The Journal of Engineering

The 2nd 2018 Asian Conference on Artificial Intelligence Technology (ACAIT 2018)

Information measure of absolute and relative quantification in double-quantitative decision-theoretic rough set model

eISSN 2051-3305
Received on 18th July 2018
Accepted on 26th July 2018
E-First on 18th October 2018
doi: 10.1049/joe.2018.8315
www.ietdl.org

Wentao Li[1], Witold Pedrycz[2], Xiaoping Xue[1] ✉, Xiaoyan Zhang[3], Bingjiao Fan[4], Binghan Long[3]

[1]Department of Mathematics, Harbin Institute of Technology, Harbin 150001, People's Republic of China
[2]Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada, T6R 2V4
[3]School of Science, Chongqing University of Technology, Chongqing 400054, People's Republic of China
[4]Faculty of Information Technology, Macau University of Science and Technology, Taipa, Macau
✉ *E-mail: xiaopingxue@263.net*

Abstract: The absolute and relative quantifications between the equivalence class and the target concept are the two important research endeavours in rough set theory. Double-quantitative decision-theoretic rough set (Dq-DTRS) models utilise both absolute quantification and relative quantification in their upper and lower approximations to reflect the distinctive degrees of quantitative information. Herein, the authors apply the information theory to Dq-DTRS model to characterise and measure these two types of quantitative information. The expressions of the information entropy with regard to the two quantifications and their corresponding information co-entropy are presented in DqI-DTRS model and DqII-DTRSmodel, respectively. This work makes a further study of Dq-DTRS models by discussing the information measures with respect to absolute and relative quantification.

1 Introduction

In many real-life applications, Pawlakrough sets [1] do not cope well with quantitative problems [2, 3]. Improving the Pawlak rough set model by incorporating quantitative information is a promising direction. The improved models are regarded as quantitative rough set models, and they include probabilistic rough sets (PRS) [4–11], graded rough sets (GRS) [12–14], and double-quantitative decision-theoretic rough set (Dq-DTRS) [14–20] models.

As a special kind of PRS model, the DTRS model uses conditional probability and Bayesian risk decision to establish three-way decisions and threshold quantitative semantics. As a result, DTRS has provided a platform for improving some basic models and a quantitative exploration. DTRS model has become increasingly popular in a variety of theoretical and practical areas, producing many thorough results [4–11]. The GRS model [13] primarily considers the absolute quantitative information between the basic concept and knowledge granules, and is also a generalisation of Pawlak rough set model. The two kinds of relative and absolute quantitative information are two quantification mythologies in certain applications. Three examples introduced in references [16, 20] have highlighted the motivation that leads to considering the relative quantification and absolute quantification, and explained the importance of these two types of quantitative information in different scenarios.

Some works related to the double quantification have been explored [14–20]. Among these studies, Li and Xu proposed a framework of Dq-DTRS model based on the Bayesian decision and GRS, and two kinds of Dq-DTRS are confirmed, which essentially indicate the relative and absolute quantification [16]. In the DqI-DTRS model, the upper approximation quantifies relative quantitative information and lower approximation quantifies absolute quantitative information; and in the DqII-DTRS model, the upper approximation quantifies absolute quantitative information, and the lower approximation quantifies relative quantitative information.

Information is an abstract concept. We often use terms such as ‘a lot of information’ or ‘less information’ to describe the quantity of information, but it is difficult to measure how much information is contained. In order to mathematically quantify the statistical nature of information loss [21], Shannon developed a general concept of information theory. Information entropy, proposed by Shannon in information theory, has been an effective and powerful mechanism for characterising the information content in diverse models [22–30]. The concept of entropy was developed in response to the observation that a certain amount of functional energy released from combustion reactions was always lost to dissipation or friction and thus not transformed into useful work. Shannon put forward the notation of information entropy, which solved the problem of a quantitative measure of information.

It is an important issue to characterise the degree of uncertainty contained in rough set models [31–37], the same to Dq-DTRS model. How to measure the absolute quantitative information and relative quantitative information is a problem to be eagerly studied. In this paper, we aim to propose a theoretical method to tackle the above-presented problem. Some information-theoretic measures of uncertainty and granularity have been investigated [26, 27, 29, 31–34, 36–38]. A common feature of these researches on the uncertainty of rough set is that they are dependent on the partitions and the cardinality of a universe. In particular, Zhu *et al.* first developed a pair of information-theoretic entropy and co-entropy functions associated to partitions and approximations [38], and then gave information-theoretic measures associated with a pair of approximation operators [38].

In this paper, the information theory is applied to the Dq-DTRS model to measure the two kinds of quantitative information. We use the information entropy and information co-entropy, at the same time, to describe the amount of information contained in the relative quantitative information and the absolute quantitative information of Dq-DTRS. It is shown that the proposed information measures provide a novel approach to evaluate the information of absolute quantification and relative quantification. The paper is organised as follows. In Section 2, basic concepts and definitions are reviewed briefly. In Section 3, we present the information entropy and information co-entropy with respect to the two kinds of quantification in both DqI-DTRS model and DqII-DTRS model. Finally, we conclude with some concluding notes and an outlook for future research in Section 4.

2 Related work and fundamentals

We review related basic concepts about Shannon entropy theory and Dq-DTRS models. Throughout this paper, the class of all subsets of the universe U is denoted by $P(U)$. An information system is a triple (U, A, F), where $U = \{x_1, x_2, \cdots, x_n\}$ is a non-empty and finite set of objects; $A = \{a_1, a_2, \cdots, a_m\}$ is a non-empty and finite set of attributes; $F = \{f_l | U \to V_l, l \le m\}$, f_l is the value of a_l on $x \in U$, V_l is the domain of a_l, $a_l \in A$. The equivalence relation R partitions U into disjoint subsets, which is $\pi = \{U_1, U_2, \cdots, U_k\}$. Such a partition of the universe is a quotient set of U and is denoted by $U/R = \{[x]_R | x \in U\}$, where $[x]_R = \{y \in U | (x, y) \in R\}$ is the equivalence class containing x.

In [22–24, 33, 34], information-theoretic measures are dependent on the size of equivalence classes and the cardinality of a universe. Shannon entropy has been used as a measure of information entropy for rough set theory.

Definition 1: Given an information system (U, A, F) and an equivalence relation R. R partitions the universe U into disjoint blocks (equivalence classes) U_i, $1 \le i \le k$. The information entropy $H(\pi)$ of the partition π is defined in the form

$$H(\pi) = -\sum_{i=1}^{k} \frac{|U_i|}{|U|} \log \frac{|U_i|}{|U|},$$

where $|U| = \sum_{i=1}^{k} |U_i|$. If $\pi = \{U\}$, the entropy function H achieves the minimum value 0; and if $\pi = \{\{x\} | x \in U\}$, it achieves the maximum value $\log |U|$.

Definition 2: Given an information system (U, A, F) and an equivalence relation R. For an arbitrary set $X \in P(U)$, a pair of upper and lower approximations of X are characterised as

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\};$$
$$\bar{R}(X) = \{x \in U | [x]_R \cap X \ne \emptyset\}.$$

For a target set (or concept) $X \in P(U)$, if $\underline{R}(X) = \bar{R}(X)$, X is called definable set in rough approximation space; and if $\underline{R}(X) \ne \bar{R}(X)$, then X is called Pawlak rough set.

The PRS (or DTRS) and the GRS are two quantitative models that measure relative and absolute quantitative information between the equivalence class and a basic concept, respectively. In [16], authors proposed a framework of Dq-DTRS, and two kinds of Dq-DTRS model are constructed, which indicate the relative and absolute quantification. Let us review the Dq-DTRS model.

Definition 3: The following upper and lower approximation operators are defined as [16]

$$\bar{R}^{I}_{(\alpha,\beta,k)}(X) = \{x \in U | \frac{|[x]_R \cap X|}{|[x]_R|} > \beta\};$$
$$\underline{R}^{I}_{(\alpha,\beta,k)}(X) = \{x \in U | \; |[x]_R| - |[x]_R \cap X| \le k\}.$$

From the above two operators, the DqI-DTRS model can be established and denoted by $(U, \bar{R}^{I}_{(\alpha,\beta,k)}, \underline{R}^{I}_{(\alpha,\beta,k)})$.

Definition 4: The model $(U, \bar{R}^{II}_{(\alpha,\beta,k)}, \underline{R}^{II}_{(\alpha,\beta,k)})$ called DqII-DTRS [16], is defined using the following two operators $\bar{R}^{II}_{(\alpha,\beta,k)}$ and $\underline{R}^{II}_{(\alpha,\beta,k)}$:

$$\bar{R}^{II}_{(\alpha,\beta,k)}(X) = \{x \in U | \; |[x]_R \cap X| > k\};$$
$$\underline{R}^{II}_{(\alpha,\beta,k)}(X) = \{x \in U | \frac{|[x]_R \cap X|}{|[x]_R|} \ge \alpha\}.$$

Inspired by the studies of Zhu *et al.* [38], we investigate the information entropy and information co-entropy of absolute and relative quantitative information in the next section, which is different from the previous studies mentioned [22–24, 26, 27, 29, 31–34, 36–38]. In this paper, we consider not only the equivalence classes of the universe of discourse but also the upper and lower approximations of all power sets of the universe.

For arbitrary $X \in P(U)$, the upper and lower approximations appear in pairs. We determine the count of all elements of $P(U)$ by every pair of double-quantitative upper and lower approximations. Compared with other types of entropy for measuring the uncertainty in rough set theory, the main feature of the entropy is that the approximation operators are taken into account. In fact, the entropies without involving approximation operators are independent of rough set theory, which rely on partitions of the universe of discourse.

3 Information measures with respect to absolute and relative quantification

In this section, we present the information entropy, information co-entropy with respect to two kinds of quantification in DqI-DTRS and DqII-DTRS, respectively. It should be pointed out that the logarithm is taken as base 2, in which case the information entropies and information co-entropies are measured in ‘bits’.

3.1 Information entropy and information co-entropy in DqI-DTRS model

In this subsection, the entropies of absolute and relative quantitative information in DqI-DTRS model and their corresponding properties are introduced.

In DqI-DTRS model, it is easy to see that every subset of U appears with the same probability $1/2^{|U|}$. We denote the upper and lower approximation operators of DqI-DTRS as $\bar{R}^{I}_{(\alpha,\beta,k), > i} = A_i$ and $\underline{R}^{I}_{(\alpha,\beta,k), > j} = B_j$. For any $X \in P(U)$, we set

$$\mathscr{A}_i = \{X \in P(U) | \bar{R}^{I}_{(\alpha,\beta,k)}(X) = A_i\};$$
$$\mathscr{B}_j = \{X \in P(U) | \underline{R}^{I}_{(\alpha,\beta,k)}(X) = B_j\}.$$

Then the upper approximation operator A_i and the lower approximation operator B_j appear with the accumulative probability $|\mathscr{A}_i|/2^{|U|}$ and $|\mathscr{B}_j|/2^{|U|}$ since the amount of all subsets of U is precisely $2^{|U|}$, respectively.

For each $X \in P(U)$, $|\mathscr{A}_i|$ $(i \in \{1, 2, \cdots, m\})$ are the number of subsets described by the relative quantification $|[x]_R \cap X| / |[x]_R|$, namely $|\mathscr{A}_1|, |\mathscr{A}_2|, \ldots, |\mathscr{A}_m|$ are the number of subsets described by the upper approximation operators $\bar{R}^{I}_{(\alpha,\beta,k),1}(X)$, $\bar{R}^{I}_{(\alpha,\beta,k),2}(X), \ldots, \bar{R}^{I}_{(\alpha,\beta,k),m}(X)$ respectively; and $\mathscr{B}_j (j \in \{1, 2, \ldots, n\})$ are the number of subsets described by the absolute quantification $|[x]_R| - |[x]_R \cap X|$, namely $|\mathscr{B}_1|, |\mathscr{B}_2|, \ldots, |\mathscr{B}_n|$ are the number of subsets described by the lower approximation operators $\underline{R}^{I}_{(\alpha,\beta,k),1}(X)$, $\underline{R}^{I}_{(\alpha,\beta,k),2}(X), \ldots, \underline{R}^{I}_{(\alpha,\beta,k),n}(X)$, respectively. Two probability distributions are obtained in the form:

$$P(\bar{R}^{I}_{(\alpha,\beta,k)}) = \left[\frac{|\mathscr{A}_1|}{2^{|U|}}, \frac{|\mathscr{A}_2|}{2^{|U|}}, \ldots, \frac{|\mathscr{A}_m|}{2^{|U|}}\right];$$
$$P(\underline{R}^{I}_{(\alpha,\beta,k)}) = \left[\frac{|\mathscr{B}_1|}{2^{|U|}}, \frac{|\mathscr{B}_2|}{2^{|U|}}, \ldots, \frac{|\mathscr{B}_n|}{2^{|U|}}\right].$$

It turns out that both $\{\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_m\}$ and $\{\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_n\}$ can give rise to a partition of $P(U)$, which means $\bigcup_{i=1}^{m} \mathscr{A}_i = P(U)$ and $\bigcup_{j=1}^{n} \mathscr{B}_j = P(U)$. Therefore, we can obtain $\sum_{i=1}^{m} |\mathscr{A}_i| = 2^{|U|}$ and $\sum_{j=1}^{n} |\mathscr{B}_j| = 2^{|U|}$.

Definition 5: Given an information system (U, A, F) and an equivalence relation R. For each $X \in P(U)$, we get partitions of $P(U)$ induced by the double quantification, which are $\{\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_m\}$ and $\{\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_n\}$, respectively. Then the information entropy of absolute and relative quantitative information in DqI-DTRS model is defined as

$$H_R^I(Rel) = -\sum_{i=1}^{m} \frac{|\mathscr{A}_i|}{2^{|U|}} \log \frac{|\mathscr{A}_i|}{2^{|U|}},$$

$$H_R^I(\text{Abs}) = -\sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log \frac{|\mathscr{B}_j|}{2^{|U|}}.$$

Definition 6: Given an information system (U, A, F) and an equivalence relation R. For each $X \in P(U)$, we get partitions of $P(U)$ induced by the double quantification, which are $\{\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_m\}$ and $\{\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n\}$, respectively. Then the information co-entropy of absolute and relative quantitative information in DqI-DTRS model is defined as

$$G_R^I(Rel) = \sum_{i=1}^{m} \frac{|\mathscr{A}_i|}{2^{|U|}} \log |\mathscr{A}_i|,$$

$$G_R^I(\text{Abs}) = \sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log |\mathscr{B}_j|.$$

Proposition 1: Let U be a universe with $|U|$ elements. Then the information entropy and information co-entropy in DqI-DTRS model satisfy the following properties:

(i) $H_R^I(\text{Abs}) + G_R^I(\text{Abs}) = |U|$,
(ii) $H_R^I(\text{Rel}) + G_R^I(\text{Rel}) = |U|$.

Proof:

(i) From Definitions 5 and 6, one has (see equation below) Then the proof of (i) is completed.
(ii) The proof of (ii) is similar to the one completed for (i). □

Example 1: Consider an example shown in Table 1, it is easy to see that $U = \{x_1, x_2, x_3, x_4\}$ and $U/IND(R) = \{\{x_1, x_2, x_3\}, \{x_4\}\}$. In this case, U has $2^4 = 16$ subsets. The parameters $\alpha = 0.7, \beta = 0.5$ and grade $k = 1$. For each subset X of U, we compute the DqI-DTRS upper approximation and lower approximation, which can be shown in Table 2.

We calculate that when $X = \varnothing$, $\{x_1\}$, $\{x_2\}$ and $\{x_3\}$, the upper approximation $\bar{R}^I_{(\alpha,\beta,k)}(X) = \varnothing$; when $X = \{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, and $\{x_1, x_2, x_3\}$, the upper approximation $\bar{R}^I_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3\}$; when $X = \{x_4\}$, $\{x_1, x_4\}$, $\{x_2, x_4\}$, and $\{x_3, x_4\}$, the upper approximation $\bar{R}^I_{(\alpha,\beta,k)}(X) = \{x_4\}$; when $X = \{x_1, x_2, x_4\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_3, x_4\}$, and U, the upper approximation $\bar{R}^I_{(\alpha,\beta,k)}(X) = U$. In addition, when $X = \{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_4\}$, $\{x_1, x_4\}$, $\{x_2, x_4\}$, and $\{x_3, x_4\}$, the lower approximation $\underline{R}^I_{(\alpha,\beta,k)}(X) = \{x_4\}$; when $X = \{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_3, x_4\}$, and U, the lower approximation $\underline{R}^I_{(\alpha,\beta,k)}(X) = U$.

From the above calculations, we obtain that

$$\begin{aligned}\{\mathscr{A}_1 &= \{\varnothing, \{x_1\}, \{x_2\}, \{x_3\}\},\\ \mathscr{A}_2 &= \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}\\ \mathscr{A}_3 &= \{\{x_4\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_3, x_4\}\},\\ \mathscr{A}_4 &= \{\{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, U\}\}\end{aligned}$$

and

$$\begin{aligned}\{\mathscr{B}_1 &= \{\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_4\}, \{x_2, x_4\}\{x_3, x_4\}\},\\ \mathscr{B}_2 &= \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\},\\ &\{x_1, x_3, x_4\}, U\}\}.\end{aligned}$$

Based on $\{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4\}$ and $\{\mathscr{B}_1, \mathscr{B}_2\}$, we can get the information entropies of absolute and relative quantification in DqI-DTRS model as follows:

$$H_R^I(\text{Rel}) = -\sum_{i=1}^{4} \frac{|\mathscr{A}_i|}{2^{|U|}} \log \frac{|\mathscr{A}_i|}{2^{|U|}} = -4 \times \frac{4}{16} \log \frac{4}{16} = 2,$$
$$H_R^I(\text{Abs}) = -\sum_{j=1}^{2} \frac{|\mathscr{B}_j|}{2^{|U|}} \log \frac{|\mathscr{B}_j|}{2^{|U|}} = -2 \times \frac{8}{16} \log \frac{8}{16} = 1.$$

The information co-entropy of absolute quantitative information and relative quantitative information in DqI-DTRSmodel are defined as

$$G_R^I(Rel) = \sum_{i=1}^{4} \frac{|\mathscr{A}_i|}{2^{|U|}} \log |\mathscr{A}_i| = 2,$$

$$G_R^I(\text{Abs}) = \sum_{j=1}^{2} \frac{|\mathscr{B}_j|}{2^{|U|}} \log |\mathscr{B}_j| = 3.$$

$-\log\left(|\mathscr{A}_i|/2^{|U|}\right)$ and $-\log\left(|\mathscr{B}_j|/2^{|U|}\right)$ in information entropies are related to the probabilities $\left(|\mathscr{A}_i|/2^{|U|}\right)$ and $\left(|\mathscr{B}_j|/2^{|U|}\right)$ of occurrence of the 'event' $\mathscr{A}_i$ and $\mathscr{B}_j$, respectively, can be interpreted as measures of the uncertainty due to the knowledge of these probabilities. Furthermore, the information entropies of probability distributions $\{\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_m\}$ and $\{\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_n\}$ can be considered as quantities which in a reasonable way measures the average uncertainty associated with their distributions and expressed as the mean values $-\sum_{i=1}^{m}\left(|\mathscr{A}_i|/2^{|U|}\right)\log\left(|\mathscr{A}_i|/2^{|U|}\right)$ and $-\sum_{i=1}^{n}\left(|\mathscr{B}_i|/2^{|U|}\right)\log\left(|\mathscr{B}_i|/2^{|U|}\right)$. That is to say, $H_R^I(\text{Rel})$ and $H_R^I(\text{Abs})$ measure the average uncertainty of relative quantitative information and absolute quantitative information with respect to the upper and lower approximation operators. As mentioned above, each pair of upper approximation operator or lower approximation operator related to absolute quantitative information and relative quantitative information induced a classification of all subsets of U, and an uncertainty measure of the classification is provided by each information entropy including $H_R^I(\text{Abs})$ and $H_R^I(\text{Rel})$. The higher the information entropy, the lower the degree of uncertainty.

The quantity $\log|\mathscr{A}_i|$ and $\log|\mathscr{B}_j|$ represent the measure of the granularity associated with the knowledge supported by the 'granule' $\mathscr{A}_i$ and $\mathscr{B}_j$. Therefore, the information co-entropies $G_R^I(\text{Abs})$ and $G_R^I(\text{Rel})$ are basically average granularity with respect to all equivalence classes in the classification carried by the

$$\begin{aligned} H_R^I(\text{Abs}) + G_R^I(\text{Abs}) &= -\sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log \frac{|\mathscr{B}_j|}{2^{|U|}} + \sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log |\mathscr{B}_j| \\ &= -\left(\sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log |\mathscr{B}_j| - \sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log 2^{|U|}\right) + \sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log |\mathscr{B}_j| \\ &= \sum_{j=1}^{n} \frac{|\mathscr{B}_j|}{2^{|U|}} \log 2^{|U|} = |U|. \end{aligned}$$

absolute quantitative information and relative quantitative information. In contrast to information entropies $H_R^I(\text{Abs})$ and $H_R^I(\text{Rel})$, the greater the information co-entropy, the coarser the classifications and the higher the degree of uncertainty of describing concepts.

3.2 Information entropy and information co-entropy in DqII-DTRS model

The entropies of absolute and relative quantitative information in DqII-DTRS model and their corresponding properties are introduced.

Similar to DqI-DTRS, we denote the upper and lower approximation operators of DqII-DTRS as $\bar{R}^{II}_{(\alpha,\beta,k),i} = C_i$ and $\underline{R}^{II}_{(\alpha,\beta,k),j} = D_j$, it is easy to see that every subset of U appears with the same probability $1/2^{|U|}$. For any $X \in P(U)$, we set

$$\mathscr{C}_i = \{X \in P(U) | \bar{R}^{II}_{(\alpha,\beta,k)}(X) = C_i\};$$
$$\mathscr{D}_j = \{X \in P(U) | \underline{R}^{II}_{(\alpha,\beta,k)}(X) = D_j\}.$$

Then the upper approximation operator C_i and the lower approximation operator D_j appear with the accumulative probability $|\mathscr{C}_i|/2^{|U|}$ and $|\mathscr{D}_j|/2^{|U|}$ since the amount of all subsets of U is $2^{|U|}$, respectively. For each $X \in P(U)$ in DqII-DTRS model, $|\mathscr{C}_i| (i \in \{1, 2, \ldots, p\})$ are the number of subsets described by the absolute quantification $|[x]_R \cap X|$, namely $|\mathscr{C}_1|, |\mathscr{C}_2|, \ldots, |\mathscr{C}_p|$ are the number of subsets described by the upper approximation operators $\bar{R}^{II}_{(\alpha,\beta,k),>1}(X)$, $\bar{R}^{II}_{(\alpha,\beta,k),2}(X), \ldots, \bar{R}^{II}_{(\alpha,\beta,k),p}(X)$, respectively; and $|\mathscr{D}_j| (j \in \{1, 2, \ldots, q\})$ are the number of subsets described by the relative quantification $|[x]_R \cap X| / |[x]_R|$, namely $|\mathscr{D}_1|, |\mathscr{D}_2|, \ldots, |\mathscr{D}_q|$ are the number of subsets described by the lower approximation operators $\underline{R}^{II}_{(\alpha,\beta,k),>1}(X)$, $\underline{R}^{II}_{(\alpha,\beta,k),2}(X), \ldots, \underline{R}^{II}_{(\alpha,\beta,k),q}(X)$, respectively. We form two probability distributions:

Table 1 Information table

U	a	b	c
x_1	1	2	1
x_2	1	2	1
x_3	1	2	1
x_4	2	1	2

Table 2 Upper and lower approximations in DqI-DTRS model

X	$\bar{R}^{I}_{(\alpha,\beta,k)}(X)$	$\underline{R}^{I}_{(\alpha,\beta,k)}(X)$
$\varnothing$	$\varnothing$	$\{x_4\}$
$\{x_1\}$	$\varnothing$	$\{x_4\}$
$\{x_2\}$	$\varnothing$	$\{x_4\}$
$\{x_3\}$	$\varnothing$	$\{x_4\}$
$\{x_4\}$	$\{x_4\}$	$\{x_4\}$
$\{x_1, x_2\}$	$\{x_1, x_2, x_3\}$	U
$\{x_1, x_3\}$	$\{x_1, x_2, x_3\}$	U
$\{x_1, x_4\}$	$\{x_4\}$	$\{x_4\}$
$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$	U
$\{x_2, x_4\}$	$\{x_4\}$	$\{x_4\}$
$\{x_3, x_4\}$	$\{x_4\}$	$\{x_4\}$
$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$	U
$\{x_1, x_2, x_4\}$	U	U
$\{x_2, x_3, x_4\}$	U	U
$\{x_1, x_3, x_4\}$	U	U
U	U	U

$$P(\bar{R}^{II}_{(\alpha,\beta,k)}) = \left[\frac{|\mathscr{C}_1|}{2^{|U|}}, \frac{|\mathscr{C}_2|}{2^{|U|}}, \ldots, \frac{|\mathscr{C}_p|}{2^{|U|}}\right];$$
$$P(\underline{R}^{II}_{(\alpha,\beta,k)}) = \left[\frac{|\mathscr{D}_1|}{2^{|U|}}, \frac{|\mathscr{D}_2|}{2^{|U|}}, \ldots, \frac{|\mathscr{D}_q|}{2^{|U|}}\right].$$

It turns out that both $\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_p\}$ and $\{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_q\}$ can give rise to a partition of $P(U)$, which means $\bigcup_{i=1}^{p} \mathscr{C}_i = P(U)$ and $\bigcup_{j=1}^{q} \mathscr{D}_j = P(U)$. Therefore, we can obtain $\sum_{i=1}^{p} |\mathscr{C}_i| = 2^{|U|}$ and $\sum_{j=1}^{q} |\mathscr{D}_j| = 2^{|U|}$.

Definition 7: Given an information system (U, A, F) and an equivalence relation R. For each $X \in P(U)$, we get two partitions of $P(U)$ induced by the double quantification, which are $\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_p\}$ and $\{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_q\}$, respectively. Then the information entropy of absolute and relative quantitative information in DqII-DTRS model is defined as

$$H_R^{II}(\text{Abs}) = -\sum_{i=1}^{p} \frac{|\mathscr{C}_i|}{2^{|U|}} \log \frac{|\mathscr{C}_i|}{2^{|U|}},$$
$$H_R^{II}(\text{Rel}) = -\sum_{j=1}^{q} \frac{|\mathscr{D}_j|}{2^{|U|}} \log \frac{|\mathscr{D}_j|}{2^{|U|}}.$$

Definition 8: Given an information system (U, A, F) and an equivalence relation R. For each $X \in P(U)$, we produce two partitions of $P(U)$ induced by double quantification, which are $\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_p\}$ and $\{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_q\}$, respectively. Then the information co-entropy of absolute and relative quantitative information in DqII-DTRS model is defined as

$$G_R^{II}(\text{Abs}) = \sum_{i=1}^{p} \frac{|\mathscr{C}_i|}{2^{|U|}} \log |\mathscr{C}_i|,$$
$$G_R^{II}(\text{Rel}) = \sum_{j=1}^{q} \frac{|\mathscr{D}_j|}{2^{|U|}} \log |\mathscr{D}_j|.$$

Proposition 2: Let U be a universe with $|U|$ elements. Then the information entropy and information co-entropy in DqII-DTRS model satisfy the following properties.

$$H_R^{II}(\text{Abs}) + G_R^{II}(\text{Abs}) = |U|,$$
$$H_R^{II}(\text{Rel}) + G_R^{II}(\text{Rel}) = |U|.$$

Proof: It is similar to the proof of Proposition 1. □

Example 2: (Continuation see Table 1): We compute the DqII-DTRS upper approximation and lower approximation, which is shown in Table 3.

When $X = \varnothing$, $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_4\}$, $\{x_1, x_4\}$, $\{x_2, x_4\}$ and $\{x_3, x_4\}$, the upper approximation $\bar{R}^{II}_{(\alpha,\beta,k)}(X) = \varnothing$; when $X = \{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_3, x_4\}$ and U, the upper approximation $\bar{R}^{II}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3\}$. Also, when $X = \varnothing$, the lower approximation $\underline{R}^{II}_{(\alpha,\beta,k)}(X) = \varnothing$; when $X = \{x_4\}$, the lower approximation $\underline{R}^{I}_{(\alpha,\beta,k)}(X) = \{x_4\}$; when $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$ and $\{x_1, x_2, x_3\}$, the lower approximation $\underline{R}^{II}_{(\alpha,\beta,k)}(X) = \{x_1, x_2, x_3\}$; when $X = \{x_1, x_4\}$, $\{x_2, x_4\}$, $\{x_3, x_4\}$, $\{x_1, x_2, x_4\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_3, x_4\}$ and U, the lower approximation $\underline{R}^{II}_{(\alpha,\beta,k)}(X) = U$.

Based on the above results, we obtain $\{\mathscr{C}_1 = \{\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_3, x_4\}\}$, $\mathscr{C}_2 = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, U\}$ and $\{\mathscr{D}_1 = \{\varnothing\}$, $\mathscr{D}_2 = \{\{x_4\}\}$, $\mathscr{D}_3 = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}$, $\mathscr{D}_4 = \{\{x_1, x_4\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_4\}, U\}$. Based on $\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$ and $\{\mathscr{D}_1, \mathscr{D}_2\}$, the absolute and relative quantification is as follows:

$$H_R^{II}(\text{Abs}) = -\sum_{i=1}^{2}\frac{|\mathscr{C}_i|}{2^{|U|}}\log\frac{|\mathscr{C}_i|}{2^{|U|}} = -2\times\frac{8}{16}\log\frac{8}{16} = 1,$$

$$H_R^{II}(\text{Rel}) = -\sum_{j=1}^{4}\frac{|\mathscr{D}_j|}{2^{|U|}}\log\frac{|\mathscr{D}_j|}{2^{|U|}} = -2\times\frac{1}{16}\log\frac{1}{16}$$
$$-2\times\frac{1}{16}\log\frac{1}{16} = 1.5436.$$

The information co-entropy of absolute quantitative information and relative quantitative information in DqII-DTRS model are as follows:

$$G_R^{II}(\text{Abs}) = \sum_{i=1}^{2}\frac{|\mathscr{C}_i|}{2^{|U|}}\log|\mathscr{C}_i| = 3,$$

$$G_R^{II}(\text{Rel}) = \sum_{j=1}^{4}\frac{|\mathscr{D}_j|}{2^{|U|}}\log|\mathscr{D}_j| = 2.4564.$$

For the proposed information measures in DqII-DTRS, $-\log\left(|\mathscr{C}_i|/2^{|U|}\right)$ and $-\log\left(|\mathscr{D}_j|/2^{|U|}\right)$ in information entropies are related to the probabilities $\left(|\mathscr{C}_i|/2^{|U|}\right)$ and $\left(|\mathscr{D}_j|/2^{|U|}\right)$ of occurrence of the 'event' $\mathscr{C}_i$ and $\mathscr{D}_j$, respectively, can be interpreted as measures of the uncertainty due to the knowledge of these probabilities. Furthermore, the information entropies of probability distributions $\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_p\}$ and $\{\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_q\}$ can be considered as quantities which in a reasonable way measures the average uncertainty associated with their distributions and expressed as the mean values $-\sum_{i=1}^{p}\left(|\mathscr{C}_i|/2^{|U|}\right)\log\left(|\mathscr{C}_i|/2^{|U|}\right)$ and $-\sum_{j=1}^{q}\left(|\mathscr{D}_i|/2^{|U|}\right)\log\left(|\mathscr{D}_j|/2^{|U|}\right)$. That is to say, $H_R^{II}(\text{Rel})$ and $H_R^{II}(\text{Abs})$ measure the average uncertainty of relative quantitative information and absolute quantitative information with respect to the upper and lower approximation operators. As mentioned above, each pair of upper approximation operator or lower approximation operator related to absolute quantitative information and relative quantitative information induced a classification of all subsets of U, and each information entropy including $H_R^{II}(\text{Abs})$ and $H_R^{II}(\text{Rel})$ provides an uncertainty measure of the classification. The greater the information entropy, the lower the degree of uncertainty.

The quantity $\log|\mathscr{C}_i|$ and $\log|\mathscr{D}_j|$ represent the measure of the granularity associated with the knowledge supported by the 'granule' $\mathscr{C}_i$ and $\mathscr{D}_j$. Therefore, the information co-entropies $G_R^{II}(\text{Abs})$ and $G_R^{II}(\text{Rel})$ are basically average granularity with respect to all equivalence classes in the classification carried by the absolute quantitative information and relative quantitative information. In contrast to information entropies $H_R^{II}(\text{Abs})$ and $H_R^{II}(\text{Rel})$, the greater the information co-entropy, the coarser the classifications and the higher the degree of uncertainty of describing vague concepts.

4 Conclusions

In the establishment of the double-quantitative rough set model, how to measure the two kinds of quantitative information is an urgent issue to be investigated. In this study, we mainly focus on the theoretical analysis of the information contained in the two kinds of quantitative information of the Dq-DTRS model. We develop the information measures of absolute quantitative information and relative quantitative information in Dq-DTRS model and further present the methods of attribute reduction based on the proposed double quantification. The proposed measures, information entropies, and information co-entropies with regard to absolute and relative quantifications perform a new direction for the study of the theory of information theory and Dq-DTRS model. This paper introduces the information theory into Dq-DTRS model, and the notations of absolute quantitative information entropy, information co-entropy, and relative quantitative information entropy, information co-entropy are discussed, respectively.

Table 3 Upper and lower approximations in DqII-DTRS model

X	$\bar{R}^{II}_{(\alpha,\beta,k)}(X)$	$\underline{R}^{II}_{(\alpha,\beta,k)}(X)$
∅	∅	∅
$\{x_1\}$	∅	$\{x_1, x_2, x_3\}$
$\{x_2\}$	∅	$\{x_1, x_2, x_3\}$
$\{x_3\}$	∅	$\{x_1, x_2, x_3\}$
$\{x_4\}$	$\{x_4\}$	$\{x_4\}$
$\{x_1, x_2\}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\{x_1, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\{x_1, x_4\}$	∅	U
$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\{x_2, x_4\}$	∅	U
$\{x_3, x_4\}$	∅	U
$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$\{x_1, x_2, x_4\}$	$\{x_1, x_2, x_3\}$	U
$\{x_2, x_3, x_4\}$	$\{x_1, x_2, x_3\}$	U
$\{x_1, x_3, x_4\}$	$\{x_1, x_2, x_3\}$	U
U	$\{x_1, x_2, x_3\}$	U

The presented information measures are based on the amount number of the power set of the universe of discourse, and it cannot get a good application in practice due to the limitations of Computer Memory. Therefore, how to develop a special algorithm for improving the Computer Memory consumption to calculate the information entropy and information co-entropy in Dq-DTRS models, and more applicable formula of the information entropy or information co-entropy is desirable. We will investigate these issues in the future work.

5 Acknowledgments

This work is supported by Natural Science Foundation of China (No. 11671111, No. 61472463, No. 61772002), Science and TechnologyResearch Program of Chongqing Municipal Education Commission (No. KJ1709221), and Wentao Li is supported by the China Scholarship Council under Grant No. 201606120161.

6 References

[1] Pawlak, Z.: 'Rough sets', *J. Comput. Inf. Sci.*, 1982, **11**, (5), pp. 341–356

[2] Lingras, P., Chen, M., Miao, D.: 'Qualitative and quantitative combinations of crisp and rough clustering schemes using dominance relations', *Int. J. Approx. Reason.*, 2014, **55**, pp. 238–258

[3] Yao, Y., Deng, X.: 'Quantitative rough sets based on subsethood measures', *Inf. Sci.*, 2014, **267**, pp. 306–322

[4] Liang, D., Liu, D.: 'Deriving three-way decisions from intuitionistic fuzzy decision-theoretic rough sets', *Inf. Sci.*, 2015, **300**, pp. 28–48

[5] Liang, D., Pedrycz, W., Liu, D.: 'Determining three-way decisions with decision-theoretic rough sets using a relative value approach', *IEEE Trans. Syst., Man, Cybern.: Syst.*, 2017, **47**, (8), pp. 1785–1799

[6] Wang, G., Ma, X., Yu, H.: 'Monotonic uncertainty measures for attribute reduction in probabilistic rough set model', *Int. J. Approx. Reason.*, 2015, **59**, pp. 41–67

[7] Yao, Y.: 'Probabilistic approaches to rough sets', *Expert Syst.*., 2003, **20**, pp. 287–297

[8] Yao, Y.: 'The superiority of three-way decisions in probabilistic rough set models', *Inf. Sci.*, 2011, **181**, pp. 1080–1096

[9] Yao, Y.: 'Three-way decisions with probabilistic rough sets', *Inf. Sci.*, 2010, **180**, pp. 341–353

[10] Yao, Y., Wong, S. K., Lingras, P.: 'A decision-theoretic rough set model'. Proc. of Int. Symp. on Methodologies for Intelligent Systems, 1990, vol. 5, pp. 17–25

[11] Zhang, Q., Zhang, Q., Wang, G.: 'The uncertainty of probabilistic rough sets in multi-granulation spaces', *Int. J. Approx. Reason.*, 2016, **77**, pp. 38–54

[12] Liu, C., Miao, D., Zhang, N.: 'Graded rough set model based on two universes and its properties', *Knowl.-Based Syst.*, 2012, **33**, pp. 65–72

[13] Yao, Y., Lin, T. Y.: 'Graded rough set approximations based on nested neighborhood systems'. Proc. of 5th European Congress on Intelligent Techniques and Soft Computing, 1997, vol. 1, pp. 196–200

[14] Zhang, X., Miao, D.: 'Two basic double-quantitative rough set models for precision and graded and their investigation using granular computing', *Int. J. Approx. Reason.*, 2013, **54**, pp. 1130–1148

[15] Fan, B., Tsang, E. C. C., Xu, W., *et al.*: 'Double-quantitative rough fuzzy set based decisions: a logical operators method', *Inf. Sci.*, 2017, **378**, pp. 264–281

[16] Li, W., Xu, W.: 'Double-quantitative decision-theoretic rough set', *Inf. Sci.*, 2015, **316**, pp. 54–67

[17] Xu, W., Guo, Y.: 'Generalized multigranulation double-quantitative decision-theoretic rough set', *Knowl.-Based Syst.*, 2016, **105**, pp. 190–205

[18] Zhang, X., Miao, D.: 'An expanded double-quantitative model regarding probabilities and grades and its hierarchical double-quantitative attribute reduction', *Inf. Sci.*, 2015, **299**, pp. 312–336

[19] Zhang, X., Miao, D.: 'Double-quantitative fusion of accuracy and importance: 'systematic measure mining, benign integration construction, hierarchical attribute reduction', *Knowl.-Based Syst.*, 2016, **91**, pp. 219–240

[20] Zhang, X., Miao, D.: 'Quantitative information architecture, granular computing and rough set models in the double-quantitative approximation space of precision and grade', *Inf. Sci.*, 2014, **268**, pp. 147–168

[21] Shannon, C. E.: 'A mathematical theory of communication', *Bell Labs Tech. J.*, 1948, **27**, pp. 379–423, 623–656

[22] Beaubouef, T., Petry, F.E., Arora, G.: 'Information-theoretic measures of uncertainty for rough sets and rough relational datasets', *Inf. Sci.*, 1998, **109**, pp. 185–195

[23] Bianucci, D., Cattaneo, G.: 'Information entropy and granulation co-entropy of partitions and coverings: a summary'. Proc. Transactions on Rough sets X, 2009 (LNCS, 5656), pp. 15–66

[24] Bianucci, D., Cattaneo, G., Ciucci, D.: 'Entropies and co-entropies of coverings with application to incomplete information systems', *Fundam. Inform.*, 2007, **75**, pp. 77–105

[25] Li, J., Huang, C., Qi, J., *et al.*: 'Three-way cognitive concept learning via multi-granularity', *Inf. Sci.*, 2017, **378**, pp. 244–263

[26] Liang, J., Shi, Z., Li, D., *et al.*: 'Information entropy, rough entropy and knowledge granulation in incomplete information systems', *Int. J. Gen. Syst.*, 2006, **35**, pp. 641–654

[27] Liang, J., Qian, Y.: 'Information granules and entropy theory in information systems, science China, series F, 2008, **51**, pp. 1427–1444

[28] Małyszko, D., Stepaniuk, J.: 'Adaptive multilevel rough entropy evolutionary thresholding', *Inf. Sci.*, 2010, **180**, (7), pp. 1138–1158

[29] Sen, D., Pal, S. K.: 'Generalized rough sets, entropy, and image ambiguity measures', *IEEE Trans. Syst., Man, Cybern., Part B*, 2009, **39**, (1), pp. 117–128

[30] Wierman, M.: 'Measuring uncertainty in rough set theory', *Int. J. Gen. Syst.*, 1999, **28**, pp. 283–297

[31] Duntsch, I., Gediga, G.: 'Uncertainty measures of rough set prediction', *Artif. Intell.*, 1998, **106**, (1), pp. 109–137

[32] Liu, D., Li, T., Li, H.: 'A multiple-category classification approach with decision-theoretic rough sets', *FundamentaInformaticae*, 2012, **115**, pp. 173–188

[33] Ma, Z., Mi, J.: 'Boundary region-based rough sets and uncertainty measures in the approximation space', *Inf. Sci.*, 2016, **370**, pp. 239–255

[34] Qian, Y., Liang, J.: 'Combination entropy and combination granulation in rough set theory', *J. Uncertain. Fuzziness Knowl.-Based Syst.*, 2008, **16**, (2), pp. 179–193

[35] Słowinski, R., Stefanowski, J.: 'Handing various types of uncertainty in the rough set approach', in Ziarko, W.P. (Ed.): '*Rough sets, fuzzy sets and knowledge discovery*' (Springer, Banff, 1993), pp. 366–376

[36] Yao, Y.: 'Information-theoretic measures for knowledge discovery and data mining', in Karmeshu, (Ed.): '*Entropy measures, maximum entropy principle and emerging applications*' (Springer-Verlag, Berlin, 2003), pp. 115–136

[37] Zhu, P., Wen, Q.: 'Entropy and co-entropy of a covering approximation space', *Int. J. Approx. Reason.*, 2012, **53**, pp. 528–540

[38] Zhu, P., Wen, Q.: 'Information-theoretic measures associated with rough set approximations', *Inf. Sci.*, 2012, **12**, pp. 33–43