Granular Computing
https://doi.org/10.1007/s41066-018-0139-9

ORIGINAL PAPER

Probabilistic decision making based on rough sets in interval-valued fuzzy information systems

Derong Shi[1,2] · Xiaoyan Zhang[1,2]

Received: 30 January 2018 / Accepted: 8 October 2018

Abstract
At present, the representative and hot research is three-way decision based on rough set theory. In addition, this topic has been applied in wide variety of specific. In the article, we aim to discuss the rough set method of decision theory in the background of interval-valued fuzzy information systems (IVFIS). First, the main work is to transform the IVFIS into two kinds of approximate spaces by the defined relations, which are fuzzy approximation space and interval-valued fuzzy approximation space, respectively. Simultaneously, fuzzy probability and IVF probability are considered in the whole process. Second, we study two kinds of decision-theoretic rough set methods by combining the Bayesian decision process. Finally, based on the above decision-making models, some illustrative examples about the credit evaluation of enterprises are introduced to deal with the real value and interval-valued data. These results show that the rough set method of decision theory we proposed has wider applications than decision-theoretic rough sets (DTRS).

Keywords Interval-valued fuzzy information system · Fuzzy approximate space · IVF Fuzzy approximate space · The rough set method of decision theory

1 Introduction

Rough set (Pawlak 1982) was proposed by Poland mathematician Pawlak, which is a theory to deal with imprecise and incomplete data. Rough set (Mandal and Ranadive 2018a, b; Agbodah 2018) theory is an indispensable mathematical tool in the field of data mining and decision theory. Compared with the classical set theory (Liang et al. 2012), the Pawlak rough set theory does not need whatever prior knowledge about information systems, for instance, the membership functions and probability distribution of the fuzzy set. Pawlak described the uncertainty of data sets through the upper and lower approximations of basic knowledge, which derived the concept of classification or decision rule on the basis of the indistinguishability of the theory of objects. As a stretch of the Pawlak's rough set model (Pawlak 1997), the decision theory rough set (DTRS) model (Xu and Li 2013, 2016; Li et al. 2014, 2016; Dai et al. 2016; Pedrycz and Chen 2011, 2015a, b; Qian et al. 2006, 2010, 2014) has been widely used in the uncertain analysis of data. As an extensive use of rough set model, DTRS not only provides an explanation of the probabilistic RST (PRST), but also develops a generalized PRST model to deal with uncertainty problem. With the help of Bayesian decision procedures (Li and Zhou 2011; Fang and Hu 2016), Yao proposed DTRS (2004). In the 1990s, the concept of DTRS includes positive rules, boundary rules, and negative rules, namely, three-way decision making. Positive rules determine acceptance, negative rules make reject and boundary domain rules that determine the non-constraint (extension). At present, three-way decision theories are applied to a wider range of fields, such as e-mail filtering, information filtering, text classification cation, clustering analysis, investment decision making, government decision making, network support system (Li et al. 2015, 2017; Pawlak 1997; Huang et al. 2017; Hao et al. 2017) and so on. The two primary coverages of DTRS are the conditional probability and thresholds. Thresholds of DTRS are controlled by loss functions.

The application of rough set theory has been researched by many scholars in the literatures. The loss function of rough set is a new research direction. The general loss function

✉ Xiaoyan Zhang
zxy19790915@163.com

1 School of Sciences, Chongqing University of Technology, Chongqing 400054, People's Republic of China

2 School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

Published online: 16 October 2018

(Lin et al. 2013, 2015; Fan et al. 2016) of supervised leaning was defined by Miao and Li. Li and zhou (Lu et al. 2016) put forward two hypotheses for the value of loss, and put forward a multi angle DTRS decision-making model. Li et al. assume that the general wastage in the classification includes the test cost and the wrong classification cost, and the algorithm is designed to find the best test attribute set at the minimum total cost (Li et al. 2012). Lingras et al. proposed a cluster based quality target according to DTRS by taking into account all kinds of losses (Lingras et al. 2009). The experiments demonstrated the effects of cluster quantification target. The revenue function and cost function are used to measure the loss function, and a profit based three element investment decision method is further proposed (Liu et al. 2011a, b). Liu et al. suggest that someone can use the object directly (such as money, energy, and time) to assess the loss or conduct some questionnaire or behavioral experiments (Liu et al. 2011a, b). On the basis of multi-agent decision preference, Yang et al. discussed some loss sets of multi-agent DTRS model (Yang and Yao 2012; Yang et al. 2013). Yao et al. proposed the threshold with the relative value of the loss and reduces the number of thresholds (2011). According to the loss of DTRS, a new clustering validity evaluation function is constructed, considering the uncertainty of inaccuracy. Liu and Liang researched the rough set theory of decision theory with interval-valued loss function and triangular fuzzy loss function, respectively (Liang and Liu 2014; Liang et al. 2013). Chen et al. used the knowledge of interval-valued fuzzy sets to deal with different applications (Chen et al. 1997; Chen and Hsiao 2000; Mondal et al. 2017; Mandal and Ranadive 2018a, b). At the same time, they put forward interval-valued decision-theoretic rough set (IVDTRS) and triangular fuzzy decision-theoretic rough set (TFDTRS) (Yao 2004; Lu et al. 2016; Berg et al. 2013; Burillo and Bustince 1996; Deng and Yao 2014). In the context of shade sets (Pedrycz 2013), Yao and Deng presented the three-way decision-theoretic approximations of fuzzy sets. Zhao and Hu (2015); Fang and Hu (2016) proposed the probability graded rough set (PGRS), which takes into account the relative quantitative information about basic concepts and knowledge particles. Tan et al. (2015); Xu and Yu (2017); Sahin and Bay (2001) discussed the generalization of Pawlak rough set and gradient rough set (GRS). In approximate space, the elements of different equivalence classes may differ greatly from each other, so that different degrees of information can be represented (Wang et al. 2016, 2017; Xu et al. 2011). The absolute quantity information of intersection of object set and equivalence class are related to GRS model. In GRS model, the intersection between basic set and equivalence class is measured by parameters. Xu brought forward generalized multi-granulation double-quantitative (Kabaila 2013; Casella et al. 1993; Allen 1990) decision-theoretic rough set. In this paper, the classical multi-granularity is extended to the generalized multi-granularity, and the upper and lower approximations are described by the degree and precision. The traditional rough set (Lin et al. 2012) method mainly deals with discrete attribute values. However, in many practical problems, due to the complexity of the problem, the attribute values in information systems are often continuous. Thus, it is necessary to discretize the real data in the range of continuous attribute values, and factually, its essence is to divide the attribute range into several discrete intervals. Therefore, knowledge acquisition and decision-making methods of IVISs are worthy of study. On the basis of fuzzy and IVF probabilities, the extension of DTRS method was put forward by Zhao and Hu (2015); Dai et al. (2016). In interval-valued approximation space, Bayesian decision theory was used to establish the interval-valued fuzzy probability decision theory model. However, Hu directly gave the similar relation between fuzzy and interval-valued fuzzy approximation spaces. The main dedications of this paper are as below: fuzzy equivalence relation and interval-valued fuzzy equivalence relation of interval-valued fuzzy information system are proposed. Furthermore, fuzzy probability approximation space and interval-valued fuzzy probability approximation space are established, respectively. By means of rough set theory, the fuzzy probability approximation space and interval-valued fuzzy probability approximation space are analyzed.

The structure of this article is as follows: in Sect. 2, we review some basic concepts such as fuzzy events, fuzzy probability and interval-valued fuzzy sets, etc. By defining a similar relation among objects, IVFIS is converted into a fuzzy approximation space. According to the fuzzy probability, fuzzy DTRS method is discussed in Sect. 3. In addition, in Sect. 4, we translated IVFIS into an IVF approximation space, and then, IVF DTRS method is established in our work. At the same time, we construct a real-life example to explain and illustrate decision-making method. Finally, some conclusions are summarized in Sect. 5.

2 Related work

For more convenience, the basic concepts of fuzzy event, interval-valued fuzzy information system etc. are reviewed in this section.

Let U be a domain and a fuzzy set $\tilde{A}$, which is seen as a mapping from U to [0, 1]. We denote all the fuzzy sets on U as $\mathcal{F}(U)$. For all $\forall o \in U$, we define the basic operation of fuzzy sets as follows (Zhang et al. 2013):

$$(\widetilde{AB})(o) = \tilde{A}(o)\tilde{B}(o), \quad \tilde{A}^c(o) = 1 - \tilde{A}(o). \tag{1}$$

Let U be object set, range of object set V, $\tilde{R}$:$U \times V \to [0, 1]$. Therefore, binary relation $\tilde{R}$ is described as a fuzzy relation from U to V. If $U = V$, then $\tilde{R}$ is a fuzzy relation on

U, and we denote all the fuzzy binary relation on $U \times U$ as $R \in \mathcal{F}(U \times U)$.

Next, we give the relevant properties of fuzzy binary relation:

1. $\tilde{R}$ is reflexive, if $\tilde{R}(o,o) = 1$ for $\forall o \in U$.
2. $\tilde{R}$ is symmetric, if $\tilde{R}(o,y) = \tilde{R}(y,o)$ for $\forall o, y \in U$.
3. $\tilde{R}$ is transitive, if $\tilde{R}(o,y) \geq \vee_{z \in U}(\tilde{R}(o,z) \wedge \tilde{R}(z,y))$ for $\forall o, y \in U$.

Definition 2.1 (Zadeh 1968) Let $(U, \mathcal{A}, P)$ be a probability space, where $\mathcal{A}$ is $\sigma-$ universes composed of fuzzy subsets on U. If a fuzzy subset $\tilde{A} = \tilde{A}(o)$ is a random variable, then $\tilde{A}$ is also a fuzzy event on U. The probability of $\tilde{A}$ is

$$P(\tilde{A}) \triangleq \int_U \tilde{A}(o)\, dP. \tag{2}$$

If U is a finite set, $U = \{o_i | i = 1, 2, \dots, n\}$, $P(o_i) = p_i$, then

$$P(\tilde{A}) \triangleq \sum_{i=1}^{n} \tilde{A}(o_i) p_i. \tag{3}$$

Definition 2.2 (Zadeh 1968) Let $(U, \mathcal{A}, P)$ be a probability space, two fuzzy set $\tilde{A}$ and $\tilde{B}$ on U. In addition, content $P(\tilde{B}) \neq 0$, then the conditional probability of A given B is defined as follows:

$$P(\tilde{A}|\tilde{B}) = \frac{P(\widetilde{AB})}{P(\tilde{B})}. \tag{4}$$

Proposition 2.3 (Zhao and Hu 2015) *Let $(U, \mathcal{A}, P)$ be a probability space, A is a classical event on U. For each fuzzy event $\tilde{B}$ on U, we have*

$$P(A|\tilde{B}) + P(A^c|\tilde{B}) = 1. \tag{5}$$

Suppose that U be a finite universe, $I[0,1] = \{[a^-, a^+] | 0 \leq a^- \leq a^+ \leq 1\}$, then interval-valued fuzzy(IVF) set is a mapping $\tilde{A}: U \to I[0,1]$ on U. From this, for all $o \in U$, $\tilde{A}(o) = [\tilde{A}^-(o), \tilde{A}^+(o)]$, then $\tilde{A}(o)$ is an interval number. A^-, A^+ are fuzzy sets on U and $\tilde{A}^-(O) \leq \tilde{A}^+(O)$. For $\forall o \in U$, we define the product and complement of fuzzy sets as follows:

$$(\widetilde{AB})(o) = \tilde{A}(o)\tilde{B}(o) = [\tilde{A}^-(o)\tilde{B}^-(o), \tilde{A}^+(o)\tilde{B}^+(o)],$$
$$\tilde{A}^c(o) = [1 - \tilde{A}^+(o), 1 - \tilde{A}^-(o)].$$

An IVF relation $\tilde{R}$ is a mapping from $U \times V$ to $I[0,1]$. If $U = V$, then $\tilde{R}$ is an IVF relation on U. Next, we give the relevant properties of IVF binary relation:

1. $\tilde{R}$ is reflexive if $\tilde{R}(o,o) = 1$ for $\forall o \in U$.
2. $\tilde{R}$ is symmetric if $\tilde{R}(o,y) = \tilde{R}(y,o)$ for $\forall o, y \in U$.
3. $\tilde{R}$ is transitive if $\tilde{R}(o,y) \geq \vee_{z \in U}(\tilde{R}(o,z) \wedge \tilde{R}(z,y))$ for $\forall o, y \in U$.

Now, the rules of two interval numbers $[a^-, a^+], [b^-, b^+]$ are given as follows:

$$[a^-, a^+] \pm [b^-, b^+] = [a^- \pm b^-, a^+ \pm b^+]$$
$$[a^-, a^+] \times [b^-, b^+] = [a^- b^-, a^+ b^+]$$

and if $b^- \neq 0$, the division is

$$\frac{[a^-, a^+]}{[b^-, b^+]} = \left[\frac{a^-}{b^-} \wedge \frac{a^+}{b^+}, \frac{a^-}{b^-} \vee \frac{a^+}{b^+}\right]$$
$$[a^-, a^+] \leq [b^-, b^+] = [a^- \leq b^-, a^+ \leq b^+].$$

Here is a customary notation, and notation $\times$ is omitted in result.

Definition 2.4 (Zhao and Hu 2015) Let $(U, \mathcal{A}, P)$ be a probability space. If

$$\tilde{A} \in I_{[0,1]}(\mathcal{A}) = \{\tilde{A} \in I_{[0,1]}(U) : \tilde{A} = [\tilde{A}^-, \tilde{A}^+]\} \tag{6}$$

then $\tilde{A}$ is an IVF event on U. The IVF probability of $\tilde{A}$ is

$$P(\tilde{A}) \triangleq \int_U \tilde{A}(o)\, dP = \left[\int_U \tilde{A}^-(o)\, dP, \int_U \tilde{A}^+(o)\, dP\right]$$
$$= [P(\tilde{A}^-), P(\tilde{A}^+)].$$

If U is a finite set, $U = \{o_i | i = 1, 2, \dots, n\}$, $P(o_i) = p_i$, then

$$P(\tilde{A}) \triangleq \sum_{i=1}^{n} \tilde{A}(o_i) p_i = \left[\sum_{i=1}^{n} \tilde{A}^-(o_i) p_i, \quad \sum_{i=1}^{n} \tilde{A}^+(o_i) p_i\right]. \tag{7}$$

Definition 2.5 Let $(U, \mathcal{A}, P)$ be a probability space, and we give two IVF events $\tilde{A}, \tilde{B}$ on U. If $P(\tilde{B}^-) \neq 0$, then the conditional probability of A given B is

$$P(\tilde{A}|\tilde{B}) = \frac{P(\widetilde{AB})}{P(\tilde{B})}. \tag{8}$$

Proposition 2.6 *Let $(U, \mathcal{A}, P)$ be a probability space. Then, for $\forall \tilde{A}, \tilde{B} \in I_{[0,1]}(\mathcal{A})$, it satisfies*

$$P(\tilde{A}|\tilde{B}) = \left[P(\tilde{A}^-|\tilde{B}^-) \wedge P(\tilde{A}^+|\tilde{B}^+), P(\tilde{A}^-|\tilde{B}^-) \vee P(\tilde{A}^+|\tilde{B}^+)\right].$$

According to this definition

$$P(\tilde{A}^c|\tilde{B}) = \left[1 - P(\tilde{A}^-|\tilde{B}^-) \vee P(\tilde{A}^+|\tilde{B}^+), 1 - P(\tilde{A}^-|\tilde{B}^-) \right.$$
$$\left. \wedge P(\tilde{A}^+|\tilde{B}^+)\right].$$

An interval-valued fuzzy information system(IVFIS) is an ordered quadruple $I = (U, AT, V, F)$, where U and AT are all non-empty finite sets; $V = \cup_{a\in AT} V_a$ and V_a is a domain of attribute a; $F = \{f : U \to V\}$ are mapping sets of object attribute value, in which $f : U \times AT \to V$ is a function such that $f(o, a) \in V_a$, for each $a \in AT, o \in U$, called an information function, and V_a is an IVF set of universe U. That is $f(o, a) = [a^L(o), a^U(o)]$ for all $a \in AT$, where $a^L(o) : U \to [0, 1]$ and $a^U(o) : U \to [0, 1]$ and satisfy $a^L(o) \le a^U(o)$ $(\forall o \in U)$. Especially, when $a^L(o) = a^U(o)$, $f(o, a)$ degenerates into a real number.

Decision-theoretic rough sets were first proposed by Yao for the Bayesian decision process. On the basis of the thoughts of three-way decisions, DTRS adopts two state sets and three action sets to depict the decision-making process. The state set is denoted by $\Omega = \{O, O^C\}$ showing that an object is part of O and is outside O, respectively. The action sets with respect to a state are given by $A = \{a_P, a_B, a_N\}$, where a_P, a_B and a_N represent three actions about deciding $o \in POS(O)$, $o \in BND(O)$, and $o \in NEG(O)$, namely, a target x belongs to O, is uncertain and not in O, respectively. The loss function concerning the loss of expected is given by the 3×2 matrix by taking various actions in the different states in Table 1.

In Table 1, λ_{PP}, λ_{BP}, and λ_{NP} express the cost happened when taking actions of a_P, a_B and a_N and an target is part of O, respectively. Similarly, λ_{PN}, λ_{BN}, and λ_{NN} indicate the cost turned up for taking previous actions when the target does is not part of O. For a target o, the expected cost on taking the actions could be expressed as

$$R(a_P|[o]_R) = \lambda_{PP}P(O|[o]_R) + \lambda_{PN}P(O^c|[o]_R); \tag{9}$$

$$R(a_B|[o]_R) = \lambda_{BP}P(O|[o]_R) + \lambda_{BN}P(O^c|[o]_R); \tag{10}$$

$$R(a_N|[o]_R) = \lambda_{NP}P(O|[o]_R) + \lambda_{NN}P(O^c|[o]_R). \tag{11}$$

By the Bayesian decision process, we can get the following minimum-risk decision rules:

(P) If o satisfied $R(a_P|[o]_R) \le R(a_B|[o]_R)$ and $R(a_P|[o]_R) \le R(a_N|[o]_R)$, then $o \in POS(O)$.
(B) If o satisfied $R(a_B|[o]_R) \le R(a_P|[o]_R)$ and $R(a_B|[o]_R) \le R(a_N|[o]_R)$, then $o \in BND(O)$.
(N) If o satisfied $R(a_N|[o]_R) \le R(a_P|[o]_R)$ and $R(a_N|[o]_R) \le R(a_B|[o]_R)$, then $o \in NEG(O)$.

Table 1 Cost function

	O: positive	O^c: negative
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

In addition, by taking into account, the loss of receiving the right things is not greater than the latency, and both of them are less than the loss of refusing the accurate things; at the same time, the loss of rejecting improper things is less than or equal to the deletion in accepting the correct things, and both shall be smaller than the loss of receiving the invalidate things. Hence, a reasonable assumption is that $0 \le \lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $0 \le \lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$.

Accordingly, the conditions of the three decision rules (P)–(N) are reducible to the following form.

(P) If o satisfied $P(O|[o]_R) \ge \alpha$ and $P(O|[o]_R) \ge \gamma$, then $o \in POS(O)$.
(B) If o satisfied $P(O|[o]_R) \le \alpha$ and $P(O|[o]_R) \ge \beta$, then $o \in BND(O)$.
(N) If o satisfied $P(O|[o]_R) \ge \beta$ and $P(O|[o]_R) \le \gamma$, then $o \in NEG(O)$.

Where the thresholds values are given by

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}; \tag{12}$$

$$\beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}; \tag{13}$$

$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}. \tag{14}$$

3 Fuzzy probability decision-theoretic rough set based on IVFIS

Since the definition of similarity given in the literature (Vanderpooten 2000), the calculation results exist a big difference when dealing with the cardinality of intervals for real numbers in the process of various index data types. Under this circumstance, it is easy to cause the loss of decision information leading to the false decision making finally. According to the decision-making problem of uncertain multi-attribute with value as interval numbers, this paper introduces a new definition of relative similarity degree of interval numbers starting from the basic ideas of the advantages and disadvantages on the decision scheme. The new method makes it more convenient to describe the links and differences between the decision-making program, which overcomes the shortcomings of the literature (Vanderpooten 2000) on the definition of interval number similarity.

Definition 3.1 Let $I = (U, AT, V, F)$ be an IVFIS, $\forall a_k \in AT$, $o_i, o_j \in U$, $f(o_i, a_k) = [a_k^L(o_i), a_k^U(o_i)]$, $f(o_j, a_k) = [a_k^L(o_j), a_k^U(o_j)]$, then

$$s_{a_k}(o_i,o_j)=\exp\left\{-\frac{1}{2}[|a_k^L(o_i)-a_k^L(o_j)|+|a_k^U(o_i)-a_k^U(o_j)|]\right\} \tag{15}$$

is called the relative similarity degree of $f(o_i,a_k)$ and $f(o_j,a_k)$. From the above formula, the relative similarity degree of $f(o_i,a_k)$ and $f(o_j,a_k)$, that is the similarity of objects o_i and o_j under the attribute a_k. In addition, the greater the value of $s_{a_k}(o_i,o_j)$, the greater the similarity degree. Especially, when $s_{a_k}(o_i,o_j)=1$, then interval number $f(o_i,a_k)$ is completely similar to $f(o_j,a_k)$. In other words, the property value of objects o_i and o_j is identical.

Theorem 3.2 *For any three intervals* $\widetilde{x}=[o^L,o^U]$, $\widetilde{y}=[y^L,y^U]$, *and* $\widetilde{z}=[z^L,z^U]$, *the following properties can be obtained.*

1. $s(\widetilde{o},\widetilde{y})$ *is bounded*, $0\le s(\widetilde{o},\widetilde{y})\le 1$.
2. $s(\widetilde{o},\widetilde{y})$ *is reflexive*, $s(\widetilde{o},\widetilde{o})=1$.
3. $s(\widetilde{o},\widetilde{y})$ *is symmetric*, $s(\widetilde{o},\widetilde{y})=s(\widetilde{y},\widetilde{o})$.
4. $s(\widetilde{o},\widetilde{y})$ *is transitive; if* $s(\widetilde{o},\widetilde{y})=1$ *and* $s(\widetilde{y},\widetilde{z})=1$, *then* $s(\widetilde{o},\widetilde{z})=1$.
5. $s(\widetilde{o},\widetilde{y})$ *is contiguous; if* $\widetilde{z}$ *is closer to* $\widetilde{y}$ *than* $\widetilde{o}$, *then* $s(\widetilde{o},\widetilde{y})\le s(\widetilde{z},\widetilde{y})$; *if* $\widetilde{z}$ *is closer to* $\widetilde{o}$ *than* $\widetilde{y}$, *then* $s(\widetilde{o},\widetilde{y})\le s(\widetilde{o},\widetilde{z})$.

Definition 3.3 Let $I=(U,AT,V,F)$ be an IVFIS, $\forall a_k\in AT$, $o_i,o_j\in U$, $f(o_i,a_k)=[a_k^L(o_i),a_k^U(o_i)]$, $f(o_j,a_k)=[a_k^L(o_j),a_k^U(o_j)]$. The similarity degree of objects o_i and o_j under attribute set AT is as follows:

$$R_{AT}(o_i,o_j)=\exp\left\{-\frac{1}{2}\left[\sum_{k=1}^{p}|a_k^L(o_i)-a_k^L(o_j)|+\sum_{k=1}^{p}|a_k^U(o_i)-a_k^U(o_j)|\right]\right\}. \tag{16}$$

Through establishing analogical relations R_{AT}, we could turn IVFIS into a fuzzy approximation space (U, R) in accordance with Definitions 3.1 and 3.2. The subscript AT will be omitted in the rear. Therefore, we can get the following properties.

(a) First, U is a non-empty classical set, a binary relation R from U to U indicates a fuzzy set $R: U\times U\to[0,1]$. Therefore, R is a fuzzy relation.
(b) Second, R is a fuzzy equivalence relation on U. The reasons are as follows:

- $\forall o\in U$, $R(o,o)=1$, R is reflexive;
- $\forall o,y\in U$, $R(o,y)=R(y,o)$, R is symmetric;
- $\forall o,y,z\in U$, $R(o,y)\wedge R(y,z)\le R(o,z)$, R is transitive.

From the above, we can find directly that these three conditions are true. Therefore, ordered pair (U, R) is a fuzzy approximation space.

Given the probability P with its description R, a fuzzy probability approximation space (U, R, P) is constructed, where U be a universe, R be a fuzzy equivalence relation, and P be a fuzzy probability.

Given a fuzzy probability approximation space (U, R, P), $\forall o\in U$, $[o]_R$ is denoted by $[o]_R(y)=R(o,y)$ for any $y\in U$. Thus, the expected costs of adopting various actions in different states for o are expressed as follows:

$$\widetilde{R}(a_P|[o]_R)=\lambda_{PP}\widetilde{P}(O|[o]_R)+\lambda_{PN}\widetilde{P}(O^c|[o]_R); \tag{17}$$

$$\widetilde{R}(a_B|[o]_R)=\lambda_{BP}\widetilde{P}(O|[o]_R)+\lambda_{BN}\widetilde{P}(O^c|[o]_R); \tag{18}$$

$$\widetilde{R}(a_N|[o]_R)=\lambda_{NP}\widetilde{P}(O|[o]_R)+\lambda_{NN}\widetilde{P}(O^c|[o]_R). \tag{19}$$

Theorem 3.4 (Zhao and Hu 2015) *The condition probability* $\widetilde{P}(O|[o]_R)$ *and* $\widetilde{P}(O^c|[o]_R)$ *are calculated by*

$$\widetilde{P}(O|[o]_R)=\frac{\sum_{o_i\in O}R(o,o_i)p_i}{\sum_{o_j\in U}R(o,o_j)p_j}, \tag{20}$$

$$\widetilde{P}(O^c|[o]_R)=\frac{\sum_{o_i\in O^c}R(o,o_i)p_i}{\sum_{o_j\in U}R(o,o_j)p_j}, \tag{21}$$

vhere $p_i=P(o_i)$. *The computing method of condition probability is not the same as we know before.*

Since $\widetilde{P}(O|[o]_R)+\widetilde{P}(O^c|[o]_R)=1$ for every $o\in U$. It is further expressed as

$$\widetilde{R}(a_P|[o]_R)=\lambda_{PP}\widetilde{P}(O|[o]_R)+\lambda_{PN}(1-\widetilde{P}(O|[o]_R)); \tag{22}$$

$$\widetilde{R}(a_B|[o]_R)=\lambda_{BP}\widetilde{P}(O|[o]_R)+\lambda_{BN}(1-\widetilde{P}(O|[o]_R)); \tag{23}$$

$$\widetilde{R}(a_N|[o]_R)=\lambda_{NP}\widetilde{P}(O|[o]_R)+\lambda_{NN}(1-\widetilde{P}(O|[o]_R)). \tag{24}$$

According to Bayesian decision process, the decision rules (P)–(N) in Sect. 2 can be rewritten as the following form:

(P_1) If o satisfied $\widetilde{R}(a_P|[o]_R)\le\widetilde{R}(a_B|[o]_R)$ and $\widetilde{R}(a_P|[o]_R)\le\widetilde{R}(a_N|[o]_R)$, then $o\in POS(O)$.
(B_1) If o satisfied $\widetilde{R}(a_B|[o]_R)\le\widetilde{R}(a_P|[o]_R)$ and $\widetilde{R}(a_B|[o]_R)\le\widetilde{R}(a_N|[o]_R)$, then $o\in BND(O)$.
(N_1) If o satisfied $\widetilde{R}(a_N|[o]_R)\le\widetilde{R}(a_P|[o]_R)$ and $\widetilde{R}(a_N|[o]_R)\le\widetilde{R}(a_B|[o]_R)$, then $o\in NEG(O)$.

The decision rules (P_1)–(N_1) are the three-way decisions, which have three regions: $POS(O)$, $BND(O)$ and $NEG(O)$. These rules mainly rely on the comparisons among $\widetilde{R}(a_P|[o]_R)$, $\widetilde{R}(a_B|[o]_R)$ and $\widetilde{R}(a_N|[o]_R)$ which are essentially computing the fuzzy probabilities.

Decision rules (P_1)–(N_1) of three-way decisions can be simplified as

(P_2) If o satisfied $\widetilde{P}(O|[o]_R) \geq \alpha$ and $\widetilde{P}(O|[o]_R) \geq \gamma$, then $o \in POS(O)$.
(B_2) If o satisfied $\widetilde{P}(O|[o]_R) < \alpha$ and $\widetilde{P}(O|[o]_R) > \beta$, then $o \in BND(O)$.
(N_2) If o satisfied $\widetilde{P}(O|[o]_R) < \gamma$ and $\widetilde{P}(O|[o]_R) \leq \beta$, then $o \in NEG(O)$.

Threshold α, β, γ are showed in the previous equation.

The additional conditions of decision rule (B_2) need that $\beta < \alpha$, namely, it follows that $0 \leq \beta < \gamma < \alpha \leq 1$. The above rules are simplified as

(P_3) If o satisfied $\widetilde{P}(O|[o]_R) \geq \alpha$, then $o \in POS(O)$.
(B_3) If o satisfied $\beta < \widetilde{P}(O|[o]_R) < \alpha$, then $o \in BND(O)$.
(N_3) If o satisfied $\widetilde{P}(O|[o]_R) \leq \beta$, then $o \in NEG(O)$.

It can be seen from the above rules, and γ is not needed any more.

Definition 3.5 According to the rules from (P_3) to (N_3), the α-lower and β-upper approximations are defined in IVFIS as follows:

$$\underline{R}^{\alpha}(O) = \{o \in U : \widetilde{P}(O|[o]_R) \geq \alpha\}, \tag{25}$$

$$\overline{R}^{\beta}(O) = \{o \in U : \widetilde{P}(O|[o]_R) > \beta\}. \tag{26}$$

Moreover, we have the positive region, boundary region, and negative region of IVFIS as follows:

$$POS^{\alpha}(O) = \{o \in U : \widetilde{P}(O|[o]_R) \geq \alpha\}; \tag{27}$$

$$NEG^{\beta}(O) = \{o \in U : \widetilde{P}(O|[o]_R) \leq \beta\}; \tag{28}$$

$$BND^{(\alpha,\beta)}(O) = \{o \in U : \beta < \widetilde{P}(O|[o]_R) < \alpha\}. \tag{29}$$

Obviously, the relationships between the approximation operator and the decision domain are as follows:

$$\underline{R}^{\alpha}(O) = POS^{\alpha}(O), \overline{R}^{\beta}(O) = (NEG^{\beta}(O))^c.$$

The order pair $(\underline{R}^{\alpha}(O), \overline{R}^{\beta}(O))$ is termed (α, β)-fuzzy probabilistic rough set of O on IVFIS.

According to DTRS, suppose the loss function satisfies $0 \leq \lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}, 0 \leq \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$ and $(\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}) \leq (\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN})$, then we can get $0 \leq \beta < \gamma < \alpha \leq 1$. There are three cases.

Case 1 When $\alpha + \beta = 1$, the loss function must satisfies $(\lambda_{BP} - \lambda_{PP})(\lambda_{NP} - \lambda_{BP}) = (\lambda_{PN} - \lambda_{BN})(\lambda_{BN} - \lambda_{NN})$.

Case 2 When $\alpha + \beta < 1$, the loss function must satisfies $(\lambda_{BP} - \lambda_{PP})(\lambda_{NP} - \lambda_{BP}) > (\lambda_{PN} - \lambda_{BN})(\lambda_{BN} - \lambda_{NN})$.

Case 3 When $\alpha + \beta > 1$, the loss function must satisfies $(\lambda_{BP} - \lambda_{PP})(\lambda_{NP} - \lambda_{BP}) < (\lambda_{PN} - \lambda_{BN})(\lambda_{BN} - \lambda_{NN})$.

Case analysis In the next, let us consider an issue of enterprise credit evaluation. The evaluation indexes are the solvency, operation ability, profit ability, development prospect and innovation ability of the enterprise. Table 2 gives an IVFIS on credit evaluation for ten enterprises by some experts, where $U = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}$ indicates ten enterprisers, $A = \{thesolvency, operationability, profitability, developmentprospect, innovationabilityoftheenterprise\}$ represents condition-attributes set, respectively. For the sake of simplicity, using a_1, a_2, a_3, a_4, a_5 to take the place of indexes.

It can be converted into a fuzzy approximation space. The details are exhibited in Table 3. It is not difficult for us to see that Table 3 shows a fuzzy equivalence relation(reflexive, symmetric and transitive) among ten classes (because of symmetric, we just give the lower triangle of the Table 3).

Table 2 IVFIS on the enterprises' credit evaluation

U	a_1	a_2	a_3	a_4	a_5
o_1	[0.69, 0.91]	[0.64, 0.66]	[0.01, 0.86]	[0.18, 0.26]	[0.18, 0.53]
o_2	[0.51, 0.70]	[0.17, 0.22]	[0.13, 0.57]	[0.42, 0.62]	[0.09, 0.14]
o_3	[0.60, 0.91]	[0.57, 0.98]	[0.01, 0.46]	[0.50, 0.71]	[0.69, 0.99]
o_4	[0.26, 0.55]	[0.05, 0.25]	[0.34, 0.83]	[0.47, 0.99]	[0.55, 0.63]
o_5	[0.55, 0.75]	[0.09, 0.41]	[0.49, 0.85]	[0.59, 0.63]	[0.05, 0.48]
o_6	[0.12, 0.67]	[0.03, 0.92]	[0.57, 0.68]	[0.26, 0.43]	[0.78, 0.92]
o_7	[0.39, 0.39]	[0.66, 0.95]	[0.56, 0.80]	[0.89, 0.92]	[0.77, 0.79]
o_8	[0.55, 0.71]	[0.06, 0.06]	[0.00, 0.76]	[0.42, 0.98]	[0.07, 0.12]
o_9	[0.24, 0.51]	[0.07, 0.63]	[0.61, 0.67]	[0.05, 0.08]	[0.17, 0.95]
o_{10}	[0.33, 0.49]	[0.37, 0.66]	[0.46, 0.89]	[0.25, 0.68]	[0.10, 0.46]

Table 3 Fuzzy equivalence relation on U

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9	o_{10}
o_1	1.0000									
o_2	0.2478	1.0000								
o_3	0.2698	0.1911	1.0000							
o_4	0.1620	0.3027	0.1773	1.0000						
o_5	0.2794	0.4584	0.1782	0.3697	1.0000					
o_6	0.1604	0.1628	0.2491	0.2698	0.2198	1.0000				
o_7	0.1381	0.1103	0.2698	0.2567	0.1854	0.2725	1.0000			
o_8	0.2112	0.5945	0.1473	0.3482	0.3867	0.1206	0.1070	1.0000		
o_9	0.2254	0.1873	0.1313	0.2322	0.2503	0.3946	0.1496	0.1360	1.0000	
o_{10}	0.3379	0.3263	0.1746	0.3345	0.4630	0.2579	0.2454	0.2491	0.3379	1.0000

Table 4 Three cases of loss function 1

	O	O^c	O	O^c	O	O^c	O	O^c
a_P	0.20	0.90	0.00	0.22	0.00	0.13	0.00	0.36
a_B	0.54	0.50	0.12	0.04	0.03	0.06	0.08	0.04
a_N	0.89	0.20	0.18	0.00	0.17	0.00	0.24	0.00

Now, let us suppose that the preference probability distribution on U is $\{0.15, 0.08, 0.1, 0.07, 0.11, 0.1, 0.16, 0.04, 0.05, 0.14\}$. Let $O = \{o_1, o_3, o_6, o_7, o_{10}\}$ represents a reliable decision class of the enterprise. In the Bayes decision process, some experts will provide values of the loss function for O, i.e., $\lambda_{iP} = \lambda(a_i|O), \lambda_{iN} = \lambda(a_i|O^c), i = P, B, N$.

In addition, for $\forall o_i \in U$, according to the formula (20) and (21), the fuzzy conditional probabilities are computed as follows:

$$\widetilde{P}(O|[o_1]_R) = 0.76, \widetilde{P}(O|[o_2]_R) = 0.42, \widetilde{P}(O|[o_3]_R) = 0.80,$$
$$\widetilde{P}(O|[o_4]_R) = 0.49, \widetilde{P}(O|[o_5]_R) = 0.47,$$
$$\widetilde{P}(O|[o_6]_R) = 0.74, \widetilde{P}(O|[o_7]_R) = 0.82, \widetilde{P}(O|[o_8]_R) = 0.41,$$
$$\widetilde{P}(O|[o_9]_R) = 0.58, \widetilde{P}(O|[o_{10}]_R) = 0.68.$$

Case 1 Take the value of the loss function 1 in Table 4 into the (12)–(14) of the threshold α, β, γ, we can get $\alpha_1 = 0.54$, $\beta_1 = 0.46$; $\alpha_2 = 0.6$, $\beta_2 = 0.4$; $\alpha_3 = 0.7$, $\beta_3 = 0.3$; $\alpha_4 = 0.8$, $\beta_4 = 0.2$. Obviously, $\alpha_i + \beta_i = 1 (i = 1, 2, 3, 4)$. When $\alpha + \beta = 1$, according to (25) and (26), we can get

$$\underline{R}^{0.54}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$$
$$\overline{R}^{0.46}(O) = \{o_1, o_3, o_4, o_5, o_6, o_7, o_9, o_{10}\}.$$
$$\underline{R}^{0.6}(O) = \{o_1, o_3, o_6, o_7, o_{10}\},$$
$$\overline{R}^{0.4}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$
$$\underline{R}^{0.7}(O) = \{o_1, o_3, o_6, o_7\},$$
$$\overline{R}^{0.3}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$
$$\underline{R}^{0.8}(O) = \{o_3, o_7\},$$
$$\overline{R}^{0.2}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$

and

$$POS^{0.54}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$$
$$NEG^{0.46}(O) = \{o_2, o_8\},$$
$$BND^{(0.54, 0.46)}(O) = \{o_4, o_5\}.$$
$$POS^{0.6}(O) = \{o_1, o_3, o_6, o_7, o_{10}\},$$
$$NEG^{0.4}(O) = \emptyset,$$
$$BND^{(0.6, 0.4)}(O) = \{o_2, o_4, o_5, o_8, o_9\}.$$
$$POS^{0.7}(O) = \{o_1, o_3, o_6, o_7\},$$
$$NEG^{0.3}(O) = \emptyset,$$
$$BND^{(0.7, 0.3)}(O) = \{o_2, o_4, o_5, o_8, o_9, o_{10}\}.$$
$$POS^{0.8}(O) = \{o_3, o_7\},$$
$$NEG^{0.2}(O) = \emptyset,$$
$$BND^{(0.8, 0.2)}(O) = \{o_1, o_2, o_4, o_5, o_6, o_8, o_9, o_{10}\}.$$

Case 2 Take the value of the loss function 2 in Table 5 into the calculation formula of the threshold α, β, γ,we can get $\alpha_1 = 0.56$, $\beta_1 = 0.42$; $\alpha_2 = 0.6$, $\beta_2 = 0.3$; $\alpha_3 = 0.7$, $\beta_3 = 0.2$; $\alpha_4 = 0.8$, $\beta_4 = 0.1$. Obviously, $\alpha_i + \beta_i < 1 (i = 1, 2, 3, 4)$. When $\alpha + \beta < 1$, according to the definition of the α-lower and β-upper approximation, we can get

$$\underline{R}^{0.56}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$$
$$\overline{R}^{0.42}(O) = \{o_1, o_3, o_4, o_5, o_6, o_7, o_9, o_{10}\}.$$
$$\underline{R}^{0.6}(O) = \{o_1, o_3, o_6, o_7, o_{10}\},$$
$$\overline{R}^{0.3}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$
$$\underline{R}^{0.7}(O) = \{o_1, o_3, o_6, o_7\},$$
$$\overline{R}^{0.2}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$
$$\underline{R}^{0.8}(O) = \{o_3, o_7\},$$
$$\overline{R}^{0.1}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$

Table 5 Three cases of loss function 2

	O	O^c	O	O^c	O	O^c	O	O^c
a_P	0.10	0.80	0.00	0.09	0.00	0.19	0.00	0.18
a_B	0.34	0.50	0.02	0.06	0.06	0.05	0.04	0.02
a_N	0.62	0.30	0.16	0.00	0.26	0.00	0.22	0.00

and

$POS^{0.56}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$
$NEG^{0.42}(O) = \{o_2, o_8\},$
$BND^{(0.56,0.42)}(O) = \{o_4, o_5\}.$
$POS^{0.6}(O) = \{o_1, o_3, o_6, o_7, o_{10}\},$
$NEG^{0.3}(O) = \emptyset,$
$BND^{(0.6,0.3)}(O) = \{o_2, o_4, o_5, o_8, o_9\}.$
$POS^{0.7}(O) = \{o_1, o_3, o_6, o_7\},$
$NEG^{0.2}(O) = \emptyset,$
$BND^{(0.7,0.2)}(O) = \{o_2, o_4, o_5, o_8, o_9, o_{10}\}.$
$POS^{0.8}(O) = \{o_3, o_7\},$
$NEG^{0.1}(O) = \emptyset,$
$BND^{(0.8,0.1)}(O) = \{o_1, o_2, o_4, o_5, o_6, o_8, o_9, o_{10}\}.$

Case 3 Take the value of the loss function 3 in Table 6 into the calculation formula of the threshold α, β, γ,we can get $\alpha_1 = 0.73, \beta_1 = 0.47$;$\alpha_2 = 0.6, \beta_2 = 0.5$; $\alpha_3 = 0.7, \beta_3 = 0.4$; $\alpha_4 = 0.8, \beta_4 = 0.3$. Obviously, $\alpha_i + \beta_i > 1 (i = 1, 2, 3, 4)$. When $\alpha + \beta > 1$, according to the definition of the α-lower and β-upper approximation,we can get

$\underline{R}^{0.73}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$
$\overline{R}^{0.47}(O) = \{o_1, o_3, o_4, o_6, o_7, o_9, o_{10}\}.$
$\underline{R}^{0.6}(O) = \{o_1, o_3, o_6, o_7, o_{10}\},$
$\overline{R}^{0.5}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\}.$
$\underline{R}^{0.7}(O) = \{o_1, o_3, o_6, o_7\},$
$\overline{R}^{0.4}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$
$\underline{R}^{0.8}(O) = \{o_3, o_7\},$
$\overline{R}^{0.3}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$

and

$POS^{0.73}(O) = \{o_1, o_3, o_6, o_7\},$
$NEG^{0.47}(O) = \{o_2, o_5, o_8\},$
$BND^{(0.73,0.47)}(O) = \{o_4, o_9, o_{10}\}.$
$POS^{0.6}(O) = \{o_1, o_3, o_6, o_7, o_{10}\},$
$NEG^{0.5}(O) = \{o_2, o_4, o_5, o_8,$
$BND^{(0.6,0.5)}(O) = \{o_9\}.$
$POS^{0.7}(O) = \{o_1, o_3, o_6, o_7\},$
$NEG^{0.4}(O) = \emptyset,$
$BND^{(0.7,0.4)}(O) = \{o_2, o_4, o_5, o_8, o_9, o_{10}\}.$
$POS^{0.8}(O) = \{o_3, o_7\},$
$NEG^{0.3}(O) = \emptyset,$
$BND^{(0.8,0.3)}(O) = \{o_1, o_2, o_4, o_5, o_6, o_8, o_9, o_{10}\}.$

4 IVF probability decision-theoretic rough set based on IVFIS

In the third part, we have considered an IVFIS which is transformed into a real-valued fuzzy approximation space. However, the expression of a constant is often imprecise in practical life. Therefore, in this section we will convert an IVFIS into an IVF probabilistic approximation space. Then, we will put forward the DTRS method under the background of IVF probabilistic approximation space.

Definition 4.1 Let $I = (U, AT, V, F)$ be an IVFIS, $\forall a_k \in AT$, $o_i, o_j \in U$, $f(o_i, a_k) = [a_k^L(o_i), a_k^U(o_i)]$, $f(o_j, a_k) = [a_k^L(o_j), a_k^U(o_j)]$, then

$$r_{ij}^- = \exp\left\{ -\frac{1}{2} \left[\sum_{k=1}^{p} |a_k^L(o_i) - a_k^L(o_j)| + \sum_{k=1}^{p} |a_k^U(o_i) - a_k^U(o_j)| \right] \right\}; \quad (30)$$

Table 6 Three cases of loss function 3

	O	O^c	O	O^c	O	O^c	O	O^c
a_P	0.40	0.48	0.00	0.13	0.00	0.13	0.00	0.19
a_B	0.43	0.40	0.06	0.04	0.03	0.06	0.04	0.03
a_N	0.72	0.14	0.10	0.00	0.12	0.00	0.11	0.00

$$r_{ij}^{+} = \exp\left\{-\frac{1}{2}\left|\sum_{k=1}^{p}|a_k^L(o_i) - a_k^L(o_j)| - \sum_{k=1}^{p}|a_k^U(o_i) - a_k^U(o_j)|\right|\right\}. \tag{31}$$

The interval-valued fuzzy relation R could be obtained on U with the above formula. As follows:

$$\widetilde{R} = \{r_{ij} : i,j \le n\}, \quad r_{ij} = [r_{ij}^{-}, r_{ij}^{+}] \in [I]. \tag{32}$$

Based on the above definition, we establish interval-valued approximation space $(U, \widetilde{R})$. Clearly, $\widetilde{R}$ is an interval-valued fuzzy relation that satisfies reflexive and symmetric.

Definition 4.2 (Pawlak 1982) Let $\widetilde{R} = \{r_{ij} : i,j \le n\}, r_{ij} \in [I]$ represent the value of interval-valued fuzzy relation $\widetilde{R}$ of the ith object and the jth object on the domain U, $\widetilde{R}\circ\widetilde{R}$ is defined as follows:

$$\widetilde{R}^2 = \widetilde{R}\circ\widetilde{R} = \{t_{ij} : i,j \le n\}, \quad t_{ij} = \vee_{k=1}^{n}(r_{ik} \wedge r_{kj}) \tag{33}$$

$\widetilde{R}\circ\widetilde{R}$ is called fuzzy compound calculation on $\widetilde{R}$.

Especially, $(U, \widetilde{R}^2)$ is obviously an interval-valued approximation space.

Theorem 4.3 *Assume that $\widetilde{R} \subseteq F(U \times U)$, then $\widetilde{R}$ has the following properties*:

1. *$\widetilde{R}$ is reflexive, $\widetilde{R}^m$ is also reflexive*;
2. *$\widetilde{R}$ is symmetric, $\widetilde{R}^m$ is also symmetric*.

Proof Below, we apply the mathematical induction method to prove the theorem.

1. If m = 2, because $\widetilde{R}$ is reflexive, so $\widetilde{R}(o, o) = 1$. In addition, $\widetilde{R}^2 = \widetilde{R}\circ\widetilde{R} = \{t_{ij} : i,j \le n\}, t_{ij} = \vee_{k=1}^{n}(r_{ik} \wedge r_{kj})$. Therefore, $\widetilde{R}^2(o, o) = (\widetilde{R}\circ\widetilde{R})(o, o) = \vee_{o\in O}(\widetilde{R}(o, o) \wedge \widetilde{R}(o, o))$ $= 1$. Now, assume $\widetilde{R}^{m-1}(o, o) = 1$, there have $\widetilde{R}^m(o, o) = \vee_{o\in O}(\widetilde{R}(o, o)^{m-1} \wedge \widetilde{R}(o, o)) = 1$. Therefore, $\widetilde{R}^m$ is also reflexive.
2. If m = 2, because $\widetilde{R}$ is symmetric, so $\widetilde{R}(o, y) = \widetilde{R}(y, o)$. In addition, $\widetilde{R}^2 = \widetilde{R}\circ\widetilde{R} = \{t_{ij} : i,j \le n\}, t_{ij} = \vee_{k=1}^{n}(r_{ik} \wedge r_{kj})$. Therefore, $\widetilde{R}^2(o, y) = (\widetilde{R}\circ\widetilde{R})(o, y) = \vee_{z\in O}(\widetilde{R}(o, z) \wedge \widetilde{R}(z, y)) = \vee_{z\in O}(\widetilde{R}(y, z) \wedge \widetilde{R}(z, o)) = \widetilde{R}^2(y, o)$. Now, assume $\widetilde{R}^{m-1}(o, y) = \widetilde{R}^{m-1}(y, o)$, there have $\widetilde{R}^m(o, y) = \vee_{z\in O}(\widetilde{R}^{m-1}(o, z) \wedge \widetilde{R}(z, y)) = \vee_{z\in O}(\widetilde{R}(y, z) \wedge \widetilde{R}^{m-1}(z, o)) = \widetilde{R}^m(y, o)$. Therefore, $\widetilde{R}^m$ is also symmetric.

□

From the above, it can conclude that $\widetilde{R}$ is an interval-valued fuzzy similarity relation. Now, the problem is to make decision for the objects with similar relation. Then, the next work is to transform the interval-valued fuzzy similarity relation into interval-valued fuzzy equivalence relation. Now, defining $\widetilde{R}^{m+1} = \widetilde{R}^m\circ\widetilde{R}$, when the interval-valued fuzzy relation R does not satisfy transitivity, then there exists $m \in N$, such that $R = \widetilde{R}^m$, and R is transitive.

Theorem 4.4 *Let $(U, \widetilde{R})$ be an interval-valued approximation space, where interval-valued fuzzy relation $\widetilde{R}$ is reflexive, symmetric, then $(U, \widetilde{R})(\widehat{R} = \widetilde{R}^m, m \in N, m \le |U|)$ must be an interval-valued fuzzy approximation space.*

Proof Because $\widetilde{R}$ is reflexive, symmetric, and $\widehat{R} = \widetilde{R}^m$, so $\widetilde{R}$ is transitive. Therefore, $(U, \widetilde{R})$ is an interval-valued fuzzy approximation space.

Now, given the probability P with its description R, then IVF probability approximation space (U, R, P) is constructed.

In the next, we will construct the basic model of IVFDTRS in accordance with the method in Sect. 2. The interval-value loss function for action cost under different conditions is shown in Table 5.

In Table 5, λ_{ij}^{-} and λ_{ij}^{+} are lower bound and upper bound of $\overline{\lambda}_{ij}(i,j = P, B, N)$. $\overline{\lambda}_{PP} = [\lambda_{PP}^{-}, \lambda_{PP}^{+}]$, $\overline{\lambda}_{BP} = [\lambda_{BP}^{-}, \lambda_{BP}^{+}]$, and $\overline{\lambda}_{NP} = [\lambda_{NP}^{-}, \lambda_{NP}^{+}]$ indicate the costs for taking actions of a_P, a_B, and a_N, respectively, when an element is in O. In other words, $\overline{\lambda}_{PN} = [\lambda_{PN}^{-}, \lambda_{PN}^{+}]$, $\overline{\lambda}_{BN} = [\lambda_{BN}^{-}, \lambda_{BN}^{+}]$ and $\overline{\lambda}_{NN} = [\lambda_{NN}^{-}, \lambda_{NN}^{+}]$ denote the losses for taking the same actions when an element belongs to O^c. On the basis of conditions in a reasonable assumption, a particular kind of loss function is considered:

$$\lambda_{PP}^{-} \le \lambda_{BP}^{-} < \lambda_{NP}^{-}, \lambda_{PP}^{+} \le \lambda_{BP}^{+} < \lambda_{NP}^{+},$$
$$\lambda_{NN}^{-} \le \lambda_{BN}^{-} < \lambda_{PN}^{-}, \lambda_{NN}^{+} \le \lambda_{BN}^{+} < \lambda_{PN}^{+}.$$

For a target o, the expected losses $\mathscr{R}(a_i|[x]_R)$ of each action are as follows:

$$\mathscr{R}(a_P|[o]_R) = \overline{\lambda}_{PP}\overline{P}(O|[o]_R) + \overline{\lambda}_{PN}\overline{P}(O^c|[o]_R); \tag{34}$$

$$\mathscr{R}(a_B|[o]_R) = \overline{\lambda}_{BP}\overline{P}(O|[o]_R) + \overline{\lambda}_{BN}\overline{P}(O^c|[o]_R); \tag{35}$$

$$\mathscr{R}(a_N|[o]_R) = \overline{\lambda}_{NP}\overline{P}(O|[o]_R) + \overline{\lambda}_{NN}\overline{P}(O^c|[o]_R). \tag{36}$$

Theorem 4.5 (Zhao and Hu 2015) *Given two IVF probabilities $\overline{P}(O|[o]_R)$ and $\overline{P}(O^c|[o]_R)$. If $O(O \subseteq U)$ is a classical event, then*

$$\overline{P}(O|[o]_R) = [p^- \wedge p^+, p^- \vee p^+]; \tag{37}$$

$$\overline{P}(O^c|[o]_R) = [1 - p^- \vee p^+, 1 - p^- \wedge p^+], \tag{38}$$

where $R = [R^-, R^+]$, $p^- = \overline{P}(O|[o]_{R^-})$ *and* $p^+ = \overline{P}(O|[o]_{R^+})$. *If* $U = \{o_1, o_2, \dots, o_n\}$ *and* $P(o_i) = p_i$, *then*

$$p^- = \frac{\sum_{o_i \in O} R^-(o, o_i) p_i}{\sum_{o_j \in U} R^-(o, o_j) p_j}, \tag{39}$$

$$p^+ = \frac{\sum_{o_i \in O^c} R^+(o, o_i) p_i}{\sum_{o_j \in U} R^+(o, o_j) p_j}. \tag{40}$$

Furthermore, if we take the value of the loss function in Table 7, then we can get

$$\begin{aligned}&\mathscr{R}(a_P|[o]_R)\\&\quad= [\lambda_{PP}^-, \lambda_{PP}^+]\overline{P}(O|[o]_R) + [\lambda_{PN}^-, \lambda_{PN}^+]\overline{P}(O^c|[o]_R);\end{aligned} \tag{41}$$

$$\begin{aligned}&\mathscr{R}(a_B|[o]_R)\\&\quad= [\lambda_{BP}^-, \lambda_{BP}^+]\overline{P}(O|[o]_R) + [\lambda_{BN}^-, \lambda_{BN}^+]\overline{P}(O^c|[o]_R);\end{aligned} \tag{42}$$

$$\begin{aligned}&\mathscr{R}(a_N|[o]_R)\\&\quad= [\lambda_{NP}^-, \lambda_{NP}^+]\overline{P}(O|[o]_R) + [\lambda_{NN}^-, \lambda_{NN}^+]\overline{P}(O^c|[o]_R).\end{aligned} \tag{43}$$

In light of Bayesian decision procedure, the decision rules $(P) - (N)$ in Sect. 2 could be re-expressed as follows:

(P') If o satisfy $\mathscr{R}(_P|[o]_R) \preceq \mathscr{R}(a_B|[o]_R)$ and $\mathscr{R}(a_P|[o]_R) \preceq \mathscr{R}(a_N|[o]_R)$, then $o \in POS(O)$.
(N') If o satisfy $\mathscr{R}(a_N|[o]_R) \preceq \mathscr{R}(a_B|[o]_R)$ and $\mathscr{R}(a_N|[o]_R) \preceq \mathscr{R}(a_P|[o]_R)$, then $o \in NEG(O)$
(B') If o satisfy neither (P') nor (N'), then $o \in BND(O)$.

Definition 4.6 Let $(U, \mathscr{R}, \mathscr{P})$ be an interval-valued fuzzy probabilistic approximation space. The loss function is the interval-value $[\overline{\lambda}]$. The $POS^{[\overline{\lambda}]}$, $BND^{[\overline{\lambda}]}$, $NED^{[\overline{\lambda}]}$ are expressed as follows:

$$\begin{aligned}POS^{[\overline{\lambda}]}(O) = \{o \in U | &\mathscr{R}(a_P|[o]_R) \preceq \mathscr{R}(a_B|[o]_R),\\ &\mathscr{R}(a_P|[o]_R) \preceq \mathscr{R}(a_N|[o]_R)\};\end{aligned} \tag{44}$$

$$\begin{aligned}NEG^{[\overline{\lambda}]}(O) = \{o \in U | &\mathscr{R}(a_N|[o]_R) \preceq \mathscr{R}(a_B|[o]_R),\\ &\mathscr{R}(a_N|[o]_R) \preceq \mathscr{R}(a_P|[o]_R)\};\end{aligned} \tag{45}$$

Table 7 Interval-valued loss function

	O : positive	O^c :negative
a_P	$\overline{\lambda}_{PP} = [\lambda_{PP}^-, \lambda_{PP}^+]$	$\overline{\lambda}_{PN} = [\lambda_{PN}^-, \lambda_{PN}^+]$
a_B	$\overline{\lambda}_{BP} = [\lambda_{BP}^-, \lambda_{BP}^+]$	$\overline{\lambda}_{BN} = [\lambda_{BN}^-, \lambda_{BN}^+]$
a_N	$\overline{\lambda}_{NP} = [\lambda_{NP}^-, \lambda_{NP}^+]$	$\overline{\lambda}_{NN} = [\lambda_{NN}^-, \lambda_{NN}^+]$

$$BND^{[\overline{\lambda}]}(O) = (POS^{[\overline{\lambda}]}(O) \cap NEG^{[\overline{\lambda}]}(O))^c. \tag{46}$$

For an interval-valued fuzzy relation R, the $[\overline{\lambda}]$-IVF probability lower approximation and the $[\overline{\lambda}]$-IVF probability upper approximation are, respectively:

$$\underline{\mathscr{R}}^{[\overline{\lambda}]}(O) = POS^{[\overline{\lambda}]}(O), \overline{\mathscr{R}}^{[\overline{\lambda}]}(O) = (NEG^{[\overline{\lambda}]}(O))^c. \tag{47}$$

The order pair $(\underline{\mathscr{R}}^{[\overline{\lambda}]}(O), \overline{\mathscr{R}}^{[\overline{\lambda}]}(O))$ is named the $[\overline{\lambda}]$-IVF probability rough set of O.

The decision rules (P')–(N') are the three-way decisions, which have three regions: $POS(O)$, $BND(O)$ and $NEG(O)$. These rules mainly rely on the comparisons among $\mathscr{R}(a_P|[o]_R)$, $\mathscr{R}(a_B|[o]_R)$ and $\mathscr{R}(a_N|[o]_R)$ which are essentially computing the IVF probabilities. Therefore, the conditions for calculating decision rules are as follows.

For the rule (P'):

$$\begin{aligned}&\mathscr{R}(a_P|[o]_R) \preceq \mathscr{R}(a_B|[o]_R)\\&\quad\Leftrightarrow \overline{P}(O|[o]_R) \succeq \frac{[\lambda_{PN}^- - \lambda_{BN}^-, \lambda_{PN}^+ - \lambda_{BN}^+]}{[\lambda_{BP}^- - \lambda_{PP}^-, \lambda_{BP}^+ - \lambda_{PP}^+]}\overline{P}(O^c|[o]_R)\\&\mathscr{R}(a_P|[o]_R) \preceq \mathscr{R}(a_N|[o]_R)\\&\quad\Leftrightarrow \overline{P}(O|[o]_R) \succeq \frac{[\lambda_{PN}^- - \lambda_{NN}^-, \lambda_{PN}^+ - \lambda_{NN}^+]}{[\lambda_{NP}^- - \lambda_{PP}^-, \lambda_{NP}^+ - \lambda_{PP}^+]}\overline{P}(O^c|[o]_R)\end{aligned}$$

For the rule (N'):

$$\begin{aligned}&\mathscr{R}(a_N|[o]_R) \preceq \mathscr{R}(a_P|[o]_R)\\&\quad\Leftrightarrow \overline{P}(O|[o]_R) \preceq \frac{[\lambda_{PN}^- - \lambda_{NN}^-, \lambda_{PN}^+ - \lambda_{NN}^+]}{[\lambda_{NP}^- - \lambda_{PP}^-, \lambda_{NP}^+ - \lambda_{PP}^+]}\overline{P}(O^c|[o]_R)\\&\mathscr{R}(a_N|[o]_R) \preceq \mathscr{R}(a_B|[o]_R)\\&\quad\Leftrightarrow \overline{P}(O|[o]_R) \preceq \frac{[\lambda_{BN}^- - \lambda_{NN}^-, \lambda_{BN}^+ - \lambda_{NN}^+]}{[\lambda_{NP}^- - \lambda_{BP}^-, \lambda_{NP}^+ - \lambda_{BP}^+]}\overline{P}(O^c|[o]_R)\end{aligned}$$

From the above, the decision rules $(P') - (N')$ can be rewritten as follows:

(P_1') If o satisfy

$$\overline{P}(O|[o]_R) \succeq \frac{[\lambda_{PN}^- - \lambda_{BN}^-, \lambda_{PN}^+ - \lambda_{BN}^+]}{[\lambda_{BP}^- - \lambda_{PP}^-, \lambda_{BP}^+ - \lambda_{PP}^+]}\overline{P}(O^c|[o]_R)$$

and

$$\overline{P}(O|[o]_R) \succeq \frac{[\lambda_{PN}^- - \lambda_{NN}^-, \lambda_{PN}^+ - \lambda_{NN}^+]}{[\lambda_{NP}^- - \lambda_{PP}^-, \lambda_{NP}^+ - \lambda_{PP}^+]}\overline{P}(O^c|[o]_R),$$

then $o \in POS(O)$.

(N_1') If o satisfy

$$\overline{P}(O|[o]_R) \preceq \frac{[\lambda^-_{PN} - \lambda^-_{NN}, \lambda^+_{PN} - \lambda^+_{NN}]}{[\lambda^-_{NP} - \lambda^-_{PP}, \lambda^+_{NP} - \lambda^+_{PP}]}\overline{P}(O^c|[o]_R)$$

and

$$\overline{P}(O|[o]_R) \preceq \frac{[\lambda^-_{BN} - \lambda^-_{NN}, \lambda^+_{BN} - \lambda^+_{NN}]}{[\lambda^-_{NP} - \lambda^-_{BP}, \lambda^+_{NP} - \lambda^+_{BP}]}\overline{P}(O^c|[o]_R),$$

then $o \in NEG(O)$.

(B'_1) If o satisfy neither (P'_1) nor (N'_1), then $o \in BND(O)$.

Therefore, from Bayesian decision procedure, the rules (P'_1) − (N'_1) could be expressed as follows:

(P'_2) If o satisfy

$$\overline{P}(O|[o]_R) \succeq [\alpha^-, \alpha^+]\overline{P}(O^c|[o]_R)$$

and

$$\overline{P}(O|[o]_R) \succeq [\gamma^-, \gamma^+]\overline{P}(O^c|[o]_R),$$

then (P'_2) $o \in POS(O)$.

(N'_2) If o satisfy

$$\overline{P}(O|[o]_R) \preceq [\gamma^-, \gamma^+]\overline{P}(O^c|[o]_R)$$

and

$$\overline{P}(O|[o]_R) \preceq [\beta^-, \beta^+]\overline{P}(O^c|[o]_R),$$

then $o \in NEG(O)$.

(B'_2) If o satisfy neither (P'_2) nor (N'_2), then $o \in BND(O)$.

In rules (P'_1) − (N'_1), the parameters are set as follows:

$$\alpha^- = \frac{\lambda^-_{PN} - \lambda^-_{BN}}{\lambda^-_{BP} - \lambda^-_{PP}} \wedge \frac{\lambda^+_{PN} - \lambda^+_{BN}}{\lambda^+_{BP} - \lambda^+_{PP}}, \tag{48}$$

$$\alpha^+ = \frac{\lambda^-_{PN} - \lambda^-_{BN}}{\lambda^-_{BP} - \lambda^-_{PP}} \vee \frac{\lambda^+_{PN} - \lambda^+_{BN}}{\lambda^+_{BP} - \lambda^+_{PP}} \tag{49}$$

$$\gamma^- = \frac{\lambda^-_{PN} - \lambda^-_{NN}}{\lambda^-_{NP} - \lambda^-_{PP}} \wedge \frac{\lambda^+_{PN} - \lambda^+_{NN}}{\lambda^+_{NP} - \lambda^+_{PP}}, \tag{50}$$

$$\gamma^+ = \frac{\lambda^-_{PN} - \lambda^-_{NN}}{\lambda^-_{NP} - \lambda^-_{PP}} \vee \frac{\lambda^+_{PN} - \lambda^+_{NN}}{\lambda^+_{NP} - \lambda^+_{PP}} \tag{51}$$

$$\beta^- = \frac{\lambda^-_{BN} - \lambda^-_{NN}}{\lambda^-_{NP} - \lambda^-_{BP}} \wedge \frac{\lambda^+_{BN} - \lambda^+_{NN}}{\lambda^+_{NP} - \lambda^+_{BP}}, \tag{52}$$

$$\beta^+ = \frac{\lambda^-_{BN} - \lambda^-_{NN}}{\lambda^-_{NP} - \lambda^-_{BP}} \vee \frac{\lambda^+_{BN} - \lambda^+_{NN}}{\lambda^+_{NP} - \lambda^+_{BP}} \tag{53}$$

Therefore, we can have the following property.

Theorem 4.7 *For convenience, if we briefly denote* $\alpha = [\alpha^-, \alpha^+], \beta = [\beta^-, \beta^+]$ *and* $\gamma = [\gamma^-, \gamma^+]$, *then IVF probability regions are simplified as follows*:

$$\begin{aligned} POS^{(\alpha,\gamma)}(O) = \{o \in U : \overline{P}(O|[o]_R) \succeq [\alpha^-, \alpha^+]\overline{P}(O^c|[o]_R),\\ \overline{P}(O|[o]_R) \succeq [\gamma^-, \gamma^+]\overline{P}(O^c|[o]_R); \end{aligned} \tag{54}$$

$$\begin{aligned} NEG^{(\gamma,\beta)}(O) = \{o \in U : \overline{P}(O|[o]_R) \preceq [\gamma^-, \gamma^+]\overline{P}(O^c|[o]_R),\\ \overline{P}(O|[o]_R) \preceq [\beta^-, \beta^+]\overline{P}(O^c|[o]_R); \end{aligned} \tag{55}$$

$$BND^{(\alpha,\gamma,\beta)}(O) = (POS^{((\alpha,\gamma))}(O) \cup NEG^{((\gamma,\beta))}(O))^c. \tag{56}$$

For the fuzzy relation R, the IVF probability upper approximation and the fuzzy probability lower approximation of O are, respectively:

$$\underline{\mathscr{R}}^{(\alpha,\gamma)}(O) = POS^{(\alpha,\gamma)}(O), \tag{57}$$

$$\overline{\mathscr{R}}^{(\gamma,\beta)}(O) = (NEG^{(\gamma,\beta)}(O))^c. \tag{58}$$

The order pair $(\underline{\mathscr{R}}^{(\alpha,\gamma)}(O), \overline{\mathscr{R}}^{(\gamma,\beta)}(O))$ is named (α, γ, β)-IVF probability rough set of O.

In reference [55], the additional conditions of decision rule (N'_2) suggest that $\beta < \alpha$, namely, it follows that $0 \le \beta < \gamma < \alpha \le 1$, and the rules are

(P'_3) If o satisfies $\overline{P}(O|[o]_R) \succeq [\alpha^-, \alpha^+]\overline{P}(O^c|[o]_R)$, then $o \in POS(O)$.

(N'_3) If o satisfies $\overline{P}(O|[o]_R) \preceq [\beta^-, \beta^+]\overline{P}(O^c|[o]_R)$, then $o \in NEG(O)$.

(B'_3) If o satisfies neither (P'_3) nor (N'_3), then $o \in BND(O)$.

Thus, we can have the following theorem.

Theorem 4.8 *The IVF probability regions are simplified as follows*:

$$\begin{aligned} &POS^{(\alpha)}(O)\\ &\quad = \{o \in U : \overline{P}(O|[o]_R) \succeq [\alpha^-, \alpha^+]\overline{P}(O^c|[o]_R)\},\\ &NEG^{(\beta)}(O)\\ &\quad = \{o \in U : \overline{P}(O|[o]_R) \preceq [\beta^-, \beta^+]\overline{P}(O^c|[o]_R)\},\\ &BND^{(\alpha,\beta)}(O) = (POS^{\alpha}(O) \cup NEG^{\beta}(O))^c. \end{aligned}$$

According to the IVF relation $\mathscr{R}$, the IVF probability lower approximation and the IVF probability upper approximation of O are shown as follows, respectively:

$$\underline{\mathscr{R}}^{(\alpha)}(O) = POS^{(\alpha)}(O), \overline{\mathscr{R}}^{(\beta)}(O) = (NEG^{(\beta)}(O))^c$$

The order pair $(\underline{\mathscr{R}}^{\alpha}(O), \overline{\mathscr{R}}^{\beta}(O))$ is named the (α, β)-IVF probability rough set of O.

According to decision-theoretic rough set, we suppose that the loss function satisfies $0 \leq \overline{\lambda}_{PP} \leq \overline{\lambda}_{BP} < \overline{\lambda}_{NP}$, $0 \leq \overline{\lambda}_{NN} \leq \overline{\lambda}_{BN} < \overline{\lambda}_{PN}$ and $(\overline{\lambda}_{BP} - \overline{\lambda}_{PP})(\overline{\lambda}_{BN} - \overline{\lambda}_{NN}) \leq (\overline{\lambda}_{NP} - \overline{\lambda}_{BP})(\overline{\lambda}_{PN} - \overline{\lambda}_{BN})$, then we can get $0 \leq \beta < \gamma < \alpha \leq 1$. Meanwhile, this paper also discusses different situations between the value of $\alpha + \beta$ and [1, 1].

Case study Now, let us continue to use case analysis 3.3 as the research object, and make the rough set theory of decision making under the interval-valued probability approximation space. On the basis of Table 2, the hypothesis $(U, \mathscr{R}, P)$ is an interval-valued fuzzy probability approximation space, including $U = \{o_1, o_2, \ldots, o_{10}\}$, R is an interval-valued fuzzy relation in Table 8.

We assume that the preference probability distribution on U is $p(o_1) = 0.15$, $p(o_2) = 0.08$, $p(o_3) = 0.10$, $p(o_4) = 0.07$, $p(o_5) = 0.11$, $p(o_6) = 0.10$, $p(o_7) = 0.16$, $p(o_8) = 0.04$, $p(o_9) = 0.05$, $p(o_{10}) = 0.14$. Let $O = \{o_1, o_3, o_6, o_7, o_{10}\}$ denotes a decision class in which the classes are excellent. Some experts will provide values of the loss function value for $O = \{o_1, o_3, o_6, o_7, o_{10}\}$, i.e., $\overline{\lambda}_{iP} = \overline{\lambda}(a_i|O), \overline{\lambda}_{iN} = \overline{\lambda}(a_i|O^c)$, $i = P, B, N$. It exhibits three cases in Table 7. Considering the loss function of Table 7, there are $\alpha = [1.9, 2.0]$, $\beta = [0.7, 1.2]$.

According to the formula (37)–(40), the interval-valued fuzzy conditional probabilities for every $o_i \in U$ are computed as follows:

$$\overline{P}(O|[o_1]_{\mathscr{R}}) = (0.43, 0.80),$$
$$\overline{P}(O|[o_2]_{\mathscr{R}}) = (0.45, 0.69),$$
$$\overline{P}(O|[o_3]_{\mathscr{R}}) = (0.45, 0.69),$$
$$\overline{P}(O|[o_4]_{\mathscr{R}}) = (0.45, 0.69),$$
$$\overline{P}(O|[o_5]_{\mathscr{R}}) = (0.40, 0.88),$$
$$\overline{P}(O|[o_6]_{\mathscr{R}}) = (0.72, 0.38),$$
$$\overline{P}(O|[o_7]_{\mathscr{R}}) = (0.59, 0.43),$$
$$\overline{P}(O|[o_8]_{\mathscr{R}}) = (0.43, 0.80),$$
$$\overline{P}(O|[o_9]_{\mathscr{R}}) = (0.39, 0.86),$$
$$\overline{P}(O|[o_{10}]_{\mathscr{R}}) = (0.73, 0.40).$$

Therefore, if we take loss functions for O in Table 9, then we can have

Case 1 When $\alpha + \beta > [1, 1]$, according to the formula (57) and (58), the IVFS lower and upper approximation about the (α, β)-IVF probability rough set are calculated as follows:

$$\underline{\mathscr{R}}^{[1.90,2.00]}(O) = \{o_1, o_3, o_7\},$$
$$\overline{\mathscr{R}}^{[1.11,1,20]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$$
$$\underline{\mathscr{R}}^{[2.00,2.33]}(O) = \emptyset,$$
$$\overline{\mathscr{R}}^{[1.5,1.94]}(O) = \{o_1, o_2, o_3, o_4, o_6, o_7, o_8, o_9, o_{10}\}.$$

Table 9 Loss functions for O

	O: positive	O^c: negative
a_P: accept	$\overline{\lambda}_{PP} = [0.30, 0.40]$	$\overline{\lambda}_{PN} = [0.99, 1.00]$
a_B: reject	$\overline{\lambda}_{BP} = [0.70, 0.75]$	$\overline{\lambda}_{BN} = [0.23, 0.30]$
a_N: defer	$\overline{\lambda}_{NP} = [0.80, 0.84]$	$\overline{\lambda}_{NN} = [0.11, 0.20]$

Table 8 Interval-valued relation on U

U	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9	o_{10}
o_1	[1.00, 1.00]									
o_2	[0.25, 0.74]	[1.00, 1.00]								
o_3	[0.27, 0.73]	[0.19, 0.69]	[1.00, 1.00]							
o_4	[0.16, 0.83]	[0.30, 0.90]	[0.18, 0.69]	[1.00, 1.00]						
o_5	[0.28, 0.65]	[0.46, 0.91]	[0.18, 0.98]	[0.37, 0.90]	[1.00, 1.00]					
o_6	[0.16, 0.55]	[0.16, 0.99]	[0.25, 0.59]	[0.27, 0.62]	[0.22, 0.89]	[1.00, 1.00]				
o_7	[0.14, 0.83]	[0.11, 0.98]	[0.27, 0.99]	[0.26, 0.79]	[0.18, 0.87]	[0.27, 0.78]	[1.00, 1.00]			
o_8	[0.21, 0.62]	[0.59, 0.80]	[0.15, 0.52]	[0.35, 0.89]	[0.39, 0.79]	[0.12, 0.81]	[0.11, 0.77]	[1.00, 1.00]		
o_9	[0.26, 0.76]	[0.19, 0.69]	[0.13, 0.67]	[0.23, 0.71]	[0.25, 0.76]	[0.39, 0.91]	[0.15, 0.72]	[0.14, 0.55]	[1.00, 1.00]	
o_{10}	[0.34, 0.86]	[0.35, 0.79]	[0.17, 0.98]	[0.33, 0.92]	[0.46, 0.86]	[0.26, 0.99]	[0.24, 0.70]	[0.25, 0.82]	[0.34, 0.76]	[1.00, 1.00].

and $(\alpha,\ \beta)$-IVF probability decision domain:

$POS^{[1.90,2.00]}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$
$NEG^{[1.11,1.20]}(O) = \{o_2, o_8\},$
$BND^{[1.90,2.00],[1.11,1.20])}(O) = \{o_4, o_5\}.$
$POS^{[2.00,2.33]}(O) = \{o_1, o_3, o_6, o_7, o_9, o_{10}\},$
$NEG^{[1.5,1.94]}(O) = \{o_2, o_8\},$
$BND^{[2.00,2.33],[1.5,1.94])}(O) = \{o_4, o_5\}.$

Case 2 When $\alpha_i^- + \beta_i^- < 1, \alpha_i^+ + \beta_i^+ < 1$, according to the formula (57) and (58), the IVFS lower and upper approximations about the $(\alpha,\ \beta)$-IVF probability rough set are calculated as follows:

$\underline{\mathscr{R}}^{[0.5,2.33]}(O) = \emptyset,$
$\overline{\mathscr{R}}^{[0.43,0.6]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$
$\underline{\mathscr{R}}^{[0.5,1.33]}(O) = \{o_1, o_2, o_3, o_4, o_6, o_7, o_8, o_9, o_{10}\},$
$\overline{\mathscr{R}}^{[0.43,0.6]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$
$\underline{\mathscr{R}}^{[0.2,0.8]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$\overline{\mathscr{R}}^{[0.2,0.8]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$

In addition, $(\alpha,\ \beta)$-IVF probability can be obtained in the following:

$POS^{[0.5,2.33]}(O) = \emptyset,$
$NEG^{[0.43,0.6]}(O) = \emptyset,$
$BND^{[0.5,2.33],[0.43,0.6])}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$
$POS^{[2.00,2.33]}(O) = \{o_1, o_2, o_3, o_4, o_6, o_7, o_8, o_9, o_{10}\},$
$NEG^{[1.5,1.94]}(O) = \emptyset,$
$BND^{[2.00,2.33],[1.5,1.94])}(O) = \{o_5\}.$
$POS^{[2.00,2.33]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$NEG^{[1.5,1.94]}(O) = \emptyset,$
$BND^{[2.00,2.33],[1.5,1.94])}(O) = \emptyset.$

Case 3 When $\alpha + \beta < [1, 1]$, according to the formula (57) and (58), the IVFS lower and upper approximation about the $(\alpha,\ \beta)$-IVF probability rough set are calculated as follows:

$\underline{\mathscr{R}}^{[0.25,0.32]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$\overline{\mathscr{R}}^{[0.4,0.67]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$
$\underline{\mathscr{R}}^{[0.45,0.46]}(O) = \emptyset,$
$\overline{\mathscr{R}}^{[0.51,0.53]}(O) = \{o_1, o_2, o_3, o_4, o_6, o_7, o_8, o_9, o_{10}\}.$

and $(\alpha,\ \beta)$-IVF probability decision domain

$POS^{[0.25,0.32]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$NEG^{[0.4,0.67]}(O) = \emptyset,$
$BND^{[0.25,0.32],[0.4,0.67])}(O) = \emptyset.$
$POS^{[0.45,0.46]}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$NEG^{[0.51,0.53]}(O) = \emptyset,$
$BND^{[0.45,0.46],[0.51,0.53])}(O) = \emptyset.$

Case 4 When $\alpha + \beta = 1$, according to the formula (57) and (58), the IVFS lower and upper approximation about the $(\alpha,\ \beta)$-IVF probability rough set are calculated as follows:

$\underline{\mathscr{R}}^{(0.5)}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$\overline{\mathscr{R}}^{(0.5)}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\}.$

and $(\alpha,\ \beta)$-IVF probability decision domain

$POS^{(0.5)}(O) = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8, o_9, o_{10}\},$
$NEG^{[0.51,0.53]}(O) = \emptyset,$
$BND^{[0.45,0.46],[0.51,0.53])}(O) = \emptyset.$

5 Conclusions

In this paper, interval-valued fuzzy information systems is converted into two kinds of approximate spaces (fuzzy approximation space and interval-valued fuzzy approximation space) using different relations. By considering fuzzy probability and interval-valued fuzzy probability, fuzzy and interval-valued fuzzy decision-theoretic rough set methods are established in this paper. The main contributions of this paper are as below. Firstly, fuzzy decision-theoretic rough set is discussed in interval fuzzy information system. Moreover, the corresponding measures and performance are considered in the approximation space. Second, interval-valued fuzzy decision-theoretic rough set method is studied to deal with actual situation. Finally, a real-life example is constructed to explain and illustrate decision-making method. Meanwhile, we find that it is necessary that how to investigate other new decision-making methods under different information systems such as inconsistent interval systems, incomplete interval systems, and so on. Whether or not there are some relationships about different decision rules among these different interval systems. These issues are important topics studied in the future.

Acknowledgements This study is supported by National Natural Science Foundation of China (No.61772002, 61472463), the Science and Technology Research Program of Chongqing Municipal Education Commission (No.KJ1709221).

References

Agbodah K (2018) The determination of three-way decisions with decision-theoretic rough sets considering the loss function evaluated by multiple experts. Granul Comput. https://doi.org/10.1007/s41066-018-0099-0

Allen AO (1990) Probablity, statistics and queueing theory with computer science applications. In: Tractional computer measurement group conference, 2nd edn. December 10-14, 1990, Orlando, Fl, USA, Proceedings. DBLP, pp 1358–1359

Berg JVD, Kaymak U, Rui JA (2013) Conditional density estimation using probabilistic fuzzy systems. IEEE Trans Fuzzy Syst 21(5):869–882

Burillo P, Bustince H (1996) Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. Fuzzy Sets Syst 78(3):305–316

Casella G, Hwang JTG, Robert C (1993) A Paradox in decision-theoretic interval estimation. Statistica Sinica 3(1):141–155

Chen SM, Hsiao WH (2000) Bidirectional approximate reasoning for rule-based systems using interval-valued fuzzy sets. Fuzzy Sets Syst 113(2):185–203

Chen SM, Hsiao WH, Jong WT (1997) Bidirectional approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst 91(3):339–353

Dai JH, Zheng GJ, Hu QH (2016) Decision-theoretic rough set approach for fuzzy decisions based on fuzzy probability measure and decision making. J Intell Fuzzy Syst 31(3):1341–1353

Deng XF, Yao YY (2014) Decision-theoretic three-way approximations of fuzzy sets. Inform Sci 279:702–715

Fan BJ, Tsang ECC, Xu WH (2016) Double-quantita -tive rough fuzzy set based decisions: a logical operations method. Inform Sci Int J 378(C):264–281

Fang BW, Hu BQ (2016) Probabilistic graded rough set and double relative quantitative decision-theoretic rough set. Int J Approx Reason 74:1–12

Hao C, Li JH, Fan M (2017) Optimal scale selection in dynamic multi-scale decision tables based on sequential three-way decisions. Inform Sci 415:213–232

Huang CC, Li JH, Mei CL (2017) Three-way concept learning based on cognitive operators: an information fusion viewpoint. Int J Approx Reason 84(1):1–20

Kabaila P (2013) Note on a paradox in decision-theoretic interval estimation. Stat Probab Lett 83(1):123–126

Li HX, Zhou XZ (2011) Risk decision making based on decision-theoretic rough set: a three-way view decision model. Int J Comput Intell Syst 4(1):1–11

Li HX, Zhou XZ, Zhao JB et al (2012) Cost-sensitive classification based on decision-theoretic rough set model. In: International conference on rough sets and knowledge technology. Springer, Berlin, Heidelberg, pp 379–388

Li WT, Zhang X, Sun WX (2014) Further study of multigranulation T-fuzzy rough sets. Sci World J 2014(2014):927014

Li JH, Mei CL, Xu WH (2015) Concept learning via granular computing: a cognitive viewpoint. Inform Sci 298(1):447–467

Li JH, Ren Y, Mei CL (2016) A comparative study of multigranulation rough sets and concept lattices via rule acquisition. Knowl Based Syst 91:152–164

Li JH, Huang CC, Qi JJ (2017) Three-way cognitive concept learning via multi-granularity. Inform Sci 378(1):244–263

Liang DC, Liu D (2014) Systematic studies on three-way decisions with interval-valued decision-theoretic rough sets. Inform Sci 276(C):186–203

Liang JY, Wang F, Dang CY (2012) An efficient rough feature selection algorithm with a multi-granulation view. Int J Approx Reason 53(6):912–926

Liang DC, Liu D, Pedrycz W (2013) Triangular fuzzy decision-theoretic rough sets. Int J Approx Reason 54(8):1087–1106

Lin GP, Qian YH, Li JJ (2012) NMGRS: neighborhood-based multigranulation rough sets. Int J Approx Reason 53(7):1080–1093

Lin GP, Liang JY, Qian YH (2013) Multigranulation rough sets:From partition to covering. Elsevier Science Inc 241(12):101–118

Lin GP, Liang JY, Qian YH (2015) An information fusion approach by combining multigranulation rough sets and evidence theory. Inform Sci 314:184–199

Lingras P, Chen M, Miao DQ (2009) Rough cluster quality index based on decision theory. IEEE Educ Act Dep 21(7):1014–1026

Liu D, Li TR, Ruan D (2011a) Probabilistic model criteria with decision-theoretic rough sets. Inform Sci 181(17):3709–3722

Liu D, Yao YY, Li TR (2011b) Three-way investment decisions with decision-theoretic rough sets. Int J Comput Intell Syst 4(1):66–74

Lu J, Li DY, Zhai YH (2016) A model for type-2 fuzzy rough sets. Inform Sci 328(C):359–377

Mandal P, Ranadive AS (2018a) Multi-granulation interval-valued fuzzy probabilistic rough sets and their corresponding three-way decisions based on interval-valued fuzzy preference relations. Gran Comput 5:1–20

Mandal P, Ranadive AS (2018b) Multi-granulation fuzzy decision-theoretic rough sets and bipolar-valued fuzzy decision-theoretic rough sets and their applications. Granul Comput. https://doi.org/10.1007/s41066-018-0111-8

Mondal SP, Mandal M, Bhattacharya D (2017) Non-linear interval-valued fuzzy numbers and their application in difference equations. Gran Comput 1:1–13

Pawlak Z (1982) Rough sets. Int J Comput Inform Sci 11:341–356

Pawlak Z (1997) Rough set approach to knowledge-based decision support. Eur J Oper Res 99(1):48–57

Pedrycz W (2013) Granular computing: analysis and design of intelligent systems. CRC Press, Boca Raton, pp 16–24

Pedrycz W, Chen SM (2011) Granular computing and intelligent systems: design with information granules of higher order and higher type. Springer, Heidelberg, Germany

Pedrycz W, Chen SM (2015a) Granular computing and decision-making: interactive and iterative approaches. Springer, Heidelberg, Germany

Pedrycz W, Chen SM (2015b) Information granularity, big data, and computational intelligence. Springer, Heidelberg, Germany

Qian YH, Liang JY (2006) Rough set method based on multi-granulations. IEEE Int Conf Cogn Inform IEEE 2006:297–304

Qian YH, Liang JY, Yao YY (2010) MGRS: a multi-granulation rough set. Inform Sci Int J 180(6):949–970

Qian YH, Li SY, Liang JY (2014) Pessimistic rough set based decisions: a multigranulation fusion strategy. Inform Sci 264(6):196–210

Sahin F, Bay JS (2001) Structural Bayesian network learning in a biological decision-theoretic intelligent agent and its application to a herding problem in the context of distributed multi-agent systems. IEEE Int Conf Syste Man Cybern IEEE Xplore 3:1606–1611

Słowiński R, Vanderpooten D (2000) A generalized defnition of rough approximationsbased on similarity. IEEE Trans Data Knowl Eng 12(2):331–336

Tan AH, Wu WZ, Li JJ (2015) Evidence-theory-based numerical characterization of multigranulation rough sets in incomplete information systems. Fuzzy Sets Syst 294(C):18–35

Wang CZ, Shao MW, He Q (2016) Feature subset selection based on fuzzy neighborhood rough sets. Knowl Based Syst 111:173–179

Wang CZ, Hu QH, Wang XZ (2017) Feature Selection Based on Neighborhood Discrimination Index. IEEE Trans Neural Netw Learn Syst PP(99):1–14

Xu WH, Li WT (2013) Multigranulation decision-theoretic rough set in ordered information system. Fundam Inform 139(1):67–89

Xu WH, Li WT (2016) Granular computing approach to two-way learning based on formal concept analysis in fuzzy datasets. IEEE Trans Cybern Fuzzy 46(2):366–379

Xu WH, Yu JH (2017) A novel approach to information fusion in multi-source datasets: a granular computing viewpoint. Inform Sci 378:410–423

Xu WH, Wang QR, Zhang XT (2011) Multi-granulation fuzzy rough sets in a fuzzy tolerance approximation space. Int J Fuzzy Syst 13(4):246–259

Yang XP, Yao JT (2012) Modelling multi-agent three-way decisions with decision-theoretic rough sets. Fundam Inform 115(2–3):157–171

Yang XB, Qi YS, Song XN (2013) Test cost sensitive multigranulation rough set: model and minimal cost selection. Inform Sci 250(11):184–199

Yao YY (2004) Information granulation and approximation in a decision-theoretical model of rough sets—rough-neural computing. Springer Berlin Heidelberg, New York, pp 491–516

Yao YY (2007) Decision-theoretic rough set models. In: International conference on rough sets and knowledge technology. Springer-Verlag, New York, pp 1–12

Yao YY (2011) Three-way decisions with probabilistic rough sets. Inform Sci 180(3):341–353

Zadeh LA (1968) Probability measures of Fuzzy events. J Math Anal Appl 23(2):421–427

Zhang XH, Pei DW, Dai JH (2013) Fuzzy mathematics and Rough set theory. Tsinghua University Press, Beijing

Zhao XR, Hu BQ (2015) Fuzzy and interval-valued fuzzy decision-theoretic rough set approaches based on fuzzy probability measure. Inform Sci 298(C):534–554