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ORIGINAL ARTICLE

Fuzziness and incremental information of disjoint regions in double-quantitative decision-theoretic rough set model

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Abstract

Double-quantitative decision-theoretic rough set (Dq-DTRS), as a new model considering double quantification to reflect the distinct degrees of quantitative information, satisfies the quantitative completeness properties and exhibits much stronger fault tolerance capabilities than decision-theoretic rough set (DTRS) and graded rough set (GRS). Since the Dq-DTRS was proposed, there have been few studies on the uncertainty analysis of the model. In this paper, we investigate the uncertainty measure of the four disjoint regions in Dq-DTRS models by introducing a fuzziness formula for rough set, and then describe the changing regularities of fuzziness of disjoint regions in DqI-DTRS model and DqII-DTRS model along with the variation of two parameters α, β and the grade k, respectively. In addition, three kinds of incremental information for Dq-DTRS model, namely useful incremental information, useless incremental information and error-correction incremental information are presented being formed with regard to the changes of boundary regions, and also the related assessment methods for these special types of incremental information are discussed in the form of several important theorems.

Keywords Decision-theoretic rough set · Double quantification · Graded rough set · Incremental information · Uncertainty measure

1 Introduction

Rough set theory [37] has been confirmed successful applications in many science and engineering fields, such as pattern recognition, data mining, image processing, medical diagnosis and others. Pawlak rough set has a severe limitation. The relationship between equivalence classes and the basic set are strict that there are no fault tolerance mechanisms, the quantitative information about the degree of overlap of the equivalence classes and the basic set is not taken into consideration. That is to say, Pawlak rough set does not cope well with quantitative problems in the real-life applications. Improving the Pawlak rough set model with quantitative information is a promising direction and expansions of the model that include such quantification are of particular relevance [29, 60]. The improved models are called quantitative rough set models, including probabilistic rough set (PRS) model [57, 59], graded rough set (GRS) model [30, 61], and double-quantitative rough set model [11, 21–23, 55, 70–73]. As pointed out in [73, 74], PRS and GRS are the two different and typical single-quantitative rough set models.

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PRS model exhibits many merits, such as the measurability of the probabilistic information, the generality and flexibility of the model and its insensitivity to noise [31, 39, 40, 58, 62]. PRS model and its generalizations can be formulated based on the notion of rough membership functions and rough inclusion. Threshold values, serving as parameters, are applied to a rough membership function or a rough inclusion to obtain probabilistic or parameterized approximations. Three kinds of PRS models have been proposed and studied intensively, which are decision-theoretic rough set (DTRS) model [16, 17, 29, 31, 43–45, 62, 67], variable precision rough set model [77], and Bayesian rough set model [76]. The main differences among these models are their different, but equivalent, formulations of probabilistic approximations and interpretations of the required parameters. Since Yao and Lin explored the relationships between rough set and modal logic, they proposed the GRS model based on graded modal logic [61]. GRS model primarily considers the absolute quantitative information regarding the basic concept and knowledge granules, and it is also a generalization of the Pawlak rough set model. The regions of the GRS model also extend the corresponding notions used in the classical rough set models.

The DTRS and GRS are two fundamental expansion models that achieve strong fault tolerance capabilities by utilizing quantitative descriptions [23]. The relative and absolute measures reflect relative accuracy and absolute accuracy or fault tolerance from two different quantitative viewpoints. Relative quantitative information and absolute quantitative information are two kinds of quantification mythologies encountered in certain applications. Double quantification regarding their fusion has visible semantic background and feasibility. For this purpose, several works related to the double-quantitative information have been explored [11, 21–23, 55, 71–74]. Zhang et al. made a comparative study of variable precision rough set model and GRS model [74], and investigated the quantitative information architecture in the double-quantitative approximation space of precision and grade [72], and defined a double-quantitative fusion of causality measure to construct a granular computing platform and hierarchical reduction system [71]. Li et al. confirmed two kinds of Dq-DTRS model based on Bayesian decision procedure and GRS, which essentially indicate the relative and absolute quantification [23], and then further presented the information measure of relative quantification and absolute quantification [22]. From the examples in [23, 73], we can see that both quantification indexes exhibit a close, supplementary, and dialectical relationship, and each one actually has its own representation virtues and application environments.

As one of the most important issues in rough set theory, uncertainty measures have been widely studied in the references [1, 2, 9, 10, 34, 35, 41, 42, 46, 53, 68, 69]. Pawlak presented several numerical measures such as accuracy and roughness of a set, and approximation accuracy of a rough classification [38]. Rough set theory may be a suitable mathematical tool for dealing with vagueness and uncertainty [38]. Many typical uncertain measures, such as roughness [2], approximation accuracy [9], rough entropy [28], fuzzy entropy [12], and fuzziness [4], had been proposed accordingly. Most recently, Zhang et al. analyzed the change rules of uncertainty of PRS model with changing knowledge spaces [69], and defined three kinds of incremental information for PRS model [68]. As reported in [68], the uncertainty for Pawlak rough set comes from the boundary region; in PRS model and its generalizations, objects belong to every region with a pair of thresholds, so the uncertainty for PRS comes from all three regions, namely positive, negative and boundary regions.

In Dq-DTRS model, the domain is classified into four disjoint regions, which are positive region, negative region, upper boundary region and lower boundary region. With the increment of attributes, the disjoint regions in Dq-DTRS models will constantly change according to the new attribute set and the uncertainty of the disjoint regions will have corresponding changes. There have been a lot of studies on dynamic information systems in the literature of rough sets, including the changes of attribute values [3, 5, 7], object sets [6, 32, 33, 65], and attribute sets [19, 20, 66]. However, there was no relevant research on the uncertainty or variation for the Dq-DTRS model with increasing new attributes. Inspired by the pioneer work [68], in this study, we focus on the fuzziness of disjoint regions in Dq-DTRS model and study the change regularities of fuzziness with changing approximation spaces. The main contents and innovation of this paper are shown as: (1) The fuzziness of four disjoint regions in Dq-DTRS is investigated by introducing a fuzziness formula presented in reference [68]. And the changing regularities of the fuzziness of disjoint regions in Dq-DTRS are discovered along with the variation of two parameters α, β and the grade k. (2) The effects of attribute increment on the variation of disjoint regions in Pawlak rough set, DTRS and Dq-DTRS models are compared and three kinds of incremental information for Dq-DTRS are presented. (3) Related judgment methods for the special types of incremental information for Dq-DTRS are discussed in the form of important theorems. These theorems are useful for understanding the variation of the four disjoint regions in Dq-DTRS model when adding new attributes.

The paper is organized as follows. Related concepts and definitions are introduced briefly in Section 2. In Section 3, we make a review of the fuzziness of the three decision regions in Pawlak rough set model and DTRS model, respectively. In Section 4, we present the fuzziness of Dq-DTRS from their four disjoint regions, and we also make comparisons on the regions changed among Pawlak rough set, DTRS and Dq-DTRS models. In Section 5, we define three kinds of incremental information for Dq-DTRS model with regions changing. Finally, Section 6 covers some conclusions.

2 Basic notions

For a non-empty and finite set U, which is called the universe, a fuzzy set A on U is characterized by a membership function $\mu_A(x)$ [64], where $\mu_A : U \to [0,1]$. $\mu_A(x)$ is called the membership degree to A of the object $x \in U$.

Definition 2.1 (*Fuzziness* [15]) Let $U = \{x_1, x_2, \dots, x_n\}$ be a finite domain, and A, B be two fuzzy sets on U. If a mapping $H : F(U) \to [0,1]$ ($F(U)$ is a set of all fuzzy subsets on U) satisfies the conditions as follows:

(1) $H(A) = 0$ if and only if $A \in P(U)$, where $P(U)$ is a power set on U.
(2) $H(A) = 1$ if and only if $(\forall\, x_i \in U)\ A(x_i) = 0.5$.
(3) For any $x_i \in U$, if $B(x_i) \le A(x_i) \le 0.5$ or $B(x_i) \ge A(x_i) \ge 0.5$, then $H(B) \le H(A)$.
(4) For any $A \in F(U)$, $H(A) = H(A^c)$, A^c denotes the complementary set of A.

$H(\bullet)$ is called fuzziness of a fuzzy set.

Any formula satisfying the above Definition 2.1 is called a fuzziness formula. Motivated by different purposes, researchers proposed various fuzziness formulas. Regarding the fuzziness modeling in the perspective of general machine learning, Wang et al. investigated a relationship between the fuzziness of a classifier and the misclassification rate of the classifier on a group of samples [48], and proposed a non-naive Bayesian classifier where the independence assumption is removed and the marginal estimation is replaced by the joint estimation [49]. In order to make the best use of the individual classifiers and their combinations, Wang et al. presented a novel approach to classifier fusion based on upper integrals [50], and offered sound evidence behind the observation that higher fuzziness of a fuzzy classifier may imply better generalization aspects of the classifier [52], then two diversity criteria are addressed for multiple-instance active learning by utilizing a support vector machine based multiple-instance learning classifier [47], they also provided some useful guidelines for improving the generalization ability of classifiers by adjusting uncertainty from the viewpoint of complexity of classification [51]. In order to better measure fuzziness of a rough set, the following fuzziness formula for rough set was presented in [4] as follows:

$$H(F_R^X) = \frac{-1}{n \ln 2} \sum_{i=1}^{n} (\mu_{F_R^X}(x_i) \ln \mu_{F_R^X}(x_i) + (1 - \mu_{F_R^X}(x_i)) \ln(1 - \mu_{F_R^X}(x_i)),$$

where $F_R^X = \sum_{i=1}^{n} (\mu_{F_R^X}(x_i)/x_i)$ is a fuzzy set on U, and $\mu_{F_R^X}(x_i) = |X \cap [x_i]_R| / |[x_i]_R|$.

Let $S = (U, A)$ be an information system. Here $U = \{x_1, x_2, \dots, x_n\}$ is a non-empty and finite set and $A = \{a_1, a_2, \dots, a_m\}$ is an attribute set. The class of all subsets of U is denoted by $P(U)$. For $X \in P(U)$, the equivalence relation R induced by A in a Pawlak approximation space (U, R) partitions the universe U into disjoint subsets. Such a partition of the universe is a quotient set of U and is denoted by $U/R = \{[x]_R | x \in U\}$, where $[x]_R = \{y \in U | (x, y) \in R\}$ is the equivalence class containing x with respect to R.

Definition 2.2 (*Pawlak rough set* [37]) Given an information system $S = (U, A)$ and an equivalence relation R. For any $X \subseteq U$, one can characterize X by a pair of upper and lower approximations which are

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \ne \emptyset\},$$
$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\}.$$

If $\overline{R}(X) = \underline{R}(X)$, X is called definable set or crisp set in rough approximation space; and if $\overline{R}(X) \ne \underline{R}(X)$, then X is called Pawlak rough set. Obviously, both upper approximation $\overline{R}(X)$ and lower approximation $\underline{R}(X)$ of a target set X are two sets. Three disjoint regions can be obtained: $P(X) = \underline{R}(X), N(X) = \sim \overline{R}(X)$ and $B(X) = \overline{R}(X) - \underline{R}(X)$ are called the positive region, negative region, and boundary region of X, respectively. These three regions constitute a partition of U, denoted as $\pi(X) = \{P(X), N(X), B(X)\}$.

In Pawlak rough set, the relationships between equivalence classes and the basic set are strict that there are no fault tolerance mechanisms. Quantitative information about the degree of overlap of the equivalence classes and the basic set is not taken into consideration. Therefore, neither wider relationships nor quantitative information can be utilized. Naturally the studies of PRS and GRS regard to relative quantitative information and absolute quantitative information are presented, respectively. As a special PRS model, DTRS based on Bayesian decision principle [8] was initially proposed by Yao [56].

Definition 2.3 (*DTRS* [56]) Given an information system $S = (U, A)$ and an equivalence relation R. For any $X \subseteq U$, the upper and lower approximations based on thresholds α, β $(0 \le \beta < \alpha \le 1)$ of the DTRS model are defined as follows.

$$\overline{R}_{(\alpha,\beta)}(X) = \left\{ x \in U | \frac{|[x]_R \cap X|}{|[x]_R|} > \beta \right\},$$
$$\underline{R}_{(\alpha,\beta)}(X) = \left\{ x \in U | \frac{|[x]_R \cap X|}{|[x]_R|} \geq \alpha \right\}.$$

If $\underline{R}_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a rough set. $P_{(\alpha,\beta)}(X) = \underline{R}_{(\alpha,\beta)}(X)$, $N_{(\alpha,\beta)}(X) = \sim \overline{R}_{(\alpha,\beta)}(X)$, $B_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_{(\alpha,\beta)}(X)$ are the positive region, negative region and boundary region, respectively.

The GRS is different from the DTRS in the description of quantification.

Definition 2.4 (*GRS* [61]) Suppose k is a non-negative integer and is called "grade",

$$\overline{R}_k(X) = \{x \in U | \ |[x]_R \cap X| > k\},$$
$$\underline{R}_k(X) = \{x \in U | \ |[x]_R| - |[x]_R \cap X| \leq k\}$$

are called grade k upper and lower approximations of X, respectively. If $\overline{R}_k(X) = \underline{R}_k(X)$, then X is called a definable set by grade k; otherwise, X is called a rough set by grade k.

Because the inclusion relation of the grade approximation does not hold any longer, positive and negative regions, upper and lower boundary regions are naturally proposed. We form the following regions:

$$P_k(X) = \overline{R}_k(X) \cap \underline{R}_k(X);$$
$$N_k(X) = \sim (\overline{R}_k(X) \cup \underline{R}_k(X));$$
$$U_k(X) = \overline{R}_k(X) - \underline{R}_k(X);$$
$$L_k(X) = \underline{R}_k(X) - \overline{R}_k(X);$$
$$B_k(X) = UbN_k(X) \cup LbN_k(X),$$

where $P_k(X), N_k(X), U_k(X), L_k(X)$ and $B_k(X)$ are called grade k positive region, negative region, upper boundary region, lower boundary region, and boundary region of X.

In reference [23], authors constructed two kinds of Dq-DTRS model, which can indicate the relative and absolute quantification simultaneously.

Definition 2.5 (*DqI-DTRS* [23]) The following upper and lower approximation operators are defined as

$$\overline{R}^I_{(\alpha,\beta,k)}(X) = \left\{ x \in U | \frac{|[x]_R \cap X|}{|[x]_R|} > \beta \right\},$$
$$\underline{R}^I_{(\alpha,\beta,k)}(X) = \left\{ x \in U | \ |[x]_R| - |[x]_R \cap X| \leq k \right\}.$$

From the above two operators, the DqI-DTRS model can be established, and denoted by $(U, \overline{R}^I_{(\alpha,\beta,k)}, \underline{R}^I_{(\alpha,\beta,k)})$. The positive region, negative region, upper boundary region and lower boundary region are obtained as

$$P^I_{(\alpha,\beta,k)}(X) = \overline{R}^I_{(\alpha,\beta,k)}(X) \cap \underline{R}^I_{(\alpha,\beta,k)}(X);$$
$$N^I_{(\alpha,\beta,k)}(X) = \sim (\overline{R}^I_{(\alpha,\beta,k)}(X) \cup \underline{R}^I_{(\alpha,\beta,k)}(X));$$
$$U^I_{(\alpha,\beta,k)}(X) = \overline{R}^I_{(\alpha,\beta,k)}(X) - \underline{R}^I_{(\alpha,\beta,k)}(X);$$
$$L^I_{(\alpha,\beta,k)}(X) = \underline{R}^I_{(\alpha,\beta,k)}(X) - \overline{R}^I_{(\alpha,\beta,k)}(X).$$

According to different parameters α, β and different grade k, we can obtain the above different regions. These disjoint four regions constitute a partition of the universe U, and this partition is denoted by

$$\pi^{I,R}_{(\alpha,\beta,k)}(X) = (P^I_{(\alpha,\beta,k)}(X), N^I_{(\alpha,\beta,k)}(X), U^I_{(\alpha,\beta,k)}(X), L^I_{(\alpha,\beta,k)}(X)).$$

Definition 2.6 (*DqII-DTRS* [23]) The model $(U, \overline{R}^{II}_{(\alpha,\beta,k)}, \underline{R}^{II}_{(\alpha,\beta,k)})$ called DqII-DTRS, is defined using the following two operators $\overline{R}^{II}_{(\alpha,\beta,k)}$ and $\underline{R}^{II}_{(\alpha,\beta,k)}$,

$$\overline{R}^{II}_{(\alpha,\beta,k)}(X) = \{x \in U || [x]_R \cap X| > k\},$$
$$\underline{R}^{II}_{(\alpha,\beta,k)}(X) = \left\{ x \in U | \ \frac{|[x]_R \cap X|}{|[x]_R|} \geq \alpha \right\}.$$

The positive region, negative region, upper boundary region and lower boundary region are obtained as

$$P^{II}_{(\alpha,\beta,k)}(X) = \overline{R}^{II}_{(\alpha,\beta,k)}(X) \cap \underline{R}^{II}_{(\alpha,\beta,k)}(X);$$
$$N^{II}_{(\alpha,\beta,k)}(X) = \sim (\overline{R}^{II}_{(\alpha,\beta,k)}(X) \cup \underline{R}^{II}_{(\alpha,\beta,k)}(X));$$
$$U^{II}_{(\alpha,\beta,k)}(X) = \overline{R}^{II}_{(\alpha,\beta,k)}(X) - \underline{R}^{II}_{(\alpha,\beta,k)}(X);$$
$$L^{II}_{(\alpha,\beta,k)}(X) = \underline{R}^{II}_{(\alpha,\beta,k)}(X) - \overline{R}^{II}_{(\alpha,\beta,k)}(X).$$

These disjoint four regions constitute a partition of the universe U, and this partition is denoted by

$$\pi^{II,R}_{(\alpha,\beta,k)}(X) = (P^{II}_{(\alpha,\beta,k)}(X), N^{II}_{(\alpha,\beta,k)}(X), U^{II}_{(\alpha,\beta,k)}(X), L^{II}_{(\alpha,\beta,k)}(X)).$$

3 A review of measuring fuzziness of Pawlak rough set and DTRS from disjoint regions

The analysis on the fuzziness of disjoint regions is helpful in improving classification quality. The DTRS has a better fault tolerance ability to compare with Pawlak rough set model. With the variation of parameters, not only the objects in the boundary region, but also the objects in the positive or negative regions may be re-classified. In this section, we review the fuzziness of Pawlak rough set and DTRS from their three regions.

Definition 3.1 [68] Let $U = \{x_1, x_2, \dots, x_n\}$ be a non-empty finite set, $P' = \{P'_1, P'_2, \dots, P'_l\}$ and $P'' = \{P''_1, P''_2, \dots, P''_l\}$ be two partitions U. If $\forall\, P'_i \in P'$, $\exists\, P''_j \in P''$ s. t. $P'_i \subseteq P''_j$, then P' is finer than P'', denoted as $P' \preceq P''$. If $P' \preceq P''$, and $\exists\, P'_i \in P', \exists\, P''_j \in P''$ s. t. $P'_i \subset P''_j$, then P' is strictly finer than P'', denoted as $P' \prec P''$.

Definition 3.2 [68] Given an information system $S = (U, A)$ and an equivalence relation R. For any $X \subseteq U$, the fuzziness of three regions in Pawlak rough set is determined in the following:

$$\begin{aligned}
H^R(P(X)) = &-\frac{|P(X)|}{|U|\ln 2}\left[\left(\frac{|P(X)\cap X|}{|P(X)|}\ln\frac{|P(X)\cap X|}{|P(X)|}\right.\right.\\
&\left.\left.+\left(1-\frac{|P(X)\cap X|}{|P(X)|}\right)\ln\left(1-\frac{|P(X)\cap X|}{|P(X)|}\right)\right)\right];\\
H^R(N(X)) = &-\frac{|N(X)|}{|U|\ln 2}\left[\left(\frac{|N(X)\cap X|}{|N(X)|}\ln\frac{|N(X)\cap X|}{|N(X)|}\right.\right.\\
&\left.\left.+\left(1-\frac{|N(X)\cap X|}{|N(X)|}\right)\ln\left(1-\frac{|N(X)\cap X|}{|N(X)|}\right)\right)\right];\\
H^R(B(X)) = &-\frac{|B(X)|}{|U|\ln 2}\left[\left(\frac{|B(X)\cap X|}{|B(X)|}\ln\frac{|B(X)\cap X|}{|B(X)|}\right.\right.\\
&\left.\left.+\left(1-\frac{|B(X)\cap X|}{|B(X)|}\right)\ln\left(1-\frac{|B(X)\cap X|}{|B(X)|}\right)\right)\right].
\end{aligned}$$

And $H^R(X) = H^R(P(X)) + H^R(N(X)) + H^R(B(X))$ is regarded as the fuzziness of the partition $\pi(X)$.

The uncertainty semantic represented by fuzziness formulas $H^R(X)$, $H^R(P(X))$, $H^R(N(X))$ and $H^R(B(X))$ are consistent with the definition of fuzziness in Definition 2.1. The greater the value of $H^R(\blacklozenge)$ (where $\blacklozenge$ represents for $P(X)$,$N(X)$, and $B(X)$, respectively), the greater the fuzziness of the represented region $\blacklozenge$, and the greater the uncertainty of $\blacklozenge$. It should be pointed out that $\frac{|P(X)\cap X|}{|P(X)|} = 1$, $\frac{|N(X)\cap X|}{|N(X)|} = 0$, $0 \le \frac{|B(X)\cap X|}{|B(X)|} \le 1$. So we can obtain $H^R(P(X)) = H^R(N(X)) = 0$ and $0 \le H^R(B(X)) \le 1$. That is to say, $H^R(X) = H^R(B(X))$.

For an information system $S = (U, A)$, where $U = \{x_1, x_2, \dots, x_n\}$ is an object set, A is an attribute set, R_1 and R_2 are two subsets of A, and $X \subseteq U$ is a target set. If $R_1 \subseteq R_2$, then $H^{R_1}(X) \ge H^{R_2}(X)$. This property shows that the uncertainty of a target concept in Pawlak rough set model will become much less fuzzy when the equivalence classes in rough approximation space (U, A) are subdivided.

DTRS and its generalizations were formed by a pair of parameters α and β $(0 \le \beta < \alpha \le 1)$ obtained from the Bayesian decision principle. In reference [68], authors made a detailed study on the uncertainty of three regions in the PRS model. As a special PRS model, the uncertainty of three regions in the DTRS model is similar to the PRS model.

Compared with Pawlak rough set model, the positive and negative regions will become larger and the boundary region will become smaller due to this pair of thresholds. These are shown in Figure 1. These three probabilistic positive, negative and boundary regions constitute a partition of U, and this partition is described as

$$\pi_{(\alpha,\beta)}(X) = \{P_{(\alpha,\beta)}(X), N_{(\alpha,\beta)}(X), B_{(\alpha,\beta)}(X)\}.$$

Definition 3.3 The fuzziness of three regions in DTRS is represented as

$$\begin{aligned}
H^R(P_{(\alpha,\beta)}(X)) = &-\frac{|P_{(\alpha,\beta)}(X)|}{|U|\ln 2}\left[\left(\frac{|P_{(\alpha,\beta)}(X)\cap X|}{|P_{(\alpha,\beta)}(X)|}\right.\right.\\
&\ln\frac{|P_{(\alpha,\beta)}(X)\cap X|}{|P_{(\alpha,\beta)}(X)|}+\left(1-\frac{|P_{(\alpha,\beta)}(X)\cap X|}{|P_{(\alpha,\beta)}(X)|}\right)\\
&\left.\left.\ln\left(1-\frac{|P_{(\alpha,\beta)}(X)\cap X|}{|P_{(\alpha,\beta)}(X)|}\right)\right)\right];\\
H^R(N_{(\alpha,\beta)}(X)) = &-\frac{|N_{(\alpha,\beta)}(X)|}{|U|\ln 2}\left[\left(\frac{|N_{(\alpha,\beta)}(X)\cap X|}{|N_{(\alpha,\beta)}(X)|}\right.\right.\\
&\ln\frac{|N_{(\alpha,\beta)}(X)\cap X|}{|N_{(\alpha,\beta)}(X)|}+\left(1-\frac{|N_{(\alpha,\beta)}(X)\cap X|}{|N_{(\alpha,\beta)}(X)|}\right)\\
&\left.\left.\ln\left(1-\frac{|N_{(\alpha,\beta)}(X)\cap X|}{|N_{(\alpha,\beta)}(X)|}\right)\right)\right];\\
H^R(B_{(\alpha,\beta)}(X)) = &-\frac{|B_{(\alpha,\beta)}(X)|}{|U|\ln 2}\left[\left(\frac{|B_{(\alpha,\beta)}(X)\cap X|}{|B_{(\alpha,\beta)}(X)|}\right.\right.\\
&\ln\frac{|B_{(\alpha,\beta)}(X)\cap X|}{|B_{(\alpha,\beta)}(X)|}+\left(1-\frac{|B_{(\alpha,\beta)}(X)\cap X|}{|B_{(\alpha,\beta)}(X)|}\right)\\
&\left.\left.\ln\left(1-\frac{|B_{(\alpha,\beta)}(X)\cap X|}{|B_{(\alpha,\beta)}(X)|}\right)\right)\right].
\end{aligned}$$

The method for quantifying fuzziness of DTRS model adheres to the following expression

$$H^R_{(\alpha,\beta)}(X) = H^R(P_{(\alpha,\beta)}(X)) + H^R(N_{(\alpha,\beta)}(X)) + H^R(B_{(\alpha,\beta)}(X)).$$

In the DTRS model, a certain level of erroneous decisions needs to be tolerated with probabilistic approximations, and this offers a new view for rule induction, which is absent from the classical rough set model. However, these extensions of rough set have a certain fault tolerance ability level to deal with uncertain objects, so the objects in three regions have different degrees of uncertainty.

Let us discuss parameter relationships for the fuzziness of three regions, and also for fuzziness of DTRS model. If the two parameters $\alpha = 1$ and $\beta = 0$ in a DTRS model, then the DTRS degenerates into Pawlak rough set model. Given (α_1, β_1) and (α_2, β_2), where $0 \leq \beta_1 \leq \beta_2 < \alpha_2 \leq \alpha_1 \leq 1$. With the existence of the relationships between fuzziness of disjoint regions: $H^R(P_{(\alpha_1,\beta_1)}(X)) \leq H^R(P_{(\alpha_2,\beta_2)}(X))$, $H^R(N_{(\alpha_1,\beta_1)}(X)) \leq H^R(N_{(\alpha_2,\beta_2)}(X))$ and $H^R(B_{(\alpha_1,\beta_1)}(X)) \geq H^R(B_{(\alpha_2,\beta_2)}(X))$, so the values of $H^R_{(\alpha_1,\beta_1)}(X)$ and $H^R_{(\alpha_2,\beta_2)}(X)$ cannot be compared. That is to say, there is no monotonicity among different pairs of parameters (α_1, β_1), $\cdots$, (α_k, β_k), $\cdots$, (α_n, β_n), where $0 \leq \beta_1 \leq \cdots \leq \beta_k \leq \cdots \leq \beta_n < \alpha_n \leq \cdots \leq \alpha_k \cdots \leq \alpha_1 \leq 1$.

Theorem 3.1 [68] *Given an information system $S = (U, A, V, f)$, and two parameters α, β. R_1 and R_2 are two subsets of A, and $X \subseteq U$ is a target set. If $R_1 \subseteq R_2$, then $H^{R_2}_{(\alpha,\ \beta)}(X) \leq H^{R_1}_{(\alpha,\ \beta)}(X)$.*

Proof The proof of Theorem 3.1 can be seen in [68]. □

This property shows that the uncertainty of a target concept in DTRS will become much less fuzzy when the equivalence classes are subdivided.

4 Measure fuzziness of Dq-DTRS from the disjoint regions

In this section, we focus on Dq-DTRS model. We will analyze the fuzziness (a kind of uncertainty) of Dq-DTRS model from their disjoint regions.

Definition 4.1 The method for quantifying fuzziness of DqI-DTRS is in the following:

$$H^{I,R}_{(\alpha,\beta,k)}(X) = H^R(P^I_{(\alpha,\beta,k)}(X)) + H^R(N^I_{(\alpha,\beta,k)}(X)) + H^R(U^I_{(\alpha,\beta,k)}(X)) + H^R(L^I_{(\alpha,\beta,k)}(X)),$$

where $H^R(P^I_{(\alpha,\beta,k)}(X))$, $H^R(N^I_{(\alpha,\beta,k)}(X))$, $H^R(U^I_{(\alpha,\beta,k)}(X))$ and $H^R(L^I_{(\alpha,\beta,k)}(X))$ are fuzziness of positive region, negative region, upper boundary region and lower boundary region in terms of DqI-DTRS model, respectively. The forms of these four fuzziness formulas are

$$H^R(P^I_{(\alpha,\beta,k)}(X)) = -\frac{|P^I_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|P^I_{(\alpha,\beta,k)}(X) \cap X|}{|P^I_{(\alpha,\beta,k)}(X)|} \ln \frac{|P^I_{(\alpha,\beta,k)}(X) \cap X|}{|P^I_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|P^I_{(\alpha,\beta,k)}(X) \cap X|}{|P^I_{(\alpha,\beta,k)}(X)|}\right) \ln\left(1 - \frac{|P^I_{(\alpha,\beta,k)}(X) \cap X|}{|P^I_{(\alpha,\beta,k)}(X)|}\right)\right],$$

$$H^R(N^I_{(\alpha,\beta,k)}(X)) = -\frac{|N^I_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|N^I_{(\alpha,\beta,k)}(X) \cap X|}{|N^I_{(\alpha,\beta,k)}(X)|} \ln \frac{|N^I_{(\alpha,\beta,k)}(X) \cap X|}{|N^I_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|N^I_{(\alpha,\beta,k)}(X) \cap X|}{|N^I_{(\alpha,\beta,k)}(X)|}\right) \ln\left(1 - \frac{|N^I_{(\alpha,\beta,k)}(X) \cap X|}{|N^I_{(\alpha,\beta,k)}(X)|}\right)\right],$$

$$H^R(U^I_{(\alpha,\beta,k)}(X)) = -\frac{|U^I_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|U^I_{(\alpha,\beta,k)}(X) \cap X|}{|U^I_{(\alpha,\beta,k)}(X)|} \ln \frac{|U^I_{(\alpha,\beta,k)}(X) \cap X|}{|U^I_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|U^I_{(\alpha,\beta,k)}(X) \cap X|}{|U^I_{(\alpha,\beta,k)}(X)|}\right) \ln\left(1 - \frac{|U^I_{(\alpha,\beta,k)}(X) \cap X|}{|U^I_{(\alpha,\beta,k)}(X)|}\right)\right],$$

$$H^R(L^I_{(\alpha,\beta,k)}(X)) = -\frac{|L^I_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|L^I_{(\alpha,\beta,k)}(X) \cap X|}{|L^I_{(\alpha,\beta,k)}(X)|} \ln \frac{|L^I_{(\alpha,\beta,k)}(X) \cap X|}{|L^I_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|L^I_{(\alpha,\beta,k)}(X) \cap X|}{|L^I_{(\alpha,\beta,k)}(X)|}\right) \ln\left(1 - \frac{|L^I_{(\alpha,\beta,k)}(X) \cap X|}{|L^I_{(\alpha,\beta,k)}(X)|}\right)\right].$$

Obviously, $H^R(P^I_{(\alpha,\beta,k)}(X)) \geq 0$, $H^R(N^I_{(\alpha,\beta,k)}(X)) \geq 0$, $H^R(U^I_{(\alpha,\beta,k)}(X)) \geq 0$ and $H^R(L^I_{(\alpha,\beta,k)}(X)) \geq 0$. So the analysis on the fuzziness of the DqI-DTRS model is much more complicated than that of Pawlak rough set model.

Theorem 4.1 *Given an information system $S = (U, A)$ and an equivalence relation R. For any target set $X \subseteq U$, for the same k, if $0 \leq \beta_1 \leq \beta_2 < \alpha_2 \leq \alpha_1 \leq 1$, then*

- $H^R(P^I_{(\alpha_1,\beta_1,k)}(X)) \geq H^R(P^I_{(\alpha_2,\beta_2,k)}(X))$;
- $H^R(N^I_{(\alpha_1,\beta_1,k)}(X)) \leq H^R(N^I_{(\alpha_2,\beta_2,k)}(X))$;
- $H^R(U^I_{(\alpha_1,\beta_1,k)}(X)) \geq H^R(U^I_{(\alpha_2,\beta_2,k)}(X))$;
- $H^R(L^I_{(\alpha_1,\beta_1,k)}(X)) \leq H^R(L^I_{(\alpha_2,\beta_2,k)}(X))$.

Proof Let $f(a, b)$ be a function with two variables a and b as follows:

$$f(a,b) = -\frac{1}{n\ln 2}[a\ln a + b\ln b - (a+b)\ln(a+b)].$$

The partial derivatives of $f(a, b)$ is

$$\frac{\partial f(a,b)}{\partial a} = \frac{-1}{n\ln 2}\ln\frac{a}{a+b} > 0,$$
$$\frac{\partial f(a,b)}{\partial b} = \frac{-1}{n\ln 2}\ln\frac{a}{a+b} > 0.$$

Therefore, $f(a, b)$ is a monotonically increasing function with variables a and b, respectively.

When $0 \le \beta_1 \le \beta_2 < \alpha_2 \le \alpha_1 \le 1$, it can be seen that $\underline{R}_{(\alpha_1,\beta_1)}(X) \le \underline{R}_{(\alpha_2,\beta_2)}(X)$ and $\overline{R}_{(\alpha_1,\beta_1)}(X) \ge \overline{R}_{(\alpha_2,\beta_2)}(X)$. According to the definition of the four regions in DqI-DTRS model, one has

$$P^I_{(\alpha,\beta,k)}(X) = \overline{R}_{(\alpha,\beta)}(X) \cap \underline{R}_k(X);$$
$$N^I_{(\alpha,\beta,k)}(X) = \sim(\overline{R}_{(\alpha,\beta)}(X) \cup \underline{R}_k(X));$$
$$U^I_{(\alpha,\beta,k)}(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_k(X);$$
$$L^I_{(\alpha,\beta,k)}(X) = \underline{R}_k(X) - \overline{R}_{(\alpha,\beta)}(X),$$

when $0 \le \beta_1 \le \beta_2 < \alpha_2 \le \alpha_1 \le 1$, it can be observed that

$$P^I_{(\alpha_1,\beta_1,k)}(X) \supseteq P^I_{(\alpha_2,\beta_2,k)}(X);$$
$$N^I_{(\alpha_1,\beta_1,k)}(X) \subseteq N^I_{(\alpha_2,\beta_2,k)}(X);$$
$$U^I_{(\alpha_1,\beta_1,k)}(X) \supseteq U^I_{(\alpha_2,\beta_2,k)}(X);$$
$$L^I_{(\alpha_1,\beta_1,k)}(X) \subseteq L^I_{(\alpha_2,\beta_2,k)}(X).$$

From the fuzziness formula of positive region, negative region, upper boundary region and lower boundary region in terms of DqI-DTRS model, we obtain

$$H^R(\blacklozenge) = -\frac{|\blacklozenge|}{|U| \times \ln 2}\left[\frac{|\blacklozenge \cap X|}{|\blacklozenge|}\ln\frac{|\blacklozenge \cap X|}{|\blacklozenge|} + \left(1 - \frac{|\blacklozenge \cap X|}{|\blacklozenge|}\right)\ln\left(1 - \frac{|\blacklozenge \cap X|}{|\blacklozenge|}\right)\right],$$

where $\blacklozenge = P^I_{(\alpha,\beta,k)}(X), N^I_{(\alpha,\beta,k)}(X), U^I_{(\alpha,\beta,k)}(X)$ and $L^I_{(\alpha,\beta,k)}(X)$, respectively.

Suppose $a = |\blacklozenge|$ and $b = |\blacklozenge| - |\blacklozenge \cap X|$. Then we have

$$\begin{aligned} H^R(\blacklozenge) &= -\frac{(a+b)}{n\ln 2}\left[\frac{a}{a+b}\ln\frac{a}{a+b} + \frac{b}{a+b}\ln\frac{b}{a+b}\right] \\ &= -\frac{1}{n\ln 2}[a\ln a + b\ln b - (a+b)\ln(a+b)]. \end{aligned}$$

Because $P^I_{(\alpha_1,\beta_1,k)}(X) \supseteq P^I_{(\alpha_2,\beta_2,k)}(X), N^I_{(\alpha_1,\beta_1,k)}(X) \subseteq N^I_{(\alpha_2,\beta_2,k)}(X)$, $U^I_{(\alpha_1,\beta_1,k)}(X) \supseteq U^I_{(\alpha_2,\beta_2,k)}(X)$, and $L^I_{(\alpha_1,\beta_1,k)}(X) \subseteq L^I_{(\alpha_2,\beta_2)}(X)$, according to the monotonicity increasing of $f(a, b)$, we can obtain that

$$H^R(P^I_{(\alpha_1,\beta_1,k)}(X)) \ge H^R(P^I_{(\alpha_2,\beta_2,k)}(X));$$
$$H^R(N^I_{(\alpha_1,\beta_1,k)}(X)) \le H^R(N^I_{(\alpha_2,\beta_2,k)}(X));$$
$$H^R(U^I_{(\alpha_1,\beta_1,k)}(X)) \ge H^R(U^I_{(\alpha_2,\beta_2,k)}(X));$$
$$H^R(L^I_{(\alpha_1,\beta_1,k)}(X)) \le H^R(L^I_{(\alpha_2,\beta_2,k)}(X)).$$

The proof has been completed. □

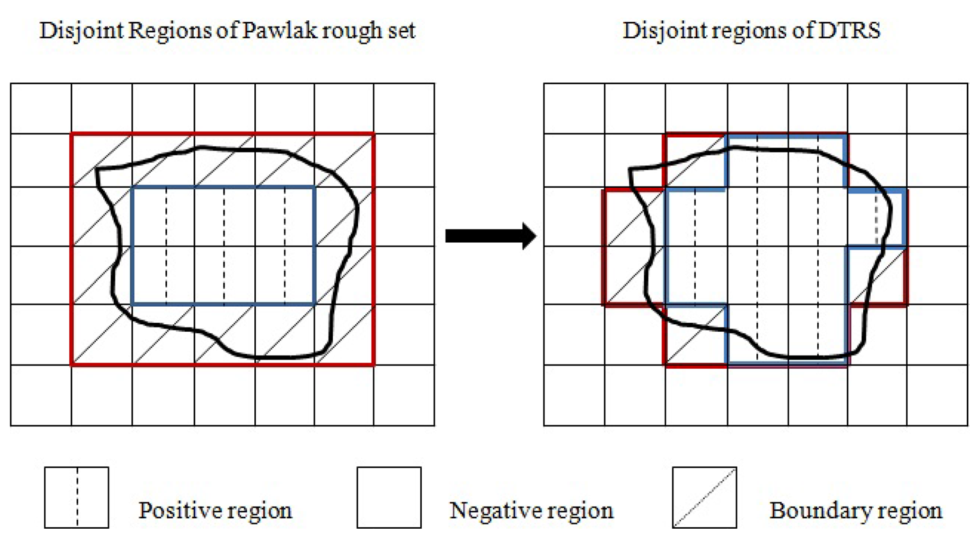

Fig. 1 Variation of disjoint regions from Pawlak rough set to DTRS

For the same grade k, it can be seen form the above Theorem 4.1 that if the two pairs of (α_1, β_1) and (α_2, β_2) satisfy $0 \le \beta_1 \le \beta_2 < \alpha_2 \le \alpha_1 \le 1$, then Theorem 4.1 provides a judgement method for comparing the fuzziness of the four disjoint region in DqI-DTRS model.

Theorem 4.2 *Given an information system $S = (U, A)$ and an equivalence relation R. For any target set $X \subseteq U$, for the same α, β, if $0 \le k_1 < k_2 \le |U|$, then for the DqI-DTRS model, we can establish the following formulas*:

- $H^R(P^I_{(\alpha,\beta,k_1)}(X)) \le H^R(P^I_{(\alpha,\beta,k_2)}(X))$;
- $H^R(N^I_{(\alpha,\beta,k_1)}(X)) \ge H^R(N^I_{(\alpha,\beta,k_2)}(X))$;
- $H^R(U^I_{(\alpha,\beta,k_1)}(X)) \ge H^R(U^I_{(\alpha,\beta,k_2)}(X))$;
- $H^R(L^I_{(\alpha,\beta,k_1)}(X)) \le H^R(L^I_{(\alpha,\beta,k_2)}(X))$.

Proof When $k_1 < k_2$, then $\underline{R}_{k_1}(X) \le \underline{R}_{k_2}(X)$ and $\overline{R}_{k_1}(X) \ge \overline{R}_{k_2}(X)$. Then for the DqI-DTRS, we can get

$$P^I_{(\alpha,\beta,k_1)}(X) \subseteq P^I_{(\alpha,\beta,k_2)}(X);$$
$$N^I_{(\alpha,\beta,k_1)}(X) \supseteq N^I_{(\alpha,\beta,k_2)}(X);$$
$$U^I_{(\alpha,\beta,k_1)}(X) \supseteq U^I_{(\alpha,\beta,k_2)}(X);$$
$$L^I_{(\alpha,\beta,k_1)}(X) \subseteq L^I_{(\alpha,\beta,k_2)}(X).$$

The next proof process of this theorem is similar to that of Theorem 4.1. □

For the same α and β, it can be seen form the above Theorem 4.2 that if k_1 and k_2 satisfy $0 \le k_1 < k_2 \le |U|$, then Theorem 4.2 provides a judgement method for comparing the fuzziness of the four disjoint region in DqI-DTRS model.

Example 4.1 (See Table 1) The medical example shown in references [23, 74] is introduced to interpret the fuzziness of the disjoint regions in DqI-DTRS. Let $S = (U, A, D)$ be a decision table, where U is composed of 36 patients, and the condition attribute and decision attribute are *fever*, *headache* and *cold*, respectively. Let R denote the equivalence relation on the condition attributes.

From Table 1, the cold patient set $X = \{x_3, x_5, x_6, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{34}\}$. We first calculate the grade $k = 1$ upper and lower approximations of X as follows

$$\overline{R}_k(X) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R,$$
$$\underline{R}_k(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R.$$

The upper and lower approximations based on parameters $\alpha = 0.5, \beta = 0.3$ with respect to R are computed as

$$\begin{aligned}\overline{R}_{(0.5,0.3)}(X) &= [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_5]_R \\ &\cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R, \\ \underline{R}_{(0.5,0.3)}(X) &= [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R.\end{aligned}$$

the upper and lower approximations of DqI-DTRS model are

$$\begin{aligned}\overline{R}_{(0.5,0.3)}(X) &= [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_5]_R \\ &\cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R, \\ \underline{R}_k(X) &= [x_5]_R \cup [x_6]_R \cup [x_{15}]_R.\end{aligned}$$

We can obtain positive region, negative region, upper boundary region and lower boundary region of DqI-DTRS:

$$P^I(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R;$$
$$N^I(X) = [x_1]_R;$$
$$U^I(X) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$L^I(X) = \emptyset.$$

Then the fuzziness of each region in DqI-DTRS could be calculated in the following

$$\begin{aligned}H^R(P^I_{(\alpha,\beta,k)}(X)) &= -\frac{6}{36 \times \ln 2}\left[\frac{5}{6}\ln\frac{5}{6} + \frac{1}{6}\ln\frac{1}{6}\right] = 0.1083, \\ H^R(N^I_{(\alpha,\beta,k)}(X)) &= 0, \\ H^R(U^I_{(\alpha,\beta,k)}(X)) &= -\frac{23}{36 \times \ln 2}\left[\frac{12}{23}\ln\frac{12}{23} + \frac{11}{23}\ln\frac{11}{23}\right] = 0.6380, \\ H^R(L^I_{(\alpha,\beta,k)}(X)) &= 0.\end{aligned}$$

Based on the DqI-DTRS model, patients $x_5, x_6, x_9, x_{15}, x_{26}$ and x_{32} belong to the positive region with fuzziness 0.1083; patients $x_1, x_7, x_{13}, x_{19}, x_{22}, x_{30}$ and x_{35} belong to the negative region with fuzziness 0; patients $x_2, x_3, x_4, x_8, x_{10}, x_{11}, x_{12}, x_{14}, x_{16}, x_{17}, x_{18}, x_{20}, x_{21}, x_{23}, x_{24}, x_{25}, x_{27}, x_{28}, x_{29}, x_{31}, x_{33}, x_{34}$ and x_{36} belongs to the upper boundary region with fuzziness 0.6380.

Definition 4.2 The method for measuring fuzziness of DqII-DTRS is in the following:

$$\begin{aligned}H^{II,R}_{(\alpha,\beta,k)}(X) = &H^R(P^{II}_{(\alpha,\beta,k)}(X)) + H^R(N^{II}_{(\alpha,\beta,k)}(X)) \\ &+ H^R(U^{II}_{(\alpha,\beta,k)}(X)) + H^R(L^{II}_{(\alpha,\beta,k)}(X)),\end{aligned}$$

where

$$H^R(P^{II}_{(\alpha,\beta,k)}(X)) = -\frac{|P^{II}_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|P^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|P^{II}_{(\alpha,\beta,k)}(X)|}\ln\frac{|P^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|P^{II}_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|P^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|P^{II}_{(\alpha,\beta,k)}(X)|}\right)\ln\left(1 - \frac{|P^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|P^{II}_{(\alpha,\beta,k)}(X)|}\right)\right],$$

$$H^R(N^{II}_{(\alpha,\beta,k)}(X)) = -\frac{|N^{II}_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|N^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|N^{II}_{(\alpha,\beta,k)}(X)|}\ln\frac{|N^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|N^{II}_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|N^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|N^{II}_{(\alpha,\beta,k)}(X)|}\right)\ln\left(1 - \frac{|N^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|N^{II}_{(\alpha,\beta,k)}(X)|}\right)\right],$$

$$H^R(U^{II}_{(\alpha,\beta,k)}(X)) = -\frac{|U^{II}_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|U^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|U^{II}_{(\alpha,\beta,k)}(X)|}\ln\frac{|U^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|U^{II}_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|U^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|U^{II}_{(\alpha,\beta,k)}(X)|}\right)\ln\left(1 - \frac{|U^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|U^{II}_{(\alpha,\beta,k)}(X)|}\right)\right],$$

$$H^R(L^{II}_{(\alpha,\beta,k)}(X)) = -\frac{|L^{II}_{(\alpha,\beta,k)}(X)|}{|U| \times \ln 2}\left[\frac{|L^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|L^{II}_{(\alpha,\beta,k)}(X)|}\ln\frac{|L^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|L^{II}_{(\alpha,\beta,k)}(X)|} + \left(1 - \frac{|L^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|L^{II}_{(\alpha,\beta,k)}(X)|}\right)\ln\left(1 - \frac{|L^{II}_{(\alpha,\beta,k)}(X) \cap X|}{|L^{II}_{(\alpha,\beta,k)}(X)|}\right)\right].$$

In Pawlak rough set model, when the granules (equivalence classes) are subdivided into finer granules by adding new attributes or information, the fuzziness of a target concept X will gradually decrease according to formula $H(F^X_R)$. In other words, the finer the granules in knowledge space, the lower the uncertainty of the target concept, and the boundary region will become smaller and narrower when the granules in Pawlak's knowledge space gradually subdivided. However, for the Dq-DTRS model, we cannot get the similar conclusions when the equivalence classes in knowledge space $S = (U, A)$ are subdivided into finer granules. Because the upper approximation and lower approximation in both DqI-DTRS and DqII-DTRS models do not contain inclusion relation, while the Pawlak rough set holds, so the upper boundary region and lower boundary region are existing in both DqI-DTRS and DqII-DTRS models.

Theorem 4.3 *Given an information system $S = (U, A)$ and an equivalence relation R. For any target set $X \subseteq U$, for the same k, if $0 \le \beta_1 \le \beta_2 < \alpha_2 \le \alpha_1 \le 1$, then*

- $H^R(P^{II}_{(\alpha_1,\beta_1,k)}(X)) \le H^R(P^{II}_{(\alpha_2,\beta_2,k)}(X))$;
- $H^R(N^{II}_{(\alpha_1,\beta_1,k)}(X)) \ge H^R(N^{II}_{(\alpha_2,\beta_2,k)}(X))$;
- $H^R(U^{II}_{(\alpha_1,\beta_1,k)}(X)) \le H^R(U^{II}_{(\alpha_2,\beta_2,k)}(X))$;
- $H^R(L^{II}_{(\alpha_1,\beta_1,k)}(X)) \ge H^R(L^{II}_{(\alpha_2,\beta_2,k)}(X))$.

Proof When $0 \le \beta_1 \le \beta_2 < \alpha_2 \le \alpha_1 \le 1$, it can be seen that $\underline{R}_{(\alpha_1,\beta_1)}(X) \le \underline{R}_{(\alpha_2,\beta_2)}(X)$ and $\overline{R}_{(\alpha_1,\beta_1)}(X) \ge \overline{R}_{(\alpha_2,\beta_2)}(X)$. According to the definition of the four regions in DqII-DTRS model,

$$P^{II}_{(\alpha,\beta,k)}(X) = \overline{R}_k(X) \cap \underline{R}_{(\alpha,\beta)}(X);$$
$$N^{II}_{(\alpha,\beta,k)}(X) = \sim(\overline{R}_k(X) \cup \underline{R}_{(\alpha,\beta)}(X));$$
$$U^{II}_{(\alpha,\beta,k)}(X) = \overline{R}_k(X) - \underline{R}_{(\alpha,\beta)}(X);$$
$$L^{II}_{(\alpha,\beta,k)}(X) = \underline{R}_{(\alpha,\beta)}(X) - \overline{R}_k(X).$$

Because $\underline{R}_{(\alpha_1,\beta_1)}(X) \le \underline{R}_{(\alpha_2,\beta_2)}(X)$, we obtain

Table 1 Initial medical data

Patient	*Fever*	*Headache*	*Cold*	Patient	*Fever*	*Headache*	*Cold*	Patient	*Fever*	*Headache*	*Cold*
x_1	0	0	0	x_{13}	0	0	0	x_{25}	0	2	0
x_2	1	1	0	x_{14}	2	1	1	x_{26}	2	2	1
x_3	0	2	1	x_{15}	0	1	1	x_{27}	1	1	0
x_4	2	1	0	x_{16}	1	1	0	x_{28}	2	0	1
x_5	1	0	1	x_{17}	0	2	0	x_{29}	2	1	1
x_6	2	2	1	x_{18}	2	1	1	x_{30}	0	0	0
x_7	0	0	0	x_{19}	0	0	0	x_{31}	1	2	0
x_8	1	2	0	x_{20}	1	2	1	x_{32}	0	1	0
x_9	2	2	1	x_{21}	2	0	1	x_{33}	2	1	1
x_{10}	1	1	1	x_{22}	0	0	0	x_{34}	1	1	1
x_{11}	1	2	1	x_{23}	2	1	0	x_{35}	0	0	0
x_{12}	2	0	0	x_{24}	1	2	1	x_{36}	2	0	0

$P^{II}_{(\alpha_1,\beta_1,k)}(X) \subseteq P^{II}_{(\alpha_2,\beta_2,k)}(X)$;
$N^{II}_{(\alpha_1,\beta_1,k)}(X) \supseteq N^{II}_{(\alpha_2,\beta_2,k)}(X)$;
$U^{II}_{(\alpha_1,\beta_1,k)}(X) \subseteq U^{II}_{(\alpha_2,\beta_2,k)}(X)$;
$L^{II}_{(\alpha_1,\beta_1,k)}(X) \supseteq L^{II}_{(\alpha_2,\beta_2,k)}(X)$.

The next proof process of this theorem is similar to that of Theorem 4.1. □

For the same grade k, it can be seen form the above Theorem 4.3 that if the two pairs of (α_1, β_1) and (α_2, β_2) satisfy $0 \le \beta_1 \le \beta_2 < \alpha_2 \le \alpha_1 \le 1$, then Theorem 4.3 provides a judgement method for comparing the fuzziness of the four disjoint region in DqII-DTRS model.

Theorem 4.4 *Given an information system $S = (U, A)$ and an equivalence relation R. For any target set $X \subseteq U$, for the same α, β, if $0 \le k_1 < k_2 \le |U|$, then for the DqII-DTRS model, we can establish the following formulas*:

- $H^R(P^{II}_{(\alpha,\beta,k_1)}(X)) \ge H^R(P^{II}_{(\alpha,\beta,k_2)}(X))$;
- $H^R(N^{II}_{(\alpha,\beta,k_1)}(X)) \le H^R(N^{II}_{(\alpha,\beta,k_2)}(X))$;
- $H^R(U^{II}_{(\alpha,\beta,k_1)}(X)) \ge H^R(U^{II}_{(\alpha,\beta,k_2)}(X))$;
- $H^R(L^{II}_{(\alpha,\beta,k_1)}(X)) \le H^R(L^{II}_{(\alpha,\beta,k_2)}(X))$.

Proof When $k_1 < k_2$, then $\underline{R}_{k_1}(X) \le \underline{R}_{k_2}(X)$ and $\overline{R}_{k_1}(X) \ge \overline{R}_{k_2}(X)$. Then for the DqII-DTRS, we have

$P^{II}_{(\alpha,\beta,k_1)}(X) \supseteq P^{II}_{(\alpha,\beta,k_2)}(X)$;
$N^{II}_{(\alpha,\beta,k_1)}(X) \subseteq N^{II}_{(\alpha,\beta,k_2)}(X)$;
$U^{II}_{(\alpha,\beta,k_1)}(X) \supseteq U^{II}_{(\alpha,\beta,k_2)}(X)$;
$L^{II}_{(\alpha,\beta,k_1)}(X) \subseteq L^{II}_{(\alpha,\beta,k_2)}(X)$.

The next proof process of this theorem is similar to that of Theorem 4.1. □

For the same α and β, it can be seen form the above Theorem 4.4 that if k_1 and k_2 satisfy $0 \le k_1 < k_2 \le |U|$, then Theorem 4.4 provides a judgement method for comparing the fuzziness of the four disjoint region in DqII-DTRS model.

Example 4.2 (Continuation of Example 4.1) The upper and lower approximations of DqII-DTRS model are

$$\overline{R}_k(X) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R,$$
$$\underline{R}_{(0.5,0.3)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R.$$

We can also get positive region, negative region, upper boundary region and lower boundary region of DqII-DTRS:

$P^{II}(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$;
$N^{II}(X) = [x_1]_R \cup [x_3]_R \cup [x_{15}]_R$;
$U^{II}(X) = [x_2]_R \cup [x_{12}]_R$;
$L^{II}(X) = [x_5]_R$.

And the fuzziness of each region in DqII-DTRS could be calculated as

$$H^R(P^{II}_{(\alpha,\beta,k)}(X)) = -\frac{14}{36 \times \ln 2}\left[\frac{11}{14}\ln\frac{11}{14} + \frac{3}{14}\ln\frac{3}{14}\right] = 0.2915,$$
$$H^R(N^{II}_{(\alpha,\beta,k)}(X)) = -\frac{12}{36 \times \ln 2}\left[\frac{2}{12}\ln\frac{2}{12} + \frac{10}{12}\ln\frac{10}{12}\right] = 0.2167,$$
$$H^R(U^{II}_{(\alpha,\beta,k)}(X)) = -\frac{9}{36 \times \ln 2}\left[\frac{4}{9}\ln\frac{4}{9} + \frac{5}{9}\ln\frac{5}{9}\right] = 0.2478,$$
$$H^R(L^{II}_{(\alpha,\beta,k)}(X)) = 0.$$

Based on the DqII-DTRS model, patients x_4, x_6, x_8, x_9, x_{11}, x_{14}, x_{18}, x_{20}, x_{23}, x_{24}, x_{26}, x_{29}, x_{31} and x_{33} belong to the positive region with the fuzziness 0.2915; patients x_1, x_3, x_7, x_{13}, x_{15}, x_{17}, x_{19}, x_{22}, x_{25}, x_{30}, x_{32} and x_{35} belong to the negative region with fuzziness 0.2167; patients x_2, x_{10}, x_{12}, x_{16}, x_{21}, x_{27}, x_{28}, x_{34} and x_{36} belongs to the upper boundary region with fuzziness 0.2478; patient x_5 belongs to the lower boundary region with the fuzziness 0.

5 Regions changing with attribute increment for Dq-DTRS model

In order to measure fuzziness of a rough set, the authors in [28] presented a method for measuring uncertainty of a target concept in rough approximation space according to information entropy. That method takes into account two kinds of uncertainty, one is coming from the objects which belong to the target concept X but they are classified into boundary, the other uncertainty is coming from the objects which do not belong to X but they are classified into boundary region. For an uncertain target concept, the uncertainty comes from four regions, namely, positive region, negative region, upper boundary region and lower boundary region with two parameters α, β, and a grade k (where $0 \le \beta < \alpha \le 1$ and $0 \le k \le |U|$) in Dq-DTRS models. In this section, we investigate the changes of boundary region in Dq-DTRS models and study the judgement theorems for three kinds of incremental information.

5.1 Incremental information in Dq-DTRS model

In this subsection, we study the effects of adding attributes on the variation of disjoint regions in Pawlak rough set, DTRS and Dq-DTRS models and make the comparative analysis on these three models. Let us see the following definition of attribute increment.

Definition 5.1 [69] Given an information system $S = (U, A)$, $R_1 \subseteq R_2 \subseteq A$. The difference $\Delta R = R_2 - R_1$ is called attribute increment.

If we want to make the final two-way decisions in Pawlak rough set model, we usually need to add some new attributes to reclassify the objects in boundary region into positive region (to make acceptance decisions) or negative region (to make rejection decisions). Then the boundary region will become smaller, and both positive region and negative region will become bigger. In the DTRS model, we cannot obtain the monotonicity of probabilistic positive regions of a target (or decision). Compare with Pawlak rough set model, the changes of three regions in DTRS are more complicated with the increase of attributes (See Figure 2).

In Figure 2, we use (i, j) $(i \in \{1, 2, 3, 4, 5, 6\};$ $j \in \{1, 2, 3, 4, 5, 6, 7\})$ to denote as the blocks of equivalence class in the upper part two small graphs, and use (m, n) $(m \in \{1, 2, \ldots, 12\}; n \in \{1, 2, \ldots, 14\})$ to denote as the blocks of equivalence class in the lower part two small graphs. It is easy to see that the positive region and negative region are much bigger and the boundary region becomes smaller in Pawlak rough set with the attribute increment, because some equivalence classes are redivided into positive region (Newly added blocks (4, 6), (4, 7), (4, 8), (4, 9), (5, 11), (6, 11), (9, 8), (9, 9), (9, 10)) and negative region (Newly added (3, 3), (3, 4), (3, 11), (3, 12), (4, 12), (5, 3), (6, 3), (7, 3), (8, 3), (8, 12), (9, 3), (9, 12), (10, 3), (10, 4), (10, 5), (10, 6), (10, 12)) from its original boundary region. For the DTRS, elements in positive region and negative region are no longer only added from the boundary region, which is different from Pawlak rough set. For the new positive region in Figure 2 (DTRS part), the block (4, 4) is added from negative region, and blocks (4, 5), (4, 6), (5, 4), (7, 11), (8, 11), (9, 6) are increased from boundary region; there are 7 blocks moved out from original positive region, which are (3, 7), (3, 8), (3, 9), (3, 10), (5, 12), (6, 12), (10, 7). For the new negative region (DTRS part again), blocks (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (5, 12), (6, 12) are added from original positive region, and blocks (5, 3), (6, 3), (7, 3), (7, 12), (8, 3), (8, 12), (9, 12), (10, 5), (10, 6), (10, 7) are increased from original boundary region; there are 3 blocks moved out from original negative region, which are (4, 4), (4, 11), (9, 11). That is to say, in DTRS model, blocks in positive region may be removed

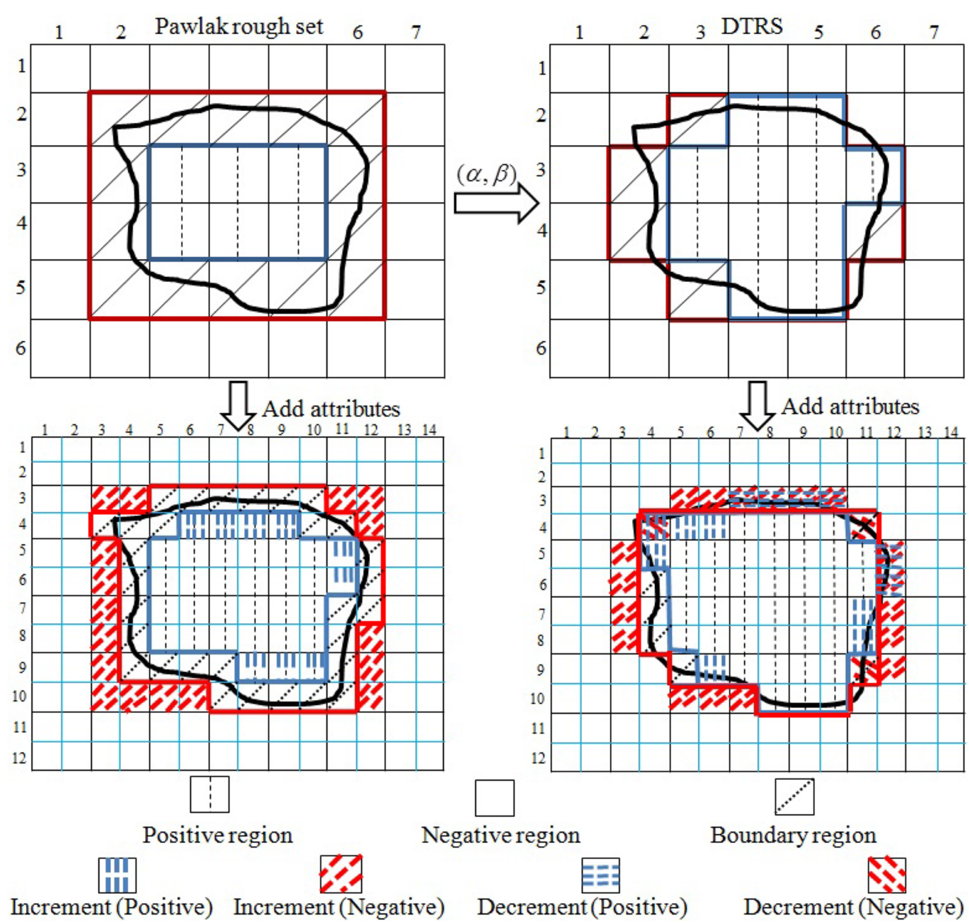

Fig. 2 Regions changing with attribute increment in Pawlak rough set and DTRS

into negative region and boundary region with attribute increment; blocks in negative region may be removed into positive region and boundary region with attribute increment; and blocks in boundary region may be removed into positive region and boundary region.

As to the situations in Dq-DTRS models (See Figure 3), it is much more complicated than the results in DTRS. We also use (i, j) $(i \in \{1,2,3,4,5,6\}; j \in \{1,2,3,4,5,6,7\})$ to denote as the blocks of equivalence class in the graphs without attribute increment, and use (m, n) $(m \in \{1,2,\ldots,12\}; n \in \{1,2,\ldots,14\})$ to denote as the blocks of equivalence class in the graphs of two kinds of Dq-DTRS model with attribute increment. Unlike DTRS, the positive and negative regions are not longer just increasing, but there are some blocks moved out from the original positive region and negative region. Let us analyze the regions changing in the following, we call the Dq-DTRS without attribute increment as original Dq-DTRS and Dq-DTRS with attribute increment as new Dq-DTRS, the same to their corresponding disjoints.

For the new positive region of DqI-DTRS (with attribute increment), the block (4, 4) is added from original lower boundary region, blocks (4, 11) and (9, 7) are increased from original negative region, and blocks (5, 4), (7, 11), (8, 4), (8, 11), (9, 5), (9, 6), (9, 7) are increase from original upper boundary region. For the new negative region (DqI-DTRS again), blocks (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (5, 12), (6, 12), (10, 7) are added from original positive region, blocks (3, 3), (3, 4), (4, 3) are increased from original lower boundary region, and blocks (5, 3), (6, 3), (7, 3), (7, 12), (8, 3), (8, 12), (10, 5) are increased from upper boundary region. The decreasing of blocks of new positive region are (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (10, 7), where blocks (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (10, 7) are redivided into new negative region of DqI-DTRS, the decreasing of blocks of new negative region are (4, 11), (9, 11), (10, 11), where (3, 5), (3, 6), are redivided into new positive region, and (3, 7) is redivided into new lower boundary region in DqI-DTRS.

For the new positive region of DqII-DTRS (with attribute increment), the blocks (9, 7), (9, 8) and (10, 8) are added from original lower boundary region, blocks (4, 4), (4, 5), (4, 6), and (9, 6) are increased from original negative region,

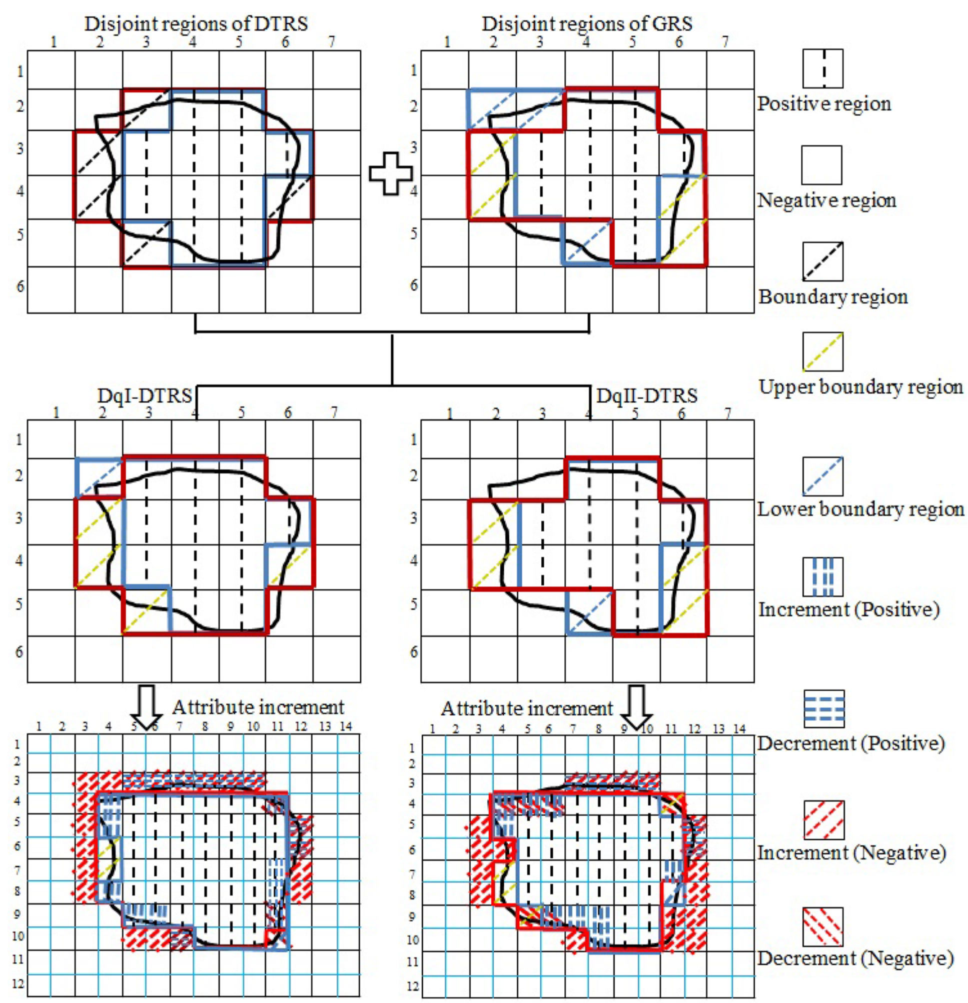

Fig. 3 Regions changing with attribute increment in Dq-DTRS

and blocks (5, 4), (7, 11) are increased from original upper boundary region. For the new negative region (DqII-DTRS again), block (10, 7) is increased from original lower boundary region, blocks (3, 7), (3, 8), (3, 9), (3, 10), (5, 12), (6, 12) are added from original positive region, and blocks (5, 3), (6, 3), (6, 4), (7, 3), (7, 12), (8, 3), (8, 12), (9, 11), (9, 12), (10, 11), (10, 12) are increased from upper boundary region. The decreasing of blocks of new positive region are (3, 7), (3, 8), (3, 9), (3, 10), (5, 12), (6, 12), where these blocks are redivided into new negative region in DqII-DTRS, the decreasing of blocks of new negative region are (4, 4), (4, 5), (4, 6), (4, 11), (9, 5), (9, 6), where (4, 4), (4, 5), (4, 6), (4, 11), (9, 6) are redivided into new positive region, and (4, 11), (9, 5) are redivided into new upper boundary region in DqII-DTRS. The special circumstance in this DqII-DTRS is that the block (8, 11) is from upper boundary region to lower boundary region.

For the three-way decisions in the real-life applications, the ultimate goal is to reach the two-way decisions. To reach this goal, we must narrow down the size of boundary region, including upper boundary region and lower boundary region. Because the boundary region of Dq-DTRS model is subdivided into two parts: upper boundary region and lower boundary region, so the changes of the regions must be much more complicated after adding the attributes. It is obvious that there are three cases of boundary region change in Dq-DTRS model with attribute increment (boundary region becomes smaller, boundary region remains unchanged, and boundary region becomes larger).

Definition 5.2 Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \leq \beta < \alpha \leq 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. In the Dq$\blacklozenge$-DTRS model ($\blacklozenge \in \{I, II\}$),

(1) If $B^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X) \supsetneqq B^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X)$ i. e. $(U^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X) \cup L^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X)) \supsetneqq (U^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X) \cup L^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X))$, then the attribute increment $\Delta R = R_2 - R_1$ is called useful incremental information.

(2) If $B^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X) = B^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X)$ i. e. $(U^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X) \cup L^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X)) = (U^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X) \cup L^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X))$, then the attribute increment $\Delta R = R_2 - R_1$ is called useless incremental information.

(3) If $B^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X) \subsetneqq B^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X)$ i. e. $(U^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X) \cup L^{\blacklozenge R_1}_{(\alpha,\beta,k)}(X)) \subsetneqq (U^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X) \cup L^{\blacklozenge R_2}_{(\alpha,\beta,k)}(X))$, then the attribute increment $\Delta R = R_2 - R_1$ is called error-correction incremental information.

- For the useful incremental information, a part of the elements in the boundary region (upper and lower boundary region) can be classified into positive region or negative region. This result indicates the useful incremental information is helpful to classify the uncertain objects in boundary region.
- For the useless incremental information, the elements in the boundary region cannot be classified into positive region or negative region. This result indicates this useless incremental information is helpless to classify the uncertain objects in boundary region.
- For the error-correction incremental information, the elements in the positive region or negative region are reclassified into boundary region. In other words, in this case, by adding some new attributes, a part of elements which are wrongly classified into the positive region or negative region due to insufficient information are reclassified into boundary region.

5.2 Assessment theorems for incremental information in Dq-DTRS model

The boundary region consists of two parts: upper boundary region and lower boundary region in the two kinds of Dq-DTRS models. In the following, we investigate the judgement methods for three kinds of incremental information with attribute increment in DqI-DTRS model and DqII-DTRS model, respectively.

Theorem 5.1 *Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \leq \beta < \alpha \leq 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. For any target concept $X \subseteq U$, if the attribute increment $\Delta R = R_2 - R_1$ only can distinguish the elements in the boundary region in DqI-DTRS model, then the attribute increment ΔR is the useful incremental information or useless incremental information.*

Proof Let $U/IND(R_1) = \{[x]_{R_1} \mid x \in U\} = \{X_1, X_2, \dots, X_l\}$ and $B^{I,R_1}_{(\alpha,\beta,k)}(X) = U^{I,R_1}_{(\alpha,\beta,k)}(X) \cup L^{I,R_1}_{(\alpha,\beta,k)}(X) = \cup\{X_{i_1}, X_{i_2}, \dots, X_{i_p}\}$, where $X_{i_1} \in U/IND(R_1), X_{i_2} \in U/IND(R_1), \dots, X_{i_p} \in U/IND(R_1)$. For more simplicity, supposing only one equivalence class is subdivided into two finer equivalence classes, without loss of generality, let $X_{i_1} = X'_{i_1} \cup X''_{i_1}$. Based on the definition of DqI-DTRS model, we have $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} > \beta) \wedge (|X_{i_1}| - |X_{i_1} \cap X| > k)$ or $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} \leq \beta) \wedge (|X_{i_1}| - |X_{i_1} \cap X| \leq k)$. There are about 16 cases regard to X'_{i_1} and X''_{i_1} (See Table 2).

As $B^{I,R_2}_{(\alpha,\beta,k)}(X) = U^{I,R_2}_{(\alpha,\beta,k)}(X) \cup L^{I,R_2}_{(\alpha,\beta,k)}(X)$, and all equivalence classes except X_{i_1} $(X_{i_1} \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X))$ keep unchanged, so we can get the following:

Table 2 Cases regard to $X_{i_1}^{'}$ and $X_{i_1}^{''}$ with incremental information

	$X_{i_1}^{'}$		$X_{i_1}^{''}$	
	$\lvert X_{i_1}^{'} \cap X\rvert / \lvert X_{i_1}^{'}\rvert$	$\lvert X_{i_1}^{'}\rvert - \lvert X_{i_1}^{'} \cap X\rvert$	$\lvert X_{i_1}^{''} \cap X\rvert / \lvert X_{i_1}^{''}\rvert$	$\lvert X_{i_1}^{''}\rvert - \lvert X_{i_1}^{''} \cap X\rvert$
(1)	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(2)	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(3)	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$\le \beta$ $\subseteq N_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(4)	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(5)	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(6)	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(7)	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$\le \beta$ $\subseteq N_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(8)	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(9)	$\le \beta$ $\subseteq N_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(10)	$\le \beta$ $\subseteq N_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(11)	$\le \beta$ $\subseteq N_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$\le \beta$ $\subseteq N_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(12)	$\le \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(13)	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$> \beta$ $\subseteq U_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(14)	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$> \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$
(15)	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$\le \beta$ $\subseteq P_{(\alpha,\beta,k)}^{I,R_2}(X)$	$> k$
(16)	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$	$\le \beta$ $\subseteq L_{(\alpha,\beta,k)}^{I,R_2}(X)$	$\le k$

(1) $B_{(\alpha,\beta,k)}^{I,R_1}(X) = B_{(\alpha,\beta,k)}^{I,R_2}(X)$; (2) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{''} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(3) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{''} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$; (4) $B_{(\alpha,\beta,k)}^{I,R_1}(X) = B_{(\alpha,\beta,k)}^{I,R_2}(X)$; (5) The same to the case in (2);

(6) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{'} - X_{i_1}^{''} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(7) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{'} - X_{i_1}^{''} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(8) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{'} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$; (9) The same to the case in (3); (10) The same to (7);

(11) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{'} - X_{i_1}^{''} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(12) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X) - X_{i_1}^{'} \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(13) The same to the case in (4); (14) The same to (8); (15) The same to (12); (16) $B_{(\alpha,\beta,k)}^{I,R_1}(X) = B_{(\alpha,\beta,k)}^{I,R_2}(X)$.

We can obtain that $B_{(\alpha,\beta,k)}^{I,R_2}(X) \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X)$, which means $B_{(\alpha,\beta,k)}^{I,R_2}(X) \subsetneqq B_{(\alpha,\beta,k)}^{I,R_1}(X)$ or $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X)$. According to the Definition 5.2, ΔR is a useful incremental information or useless incremental information. □

In Theorem 5.1, take the elements in upper boundary region for example, if the attribute increment ΔR only distinguish the elements in the upper boundary region, there are three possible directions for the reclassified elements: (1) lower boundary region; (2) positive region; and (3) negative region. So the boundary region becomes unchanged or smaller. Which means ΔR is the useless incremental information or useful incremental information in DqI-DTRS.

Theorem 5.2 *Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \le \beta < \alpha \le 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. For any target concept $X \subseteq U$, if the attribute increment $\Delta R = R_2 - R_1$ only can distinguish the elements in the positive region in DqI-DTRS model, then the attribute increment ΔR is the error-correction incremental information or useless incremental information.*

Proof Let $U/IND(R_1) = \{[x]_{R_1} \mid x \in U\} = \{X_1, X_2, \ldots, X_l\}$ and $P_{(\alpha,\beta,k)}^{I,R_1}(X) = \cup\{X_{i_1}, X_{i_2}, \ldots, X_{i_q}\}$, where $X_{i_1} \in U/IND$ $X_{i_1} \in U/IND(R_1), X_{i_2} \in U/IND(R_1), \ldots, X_{i_q} \in U/IND(R_1)$. Supposing only one equivalence class is subdivided into two finer equivalence classes, without loss of generality, let $X_{i_1} = X_{i_1}^{'} \cup X_{i_1}^{''}$. Based on the definition of DqI-DTRS model, we can obtain $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} > \beta) \wedge (|X_{i_1}| - |X_{i_1} \cap X| \le k)$. There are about 16 cases regard to $X_{i_1}^{'}$ and $X_{i_1}^{''}$ (See Table 2).

As $B_{(\alpha,\beta,k)}^{I,R_2}(X) = U_{(\alpha,\beta,k)}^{I,R_2}(X) \cup L_{(\alpha,\beta,k)}^{I,R_2}(X)$, and all equivalence classes except X_{i_1} $(X_{i_1} \subseteq P_{(\alpha,\beta,k)}^{I,R_1}(X))$ keep unchanged, so we can get the following:

(1) $B_{(\alpha,\beta,k)}^{I,R_1}(X) \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X) \cup X_{i_1}^{'} \cup X_{i_1}^{''} = B_{(\alpha,\beta,k)}^{I,R_2}(X)$;

(2) $B_{(\alpha,\beta,k)}^{I,R_1}(X) \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X) \cup X_{i_1}^{'} = B_{(\alpha,\beta,k)}^{I,R_2}(X)$;

(3) $B_{(\alpha,\beta,k)}^{I,R_1}(X) \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X) \cup X_{i_1}^{'} = B_{(\alpha,\beta,k)}^{I,R_2}(X)$;

(4) $B_{(\alpha,\beta,k)}^{I,R_1}(X) \subseteq B_{(\alpha,\beta,k)}^{I,R_1}(X) \cup X_{i_1}^{'} \cup X_{i_1}^{''} = B_{(\alpha,\beta,k)}^{I,R_2}(X)$;

(5) The same to the case in (2); (6) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(7) $B_{(\alpha,\beta,k)}^{I,R_2}(X) = B_{(\alpha,\beta,k)}^{I,R_1}(X)$;

(8) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$; (9) The same to the case in (3); (10) The same to (7);

(11) $B^{I,R_2}_{(\alpha,\beta,k)}(X) = B^{I,R_1}_{(\alpha,\beta,k)}(X)$; (12) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$; (13) The same to (4);

(14) The same to (8); (15) The same to (12); (16) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X'_{i_1} \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$.

From the above 16 cases, we can obtain that $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_2}_{(\alpha,\beta,k)}(X)$, which means $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subsetneqq B^{I,R_2}_{(\alpha,\beta,k)}(X)$ or $B^{I,R_1}_{(\alpha,\beta,k)}(X) = B^{I,R_2}_{(\alpha,\beta,k)}(X)$. According to the Definition 5.2, ΔR is an error-correction incremental information or useless incremental information. □

In Theorem 5.2, if the attribute increment ΔR only distinguish the elements in the positive region in DqI-DTRS, there are three possible directions for the reclassified elements: (1) upper boundary region; (2) lower boundary region; and (3) negative region. So the boundary region becomes larger or unchanged. Which means ΔR is the error-correction incremental information or useless incremental information in DqI-DTRS.

Theorem 5.3 *Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \le \beta < \alpha \le 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. For any target concept $X \subseteq U$, if the attribute increment $\Delta R = R_2 - R_1$ only can distinguish the elements in the negative region in DqI-DTRS model, then the attribute increment ΔR is the error-correction incremental information or useless incremental information.*

Proof Let $U/IND(R_1) = \{[x]_{R_1} \mid x \in U\} = \{X_1, X_2, \dots, X_l\}$ and $P^{I,R_1}_{(\alpha,\beta,k)}(X) = \cup\{X_{i_1}, X_{i_2}, \dots, X_{i_r}\}$, where $X_{i_1} \in U/IND(R_1), X_{i_2} \in U/IND(R_1), \dots, X_{i_r} \in U/IND(R_1)$. Supposing only one equivalence class is subdivided into two finer equivalence classes, without loss of generality, let $X_{i_1} = X'_{i_1} \cup X''_{i_1}$. Based on the definition of DqI-DTRS model, we can get $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} \le \beta) \wedge (|X_{i_1}| - |X_{i_1} \cap X| > k)$. There are about 16 cases regard to X'_{i_1} and X''_{i_1} (See Table 2).

As $B^{I,R_2}_{(\alpha,\beta,k)}(X) = U^{I,R_2}_{(\alpha,\beta,k)}(X) \cup L^{I,R_2}_{(\alpha,\beta,k)}(X)$, and all equivalence classes except X_{i_1} $(X_{i_1} \subseteq N^{I,R_1}_{(\alpha,\beta,k)}(X))$ keep unchanged, so we can get the same results as the 16 situations in Theorem 5.2. We can obtain that $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_2}_{(\alpha,\beta,k)}(X)$, which means $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subsetneqq B^{I,R_2}_{(\alpha,\beta,k)}(X)$ or $B^{I,R_1}_{(\alpha,\beta,k)}(X) = B^{I,R_2}_{(\alpha,\beta,k)}(X)$. According to the Definition 5.2, ΔR is an error-correction incremental information or useless incremental information. □

In Theorem 5.3, if the attribute increment ΔR only distinguish the elements in the negative region in DqI-DTRS, there are three possible directions for the reclassified elements: (1) upper boundary region; (2) lower boundary region; and (3) positive region. So the boundary region becomes larger or unchanged. Which means ΔR is the error-correction incremental information or useless incremental information in DqI-DTRS.

Theorem 5.4 *Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \le \beta < \alpha \le 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. For any target concept $X \subseteq U$, if the attribute increment $\Delta R = R_2 - R_1$ only can distinguish the elements in the boundary region in DqII-DTRS model, then attribute increment ΔR is the useful incremental information or useless incremental information.*

Proof Let $U/IND(R_1) = \{[x]_{R_1} \mid x \in U\} = \{X_1, X_2, \dots, X_l\}$ and $B^{II,R_1}_{(\alpha,\beta,k)}(X) = U^{II,R_1}_{(\alpha,\beta,k)}(X) \cup L^{II,R_1}_{(\alpha,\beta,k)}(X) = \cup\{X_{i_1}, X_{i_2}, \dots, X_{i_p}\}$, where $X_{i_1} \in U/IND(R_1), X_{i_2} \in U/IND(R_1), \dots, X_{i_p} \in U/IND(R_1)$. Supposing only one equivalence class is subdivided into two finer equivalence classes, without loss of generality, let $X_{i_1} = X'_{i_1} \cup X''_{i_1}$. Based on the definition of DqII-DTRS model, we can get $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} < \alpha) \wedge (|X_{i_1} \cap X| > k)$ or $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} \ge \alpha) \wedge (|X_{i_1} \cap X| \le k)$. There are about 16 cases regard to X'_{i_1} and X''_{i_1} (See Table 3):

As $B^{II,R_2}_{(\alpha,\beta,k)}(X) = U^{II,R_2}_{(\alpha,\beta,k)}(X) \cup L^{II,R_2}_{(\alpha,\beta,k)}(X)$, and all equivalence classes except X_{i_1} $(X_{i_1} \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X))$ keep unchanged, so we can get the following:

(1) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X) - X'_{i_1} - X''_{i_1} \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$;
(2) $B^{I,R_2}_{(\alpha,\beta,k)}(X) = B^{I,R_1}_{(\alpha,\beta,k)}(X) - X'_{i_1} \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X)$;

(3) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X) - X'_{i_1} \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$;
(4) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X) - X'_{i_1} - X''_{i_1} \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$;

(5) The same to the case in (2); (6) $B^{II,R_1}_{(\alpha,\beta,k)}(X) = B^{II,R_2}_{(\alpha,\beta,k)}(X)$;
(7) $B^{II,R_1}_{(\alpha,\beta,k)}(X) = B^{II,R_2}_{(\alpha,\beta,k)}(X)$;

(8) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X) - X''_{i_1} \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$; (9) The same to the case in (3); (10) The same to (7);

(11) $B^{II,R_1}_{(\alpha,\beta,k)}(X) = B^{II,R_2}_{(\alpha,\beta,k)}(X)$; (12) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X) - X''_{i_1} \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$; (13) The same to (4);

(14) The same to (8); (15) The same to (12); (16) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X) - X'_{i_1} - X''_{i_1} \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$.

From the above 16 cases, we can obtain that $B^{II,R_2}_{(\alpha,\beta,k)}(X) \subseteq B^{II,R_1}_{(\alpha,\beta,k)}(X)$, which means $B^{II,R_2}_{(\alpha,\beta,k)}(X) \subsetneqq B^{II,R_1}_{(\alpha,\beta,k)}(X)$

Table 3 Cases regard to X'_{i_1} and X''_{i_1} with incremental information

	X'_{i_1}		X''_{i_1}	
	$\lvert X'_{i_1}\cap X\rvert/\lvert X'_{i_1}\rvert$	$\lvert X'_{i_1}\cap X\rvert$	$\lvert X''_{i_1}\cap X\rvert/\lvert X''_{i_1}\rvert$	$\lvert X''_{i_1}\cap X\rvert$
(1)	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(2)	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(3)	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(4)	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(5)	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(6)	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(7)	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(8)	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(9)	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(10)	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(11)	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(12)	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(13)	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$\geq \alpha$ $\subseteq P^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(14)	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$\geq \alpha$ $\subseteq L^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$
(15)	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$< \alpha$ $\subseteq U^{II,R_2}_{(\alpha,\beta,k)}(X)$	$> k$
(16)	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$	$< \alpha$ $\subseteq N^{II,R_2}_{(\alpha,\beta,k)}(X)$	$\leq k$

or $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X)$. According to the Definition 5.2, ΔR is a useful incremental information or useless incremental information. □

In Theorem 5.4, take the elements in upper boundary region for example, if the attribute increment ΔR only distinguish the elements in the upper boundary region, there are three possible directions for the reclassified elements: (1) lower boundary region; (2) positive region; and (3) negative region. So the boundary region becomes unchanged or smaller. Which means ΔR is the useless incremental information or useful incremental information in DqII-DTRS.

Theorem 5.5 *Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \leq \beta < \alpha \leq 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. For any target concept $X \subseteq U$, if the attribute increment $\Delta R = R_2 - R_1$ only can distinguish the elements in the positive region in DqII-DTRS model, then the attribute increment ΔR is the error-correction incremental information or useless incremental information.*

Proof Let $U/IND(R_1) = \{[x]_{R_1} \mid x \in U\} = \{X_1, X_2, \dots, X_l\}$ and $B^{II,R_1}_{(\alpha,\beta,k)}(X) = U^{II,R_1}_{(\alpha,\beta,k)}(X) \cup L^{II,R_1}_{(\alpha,\beta,k)}(X) = \cup\{X_{i_1}, X_{i_2}, \dots, X_{i_q}\}$, where $X_{i_1} \in U/IND(R_1), X_{i_2} \in U/IND(R_1), \dots, X_{i_q} \in U/IND(R_1)$. Supposing only one equivalence class is subdivided into two finer equivalence classes, without loss of generality, let $X_{i_1} = X'_{i_1} \cup X''_{i_1}$. Based on the definition of DqII-DTRS model, we can get $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} \geq \alpha) \wedge (|X_{i_1} \cap X| > k)$. There are also 16 situations regard to X'_{i_1} and X''_{i_1} (See Table 3).

As $B^{II,R_2}_{(\alpha,\beta,k)}(X) = U^{II,R_2}_{(\alpha,\beta,k)}(X) \cup L^{I,R_2}_{(\alpha,\beta,k)}(X)$, and all equivalence classes except X_{i_1} $(X_{i_1} \subseteq P^{I,R_1}_{(\alpha,\beta,k)}(X))$ keep unchanged, so we can get the following results:

(1) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X)$; (2) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$;

(3) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$; (4) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X)$;

(5) The same to the case in (2); (6) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X'_{i_1} \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$;

(7) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X'_{i_1} \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$; (8) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X'_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$;

(9) The same to the case in (3); (10) The same to (7); (11) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X'_{i_1} \cup X''_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$;

(12) $B^{I,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{I,R_1}_{(\alpha,\beta,k)}(X) \cup X'_{i_1} = B^{I,R_2}_{(\alpha,\beta,k)}(X)$; (13) The same to the case in (4);

(14) The same to (8); (15) The same to (12); (16) $B^{II,R_2}_{(\alpha,\beta,k)}(X) = B^{II,R_1}_{(\alpha,\beta,k)}(X)$.

From the above 16 cases, we can obtain that $B^{II,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{II,R_2}_{(\alpha,\beta,k)}(X)$, which means $B^{II,R_1}_{(\alpha,\beta,k)}(X) \subsetneq B^{II,R_2}_{(\alpha,\beta,k)}(X)$ or $B^{II,R_1}_{(\alpha,\beta,k)}(X) = B^{II,R_2}_{(\alpha,\beta,k)}(X)$. According to the Definition 5.2,

ΔR is an error-correction incremental information or useless incremental information. □

In Theorem 5.5, if the attribute increment ΔR only distinguish the elements in the positive region in DqI-DTRS, there are three possible directions for the reclassified elements: (1) upper boundary region; (2) lower boundary region; and (3) negative region. So the boundary region becomes larger or unchanged. Which means ΔR is the error-correction incremental information or useless incremental information in DqII-DTRS.

Theorem 5.6 *Given an information system $S = (U, A)$ with a pair of thresholds α, β $(0 \le \beta < \alpha \le 1)$ and the grade k, $R_1 \subseteq R_2 \subseteq A$. For any target concept $X \subseteq U$, if the attribute increment $\Delta R = R_2 - R_1$ only can distinguish the elements in the negative region in DqII-DTRS model, then attribute increment ΔR is the error-correction incremental information or useless incremental information.*

Proof Let $U/IND(R_1) = \{[x]_{R_1} \mid x \in U\} = \{X_1, X_2, \dots, X_l\}$ and $B^{II,R_1}_{(\alpha,\beta,k)}(X) = U^{II,R_1}_{(\alpha,\beta,k)}(X) \cup L^{II,R_1}_{(\alpha,\beta,k)}(X) = \cup\{X_{i_1}, X_{i_2}, \dots, X_{i_r}\}$, where $X_{i_1} \in U/IND(R_1), X_{i_2} \in U/IND(R_1), \dots, X_{i_r} \in U/IND(R_1)$. Supposing only one equivalence class is subdivided into two finer equivalence classes, without loss of generality, let $X_{i_1} = X'_{i_1} \cup X''_{i_1}$. Based on the definition of DqII-DTRS model, we can get $(\frac{|X_{i_1} \cap X|}{|X_{i_1}|} < \alpha) \wedge (|X_{i_1} \cap X| \le k)$. There are also 16 situations regard to X'_{i_1} and X''_{i_1} (See Table 3).

As $B^{IIR_2}_{(\alpha,\beta,k)}(X) = U^{II,R_2}_{(\alpha,\beta,k)}(X) \cup L^{I,R_2}_{(\alpha,\beta,k)}(X)$, and all equivalence classes except X_{i_1} $(X_{i_1} \subseteq N^{II,R_1}_{(\alpha,\beta,k)}(X))$ keep unchanged, so we can get the same results as the 16 situations in Theorem 5.5. We can obtain that $B^{II,R_1}_{(\alpha,\beta,k)}(X) \subseteq B^{II,R_2}_{(\alpha,\beta,k)}(X)$, which means $B^{II,R_1}_{(\alpha,\beta,k)}(X) \subsetneqq B^{II,R_2}_{(\alpha,\beta,k)}(X)$ or $B^{II,R_1}_{(\alpha,\beta,k)}(X) = B^{II,R_2}_{(\alpha,\beta,k)}(X)$. According to the Definition 5.2, ΔR is an error-correction incremental information or useless incremental information. □

In Theorem 5.6, if the attribute increment ΔR only distinguish the elements in the negative region in DqI-DTRS, there are three possible directions for the reclassified elements: (1) upper boundary region; (2) lower boundary region; and (3) positive region. So the boundary region becomes larger or unchanged. Which means ΔR is the error-correction incremental information or useless incremental information in DqII-DTRS.

From the above Theorems 5.1-5.6, we can see that (1) if the attribute increment ΔR can only distinguish the objects in the positive region or negative region in both DqI-DTRS and DqII-DTRS, then ΔR is error-correction incremental information or useless incremental information, while this kind of attribute increment is useless for further classification in Pawlak rough set model; (2) if the attribute increment ΔR can only distinguish the objects in boundary region in both DqI-DTRS and DqII-DTRS, then ΔR is useful incremental information or useless incremental information.

5.3 An illustrative example for incremental information in Dq-DTRS

Example 5.1 Table 4 is an information system $S = (U, A)$, where U is a universe of discourse which consists of 18 patients with the clinical features, the attributes a, b, c, d and e are *Cough*, *Rhinorrhoea*, *Myodynia*, *Diarrhea* and *Nausea*, respectively. Consider a target concept $X = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{14}\}$, which represents the initial diagnosis of each patient suffering from a cold. Suppose $R_1 = \{a, b\}$, $R_2 = \{a, b, c\}$, $R_3 = \{a, b, d\}$ and $R_4 = \{a, b, e\}$ are four attribute subsets of A.

From the Table 4, we can easily obtain the partitions as follows,

Table 4 Information table $S = (U, A)$

U	a	b	c	d	e	U	a	b	c	d	e
x_1	2	2	3	1	0	x_{10}	1	0	2	2	1
x_2	2	2	0	1	0	x_{11}	1	0	2	3	1
x_3	2	2	3	1	0	x_{12}	1	0	2	2	1
x_4	2	2	0	1	0	x_{13}	1	0	2	2	1
x_5	2	2	0	1	0	x_{14}	0	3	1	2	0
x_6	3	1	2	0	2	x_{15}	0	3	1	2	3
x_7	3	1	2	0	2	x_{16}	0	3	1	2	0
x_8	1	0	2	3	1	x_{17}	0	3	1	2	3
x_9	1	0	2	3	1	x_{18}	0	3	1	2	3

$$U/IND(R_1) = \{\{x_1, x_2, x_3, x_4, x_5\},\ \{x_6, x_7\},\ \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},\ \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\}\},$$
$$U/IND(R_2) = \{\{x_1, x_3\},\ \{x_2, x_4, x_5\},\ \{x_6, x_7\},\ \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},\ \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\}\},$$
$$U/IND(R_3) = \{\{x_1, x_2, x_3, x_4, x_5\},\ \{x_6, x_7\},\ \{x_8, x_9, x_{11}\},\ \{x_{10}, x_{12}, x_{13}\},\ \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\}\},$$
$$U/IND(R_4) = \{\{x_1, x_2, x_3, x_4, x_5\},\ \{x_6, x_7\},\ \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},\ \{x_{14}, x_{16}\},\ \{x_{15}, x_{17}, x_{18}\}\}.$$

It is obvious that $U/IND(R_2) \preceq U/IND(R_1)$, $U/IND(R_3) \preceq U/IND(R_1)$, and $U/IND(R_4) \preceq U/IND(R_1)$. Given two parameters ($\alpha = 0.60, \beta = 0.30$) and the grade $k = 2$. We can obtain upper and lower approximations of DqI-DTRS and DqII-DTRS models with respect to $R_1,\ R_2,\ R_3$ and R_4 in the following, respectively.

For the DqI-DTRS model with respect to R_1:

$$\overline{R_1}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$
$$\underline{R_1}_k(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}.$$

The four disjoint regions of X with respect to R_1 in DqI-DTRS which are

$$P^{I,R_1}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\};$$
$$N^{I,R_1}_{(\alpha,\beta,k)}(X) = \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{I,R_1}_{(\alpha,\beta,k)}(X) = \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$L^{I,R_1}_{(\alpha,\beta,k)}(X) = \emptyset.$$

For the DqI-DTRS model with respect to R_2:

$$\overline{R_2}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$
$$\underline{R_2}_k(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}.$$

The four disjoint regions of X with respect to R_2 in DqI-DTRS which are

$$P^{I,R_2}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\};$$
$$N^{I,R_2}_{(\alpha,\beta,k)}(X) = \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{I,R_2}_{(\alpha,\beta,k)}(X) = \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$L^{I,R_2}_{(\alpha,\beta,k)}(X) = \emptyset.$$

For the DqI-DTRS model with respect to R_3:

$$\overline{R_3}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$
$$\underline{R_3}_k(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\}.$$

The four disjoint regions of X with respect to R_3 in DqI-DTRS which are

$$P^{I,R_3}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$N^{I,R_3}_{(\alpha,\beta,k)}(X) = \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{I,R_3}_{(\alpha,\beta,k)}(X) = \emptyset;$$
$$L^{I,R_3}_{(\alpha,\beta,k)}(X) = \emptyset.$$

For the DqI-DTRS model with respect to R_4:

$$\overline{R_4}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{16}\},$$
$$\underline{R_4}_k(X) = \{x_1, x_2, x_3, x_4, x_5, x_{14}, x_{16}\}.$$

The four disjoint regions of X with respect to R_4 in DqI-DTRS which are

$$P^{I,R_4}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_{14}, x_{16}\};$$
$$N^{I,R_4}_{(\alpha,\beta,k)}(X) = \{x_{15}, x_{17}, x_{18}\};$$
$$U^{I,R_4}_{(\alpha,\beta,k)}(X) = \{x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$L^{I,R_4}_{(\alpha,\beta,k)}(X) = \emptyset.$$

It can be seen from the disjoint regions of X in DqI-DTRS with respect to R_1, R_2, R_3 and R_4:

- The boundary region of R_2 is equal to the boundary region of R_1 when the equivalence classes are subdivided into many finer equivalence classes, which including both upper boundary region and lower boundary region. The positive and negative regions of R_2 are equal to ones of R_1. Then the attribute increment $\Delta R = R_2 - R_1 = \{c\}$ is useless incremental information.
- Compared with regions of R_1, the boundary region becomes smaller and the positive region becomes much bigger of R_3 when the equivalence classes contained in the boundary region are subdivided into many finer equivalence classes. Then the attribute increment $\Delta R = R_3 - R_1 = \{d\}$ is useful incremental information.

- Compared with regions of R_1, the boundary region becomes bigger, the negative region becomes smaller of R_4. The positive regions between R_1 and R_4 cannot be compared. Then the attribute increment $\Delta R = R_4 - R_1 = \{e\}$ is error-correction incremental information.

For the DqII-DTRS model with respect to R_1:

$$\overline{R_1}_k(X) = \{x_1, x_2, x_3, x_4, x_5, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$
$$\underline{R_1}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}.$$

The four disjoint regions of X with respect to R_1 in DqII-DTRS which are

$$P^{II,R_1}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5\};$$
$$N^{II,R_1}_{(\alpha,\beta,k)}(X) = \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{II,R_1}_{(\alpha,\beta,k)}(X) = \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$L^{II,R_1}_{(\alpha,\beta,k)}(X) = \{x_6, x_7\}.$$

For the DqII-DTRS model with respect to R_2:

$$\overline{R_2}_k(X) = \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$
$$\underline{R_2}_{(\alpha,\beta)}(X) = \{x_2, x_4, x_5, x_6, x_7\}.$$

The four disjoint regions of X with respect to R_2 in DqII-DTRS which are

$$P^{II,R_2}_{(\alpha,\beta,k)}(X) = \emptyset;$$
$$N^{II,R_2}_{(\alpha,\beta,k)}(X) = \{x_1, x_3, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{II,R_2}_{(\alpha,\beta,k)}(X) = \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$L^{II,R_2}_{(\alpha,\beta,k)}(X) = \{x_2, x_4, x_5, x_6, x_7\}.$$

For the DqII-DTRS model with respect to R_2:

$$\overline{R_3}_k(X) = \{x_1, x_2, x_3, x_4, x_5\},$$
$$\underline{R_3}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}\}.$$

The four disjoint regions of X with respect to R_3 in DqII-DTRS which are

$$P^{II,R_3}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5\};$$
$$N^{II,R_3}_{(\alpha,\beta,k)}(X) = \{x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{II,R_3}_{(\alpha,\beta,k)}(X) = \emptyset;$$
$$L^{II,R_3}_{(\alpha,\beta,k)}(X) = \{x_6, x_7, x_8, x_9, x_{11}\}.$$

For the DqII-DTRS model with respect to R_2:

$$\overline{R_4}_k(X) = \{x_1, x_2, x_3, x_4, x_5, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\},$$
$$\underline{R_4}_{(\alpha,\beta)}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}.$$

The four disjoint regions of X with respect to R_4 in DqII-DTRS which are

$$P^{II,R_4}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3, x_4, x_5\};$$
$$N^{II,R_4}_{(\alpha,\beta,k)}(X) = \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}\};$$
$$U^{II,R_4}_{(\alpha,\beta,k)}(X) = \{x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}\};$$
$$L^{II,R_4}_{(\alpha,\beta,k)}(X) = \{x_6, x_7\}.$$

It can be seen from the disjoint regions of X in DqII-DTRS with respect to R_1, R_2, R_3 and R_4:

- Compared with regions of R_1, both the boundary region and negative region of R_2 become much bigger, and the positive region of R_2 becomes smaller. Then the attribute increment $\Delta R = R_2 - R_1 = \{c\}$ is error-correction incremental information.
- Compared with regions of R_1, the boundary region of R_3 becomes smaller and the negative region of R_3 becomes much bigger when the equivalence classes contained in the boundary region are subdivided into many finer equivalence classes. And the positive region of R_3 is equal to the one of R_1. Then the attribute increment $\Delta R = R_3 - R_1 = \{d\}$ is useful incremental information.
- The boundary region of R_4 is equal to the boundary region of R_1 when the equivalence classes are subdivided into many finer equivalence classes, which including both upper boundary region and lower boundary region. The positive and negative regions of R_4 are equal to ones of R_1. Then the attribute increment $\Delta R = R_4 - R_1 = \{e\}$ is useless incremental information.

The comparisons on disjoint regions changed with incremental attributes among Pawlak rough set, DTRS, Dq-DTRS (including DqI-DTRS and DqII-DTRS) are shown in Table 5. Based on the analysis in reference [68], the attribute increments $\{b\}$ and $\{c\}$ are attributed to useless incremental information, and the attribute increment $\{d\}$ is attributed to useful incremental information in Pawlak rough set model; the attribute increments $\{b\}$ and $\{d\}$ are attributed to error-correction incremental information, and the attribute increment $\{c\}$ is attributed to useful incremental information in DTRS model. In DqI-DTRS model, $\{b\}$, $\{c\}$ and $\{d\}$ are attributed to useless incremental information, useful incremental information and error-correction incremental information, respectively. In DqII-DTRS model, $\{b\}$, $\{c\}$ and

Table 5 Comparisons on regions changed with attribute increment

		Pawlak rough set				DTRS				Dq-DTRS							
										I				II			
		R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4	R_1	R_2	R_3	R_4
	$P(X)$	$x_{6,7}$	$x_{6,7}$	$x_{6,7}$	$x_{6,7}$	x_{1-7}	x_{2-7}	$x_{1-9,11}$	x_{1-7}	x_{1-7}	x_{1-7}	x_{1-13}	$x_{1-5,14,16}$	x_{1-5}	$\emptyset$	x_{1-5}	x_{1-5}
	$N(X)$	$\emptyset$	$\emptyset$	$\emptyset$	$x_{15,17,18}$	x_{14-18}	x_{14-18}	x_{14-18}	$x_{15,17,18}$	x_{14-18}	x_{14-18}	x_{14-18}	$x_{15,17,18}$	x_{14-18}	$x_{1,3,14-18}$	$x_{10,12-18}$	x_{14-18}
$B(X)$	$U(X)$	$x_{1-5,8-18}$	$x_{1-5,8-18}$	$x_{1-5,8-18}$	$x_{1-5,8-14,16}$	x_{8-13}	$x_{1,8-13}$	$x_{10,12,13}$	$x_{10,12-14,16}$	x_{8-13}	x_{8-13}	$\emptyset$	x_{6-13}	x_{8-13}	x_{8-13}	$\emptyset$	x_{8-13}
	$L(X)$									$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$x_{6,7}$	x_{2-7}	x_{8-13}	$x_{6,7}$

$\{d\}$ are attributed to error-correction incremental information, useful incremental information and useless incremental information, respectively.

6 Conclusions

In this paper, we make further studies of Dq-DTRS models by discussing the uncertainty of disjoint regions and also by exploring the disjoint regions changing with attribute increment. The research on uncertainty in rough set theory plays an important role in the knowledge acquisition and approximate reasoning. The superiority of Dq-DTRS model for the decision-making applications has been explained in the previous work [23]. It shows that although Dq-DTRS model is a directional expansion of Pawlak rough set with double quantification of the relative information and absolute information, it is very different from Pawlak rough set in many aspects, such as: the formation of disjoint regions and the extraction of decision rules. So far, there are many typical measure methods for the Pawlak rough set and PRS [2, 4, 9, 12, 28, 68], but there are few studies conducted on the uncertainty measure for Dq-DTRS model. Among these typical measure methods, Zhang et al. [68] presented a novel uncertainty measure for the PRS model from three regions and defined three kinds of incremental information for the PRS model and its generalizations, which provides us a research direction to study uncertainty measure for the Dq-DTRS model. We know that for the different generalized rough set model, even the same measure method will have different properties and presentation characteristics. This paper mainly analyzes the fuzziness of four disjoint regions in Dq-DTRS and addresses the change of uncertainty of four regions along with the change of parameters and the grade, and then the change regularities of disjoint regions in Dq-DTRS with changing approximation spaces are studied correspondingly. In the future work, how to use the fuzziness and three kinds of classification of incremental information to explore the context of attribute reduction and how to embody the two kinds of quantitative information in the fuzziness of four disjoint regions are worth investigating.

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