ **then**
        $\underline{MS\text{-}GM}^{\text{II}}_{MS}(D_g) \leftarrow \underline{MS\text{-}GM}^{\text{II}}_{MS}(D_g) \cup \{x_i\}$;
      end
      if $\frac{MS\text{-}USII(i)}{s} > 1 - \varphi$ **then**
        $\overline{MS\text{-}GM}^{\text{II}}_{MS}(D_g) \leftarrow \overline{MS\text{-}GM}^{\text{II}}_{MS}(D_g) \cup \{x_i\}$;
      end
    end
  end
  return : $\alpha_{IIMDS}(U/D) \leftarrow \frac{\sum_{k=1}^{n} \left|\underline{MS\text{-}GM}^{\text{II}}_{MS}(D_g)\right|}{\sum_{k=1}^{n} \left|\overline{MS\text{-}GM}^{\text{II}}_{MS}(D_g)\right|}$.
end

(N^{II}) If $|IS_i : |[x]_{IS_i} \cap X| \leq k| \geq 1$, $|IS_i : P(X|[x]_{IS_i}) < \alpha| = s$, then $x \in NEG^{\text{II}}(X)$;
(UB^{II}) If $|IS_i : |[x]_{IS_i} \cap X| \leq k| = s$, $|IS_i : P(X|[x]_{IS_i}) < \alpha| = s$, then $x \in UBN^{\text{II}}(X)$;
(LB^{II}) If $|IS_i : |[x]_{IS_i} \cap X| \leq k| \geq 1$, $|IS_i : P(X|[x]_{IS_i}) \geq \alpha| \geq 1$, then $x \in LBN^{\text{II}}(X)$.

Definition 3.13. Let $MDS = \{DS_1, DS_2, \ldots, DS_s\}$ be a MsDS, where $DS_i = (U, AT \cup DT, V_i, f_i)$, $U/DT = \{D_1, D_2, \cdots, D_n\}$ is a set of decision classes. In IIMS-GMDQ-DTRS model, the approximation accuracy of U/DT is defined by

$$\alpha_{IIMDS}(U/DT) = \frac{\sum_{j=1}^{n} \left|\underline{MS\text{-}GM}^{\text{II}}_{MS}(D_j)\right|}{\sum_{j=1}^{n} \left|\overline{MS\text{-}GM}^{\text{II}}_{MS}(D_j)\right|}. \tag{31}$$

The following the Algorithm 3 is designed to calculate the approximation accuracy of IIMS-GMDQ-DTRS model, and its time complexity is similar to that of Algorithm 1.

3.4. The relations between models

In this subsection, we discuss the inner relations between the MS-GMRS model and a pair of MS-GMDQ-DTRS models. Further, the relations between the decision regions of two MS-GMDQ-DTRS models are explored.

(1) The inner relations between the proposed models.

a. When $\beta = 0, k = 0$, the IMS-GMDQ-DTRS model degenerates to MS-GMRS model, i.e.,

$$(\overline{MS\text{-}GM}^{\text{I}}_{MS}(X), \underline{MS\text{-}GM}^{\text{I}}_{MS}(X)) \xrightarrow{\beta=0,k=0} (\overline{MS\text{-}GM}_{MS}(X), \underline{MS\text{-}GM}_{MS}(X)).$$

Since $\beta = 0, k = 0$, the upper support feature function $MS\text{-}USI^{IS_i}_X(x)$ degenerates to $1 - MS\text{-}S^{IS_i}_{X^c}(x)$, and the lower support feature function $MS\text{-}LSI^{IS_i}_X(x)$ degenerates to $MS\text{-}S^{IS_i}_X(x)$. Therefore, we have $\overline{MS\text{-}GM}^{\text{I}}_{MS}(X) = \overline{MS\text{-}GM}_{MS}(X)$ and $\underline{MS\text{-}GM}^{\text{I}}_{MS}(X) = \underline{MS\text{-}GM}_{MS}(X)$. So the relation **a.** holds. That is to say, IMS-GMDQ-DTRS model is an extension of MS-GMRS model.

When $\beta > 0, k > 0$, the fault tolerance capability of the IMS-GMDQ-DTRS model is higher than that of the MS-GMRS model, i.e., $\alpha_{IMDS}(U/DT) \geq \alpha_{MDS}(U/DT)$. Since for any $X \in U/DT$, according to Eqs. (3), (4), (12), (13), we can get $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \subseteq \overline{MS\text{-}GM}_{MS}(X), \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \supseteq \underline{MS\text{-}GM}_{MS}(X)$. According to Eqs. (11), (21), it is easy to get $\alpha_{IMDS}(U/DT) \geq \alpha_{MDS}(U/DT)$. The higher the approximation accuracy of the model, the stronger the fault tolerance capability of the model. Thus the fault tolerance of IMS-GMDQ-DTRS model is superior to MS-GMRS model.

b. When $\alpha = 1, k = 0$, the IIMS-GMDQ-DTRS model also degenerates to MS-GMRS model, i.e.,

$$(\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X), \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X)) \xrightarrow{\alpha=1,k=0} (\overline{MS\text{-}GM}_{MS}(X), \underline{MS\text{-}GM}_{MS}(X)).$$

Because $\alpha = 1, k = 0$, the upper support feature function $MS\text{-}USI\mathrm{I}^{IS_i}_X(x)$ degenerates to $1 - MS\text{-}S^{IS_i}_{X^c}(x)$, and the lower support feature function $MS\text{-}LSI\mathrm{I}^{IS_i}_X(x)$ also degenerates to $MS\text{-}S^{IS_i}_X(x)$. Thus, we have $\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \overline{MS\text{-}GM}_{MS}(X)$ and $\underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \underline{MS\text{-}GM}_{MS}(X)$. So the relation **b.** holds. Similarly, IIMS-GMDQ-DTRS model is also an extension of MS-GMRS model.

When $\alpha < 1, k > 0$, the fault tolerance capability IIMS-GMDQ-DTRS model is higher than that of the MS-GMRS model, i.e., $\alpha_{IIMDS}(U/DT) \geq \alpha_{MDS}(U/DT)$. Since for any $X \in U/DT$, according to Eqs. (3), (4), (22), (23), we obtain the conclusion that $\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) \subseteq \overline{MS\text{-}GM}_{MS}(X), \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) \supseteq \underline{MS\text{-}GM}_{MS}(X)$. Based on Eqs. (11), (31), we obtain the conclusion that $\alpha_{IIMDS}(U/DT) \geq \alpha_{MDS}(U/DT)$. Similarly, the fault tolerance of IIMS-GMDQ-DTRS model is also better than that of MS-GMRS model.

c. When $\alpha = 1, \beta = 0, k = 0$, two MS-GMDQ-DTRS models degenerate to MS-GMRS model.

When $\alpha = 1, \beta = 0, k = 0$, based on the relations **a.** and **b.**, these three models are equivalent, i.e.,

$$(\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X), \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X))$$
$$\xrightarrow{\beta=0,k=0} (\overline{MS\text{-}GM}_{MS}(X), \underline{MS\text{-}GM}_{MS}(X)) \xleftarrow{\alpha=1,k=0} (\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X), \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X)).$$

For the same reason as **a.** and **b.**, the IMS-PMDQ-DTRS model degenerates to MS-PMRS model and IMS-OMDQ-DTRS model degenerates to MS-OMRS model. The IIMS-PMDQ-DTRS model degenerates to MS-PMRS model and IIMS-OMDQ-DTRS model degenerates to MS-OMRS model.

When $\alpha < 1, \beta > 0, k > 0$, the fault tolerance capability of two MS-GMDQ-DTRS models are higher than that of the MS-GMRS model, i.e., $\alpha_{IMDS}(U/DT) \geq \alpha_{MDS}(U/DT)$ and $\alpha_{IIMDS}(U/DT) \geq \alpha_{MDS}(U/DT)$. Based on the conclusions of **a.** and **b.**, the conclusion can be easily proved. Therefore, the fault tolerance capability of two MS-GMDQ-DTRS models are superior to MS-GMRS model in MsDS.

(2) The relations between the decision regions of two MS-GMDQ-DTRS models.

Based on DTRS model, if loss function satisfies $(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) > (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, we have $0 \leq \beta < \alpha \leq 1$. We discuss the relations between different regions in two MS-GMDQ-DTRS models under different conditions of α and β while k keeps unchanged.

a. When $\alpha + \beta = 1$, the relations of the decision regions are

$$\text{(I) } POS^{\mathrm{I}}(X) = NEG^{\mathrm{II}}(X^c);\ \text{(II) } NEG^{\mathrm{I}}(X) = POS^{\mathrm{II}}(X^c);\ \text{(III) } UBN^{\mathrm{I}}(X) = UBN^{\mathrm{II}}(X^c);\ \text{(IV) } LBN^{\mathrm{I}}(X) = LBN^{\mathrm{II}}(X^c).$$

Proof. First, by considering the support feature function, we can get $P(X|[x]_{IS_i}) > \beta \Leftrightarrow P(X^c|[x]_{IS_i}) < 1 - \beta$, $|[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k \Leftrightarrow |[x]_{IS_i} \cap X^c| \leq k$. Due to $\beta = 1 - \alpha$, $P(X|[x]_{IS_i}) > \beta \Leftrightarrow P(X^c|[x]_{IS_i}) < \alpha$ can be obtained. So we can get $|IS_i : P(X|[x]_{IS_i}) > \beta| > s(1 - \varphi) \Leftrightarrow |IS_i : P(X^c|[x]_{IS_i}) < \alpha| > s(1 - \varphi) \Leftrightarrow |IS_i : P((X^c)|[x]_{IS_i}) \geq \alpha| < s\varphi$ and $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap (X)| \leq k| \geq s\varphi \Leftrightarrow |IS_i : |[x]_{IS_i} \cap (X^c)| \leq k| \geq s\varphi \Leftrightarrow |IS_i : |[x]_{IS_i} \cap (X^c)| > k| \leq s(1 - \varphi)$. So we can get $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \cap \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \Leftrightarrow (\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) \cup \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c))^c$. Therefore, $POS^{\mathrm{I}}(X) = NEG^{\mathrm{II}}(X^c)$ is certified. The proof of (II), (III), and (VI) are similar to that of (I).

b. When $\alpha + \beta < 1$, the relations of the decision regions are

$$\text{(I) } POS^{\mathrm{I}}(X) \supseteq NEG^{\mathrm{II}}(X^c);\ \text{(II) } NEG^{\mathrm{I}}(X) \subseteq POS^{\mathrm{II}}(X^c);\ \text{(III) } UBN^{\mathrm{I}}(X) \supseteq UBN^{\mathrm{II}}(X^c);\ \text{(IV) } LBN^{\mathrm{I}}(X) \subseteq LBN^{\mathrm{II}}(X^c).$$

Proof. Similarly, based on the support feature function, we can get $P(X|[x]_{IS_i}) > \beta \Leftrightarrow P(X^c|[x]_{IS_i}) < 1 - \beta$, $|[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k \Leftrightarrow |[x]_{IS_i} \cap X^c| \leq k$. Then for $|IS_i : P(X|[x]_{IS_i)>\beta}| > s(1 - \varphi) \Leftrightarrow |IS_i : P(X^c|[x]_{IS_i}) < 1 - \beta| > s(1 - \varphi)$ and $|IS_i : |[x]_{IS_i}| - |[x]_{IS_i} \cap X| \leq k| \geq s\varphi \Leftrightarrow |IS_i : |[x]_{IS_i} \cap (X^c)| \leq k| \geq s\varphi$ are obtained. Simultaneously, $|IS_i : P(X^c|[x]_{IS_i}) < 1 - \beta| > s(1 - \varphi) \Leftrightarrow |IS_i : P(X^c|[x]_{IS_i}) \geq 1 - \beta| < s\varphi$ and $|IS_i : |[x]_{IS_i} \cap (X^c)| \leq k| \geq s\varphi \Leftrightarrow |IS_i : |[x]_{IS_i} \cap (X^c)| > k| \leq s(1 - \varphi)$ hold. Since $\alpha < 1 - \beta$, $|IS_i : P((X^c)|[x]_{IS_i}) \geq \alpha| < s\varphi \Rightarrow |IS_i : P(X^c|[x]_{IS_i}) \geq 1 - \beta| < s\varphi$ and $|IS_i : |[x]_{IS_i} \cap (X^c)| > k| \leq s(1 - \varphi) \Leftrightarrow |IS_i : |[x]_{IS_i} \cap (X^c)| > k| \leq s(1 - \varphi)$ hold. So we get $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \cap \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \Leftarrow (\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) \cup \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c))^c$. Thus the $POS^{\mathrm{I}}(X) \supseteq NEG^{\mathrm{II}}(X^c)$ is certified. The (II), (III), and (VI) may be proofed similarly as (I).

Table 1
The decision regions relations of two MS-GMDQ-DTRS models.

Cases	Relations			
$\alpha+\beta=1$	$POS^{I}(X)=NEG^{II}(X^c)$	$NEG^{I}(X)=POS^{II}(X^c)$	$UBN^{I}(X)=UBN^{II}(X^c)$	$LBN^{I}(X)=LBN^{II}(X^c)$
$\alpha+\beta<1$	$POS^{I}(X)\supseteq NEG^{II}(X^c)$	$NEG^{I}(X)\subseteq POS^{II}(X^c)$	$UBN^{I}(X)\supseteq UBN^{II}(X^c)$	$LBN^{I}(X)\subseteq LBN^{II}(X^c)$
$\alpha+\beta>1$	$POS^{I}(X)\subseteq NEG^{II}(X^c)$	$NEG^{I}(X)\supseteq POS^{II}(X^c)$	$UBN^{I}(X)\subseteq UBN^{II}(X^c)$	$LBN^{I}(X)\supseteq LBN^{II}(X^c)$

Table 2
A multi-source decision system.

U	DS_1				DS_2				DS_3				DS_4				
	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	a_1	a_2	a_3	a_4	d
x_1	1	2	2	1	1	2	2	1	1	2	1	1	1	2	2	1	1
x_2	1	2	1	1	1	2	2	1	1	2	1	1	1	2	1	1	1
x_3	1	1	2	1	1	1	1	1	1	2	1	1	1	1	2	1	0
x_4	0	1	1	1	1	1	1	1	0	1	2	1	0	1	2	0	1
x_5	2	1	1	2	0	1	1	1	1	1	1	1	2	2	1	1	0
x_6	0	1	1	0	0	1	1	1	0	1	2	1	1	1	2	0	1
x_7	1	1	2	1	2	2	1	1	1	2	1	1	1	2	1	1	0
x_8	1	1	1	0	2	2	1	1	1	1	1	1	1	1	1	0	1
x_9	2	1	1	0	2	2	1	1	2	1	2	1	2	1	2	1	0
x_{10}	1	1	1	0	1	1	1	1	2	1	2	1	0	1	2	0	0

c. When $\alpha+\beta>1$, the relations of the decision regions are

$$\text{(I) } POS^{\mathrm{I}}(X)\subseteq NEG^{\mathrm{II}}(X^c);\ \text{(II) } NEG^{\mathrm{I}}(X)\supseteq POS^{\mathrm{II}}(X^c);\ \text{(III) } UBN^{\mathrm{I}}(X)\subseteq UBN^{\mathrm{II}}(X^c);\ \text{(IV) } LBN^{\mathrm{I}}(X)\supseteq LBN^{\mathrm{II}}(X^c).$$

Proof. Similarly, according to the support feature function, we can get $P(X|[x]_{IS_i})>\beta \Leftrightarrow P(X^c|[x]_{IS_i})<1-\beta$, $|[x]_{IS_i}|-|[x]_{IS_i}\cap X|\le k \Leftrightarrow |[x]_{IS_i}\cap X^c|\le k$. Next, for $|IS_i : P(X|[x]_{IS_i})>\beta|>s(1-\varphi) \Leftrightarrow |IS_i : P(X^c|[x]_{IS_i})<1-\beta|>s(1-\varphi)$, $|IS_i : |[x]_{IS_i}|-|[x]_{IS_i}\cap X|\le k|\ge s\varphi \Leftrightarrow |IS_i : |[x]_{IS_i}\cap (X^c)|\le k|\ge s\varphi$ can be obtained. Simultaneously, $|IS_i : P(X^c|[x]_{IS_i})<1-\beta|>s(1-\varphi) \Leftrightarrow |IS_i : P(X^c|[x]_{IS_i})\ge 1-\beta|<s\varphi$, and $|IS_i : |[x]_{IS_i}\cap X^c|\le k|\ge s\varphi \Leftrightarrow |IS_i : |[x]_{IS_i}\cap (X^c)|>k|\le s(1-\varphi)$. Since $\alpha>1-\beta$, the $|IS_i : P(X^c|[x]_{IS_i})\ge 1-\beta|<s\varphi \Rightarrow |IS_i : P((X^c)|[x]_{IS_i})\ge\alpha|<s\varphi$ and $|IS_i : |[x]_{IS_i}\cap (X^c)|>k|\le s(1-\varphi) \Leftrightarrow |IS_i : |[x]_{IS_i}\cap (X^c)|>k|\le s(1-\varphi)$ can be obtained. So we can get $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X)\cap \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \Rightarrow (\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c)\cup \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c))^c$. Therefore, $POS^{\mathrm{I}}(X)\subseteq NEG^{\mathrm{II}}(X^c)$ is certified. The proof of (II), (III), and (VI) is similar to that of (I).

Intuitively, the relations between the decision regions of two MS-GMDQ-DTRS models in different cases are shown in Table 1.

4. Case study

In this section, the conclusions of **3.4** are verified by an case of car detection. The fault tolerance between the proposed models is compared by calculating the approximation accuracy values of the proposed models. There are four automobile evaluation factories that evaluate 10 cars in terms of fuel consumption, machinery, appearance, and safety performance. Then the evaluation grade is divided into upper, middle, and lower. Finally, each car is evaluated as a high quality car or a general car. Therefore, the MsDS is constituted by evaluation results.

Let $MDS=\{DS_1, DS_2, DS_3, DS_4\}$ be a MsDS, where $DS_i=(U, AT\cup d, V_i, f_i)$. The $U=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ stands for ten cars. The $AT=\{a_1, a_2, a_3, a_4\}$, where a_1 stands for fuel consumption, a_2 stands for machinery, a_3 stands for appearance, and a_4 stands for safety performance. The $V^i_{AT}=\{0,1,2\}$, where "0" stands for lower grade, "1" stands for middle grade, and "2" stands for upper grade. The d is decision attribute which stands for quality of car. The $D^i_d=\{0,1\}$, where "0" stands for general car and "1" stands for high quality car. The decision class $X=\{x_1, x_2, x_4, x_6, x_8\}$ is selected as the target set. Let $k=1, \varphi=0.65$. DS_1, DS_2, DS_3, DS_4 are the test results of four automobile evaluation factories. The MsDS is shown as Table 2.

Table 3 lists the equivalence classes of each object under each source.

4.1. Verification of 3.4 (1)

(1) Verification the conclusion 3.4 (1)-a

According to Eqs. (3), (4), the upper and lower approximations of X in MS-GMRS model are

$$\overline{MS\text{-}GM}_{MS}(X)=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\},\ \underline{MS\text{-}GM}_{MS}(X)=\{x_1, x_6\}.$$

Assume $\beta=0.4$, $k=1$, according to Eqs. (12), (13), the upper and lower approximations of X in the IMS-GMDQ-DTRS model are

Table 3
Statistical results of equivalence classes under each source.

U	$[x]_{DS_1}$	$[x]_{DS_2}$	$[x]_{DS_3}$	$[x]_{DS_4}$
x_1	x_1	x_1, x_2	x_1, x_2, x_3, x_7	x_1
x_2	x_2	x_1, x_2	x_1, x_2, x_3, x_7	x_2, x_7
x_3	x_3, x_7	x_3, x_4, x_{10}	x_1, x_2, x_3, x_7	x_3
x_4	x_4	x_3, x_4, x_{10}	x_4, x_6	x_4, x_{10}
x_5	x_5	x_5, x_6	x_5, x_8	x_5
x_6	x_6	x_5, x_6	x_4, x_6	x_6
x_7	x_3, x_7	x_7, x_8, x_9	x_1, x_2, x_3, x_7	x_2, x_7
x_8	x_8, x_{10}	x_7, x_8, x_9	x_5, x_8	x_8
x_9	x_9	x_7, x_8, x_9	x_9, x_{10}	x_9
x_{10}	x_8, x_{10}	x_3, x_4, x_{10}	x_9, x_{10}	x_4, x_{10}

Table 4
Three cases of loss function.

	$\alpha = 0.6, \beta = 0.4$	$\alpha = 0.6, \beta = 0.3$	$\alpha = 0.7, \beta = 0.4$
a_P: accept	$\lambda_{PP} = 0$, $\lambda_{PN} = 22$	$\lambda_{PP} = 0$, $\lambda_{PN} = 9$	$\lambda_{PP} = 0$, $\lambda_{PN} = 13$
a_B: defer	$\lambda_{BP} = 12$, $\lambda_{BN} = 4$	$\lambda_{BP} = 2$, $\lambda_{BN} = 6$	$\lambda_{BP} = 3$, $\lambda_{BN} = 6$
a_N: reject	$\lambda_{NP} = 18$, $\lambda_{NN} = 0$	$\lambda_{NP} = 16$, $\lambda_{NN} = 0$	$\lambda_{NP} = 12$, $\lambda_{NN} = 0$

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\}.$$

The above results show that $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \subseteq \overline{MS\text{-}GM}_{MS}(X)$, $\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) \supseteq \underline{MS\text{-}GM}_{MS}(X)$.

Assume $\beta = 0$, $k = 0$, according to Eqs. (12), (13), the upper and lower approximations of X in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_6\}.$$

This indicates $\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \overline{MS\text{-}GM}_{MS}(X)$, $\underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \underline{MS\text{-}GM}_{MS}(X)$. Thus the conclusion 3.4 (1)-a is verified by the calculation results.

(2) Verification the conclusion 3.4 (1)-b

Assume $\alpha = 0.6$, $k = 1$, according to Eqs. (22), (23), the upper and lower approximations of X in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_2\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_6\}.$$

The results show that $\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) \subseteq \overline{MS\text{-}GM}_{MS}(X)$, $\underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) \supseteq \underline{MS\text{-}GM}_{MS}(X)$.

Assume $\alpha = 1$, $k = 0$, according to Eqs. (22), (23), the upper and lower approximations of X in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_6\}.$$

This indicates $\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \overline{MS\text{-}GM}_{MS}(X)$, $\underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \underline{MS\text{-}GM}_{MS}(X)$. Therefore, the conclusion 3.4 (1)-b is verified by the calculation results.

(3) Verification the conclusion 3.4 (1)-c.

When $\alpha = 1$, $\beta = 0$, $k = 0$, according to the calculation results of (1) and (2), the upper and lower approximations of the proposed models are equal, i.e.,

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \overline{MS\text{-}GM}_{MS}(X) = \overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X), \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \underline{MS\text{-}GM}_{MS}(X) = \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X).$$

That is say that two MS-GMDQ-DTRS models degenerate to MS-GMRS model, that is, the conclusion 3.4 (1)-c is verified.

4.2. *Verification of 3.4 (2)*

In the Bayesian decision procedure, the expert gives the loss function values for three cases in Table 4.

(1) When $\alpha + \beta = 1$, $\alpha = 0.6$, $\beta = 0.4$, we verify the conclusion 3.4 (2)-a.

According to Eqs. (12), (13), (16), the upper and lower approximations and decision regions of X in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\};$$
$$POS^{\mathrm{I}}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\}, NEG^{\mathrm{I}}(X) = \{x_3, x_9\}, UBN^{\mathrm{I}}(X) = \{x_7, x_{10}\}, LBN^{\mathrm{I}}(X) = \emptyset.$$

According to Eqs. (22), (23), (26), the upper and lower approximations and decision regions of X^c in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_7, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_9\};$$
$$POS^{\mathrm{II}}(X^c) = \{x_3, x_9\}, NEG^{\mathrm{II}}(X^c) = \{x_1, x_2, x_4, x_5, x_6, x_8\}, UBN^{\mathrm{II}}(X^c) = \{x_7, x_{10}\}, LBN^{\mathrm{II}}(X^c) = \emptyset.$$

The above results show that $POS^{\mathrm{I}}(X) = NEG^{\mathrm{II}}(X^c)$, $NEG^{\mathrm{I}}(X) = POS^{\mathrm{II}}(X^c)$, $UBN^{\mathrm{I}}(X) = UBN^{\mathrm{II}}(X^c)$, $LBN^{\mathrm{I}}(X) = LBN^{\mathrm{II}}(X^c)$. Thus the conclusion 3.4 (2)-a is verified.

(2) When $\alpha + \beta < 1$, $\alpha = 0.6$, $\beta = 0.3$, we verify the conclusion 3.4 (2)-b.

According to Eqs. (12), (13), (16), the upper and lower approximations and decision regions of X in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\};$$
$$POS^{\mathrm{I}}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\}, NEG^{\mathrm{I}}(X) = \{x_9\}, UBN^{\mathrm{I}}(X) = \{x_3, x_7, x_{10}\}, LBN^{\mathrm{I}}(X) = \emptyset.$$

According to Eqs. (22), (23), (26), the upper and lower approximations and decision regions of X^c in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_7, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_9\};$$
$$POS^{\mathrm{II}}(X^c) = \{x_3, x_9\}, NEG^{\mathrm{II}}(X^c) = \{x_1, x_2, x_4, x_5, x_6, x_8\}, UBN^{\mathrm{II}}(X^c) = \{x_7, x_{10}\}, LBN^{\mathrm{II}}(X^c) = \emptyset.$$

The results indicate that $POS^{\mathrm{I}}(X) \supseteq NEG^{\mathrm{II}}(X^c)$, $NEG^{\mathrm{I}}(X) \subseteq POS^{\mathrm{II}}(X^c)$, $UBN^{\mathrm{I}}(X) \supseteq UBN^{\mathrm{II}}(X^c)$, $LBN^{\mathrm{I}}(X) \subseteq LBN^{\mathrm{II}}(X^c)$. Thus the conclusion 3.4 (2)-b is verified.

(3) When $\alpha + \beta > 1$, $\alpha = 0.7$, $\beta = 0.4$, we verify the conclusion 3.4 (2)-c.

According to Eqs. (12), (13), (16), the upper and lower approximations and decision regions of X in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\};$$
$$POS^{\mathrm{I}}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\}, NEG^{\mathrm{I}}(X) = \{x_3, x_9\}, UBN^{\mathrm{I}}(X) = \{x_7, x_{10}\}, LBN^{\mathrm{I}}(X) = \emptyset.$$

According to Eqs. (22), (23), (26), the upper and lower approximations and decision regions of X^c in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_7, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_9\};$$
$$POS^{\mathrm{II}}(X^c) = \{x_9\}, NEG^{\mathrm{II}}(X^c) = \{x_1, x_2, x_4, x_5, x_6, x_8\}, UBN^{\mathrm{II}}(X^c) = \{x_3, x_7, x_{10}\}, LBN^{\mathrm{II}}(X^c) = \emptyset.$$

The above results show that $POS^{\mathrm{I}}(X) \subseteq NEG^{\mathrm{II}}(X^c)$, $NEG^{\mathrm{I}}(X) \supseteq POS^{\mathrm{II}}(X^c)$, $UBN^{\mathrm{I}}(X) \subseteq UBN^{\mathrm{II}}(X^c)$, $LBN^{\mathrm{I}}(X) \supseteq LBN^{\mathrm{II}}(X^c)$. Thus the conclusion 3.4 (2)-c is verified.

4.3. The comparison of approximation accuracy

In this subsection, we verify that the fault tolerance capability of the two MS-GMDQ-DTRS models is better than the MS-GMRS model.

(1) The approximation accuracy of the MS-GMRS mode is calculated.

According to Eqs. (3), (4), the upper and lower approximations of X and X^c in the MS-GMRS model are

$$\overline{MS\text{-}GM}_{MS}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}_{MS}(X) = \{x_1, x_6\};$$
$$\overline{MS\text{-}GM}_{MS}(X^c) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}, \underline{MS\text{-}GM}_{MS}(X^c) = \{x_9\}.$$

Based on Eq. (11), the approximation accuracy of MS-GMRS is $\alpha_{MDS}(U/d) = 0.1765$.

(2) When $\alpha = 0.6$, $\beta = 0.4$, the approximation accuracy of the two MS-GMDQ-DTRS models are calculated.

According to Eqs. (12), (13), the upper and lower approximations of X and X^c in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\};$$
$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X^c) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X^c) = \{x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

Based on Eq. (21), the approximation accuracy of IMS-GMDQ-DTRS model is $\alpha_{IMDS}(U/d) = 0.8125$.

Similarly, according to Eqs. (22), (23), the upper and lower approximations of X and X^c in the IIMS-GMDQ-DTRS model are

Table 5
Specific information about the data sets.

No.	Name	Objects	Attributes	Decision classes	Number of sources
1	Liver Disorders	345	7	2	10
2	Balance Scale	625	4	3	10
3	Wireless Indoor Localization	2000	7	4	10
4	Abalone	4177	8	3	10

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_2\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_6\};$$
$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_7, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_9\}.$$

Based on Eq. (31), the approximation accuracy of IIMS-GMDQ-DTRS model is $\alpha_{IIMDS}(U/d) = 0.6667$.

The above results indicate that $\alpha_{IMDS}(U/d) > \alpha_{MDS}(U/d)$ and $\alpha_{IIMDS}(U/d) > \alpha_{MDS}(U/d)$. Therefore, when $\alpha + \beta = 1$, the conclusion that the fault tolerance capability of the two MS-GMDQ-DTRS models is better than the MS-GMRS model is verified.

(3) When $\alpha = 0.6$, $\beta = 0.3$, the approximation accuracy of the two MS-GMDQ-DTRS models are calculated.

According to Eqs. (12), (13), the upper and lower approximations of X and X^c in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\};$$
$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X^c) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X^c) = \{x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

Based on Eq. (21), the approximation accuracy of IMS-GMDQ-DTRS model is $\alpha_{IMDS}(U/d) = 0.7647$.

Similarly, according to Eqs. (22), (23), the upper and lower approximations of X and X^c in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_2\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_6\};$$
$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_7, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_9\}.$$

Based on Eq. (31), the approximation accuracy of IIMS-GMDQ-DTRS model is $\alpha_{IIMDS}(U/d) = 0.6667$.

The above results show that $\alpha_{IMDS}(U/d) > \alpha_{MDS}(U/d)$ and $\alpha_{IIMDS}(U/d) > \alpha_{MDS}(U/d)$. Therefore, when $\alpha + \beta < 1$, this conclusion is also verified.

(4) When $\alpha = 0.7$, $\beta = 0.4$, the approximation accuracy of two MS-GMDQ-DTRS models are calculated.

According to Eqs. (12), (13), the upper and lower approximations of X and X^c in the IMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X) = \{x_1, x_2, x_4, x_5, x_6, x_8\};$$
$$\overline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X^c) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{I}}_{MS}(X^c) = \{x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

Based on Eq. (21), the approximation accuracy of IMS-GMDQ-DTRS model is $\alpha_{IMDS}(U/d) = 0.8125$.

Similarly, according to Eqs. (22), (23), the upper and lower approximations of X and X^c in the IIMS-GMDQ-DTRS model are

$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_2\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X) = \{x_1, x_6\};$$
$$\overline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_3, x_7, x_9, x_{10}\}, \underline{MS\text{-}GM}^{\mathrm{II}}_{MS}(X^c) = \{x_9\}.$$

Based on Eq. (31), the approximation accuracy of IIMS-GMDQ-DTRS model is $\alpha_{IIMDS}(U/d) = 0.5$.

The above results show that $\alpha_{IMDS}(U/d) > \alpha_{MDS}(U/d)$ and $\alpha_{IIMDS}(U/d) > \alpha_{MDS}(U/d)$. Therefore, when $\alpha + \beta > 1$, this conclusion is also verified.

5. Experimental analysis

In this section, a series of experiments are conducted to show two MS-GMDQ-DTRS models are superior to MS-GMRS model in terms of fault tolerance by calculating the approximation accuracy. In this experiment, four data sets were downloaded from UCI, which are shown in Table 5. In this paper, all algorithms are coded in MATLAB. The specific operating environment (including hardware and software) is shown in Table 6.

In machine learning databases, multi-source data set is not easily available directly. The following two methods are proposed to construct multi-source data set by adding noise.

In the original decision system, the value of object x under attribute a is denoted as $DS(x, a)$. The corresponding value of the ith decision system is denoted as $DS_i(x, a)$. First, generate q numbers $(n_1, n_2, \ldots, n_q)$ that satisfy the normal distribution. The first method is to add white noise by

Table 6
Specific information about the operating environment.

Name	Model	Parameter
CPU	Intel(R) Core(TM) i7-8700	3.20 GHz
Platform	MATLAB	R2016b
System	Windows 10	64 bit
Memory	DDR3	16.0 GB; 1600 MHz
Hard DSsk	MQ01ABD050	500 GB

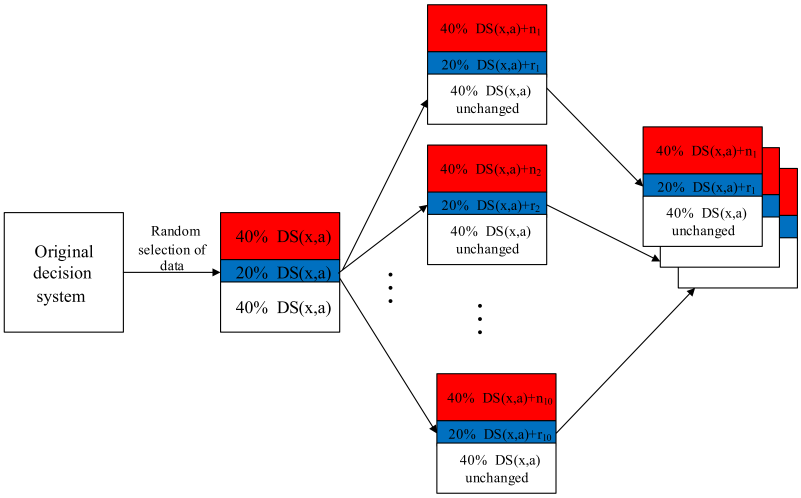

Fig. 2. The generation process of Multi-source decision system.

$$DS_i(x,a) = \begin{cases} DS(x,a) + n_i, & if \quad 0 \le |n_i| \le 1, \\ DS(x,a), & otherwise. \end{cases}$$

Similarly, the second method is to add random noise by

$$DS_i(x,a) = \begin{cases} DS(x,a) + r_i, & if \quad 0 \le |r_i| \le 1, \\ DS(x,a), & otherwise. \end{cases}$$

We randomly select 40% of the original data to add white noise, the remaining 20% to add random noise, and the rest is unchanged. Then, through the above approach, a MsIS can be obtained. The process of generating MsDS is shown in Fig. 2. Every time we generate a MsDS, we have to re-randomly select the noise-added data once in the original data. The experiment generates ten MsDSs on the basis of each data set in Table 5.

For each data set in Table 5, the approximation accuracy of the proposed models are calculated by Algorithms 1-3, respectively. Let $\varphi = 0.65$, $k = 1$, $\beta \in [0, 0.5)$, $\alpha \in (0.5, 1]$, the experimental results are shown in Tables 7, 8, 9, 10. The maximum values are highlighted in bold-face. In order to facilitate the expression, MS-GMRS model, IMS-GMDQ-DTRS model, and IIMS-GMDQ-DTRS model are abbreviated as MS, IMS, and IIMS, respectively. Under different α, β, more detailed change trend lines of the approximate accuracy of the proposed models are shown in Figs. 3, 4, 5. In each figure, x-axis is the number of MsDS and y-axis is value of approximation accuracy.

From Figs. 3, 4, under different α, β, we observe that for each data sets in Table 5, the approximation accuracy of two MS-GMDQ-DTRS models are higher than that of MS-GMRS model. This indicates that the fault tolerance capability of the two MS-GMDQ-DTRS models is better than the MS-GMRS model. Furthermore, from Fig. 3, we find a rule that the approximate accuracy of IMS-GMDQ-DTRS model decreases as β decreases. This shows that the fault tolerance capability of the IMS-GMDQ-DTRS model is related to β and is monotonic. When $\beta = 0.4$, the fault tolerance capability reaches the maximum. From Fig. 4, we also find a rule that the approximate accuracy of IIMS-GMDQ-DTRS model decreases as α increases. This also indicates that the fault tolerance capability of the IIMS-GMDQ-DTRS model is monotonic with α. When $\alpha = 0.6$, the fault tolerance capability reaches the maximum. From Fig. 5, for each data set in Table 5, when $\alpha = 0.6$, $\beta = 0.4$, we find that the approximation accuracy of IMS-GMDQ-DTRS model is the highest. This indicates that when the fault tolerance capability of all models reaches its maximum, the IMS-GMDQ-DTRS model is higher than that of the other two models. Therefore, from the perspective of fault tolerance, the IMS-GMDQ-DTRS model should be preferred to deal with the classification and decision-making of multi-source data set.

Table 7
The approximation accuracy of Liver Disorders data set.

No.	MS	IMS					IIMS				
		$\beta=0.4$	$\beta=0.3$	$\beta=0.2$	$\beta=0.1$	$\beta=0$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
1	0.3168	**0.9438**	0.8742	0.8092	0.7691	0.7691	0.9146	0.7979	0.6690	0.5784	0.5784
2	0.2321	**0.7691**	0.6755	0.6295	0.6044	0.5946	0.7199	0.5518	0.4510	0.3894	0.3641
3	0.2256	**0.7199**	0.6360	0.6004	0.5524	0.5524	0.6807	0.5303	0.4538	0.3351	0.3351
4	0.2707	**0.8904**	0.8008	0.7490	0.7221	0.7035	0.8474	0.6916	0.5844	0.5227	0.4773
5	0.2278	**0.7670**	0.6780	0.6360	0.6108	0.6032	0.7066	0.5413	0.4473	0.3846	0.3647
6	0.2946	**0.9335**	0.8525	0.7829	0.7429	0.7373	0.9057	0.7710	0.6330	0.5421	0.5286
7	0.2212	**0.6953**	0.6197	0.5822	0.5451	0.5451	0.6466	0.5052	0.4215	0.3272	0.3272
8	0.2432	**0.8174**	0.7397	0.6911	0.6450	0.6450	0.7590	0.6205	0.5181	0.4066	0.4066
9	0.3244	**0.9904**	0.8996	0.8565	0.7908	0.7908	0.9856	0.8345	0.7518	0.6079	0.6079
10	0.2897	**0.9416**	0.8217	0.7679	0.7234	0.7234	0.9208	0.7228	0.6139	0.5116	0.5116

Table 8
The approximation accuracy of Balance Scale data set.

No.	MS	IMS					IIMS				
		$\beta=0.4$	$\beta=0.3$	$\beta=0.2$	$\beta=0.1$	$\beta=0$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
1	0.0939	**0.8564**	0.7383	0.6542	0.6542	0.6542	0.4867	0.3669	0.2148	0.2148	0.2148
2	0.1008	**0.8827**	0.8266	0.7021	0.6891	0.6891	0.5430	0.4959	0.2705	0.2459	0.2459
3	0.2301	**0.8939**	0.7924	0.6862	0.6513	0.6464	0.6990	0.6344	0.5170	0.4320	0.4184
4	0.3416	**0.7654**	0.6458	0.5254	0.5072	0.4911	0.6378	0.6247	0.5433	0.4685	0.4528
5	0.0923	**0.8785**	0.8079	0.6619	0.6614	0.6614	0.5521	0.4794	0.2220	0.2181	0.2181
6	0.1096	**0.8439**	0.7740	0.6397	0.6109	0.6109	0.5778	0.5056	0.3093	0.2389	0.2389
7	0.1058	**0.7742**	0.7051	0.5900	0.5690	0.5685	0.4975	0.4420	0.2655	0.2101	0.2101
8	0.0968	**0.8539**	0.7942	0.6703	0.6636	0.6636	0.5129	0.4692	0.2425	0.2306	0.2306
9	0.1230	**0.9187**	0.8118	0.6839	0.6609	0.6604	0.6219	0.5085	0.3195	0.2722	0.2703
10	0.0989	**0.7758**	0.6983	0.6009	0.5859	0.5859	0.5080	0.4100	0.2692	0.2103	0.2103

Table 9
The approximation accuracy of Wireless Indoor Localization data set.

No.	MS	IMS					IIMS				
		$\beta=0.4$	$\beta=0.3$	$\beta=0.2$	$\beta=0.1$	$\beta=0$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
1	0.2871	**0.6978**	0.6773	0.6264	0.5849	0.4923	0.6164	0.5935	0.5045	0.4703	0.3438
2	0.3371	**0.7558**	0.6843	0.6409	0.5852	0.4817	0.6228	0.5957	0.5459	0.4964	0.3967
3	0.2593	**0.6172**	0.6005	0.5397	0.4781	0.3541	0.5846	0.5654	0.4711	0.4006	0.3019
4	0.2792	**0.6430**	0.6219	0.5666	0.4998	0.3730	0.5763	0.5480	0.4507	0.3998	0.3177
5	0.3370	**0.7527**	0.7247	0.6469	0.6026	0.4893	0.6854	0.6573	0.5507	0.4922	0.3920
6	0.2956	**0.6742**	0.6485	0.5803	0.5309	0.4175	0.6570	0.6282	0.5306	0.4570	0.3560
7	0.2637	**0.6549**	0.6350	0.5572	0.4881	0.3704	0.5609	0.5072	0.4306	0.3863	0.3081
8	0.2619	**0.6308**	0.6057	0.5684	0.4934	0.3699	0.5910	0.5655	0.4860	0.4113	0.3168
9	0.3515	**0.7109**	0.6860	0.6359	0.5646	0.4256	0.6197	0.5955	0.5325	0.4334	0.3481
10	0.2685	**0.6113**	0.5917	0.5917	0.4766	0.3580	0.5766	0.5514	0.4523	0.4004	0.3040

Table 10
The approximation accuracy of Abalone data set.

No.	MS	IMS					IIMS				
		$\beta=0.4$	$\beta=0.3$	$\beta=0.2$	$\beta=0.1$	$\beta=0$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
1	0.0798	**0.5431**	0.4216	0.3532	0.2989	0.2691	0.2937	0.1937	0.1588	0.1068	0.1016
2	0.0799	**0.5422**	0.4226	0.3605	0.3046	0.2726	0.2954	0.2051	0.1617	0.1082	0.1018
3	0.0805	**0.5692**	0.4361	0.3612	0.3065	0.2720	0.3152	0.2119	0.1653	0.1091	0.1028
4	0.0847	**0.5587**	0.4450	0.3808	0.3215	0.2918	0.2941	0.2016	0.1666	0.1157	0.1113
5	0.0796	**0.5415**	0.4241	0.3546	0.3002	0.2696	0.2923	0.1966	0.1556	0.1068	0.1012
6	0.0796	**0.5440**	0.4275	0.3599	0.3068	0.2722	0.3022	0.2060	0.1628	0.1075	0.1020
7	0.0796	**0.5411**	0.4278	0.3579	0.3050	0.2705	0.2982	0.2027	0.1567	0.1071	0.1014
8	0.0805	**0.5512**	0.4317	0.3620	0.3062	0.2722	0.3040	0.2104	0.1645	0.1091	0.1028
9	0.0800	**0.5547**	0.4291	0.3596	0.3055	0.2710	0.3040	0.2061	0.1626	0.1081	0.1024
10	0.0811	**0.5325**	0.4239	0.3600	0.3052	0.2761	0.2852	0.1895	0.1592	0.1083	0.1043

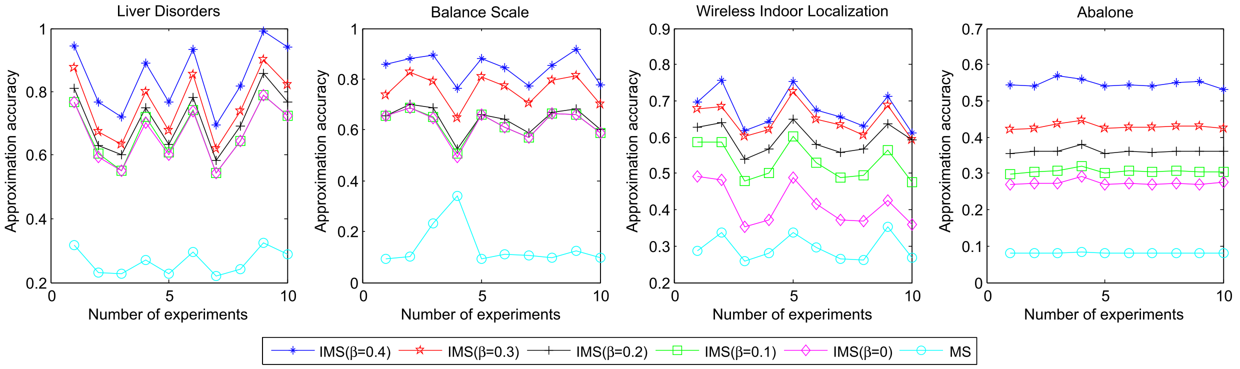

Fig. 3. The approximation accuracies of MS-GMRS and IMS-GMDQ-DTRS.

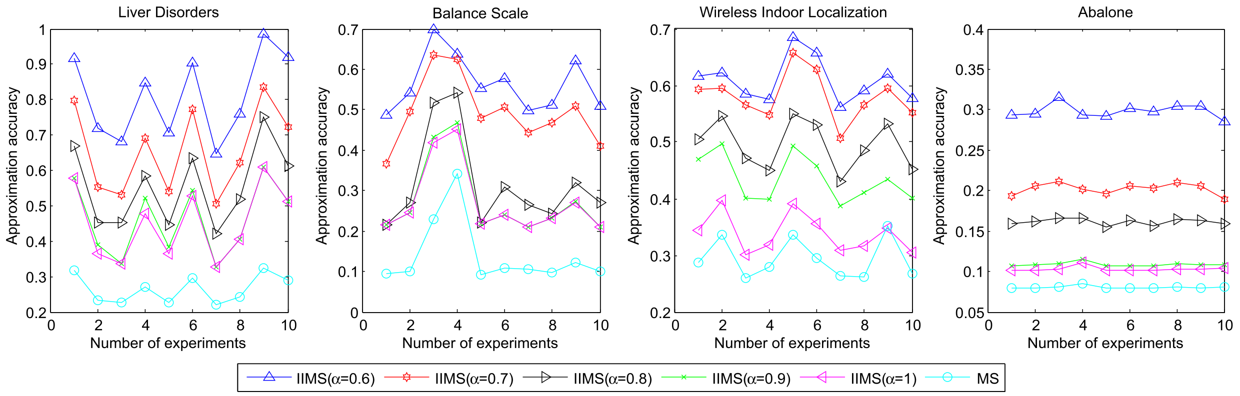

Fig. 4. The approximation accuracies of MS-GMRS and IIMS-GMDQ-DTRS.

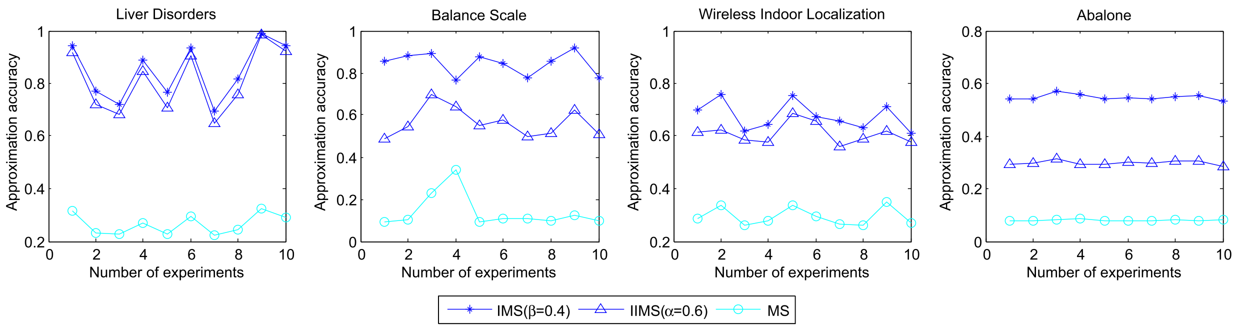

Fig. 5. The approximation accuracies of MS-GMRS, IMS-GMDQ-DTRS ($\beta = 0.4$), and IIMS-GMDQ-DTRS ($\alpha = 0.6$).

6. Conclusion

In this paper, we proposed a new method to discover knowledge directly from MsIS without information loss. Inspired by the generalized multi-granulation rough set theory, this method regards each information system in MsIS as a granular structure, and then builds a rough set model to directly generate approximations of the target concept. First, as the basis of other models, the generalized multi-granulation rough set model for MsIS (MS-GMRS) was proposed. Second, we combined MS-GMRS with double-quantitative decision-theoretic rough set to obtain two new models, called two kinds of generalized multi-granulation double-quantization decision-theoretic rough set model of MsIS (MS-GMDQ-DTRS). Simultaneously, we respectively proposed their decision rules. Third, by discussing the relations between the three models, we find that the two MS-GMDQ-DTRS models can be degraded to MS-GMRS model. In other word, the two models are two extended forms of the MS-GMRS model. We proved that these two new models are more fault tolerant than the MS-GMRS model. Final, the experiment compares fault tolerance of the proposed models by calculating respective approximate accuracy. The experimental results show that the two MS-GMDQ-DTRS models have better fault tolerance in acquiring MsIS knowledge compared with

MS-GMRS model, especially the first type. In the future, we will further study the attribute reduction approaches and logic operations of these three models in MsIS.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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