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Distance-based double-quantitative rough fuzzy sets with logic operations

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ABSTRACT

Based on various requirements, many generalized rough set models have been developed to alleviate the limitations of generic Pawlak rough set theory and tackle different categories of information systems. One of the limitations is that rough set models based on equivalence relation are only applicable to discrete data information systems, and not suitable for dealing with real-valued continuous data without any prior processing. Another limitation is that "classical" rough sets do not consider the quantitative information about the degree of overlap between equivalence classes and the basic set, so they cannot cope well with the quantification problems. In this paper, we propose a framework of distance-based double-quantitative rough fuzzy set (Db-Dq-RFS) with logic operation by forming a distance-based fuzzy similarity relation in an information system with continuous data to simultaneously solve the two limitations. It is presented how to construct the distance-based fuzzy similarity relation in a normalized information system, and how to use this fuzzy similarity relation to generate distance-based single-quantitative rough fuzzy set (Db-Sq-RFS) models and the Db-Dq-RFS models with logic operation. The proposed Db-Dq-RFS models can overcome certain limitations of the classical rough set model. Following further studies to discuss the decision rules with parameters variation in the four kinds of Db-Dq-RFS models, we present an illustrative example to interpret the proposed developments and to verify the effect of parameters variation on decision rules. To illustrate the effectiveness of the parameters variation on decision rules, experimental evaluation is performed using five datasets coming from the University of California–Irvine (UCI) repository.

1. Introduction

Rough set theory [39] is an extension of set theory and a such could be regarded as a mathematical tool to handle imprecision, vagueness and uncertainty in data analysis. This relatively new soft computing methodology has received great attention in recent years, and its usefulness has been confirmed through successful applications in many areas science and engineering, such as pattern recognition, data mining, image processing, and medical diagnosis. Rough set theory is

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built on the basis of the classification mechanism, it is classified as the equivalence relation in a specific universe, and the equivalence relation constitutes a partition of the universe. A concept, or more precisely the extension of a concept, is represented by a subset of a universe of objects and is approximated by a pair of definable concepts of a logic language. However, classical rough set models exhibit limitations in real-life applications. These limitations are mainly embodied and concluded as follows. Firstly, Pawlak rough sets do not deal well with the quantification problems. The relationship between equivalence classes and the basic set is so strict that there are no fault tolerance mechanisms available, and the quantitative information about the degree of overlap of the equivalence classes and the basic set is not taken into consideration. In fact, there are some degrees of inclusion relations between sets, and the extent of overlap of sets is important information to consider in applications. Secondly, the requirement imposed on an equivalence relation in Pawlak rough set model is a stringent condition that has limited the application domains of the theory. When we use the rough set to deal with numerical data, the rough set theory based on equivalence relations is mainly applicable to the information systems with discrete data. In case of continuous numerical data, a discretization process has to be completed, and this preprocessing will commonly result in loss of information, reducing the classification accuracy.

To overcome the first limitation, improving the Pawlak rough set model by incorporating quantitative information is a promising direction and model expansions that include such quantification are of particular relevance [26,34,61,67,72,73]. The improved models are called quantitative rough set models; they include probabilistic rough set (PRS) model [59,60] and graded rough set (GRS) model [62]. As pointed out in [72,73], PRS and GRS are the two different and typical single-quantitative rough set models. As to the second limitation, one of the main directions of research is to develop generalized rough sets by using the non-equivalence relation to take the place of the equivalence relation [4,29]. Many researchers have presented the notion of approximation operators by using tolerance relation [21,57], neighborhood relation [17,51], similarity relation [3,44], and others [18,19,25–28,31,52] to solve the second limitation. Pawlak rough set model can be extended to a fuzzy domain by replacing the equivalence relation with fuzzy equivalence relation [18,37]. Fuzzy equivalence relation satisfies reflexivity, symmetry and transitivity. As a more general and extensive relation than fuzzy equivalence relation, the similarity relation only satisfies the properties of reflexivity and symmetry.

PRS model and its generalizations can be formulated based on the notion of rough membership functions and rough inclusion. Threshold values, serving as parameters, are applied to a rough membership function or a rough inclusion to obtain probabilistic or parameterized approximations. Three probabilistic rough set models have been proposed and studied intensively, which are decision-theoretic rough set (DTRS) model [2,9,22,23,33,43,46,63], variable precision rough set model [76], and Bayesian rough set model [75]. The main differences among these models are their different, but equivalent, formulations of probabilistic approximations and interpretations of the required parameters. Since Yao and Lin explored the relationships between rough sets and modal logics, they proposed the GRS model based on graded modal logics [62]. GRS model primarily considers the absolute quantitative information regarding the basic concept and knowledge granules, and it is also a generalization of the Pawlak rough set model. The regions of the GRS model also extend the corresponding notions used in the classical rough set models. Because the inclusion relation of the grade approximations does not hold any longer, positive and negative regions, upper and lower boundary regions are naturally proposed. They classify the universe more precisely and exhibit their own logical meanings related to the grade quantitative index. GRS model considers absolute quantitative information between equivalence classes and the basic concept [32,72,73]. PRS model and GRS model can reflect relative quantitative information and absolute quantitative information about the degree of overlap between equivalence classes and a basic concept, respectively. The relative and absolute quantitative information are two distinct objective sides that describe approximate space, and each has its own virtues and pertinent application environments, so that none can be neglected. Relative quantitative information and absolute quantitative information are two kinds of quantification mythologies encountered in certain applications. From the examples reported in [26,72], both quantification indexes exhibit a close, supplementary, and dialectical relationship, and each one actually has its own representation virtues and application environments. It should be noted that the existing models regarding to double quantification studied in [11,26,54,64,68–73] are all based on equivalence relations. That is to say, these studies can only overcome the first limitation, while cannot address the second limitation.

As it has been pointed out before, the second limitation can be eliminated by replacing the equivalence relation with similarity relation. From the results presented in [5,15,17,53], the distance provides a comprehensible perspective for characterizing the difference between two objects in a metric space. In other words, distances between objects can describe similarities between them, which means that the distance matrices between objects can be used to induce similarity relations. Many different kinds of distance functions have been proposed to work for numerical attribute values in the research field of statistics [6], pattern cognition [8], and cognitive computing [38,50,55]. Recently, Cook et al. presented a general framework for distance-based consensus in ordinal ranking models [7]. Gesu and Starovoitov investigated the distance function for the application of image comparison [13]. Angiulli et al. studied a distance-based detection and prediction of outliers [1]. Khalifeh et al. covered some new results on distance-based graph invariants [20]. Luxburg and Bousquet used the Lischitz functions to make the distance-based classifications in information system [36]. Song et al. investigated the self-similarity of objects in the complex networks [45]. Leu et al. proposed a distance-based fuzzy time series model for exchange rates forecasting [24]. Yu et al. developed a distance-based group decision-making methodology for multi-person multi-criteria emergency decision support [65]. Luukka proposed the similarity classifier based on modified probabilistic equivalence relations [35]. Liang et al. introduced a set distance to understand measures from rough set theory from the viewpoint of distance [30]. Inspired by the above distance-based studies, a novel fuzzy similarity relation based on the distance matrix is proposed in this paper to characterize the similarity between objects in an information system, which is called distance-based fuzzy similarity relation. The distance-based fuzzy similarity relation categorizes objects into the classes with fuzzy boundaries depend on their similarity, and it is presented to address the second limitation.

It should be noted that the models (Db-Sq-RFS and Db-Dq-RFS) proposed in this study are quite different from the typical fuzzy rough set model [10] addressed by Dubois and Prade and its generalizations [3,4,12,16,18,47–49,51]. The main differences are shown as follows. (1) The method of how to obtain the fuzzy similarity relation is not discussed in Dubois's fuzzy rough set model. However, this study provides a visual and systematic formation process for obtaining the distance-based fuzzy similarity relation. (2) The set type of the obtained approximations in this paper is completely different from the set type of the approximations in Dubois's fuzzy rough set model. In Dubois's fuzzy rough set model, the membership functions for upper and lower approximations are obtained firstly, which means that the upper and lower approximations are fuzzy sets. However, the upper and lower approximations defined in this study are classical sets, and it means that we use two classical sets (upper and lower approximations) to approximate a given fuzzy set. (3) The quantitative information is not reflected on Dubois's fuzzy rough set model. While the Db-Sq-RFS models can reflect one kind of quantitative information, and the Db-Dq-RFS models can reflect two kinds of quantitative information in their upper and lower approximations.

From the above analysis, we observe that only one of the two limitations mentioned above are considered in most existing works on generalized rough set models, there are few models that could handle both of these two issues at the same time. The models presented in this study are aimed to address these two issues simultaneously. This is the motivation behind the research presented here. By considering the distance-based fuzzy similarity relation in rough set models with double quantification, our objective is to introduce a framework of Db-Dq-RFS model with logic operation to solve these two issues simultaneously. We tend to classify data for simplification and memorization of information in our cognition and use the Db-Dq-RFS to classify the vague and fuzzy data. The paper is organized as follows. Related concepts and definitions are reviewed briefly in Section 2. In Section 3, we discuss the distance matrix between any two objects, and present the formation of distance-based fuzzy similarity relation to deal with categorical data in a normalized information system. In Section 4, we first investigate two kinds of Db-Sq-RFS models, which are Db-PRFS and Db-GRFS, then we present Db-Dq-RFS models by building upper and lower approximations with fuzzy similarity classes. In Section 5, we discuss decision rules with parameters variation for each Db-Dq-RFS model, and a detailed example is presented to illustrate the provided models and their corresponding decision rules. In Section 6, we do the experimental testing by five datasets from the UCI datasets, and relevant comparisons are made with existing models. Finally, Section 7 covers some conclusions.

2. Related work and fundamentals

In this section, some basic preliminaries and necessary concepts are briefly introduced and reviewed. For the non-empty set U, the class of all subsets of U is denoted by $\mathcal{P}(U)$, and the class of all fuzzy subsets of U is denoted by $\mathcal{F}(U)$. The complementary set of X is denoted by $\sim X$.

Zadeh introduced fuzzy sets [66] in which a fuzzy subset $\widetilde{X}$ of U is defined as a membership function assigning to each element x of U a certain degree of membership. The value $\widetilde{X}(x) \in [0,1]$ and $\widetilde{X}(x)$ is referred to as the membership degree of x to the fuzzy set $\widetilde{X}$. For any fuzzy concepts $\widetilde{X}, \widetilde{Y} \in \mathcal{F}(U)$, we say that $\widetilde{X}$ is contained in $\widetilde{Y}$, denoted by $\widetilde{X} \subseteq \widetilde{Y}$, if $\widetilde{X}(x) \le \widetilde{Y}(x)$ for all $x \in U$; we say that $\widetilde{X} = \widetilde{Y}$ if and only if $\widetilde{X} \subseteq \widetilde{Y}$ and $\widetilde{X} \supseteq \widetilde{Y}$. The basic operations on fuzzy set are described as follows.

$$(\widetilde{X} \cup \widetilde{Y})(x) = max\{\widetilde{X}(x), \widetilde{Y}(x)\} = \widetilde{X}(x) \vee \widetilde{Y}(x);$$
$$(\widetilde{X} \cap \widetilde{Y})(x) = min\{\widetilde{X}(x), \widetilde{Y}(x)\} = \widetilde{X}(x) \wedge \widetilde{Y}(x);$$
$$\sim \widetilde{X}(x) = 1 - \widetilde{X}(x),$$

where "$\vee$" and "$\wedge$" are the maximum operator and minimum operator, respectively.

The notion of information system provides a convenient basis for the representation of objects in terms of their attributes.

Definition 2.1 *(Information system).* An information system is a tuple (U, A, V, f), where U is a non-empty and finite set of objects, and $U = \{x_1, x_2, \cdots, x_n\}$; A is a non-empty and finite set of attributes, and $A = \{a_1, a_2, \cdots, a_m\}$; $f = \{f_l | U \to V_l, l \le m\}$, f_l is the value of a_l on $x \in U$, V_l is the domain of a_l, $a_l \in A$.

A decision information system is an information system $(U, A \cup D, V, f)$, where $A \cap D = \emptyset$, A is the condition attribute set, while D is called the decision attribute set. In the decision information system, R_A and R_D are equivalence relations induced by A and D, respectively. The constructions of R_A and R_D are expressed as follows. $R_A = \{(x, y) \in U \times U | f_l(x) = f_l(y), \forall a_l \in A\}$ and $R_D = \{(x, y) \in U \times U | f_k(x) = f_k(y), \forall d_k \in D\}$. It is easy to see that R_A partitions the universe U into disjoint subsets, the same to R_D. Such a partition of the universe is a quotient set of U and is denoted by $U/R_A = \{[x]_{R_A} | x \in U\}$, where $[x]_{R_A}$ is called equivalence class containing x with respect to R_A, and $[x]_{R_A} = \{y \in U | (x, y) \in R_A\}$. If $R_A \subseteq R_D$, then we say that $(U, A \cup D, V, f)$ is consistent, otherwise it is inconsistent. For the sake of simplicity, in the sequel, we set $D = \{d\}$, $V_d = \{1, 2, \cdots, r\}$, and $U/R_D = \{D_1, D_2, \cdots, D_r\}$. D_j is the decision class $D_j = \{x \in U | d(x) = j\}$.

Definition 2.2 *(Pawlak rough set [39]).* Let $I = (U, A, V, f)$ be an information system and R be an equivalence relation. For any $X \subseteq U$, one can characterize X by a pair of upper and lower approximations which are

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\},$$
$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\}.$$

For a target concept $X \subseteq U$, if $\overline{R}(X) = \underline{R}(X)$, X is called definable set or set in rough approximation space; and if $\overline{R}(X) \neq \underline{R}(X)$, then X is called Pawlak rough set. Obviously, both upper approximation $\overline{R}(X)$ and lower approximation $\underline{R}(X)$ of a target set X are two sets. Three disjoint regions can be obtained as: $Pos(X) = \underline{R}(X), Neg(X) =\sim \overline{R}(X)$ and $Bn(X) = \overline{R}(X) - \underline{R}(X)$ are called the positive region, negative region, and boundary region of X, respectively.

In Pawlak rough sets, the relationships between equivalence classes and the basic set are strict that there are no fault tolerance mechanisms. Quantitative information about the degree of overlap of the equivalence classes and the basic set is not taken into consideration. Therefore, neither wider relationships nor quantitative information can be utilized. Naturally the study of PRS and GRS regard to relative quantitative information and absolute quantitative information are presented, respectively.

Let U be a non-empty and finite set of objects, one can define P as probability measure if the set-valued function P maps from 2^U to $[0, 1]$. P satisfies the two conditions: $P(U) = 1$; if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$. Then P is a probability measure of σ-algebra which is combined by the family subset of U. In the following definition, we introduce the notion of the PRS model.

Definition 2.3 *(PRS model [58,63,74]).* Let $I = (U, A, V, f)$ be an information system and R be an equivalence relation. Given two parameters α, β $(0 \leq \beta < \alpha \leq 1)$, for any $X \subseteq U$, the upper and lower approximations based on thresholds α, β are defined as follows

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\} = \cup\{[x]_R | P(X|[x]_R) > \beta\},$$
$$\underline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\} = \cup\{[x]_R | P(X|[x]_R) \geq \alpha\}.$$

If $\overline{R}_{(\alpha,\beta)}(X) = \underline{R}_{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a PRS. Accordingly, the positive, negative and boundary regions of PRS model are

$$Pos_{(\alpha,\beta)}(X) = \underline{R}_{(\alpha,\beta)}(X);$$
$$Neg_{(\alpha,\beta)}(X) =\sim \overline{R}_{(\alpha,\beta)}(X);$$
$$Bn_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_{(\alpha,\beta)}(X).$$

The acceptance of PRS is merely due to the fact that they are defined by using probabilistic information and are more general and flexible [43]. The PRS model uses conditional probability to quantify the degree of set inclusion. The conditional probability is calculated by the rough membership function $P(X|[x]_R) = |[x]_R \cap X|/|[x]_R|$, which implies the relative quantitative information about the degree of overlap between equivalence classes and a basic set. Thresholds imposed on the probability are used to define rough set approximations. The threshold values, known as parameters, are applied to a rough membership [40] or a rough inclusion [42] to obtain probabilistic or parameterized approximations. In the rough set theory literature, the notion of rough inclusion, introduced explicitly by Polkowski and Skowron [41,42], has been studied using other names, including relative degree of misclassification [76], majority inclusion relation [76], inclusion degrees [56], and so on. Rough inclusion functions are mappings with which one can measure the degree of inclusion of a set in a set [14]. As a special case of rough inclusion, the rough membership is calculated from data as the ratio of objects from elementary set $[x]_R$ that belongs to X.

Yao and Lin explored the relationship between rough sets and modal logics and proposed the GRS model based on graded modal logics [62,73]. The GRS is different from the PRS in the description of this quantification.

Definition 2.4 *(GRS model [62,73]).* Let $I = (U, A, V, f)$ be an information system and R be an equivalence relation. Suppose k is a non-negative integer called "grade", for any $X \subseteq U$,

$$\overline{R}_k(X) = \{x \in U|\ |[x]_R \cap X| > k\} = \cup\{[x]_R|\ |[x]_R \cap X| > k\},$$
$$\underline{R}_k(X) = \{x \in U|\ |[x]_R| - |[x]_R \cap X| \leq k\} = \cup\{[x]_R|\ |[x]_R| - |[x]_R \cap X| \leq k\}$$

are called grade k upper and lower approximations of X, respectively. If $\overline{R}_k(X) = \underline{R}_k(X)$, then X is called a definable set by grade k; otherwise, X is called a rough set by grade k. $\overline{R}_k$ and $\underline{R}_k$ are called grade k upper and lower approximation operators, respectively. If $k = 0$, then $\overline{R}_k(X) = \overline{R}(X)$, $\underline{R}_k(X) = \underline{R}(X)$. Therefore, the classical rough set model is a special case of the GRS model.

It should be noted that the upper approximation $\overline{R}_k(X)$ is the union of the equivalence classes whose number of elements inside X exceed k; $\underline{R}_k(X)$ is the union of the equivalence classes whose numbers of elements outside X are at most k. GRS model primarily considers the absolute quantitative information regarding the equivalence classes and the basic concept. The regions of the GRS model are extensions of grade approximations. Because the inclusion relation of the grade approximation does not hold any longer, positive and negative regions, upper and lower boundary regions are naturally proposed. We form the following regions:

$$Pos_k(X) = \overline{R}_k(X) \cap \underline{R}_k(X);$$
$$Neg_k(X) = \sim (\overline{R}_k(X) \cup \underline{R}_k(X));$$
$$UBn_k(X) = \overline{R}_k(X) - \underline{R}_k(X);$$
$$LBn_k(X) = \underline{R}_k(X) - \overline{R}_k(X);$$
$$Bn_k(X) = UBn_k(X) \cup LBn_k(X),$$

where $Pos_k(X), Neg_k(X), UBn_k(X), LBn_k(X)$ and $Bn_k(X)$ are called grade k positive region, negative region, upper boundary region, lower boundary region, and boundary region of X.

PRS model and GRS model can reflect relative quantitative information and absolute quantitative information about the degree of overlap between equivalence classes and a basic concept, respectively. The relative and absolute quantitative information are the two distinct objective sides that describe approximate space, and each has its own virtues and pertinent application environments.

3. Formation of distance-based fuzzy similarity relation

The classical rough set models can only be used to deal with discrete numerical data in information systems, while cannot be used to handle continuous numerical data. In this section, we present a formation of distance-based fuzzy similarity relation to tackle categorical data in a numerical information system. The distance-based fuzzy similarity relation is established on a basis of a distance matrix, then the distance-based fuzzy similarity classes are obtained based on the formed distance-based fuzzy similarity relation. It should be noted that the proposed formation processes of distance-based fuzzy similarity relation is also applicable to discrete numerical data. The information systems studied in this paper are all numerical information systems.

Definition 3.1. [5] Let $I = (U, A, V, f)$ be an information system. For $x_i \in U$ and $a_j \in A$, the values of the information system are normalized as

$$f(x_i, a_j) = \frac{\upsilon(x_i, a_j) - min(\upsilon(x_k, a_j))}{max(\upsilon(x_k, a_j)) - min(\upsilon(x_k, a_j))},$$

where $\upsilon(x_i, a_j)$ is the value of x_i on a_j, $max(\upsilon(x_k, a_j))$ is the maximal value on a_j, and $min(\upsilon(x_k, a_j))$ is the minimal value on a_j.

After normalizing the information system, we introduce the distance between two objects in an information system in the following Definition 3.2. A detailed description on the distance function can be found in [53].

Definition 3.2. [53] Let $I = (U, A, V, f)$ be an information system. x_i and x_j are the two objects in U. For a subset $B \subseteq A$, the distance metric d_{ij} on B is calculated as

$$d_{ij} = (\sum_{k=1}^{m} |f(x_i, a_k) - f(x_j, a_k)|^p)^{1/p},$$

where $f(x_i, a_k)$ is the value of x_i on a_k and $m = |B|$. The variable values of the parameter p represent different kinds of distances. If $p = 1$, it is a Manhattan distance; if $p = 2$, it is an Euclidean distance; if $p = \infty$, it is a Chebychev distance.

Definition 3.3. Let $I = (U, A, V, f)$ be an information system and $B \subseteq A$. A distance matrix of B is defined by

$$D(B) = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & d_{ij} & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix},$$

where $d_{ij} \in [0, 1]$ is the distance between two objects x_i and x_j on B.

The above obtained distance matrix describes how far they are between two objects in an information system. The following Definition 3.4 is about the comparison between two distance matrices.

Definition 3.4. Let $I = (U, A, V, f)$ be an information system and $B, C \subseteq A$. $D(B)$ and $D(C)$ are two distance matrices on B and C. b_{ij} and c_{ij} are elements of $D(B)$ and $D(C)$, respectively. $\forall i, j \in \{1, 2, \cdots, n\}$, if $b_{ij} \le c_{ij}$, then the distance matrix $D(B)$ is not larger than $D(C)$, denoted as $D(B) \le D(C)$.

Theorem 3.1. *Let $I = (U, A, V, f)$ be an information system and $B, C \subseteq A$. $D(B)$ and $D(C)$ are two distance matrices on B and C. If $B \subseteq C$, then $D(B) \le D(C)$.*

Proof. Suppose $|B| = m$ and $|C| = n$. Because $B \subseteq C$, then we can get that $m \le n$. By Definition 3.2, we obtain $b_{ij} = (\sum_{k=1}^{m} |f(x_i, a_k) - f(x_j, a_k)|^p)^{1/p}$ and $c_{ij} = (\sum_{k=1}^{n} |f(x_i, a_k) - f(x_j, a_k)|^p)^{1/p}$. That is to say $b_{ij} \le c_{ij}$. Then we have $D(B) \le D(C)$. □

It can be seen from the above Theorem 3.1 that if the two attribute subsets satisfy $B \subseteq C$, then Theorem 3.1 provides a basic judgment method for comparing the two distance matrices $D(B)$ and $D(C)$, which is $D(B) \le D(C)$.

Definition 3.5. Let $I = (U, A, V, f)$ be an information system and $B \subseteq A$. $D(A)$ and $D(B)$ are two distance matrices on A and B. The $D'(B)$ is normalized as

$$D'(B) = \frac{D(B)}{max(D(A))},$$

the $max(D(A))$ represents for the maximal value of elements of distance matrix $D(A)$.

Definition 3.6. Let $I = (U, A, V, f)$ be an information system and a subset $B \subseteq A$. Suppose $D'(B)$ is a normalized distance matrix on B. The element d_{ij} of $D'(B)$ is the distance between two objects x_i and x_j on B. Then $\widetilde{B}$ is a distance-based fuzzy similarity relation on B, denoted by the distance-based fuzzy similarity relation matrix $S(\widetilde{B})$ as follows.

$$S(\widetilde{B}) = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & s_{ij} & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix},$$

where $0 \le s_{ij} = 1 - d_{ij} \le 1$ is the similarity value obtained between two objects x_i and x_j on attribute set B.

We can see that the distance-based fuzzy similarity relation matrix is derived from a distance matrix by the formula $S(\widetilde{B}) = 1 - D'(B)$. The distance-based fuzzy similarity relation matrix represents how similar they are between two objects. It shows the similarity relation between the two objects. It is easy to see that $\widetilde{B}$ is a fuzzy similarity relation, which means $\forall x, y \in U$, $\widetilde{B}$ satisfies the following two properties,

(1) Reflexivity: $\widetilde{B}(x, x) = 1$.
(2) Symmetry: $\widetilde{B}(x, y) = \widetilde{B}(y, x)$.

Theorem 3.2. *Let $I = (U, A, V, f)$ be an information system and $B, C \subseteq A$. The $\widetilde{B}, \widetilde{C}$ are two distance-based fuzzy similarity relations on B, C. If $B \subseteq C$, then $\widetilde{B} \supseteq \widetilde{C}$.*

Proof. Since $B \subseteq C$, according to Theorem 3.1, we can obtain $D(B) \le D(C)$. By Definition 3.6, we know $S(\widetilde{B}) \ge S(\widetilde{C})$, then $\widetilde{B}(x, y) \ge \widetilde{C}(x, y)$. Therefore, $\widetilde{B} \supseteq \widetilde{C}$. □

Given two distance-based fuzzy similarity relations $\widetilde{B}$ and $\widetilde{C}$, the complement, intersection, union and inclusion operators are defined as follows.

(1) Complement: $\widetilde{E} = \sim \widetilde{B} \Leftrightarrow \widetilde{E}(x, y) = 1 - \widetilde{B}(x, y)$.
(2) Intersection: $\widetilde{E} = \widetilde{B} \cap \widetilde{C} \Leftrightarrow \widetilde{E}(x, y) = min\{\widetilde{B}(x, y), \widetilde{C}(x, y)\}$.
(3) Union: $\widetilde{E} = \widetilde{B} \cup \widetilde{C} \Leftrightarrow \widetilde{E}(x, y) = max\{\widetilde{B}(x, y), \widetilde{C}(x, y)\}$.
(4) Inclusion: $\widetilde{B} \subseteq \widetilde{C} \Leftrightarrow \widetilde{B}(x, y) \le \widetilde{C}(x, y)$.

Table 3.1
An information system.

U	a	b	c	d	e
x_1	0.54	0.21	0.64	0.13	0.38
x_2	0.06	0.63	0.73	0.62	0.43
x_3	0.38	0.44	0.51	0.74	0.67
x_4	0.58	0.57	0.73	0.37	0.07
x_5	0.58	0.58	0.34	0.71	0.73
x_6	0.56	0.56	0.87	0.60	0.68
x_7	0.21	0.51	0.31	0.45	0.29
x_8	0.51	0.48	0.84	0.48	0.91

Table 3.2
The normalized information system.

U	a	b	c	d	e
x_1	0.92	0.00	0.59	0.00	0.37
x_2	0.00	1.00	0.75	0.80	0.43
x_3	0.62	0.55	0.36	1.00	0.71
x_4	1.00	0.86	0.75	0.39	0.00
x_5	1.00	0.88	0.05	0.95	0.79
x_6	0.96	0.83	1.00	0.77	0.73
x_7	0.29	0.71	0.00	0.52	0.26
x_8	0.87	0.64	0.95	0.57	1.00

Being different distance-based fuzzy similarity relation may be formed by different distance metrics of objects introduced in Definition 3.2. It can be easily verified that these operators satisfy the following properties

(1) $\widetilde{B} \cap \widetilde{C} = \widetilde{C} \cap \widetilde{B}, \widetilde{B} \cup \widetilde{C} = \widetilde{C} \cup \widetilde{B}$;
(2) $(\widetilde{B} \cap \widetilde{C}) \cap \widetilde{E} = \widetilde{B} \cap (\widetilde{C} \cap \widetilde{E}), (\widetilde{B} \cup \widetilde{C}) \cup \widetilde{E} = \widetilde{B} \cup (\widetilde{C} \cup \widetilde{E})$;
(3) $\sim (\widetilde{B} \cap \widetilde{C}) =\sim \widetilde{B} \cup \sim \widetilde{C}, \sim (\widetilde{B} \cup \widetilde{C}) =\sim \widetilde{B} \cap \sim \widetilde{C}$;
(4) $\sim (\sim \widetilde{B}) = \widetilde{B}$.

Different from the equivalence relation of generating a partition of the domain on an information system, a similarity relation induces intersecting blocks, which constitute a covering of the domain. Correspondingly, a distance-based fuzzy similarity relation induces a fuzzy covering of the domain and some fuzzy similarity classes. The distance-based fuzzy similarity classes are also called distance-based fuzzy similarity granules.

Definition 3.7. Let $I = (U, A, V, f)$ be an information system, $B \subseteq A$. The $\widetilde{B}$ is a distance-based fuzzy similarity relation on B. A fuzzy cover of the domain induced by a distance-based fuzzy similarity relation is defined by

$$\frac{U}{\widetilde{B}} = \{[x_i]_{\widetilde{B}}\}_{i=1}^{n},$$

where $[x_i]_{\widetilde{B}} = \{s_{i1}/x_1 + s_{i2}/x_2 + \cdots + s_{in}/x_n\}$. $[x_i]_{\widetilde{B}}$ is the distance-based fuzzy similarity class belonging to x_i. s_{ij} is a similarity value between x_i and x_j. The symbol "+" denotes a union of elements. The cardinality of a distance-based fuzzy similarity granule $[x_i]_{\widetilde{B}}$ is defined as

$$|[x_i]_{\widetilde{B}}| = \sum_{j=1}^{n} s_{ij}.$$

The above process provides us with a visible way to form the distance-based fuzzy similarity relation and the corresponding distance-based fuzzy similarity classes (granules). Here we present an example to introduce and interpret the above processes.

Example 3.1. Table 3.1 is an information system from the reference [5], where the object set $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and the attribute set $A = \{a, b, c, d, e\}$. Table 3.2 is a normalized information system from Table 3.1.

Table 3.2 presents a normalized information system resulting from the initial information system by using the method outlined in Definition 3.1.

We use the Euclidean distance to granulate objects in the normalized information system. Two distance matrices of the attribute sets $A_1 = \{a, b, c\}$ and $A = \{a, b, c, d, e\}$ are formed as

$$D(A_1)=\begin{pmatrix}0.00&1.37&0.67&0.88&1.03&0.93&1.12&0.74\\1.37&0.00&0.86&1.01&1.22&1.01&0.85&0.96\\0.67&0.86&0.00&0.63&0.59&0.78&0.51&0.65\\0.88&1.01&0.63&0.00&0.70&0.25&1.04&0.32\\1.03&1.22&0.59&0.70&0.00&0.95&0.73&0.93\\0.93&1.01&0.78&0.25&0.95&0.00&1.21&0.22\\1.12&0.85&0.51&1.04&0.73&1.21&0.00&1.11\\0.74&0.96&0.65&0.32&0.93&0.22&1.11&0.00\end{pmatrix},$$

$$D(A)=\begin{pmatrix}0.00&1.59&1.25&1.03&1.47&1.26&1.24&1.13\\1.59&0.00&0.93&1.17&1.28&1.05&0.91&1.14\\1.25&0.93&0.00&1.13&0.60&0.82&0.83&0.83\\1.03&1.17&1.13&0.00&1.19&0.86&1.08&1.07\\1.47&1.28&0.60&1.19&0.00&0.97&1.00&1.03\\1.26&1.05&0.82&0.86&0.97&0.00&1.32&0.40\\1.24&0.91&0.83&1.08&1.00&1.32&0.00&1.33\\1.13&1.14&0.83&1.07&1.03&0.40&1.33&0.00\end{pmatrix}.$$

We can easily obtain that $max(D(A))=1.59$, from Definition 3.5, the above two distance matrices $D(A_1)$ and $D(A)$ are normalized as follows.

$$D'(A_1)=\begin{pmatrix}0.00&0.86&0.42&0.55&0.65&0.59&0.71&0.46\\0.86&0.00&0.54&0.64&0.77&0.63&0.54&0.60\\0.42&0.54&0.00&0.40&0.37&0.49&0.32&0.41\\0.55&0.64&0.40&0.00&0.44&0.16&0.66&0.20\\0.65&0.77&0.37&0.44&0.00&0.60&0.46&0.59\\0.59&0.63&0.49&0.16&0.60&0.00&0.76&0.14\\0.71&0.54&0.32&0.66&0.46&0.76&0.00&0.70\\0.46&0.60&0.41&0.20&0.59&0.14&0.70&0.00\end{pmatrix},$$

$$D'(A)=\begin{pmatrix}0.00&1.00&0.79&0.65&0.92&0.79&0.78&0.71\\1.00&0.00&0.58&0.74&0.81&0.66&0.57&0.72\\0.79&0.58&0.00&0.71&0.38&0.51&0.52&0.53\\0.65&0.74&0.71&0.00&0.75&0.54&0.68&0.67\\0.92&0.81&0.38&0.75&0.00&0.61&0.63&0.65\\0.79&0.66&0.51&0.54&0.61&0.00&0.83&0.25\\0.78&0.57&0.52&0.68&0.63&0.83&0.00&0.84\\0.71&0.72&0.52&0.67&0.65&0.25&0.84&0.00\end{pmatrix}.$$

Based on the formation of distance-based fuzzy similarity relation matrix in Definition 3.6, we construct distance-based fuzzy similarity relation matrices as follows.

$$S(\widetilde{A}_1)=\begin{pmatrix}1.00&0.14&0.58&0.45&0.35&0.42&0.29&0.54\\0.14&1.00&0.46&0.36&0.23&0.37&0.46&0.40\\0.58&0.46&1.00&0.60&0.63&0.51&0.68&0.59\\0.45&0.36&0.60&1.00&0.56&0.84&0.34&0.80\\0.35&0.23&0.63&0.56&1.00&0.40&0.54&0.41\\0.42&0.37&0.51&0.84&0.40&1.00&0.24&0.86\\0.29&0.46&0.68&0.34&0.54&0.24&1.00&0.30\\0.54&0.40&0.59&0.80&0.41&0.86&0.30&1.00\end{pmatrix},$$

$$S(\widetilde{A})=\begin{pmatrix}1.00&0.00&0.21&0.35&0.08&0.21&0.22&0.29\\0.00&1.00&0.42&0.26&0.19&0.34&0.43&0.28\\0.21&0.42&1.00&0.29&0.62&0.49&0.48&0.48\\0.35&0.26&0.29&1.00&0.25&0.46&0.32&0.33\\0.08&0.19&0.62&0.25&1.00&0.39&0.37&0.35\\0.21&0.34&0.49&0.46&0.39&1.00&0.17&0.75\\0.22&0.43&0.48&0.32&0.37&0.17&1.00&0.16\\0.29&0.28&0.48&0.33&0.35&0.75&0.16&1.00\end{pmatrix}.$$

Then, the distance-based fuzzy similarity classes with regard to the distance-based fuzzy similarity relation $\widetilde{A}$ are listed as follows.

$$[x_1]_{\widetilde{A}} = \{\frac{1}{x_1} + \frac{0}{x_2} + \frac{0.21}{x_3} + \frac{0.35}{x_4} + \frac{0.08}{x_5} + \frac{0.21}{x_6} + \frac{0.22}{x_7} + \frac{0.29}{x_8}\},$$
$$[x_2]_{\widetilde{A}} = \{\frac{0}{x_1} + \frac{1}{x_2} + \frac{0.42}{x_3} + \frac{0.26}{x_4} + \frac{0.19}{x_5} + \frac{0.34}{x_6} + \frac{0.43}{x_7} + \frac{0.28}{x_8}\},$$
$$[x_3]_{\widetilde{A}} = \{\frac{0.21}{x_1} + \frac{0.42}{x_2} + \frac{1}{x_3} + \frac{0.29}{x_4} + \frac{0.62}{x_5} + \frac{0.49}{x_6} + \frac{0.48}{x_7} + \frac{0.48}{x_8}\},$$
$$[x_4]_{\widetilde{A}} = \{\frac{0.35}{x_1} + \frac{0.26}{x_2} + \frac{0.29}{x_3} + \frac{1}{x_4} + \frac{0.25}{x_5} + \frac{0.46}{x_6} + \frac{0.32}{x_7} + \frac{0.33}{x_8}\},$$
$$[x_5]_{\widetilde{A}} = \{\frac{0.08}{x_1} + \frac{0.19}{x_2} + \frac{0.62}{x_3} + \frac{0.25}{x_4} + \frac{1}{x_5} + \frac{0.39}{x_6} + \frac{0.37}{x_7} + \frac{0.35}{x_8}\},$$
$$[x_6]_{\widetilde{A}} = \{\frac{0.21}{x_1} + \frac{0.34}{x_2} + \frac{0.49}{x_3} + \frac{0.46}{x_4} + \frac{0.39}{x_5} + \frac{1}{x_6} + \frac{0.17}{x_7} + \frac{0.75}{x_8}\},$$
$$[x_7]_{\widetilde{A}} = \{\frac{0.22}{x_1} + \frac{0.43}{x_2} + \frac{0.48}{x_3} + \frac{0.32}{x_4} + \frac{0.37}{x_5} + \frac{0.17}{x_6} + \frac{1}{x_7} + \frac{0.16}{x_8}\},$$
$$[x_8]_{\widetilde{A}} = \{\frac{0.29}{x_1} + \frac{0.28}{x_2} + \frac{0.48}{x_3} + \frac{0.33}{x_4} + \frac{0.35}{x_5} + \frac{0.75}{x_6} + \frac{0.16}{x_7} + \frac{1}{x_8}\}.$$

After discussing the distance-based fuzzy similarity classes, it is important to introduce the following operators between these distance-based fuzzy similarity classes.

Definition 3.8. Given two distance-based fuzzy similarity classes $[x_i]_{\widetilde{A}}$ and $[x_i]_{\widetilde{B}}$. The complement, intersection, union and inclusion operations between these classes are defined as

- $\sim [x_i]_{\widetilde{A}} = \{(1-a_{i1})/x_1 + (1-a_{i2})/x_2 + \cdots + (1-a_{in})/x_n\} = \sum_{j=1}^{n} \frac{1-a_{ij}}{x_j}$,
- $[x_i]_{\widetilde{A}} \cap [x_i]_{\widetilde{B}} = \{min(a_{i1}, b_{i1})/x_1 + min(a_{i2}, b_{i2})/x_2 + \cdots + min(a_{in}, b_{in})/x_n\} = \sum_{j=1}^{n} \frac{a_{ij} \wedge b_{ij}}{x_j}$,
- $[x_i]_{\widetilde{A}} \cup [x_i]_{\widetilde{B}} = \{max(a_{i1}, b_{i1})/x_1 + max(a_{i2}, b_{i2})/x_2 + \cdots + max(a_{in}, b_{in})/x_n\} = \sum_{j=1}^{n} \frac{a_{ij} \vee b_{ij}}{x_j}$,
- $[x_i]_{\widetilde{A}} \subseteq [x_i]_{\widetilde{B}} \Leftrightarrow \forall j \in \{1, 2, \cdots, n\}, a_{ij} \le b_{ij} \Leftrightarrow \sum_{j=1}^{n} \frac{a_{ij}}{x_j} \le \sum_{j=1}^{n} \frac{b_{ij}}{x_j}$,

where a_{ij} is the similarity value between x_i and x_j on $\widetilde{A}$, b_{ij} is the similarity value between x_i and x_j on $\widetilde{B}$.

In an information system $I = (U, A, V, f)$, for two subsets $B, C \subseteq A$, $\frac{U}{\widetilde{B}}$ and $\frac{U}{\widetilde{C}}$ are two fuzzy covers of the domain induced by on $\widetilde{B}$ and $\widetilde{C}$, respectively.

Definition 3.9. We say that $\frac{U}{\widetilde{B}}$ is thinner than $\frac{U}{\widetilde{C}}$ if it satisfies the following partial order

$$\frac{U}{\widetilde{B}} \preceq \frac{U}{\widetilde{C}} \Leftrightarrow \forall\, [x_i]_{\widetilde{B}} \in \frac{U}{\widetilde{B}},\ \exists\, [x_i]_{\widetilde{C}} \in \frac{U}{\widetilde{C}} \text{ s.t. } [x_i]_{\widetilde{B}} \subseteq [x_i]_{\widetilde{C}}.$$

Theorem 3.3. *Let $I = (U, A, V, f)$ be an information system. For two distance-based subsets $B, C \subseteq A$, $\widetilde{B}, \widetilde{C}$ be two fuzzy similarity relations on B, C, respectively, then we have*

$$C \subseteq B \Rightarrow \frac{U}{\widetilde{B}} \preceq \frac{U}{\widetilde{C}}.$$

Proof. From Theorem 3.2, we obtain $\widetilde{B} \subseteq \widetilde{C}$, which means for each i and j, $b_{ij} \le c_{ij}$. From Definition 3.8, we get $[x_i]_{\widetilde{B}} \subseteq [x_i]_{\widetilde{C}}$. That is to say $\frac{U}{\widetilde{B}} \preceq \frac{U}{\widetilde{C}}$. □

To form the distance-based fuzzy similarity relation discussed in this section, we need to normalize the information system and then define the normalized distance matrix in the normalized information system. For simplicity and without loss of generality, we suppose all the values of the following discussed information systems and all the distance matrices are normalized.

4. Distance-based double-quantitative approximation spaces with logic operations

We present the formation of distance-based fuzzy similarity relation in the previous section, and also provide the distance-based fuzzy similarity classes based on the proposed distance-based fuzzy similarity relation. In this section, we investigate two kinds of Db-Sq-RFS (distance-based single-quantitative rough fuzzy set) models and four kinds of Db-Dq-RFS models (distance-based double-quantitative rough fuzzy set) with logic operation.

4.1. Db-Sq-RFS model

The Db-Sq-RFS model contains two models, which are distance-based probabilistic rough fuzzy set (Db-PRFS) and distance-based graded rough fuzzy set (Db-GRFS). The Db-PRFS model considers the relative quantitative information in their upper approximation and lower approximation, and the Db-GRFS model considers the absolute quantitative information in their upper and lower approximations.

Definition 4.1 *(Db-PRFS).* Given an information system $I=(U,A,V,f)$, for a subset $R\subseteq A$, the $\widetilde{R}$ is a distance-based fuzzy similarity relation on R. For any $\widetilde{X}\in\mathcal{F}(U)$ and $0\le\beta<\alpha\le 1$. The distance-based probabilistic upper and lower approximation operators are defined as follows.

$$\overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X})=\{x\in U\,|\,\frac{|[x]_{\widetilde{R}}\cap\widetilde{X}|}{|[x]_{\widetilde{R}}|}>\beta\},$$
$$\underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X})=\{x\in U\,|\,\frac{|[x]_{\widetilde{R}}\cap\widetilde{X}|}{|[x]_{\widetilde{R}}|}\ge\alpha\}.$$

Based on the distance-based probabilistic upper and lower approximation operators, we determine a rough set model called the Db-PRFS model, which is denoted by $(U,\overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}),\underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}))$. Accordingly, the positive, negative and boundary regions of Db-PRFS model are

$$Pos_{(\alpha,\beta)}(\widetilde{X})=\underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X});$$
$$Neg_{(\alpha,\beta)}(\widetilde{X})=\sim\overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X});$$
$$Bn_{(\alpha,\beta)}(\widetilde{X})\;=\overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X})-\underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}).$$

Example 4.1 *(Continuation of Example 3.1).* In order to make the presented model more comprehensible, we give a real world application background to Table 3.1, where U is a universe which consists of 8 patients with the clinical features degree; the attributes a, b, c, d, and e are *Cough*, *Rhinorrhoea*, *Myodynia*, *Diarrhea* and *Nausea*, respectively. Consider a fuzzy set $\widetilde{X}=\{0.30/x_1+0.30/x_2+0.10/x_3+0.50/x_4+0.00/x_5+0.20/x_6+0.30/x_7+0.30/x_8\}$, which represents the initial diagnosis of each patient suffering from a cold. The two parameters are set as $\alpha=0.60$, $\beta=0.45$. For the attribute set R, we get

$$|[x_1]_{\widetilde{R}}\cap\widetilde{X}|=0.30+0.00+0.10+0.35+0.00+0.20+0.22+0.29=1.36,$$
$$|[x_2]_{\widetilde{R}}\cap\widetilde{X}|=0.00+0.30+0.10+0.26+0.00+0.20+0.30+0.28=1.44,$$
$$|[x_3]_{\widetilde{R}}\cap\widetilde{X}|=0.21+0.30+0.10+0.29+0.00+0.20+0.30+0.30=1.70,$$
$$|[x_4]_{\widetilde{R}}\cap\widetilde{X}|=0.30+0.26+0.10+0.50+0.00+0.20+0.30+0.30=1.96,$$
$$|[x_5]_{\widetilde{R}}\cap\widetilde{X}|=0.08+0.19+0.10+0.25+0.00+0.20+0.30+0.30=1.42,$$
$$|[x_6]_{\widetilde{R}}\cap\widetilde{X}|=0.21+0.30+0.10+0.46+0.00+0.20+0.17+0.30=1.74,$$
$$|[x_7]_{\widetilde{R}}\cap\widetilde{X}|=0.22+0.30+0.10+0.32+0.00+0.17+0.30+0.16=1.57,$$
$$|[x_8]_{\widetilde{R}}\cap\widetilde{X}|=0.29+0.28+0.10+0.33+0.00+0.20+0.16+0.30=1.66,$$

and

$$|[x_1]_{\widetilde{R}}|=2.36,\;|[x_2]_{\widetilde{R}}|=2.92,\;|[x_3]_{\widetilde{R}}|=3.99,\;|[x_4]_{\widetilde{R}}|=3.26,$$
$$|[x_5]_{\widetilde{R}}|=3.25,\;|[x_6]_{\widetilde{R}}|=3.81,\;|[x_7]_{\widetilde{R}}|=3.15,\;|[x_8]_{\widetilde{R}}|=3.64.$$

Then the conditional probabilities of each object are calculated as

$$\frac{|[x_1]_{\widetilde{R}}\cap\widetilde{X}|}{|[x_1]_{\widetilde{R}}|}=\frac{1.46}{2.36}=0.617,\quad\frac{|[x_2]_{\widetilde{R}}\cap\widetilde{X}|}{|[x_2]_{\widetilde{R}}|}=\frac{1.44}{2.92}=0.493,$$

$$\frac{|[x_3]_{\widetilde{R}} \cap \widetilde{X}|}{|[x_3]_{\widetilde{R}}|} = \frac{1.70}{3.99} = 0.426, \quad \frac{|[x_4]_{\widetilde{R}} \cap \widetilde{X}|}{|[x_4]_{\widetilde{R}}|} = \frac{1.96}{3.26} = 0.601,$$
$$\frac{|[x_5]_{\widetilde{R}} \cap \widetilde{X}|}{|[x_5]_{\widetilde{R}}|} = \frac{1.42}{3.25} = 0.437, \quad \frac{|[x_6]_{\widetilde{R}} \cap \widetilde{X}|}{|[x_6]_{\widetilde{R}}|} = \frac{1.74}{3.81} = 0.457,$$
$$\frac{|[x_7]_{\widetilde{R}} \cap \widetilde{X}|}{|[x_7]_{\widetilde{R}}|} = \frac{1.57}{3.15} = 0.498, \quad \frac{|[x_8]_{\widetilde{R}} \cap \widetilde{X}|}{|[x_8]_{\widetilde{R}}|} = \frac{1.66}{3.64} = 0.456.$$

From Definition 4.1, the distance-based probabilistic upper approximation and lower approximation are calculated as follows.

$$\overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) = \{x_1, x_2, x_4, x_6, x_7, x_8\},$$
$$\underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) = \{x_1, x_4\}.$$

The corresponding three disjoint regions of Db-PRFS model are obtained as

$$Pos_{(\alpha,\beta)}(\widetilde{X}) = \{x_1, x_4\};$$
$$Neg_{(\alpha,\beta)}(\widetilde{X}) = \{x_3, x_5\};$$
$$Bn_{(\alpha,\beta)}(\widetilde{X}) \ = \{x_2, x_6, x_7, x_8\}.$$

The patients x_1 and x_4 belong to the positive region meaning that these two patients really need to receive treatment; x_3 and x_5 belong to the negative region means that they need not to get treatments; x_2, x_6, x_7, and x_8 belong to the boundary region means that they need further observations to make the decision.

Definition 4.2 *(Db-GRFS).* Given an information system $I = (U, A, V, f)$, for a subset $R \subseteq A$, the $\widetilde{R}$ is a distance-based fuzzy similarity relation on R. For any $\widetilde{X} \in \mathcal{F}(U)$, $k \in \mathbf{R}$ and $0 \le k \le |U|$, where $\mathbf{R}$ is a real number. The distance-based graded upper and lower approximation operators are defined as follows.

$$\overline{\widetilde{R}_k}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\},$$
$$\underline{\widetilde{R}_k}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k\}.$$

If $\overline{\widetilde{R}_k}(\widetilde{X}) = \underline{\widetilde{R}_k}(\widetilde{X})$, then $\widetilde{X}$ is called a distance-based definable fuzzy set by grade k; otherwise, $\widetilde{X}$ is called a distance-based rough fuzzy set by grade k. $\overline{\widetilde{R}_k}$ and $\underline{\widetilde{R}_k}$ are called distance-based grade k upper and lower approximation operators, respectively. Accordingly the positive region, negative region, upper boundary region, lower boundary region and boundary region of Db-GRFS are

$$Pos_k(\widetilde{X}) = \overline{\widetilde{R}}_k(\widetilde{X}) \cap \underline{\widetilde{R}}_k(\widetilde{X});$$
$$Neg_k(\widetilde{X}) = \sim (\overline{\widetilde{R}}_k(\widetilde{X}) \cup \underline{\widetilde{R}}_k(\widetilde{X}));$$
$$UBn_k(\widetilde{X}) = \overline{\widetilde{R}}_k(\widetilde{X}) - \underline{\widetilde{R}}_k(\widetilde{X});$$
$$LBn_k(\widetilde{X}) = \underline{\widetilde{R}}_k(\widetilde{X}) - \overline{\widetilde{R}}_k(\widetilde{X});$$
$$Bn_k(\widetilde{X}) = UBn_k(\widetilde{X}) \cup LBn_k(\widetilde{X}).$$

It should be noted that $k \in \mathbf{R}$ is different from the ones encountered in the previous works [11,26,54,68–73], where k satisfies the condition $k \in \mathbf{N}$.

Example 4.2 *(Continuation of Examples 3.1 and 4.1).* Consider grade $k = 1.6$; we obtain

$$|[x_1]_{\widetilde{R}}| - |[x_1]_{\widetilde{R}} \cap \widetilde{X}| = 1.00, \ |[x_2]_{\widetilde{R}}| - |[x_2]_{\widetilde{R}} \cap \widetilde{X}| = 1.48,$$
$$|[x_3]_{\widetilde{R}}| - |[x_3]_{\widetilde{R}} \cap \widetilde{X}| = 2.27, \ |[x_4]_{\widetilde{R}}| - |[x_4]_{\widetilde{R}} \cap \widetilde{X}| = 1.30,$$
$$|[x_5]_{\widetilde{R}}| - |[x_5]_{\widetilde{R}} \cap \widetilde{X}| = 1.83, \ |[x_6]_{\widetilde{R}}| - |[x_6]_{\widetilde{R}} \cap \widetilde{X}| = 2.07,$$
$$|[x_7]_{\widetilde{R}}| - |[x_7]_{\widetilde{R}} \cap \widetilde{X}| = 1.58, \ |[x_8]_{\widetilde{R}}| - |[x_8]_{\widetilde{R}} \cap \widetilde{X}| = 1.98.$$

Then the distance-based graded upper approximation and lower approximation are calculated as

$$\overline{\widetilde{R}_k}(\widetilde{X}) = \{x_3, x_4, x_6, x_8\},$$
$$\underline{\widetilde{R}_k}(\widetilde{X}) = \{x_1, x_2, x_4, x_7\}.$$

Table 4.1
Distance-based double-quantitative approximations of fuzzy set with logic operations.

Model	Double quantification	
I	$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$	$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$
II	$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$	$\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$
III	$\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$	$\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$
IV	$\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$	$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$

The above two distance-based graded upper and lower approximations are obtained directly according to Definition 4.2. For any $x_i \in U$, if the cardinality of $[x_i]_{\widetilde{R}} \cap \widetilde{X}$ is greater than grade k, then x_i belongs to distance-based graded upper approximation; if $|[x_i]_{\widetilde{R}}| - |[x_i]_{\widetilde{R}} \cap \widetilde{X}|$ is no more than k, then x_i belongs to the distance-based graded lower approximation. It should be noted that there may exist different objects belong to these distance-based graded upper approximation and lower approximation, respectively. In Example 4.2, the objects x_3, x_6 and x_8 belong to $\overline{\widetilde{R}_k}(\widetilde{X})$, but do not belong to $\underline{\widetilde{R}_k}(\widetilde{X})$; and x_1, x_2 and x_7 belong to $\underline{\widetilde{R}_k}(\widetilde{X})$, but do not belong to $\overline{\widetilde{R}_k}(\widetilde{X})$. This leads to the circumstance that there is no inclusion relation between distance-based graded upper approximation and distance-based graded lower approximation.

The corresponding three disjoint regions of Db-GRFS model are obtained in the form.

$$Pos_k(\widetilde{X}) = \{x_4\};$$
$$Neg_k(\widetilde{X}) = \{x_5\};$$
$$UBn_k(\widetilde{X}) = \{x_3, x_6, x_8\};$$
$$LBn_k(\widetilde{X}) = \{x_1, x_2, x_7\}.$$

The patient x_4 belongs to the positive region means that x_4 really needs to receive treatment; x_5 belongs to the negative region means that he (or she) does not need to get treatments; x_3, x_6 and x_8 belong to the upper boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely not to receive treatment; x_1, x_2 and x_7 belong to the lower boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely to receive treatment.

4.2. Db-Dq-RFS model

When we consider both relative quantitative information and absolute quantitative information in the upper and lower approximations, we obtain four kinds of distance-based double-quantitative rough approximation operators.

Suppose $I = (U, A, V, f)$ is an information system, $R \subseteq A$. For any $\widetilde{X} \in \mathcal{F}(U)$, $0 \le \beta < \alpha \le 1$, $k \in \mathbf{R}$, $0 \le k \le |U|$, and $x \in U$. When we consider relative quantitative information and absolute quantitative information in the upper and lower approximations at the same time, there are four cases, which can be shown as follows.

(1) Double-quantitative *upper approximation* with *logic conjunction*.
(2) Double-quantitative *upper approximation* with *logic disjunction*.
(3) Double-quantitative *lower approximation* with *logic conjunction*.
(4) Double-quantitative *lower approximation* with *logic disjunction*.

We obtain four kinds of Db-Dq-RFS models by combining the above four cases, where the forms of combination are shown in **Table 4.1.**

In what follows, we investigate four kinds of distance-based double-quantitative approximations of fuzzy set with logic operations. These four models have their own specific application background, and we should decide which model to use according to the actual application requirements.

If the upper approximation must contain two kinds of quantitative information, and the lower approximation must also contain two kinds of quantitative information, the following model in Definition 4.3 can be applied.

Definition 4.3. Suppose $I = (U, A, V, f)$ is an information system, $R \subseteq A$. For any $\widetilde{X} \in \mathcal{F}(U)$, $0 \le \beta < \alpha \le 1$, $k \in \mathbf{R}$, $0 \le k \le |U|$ and $x \in U$. The first kind of distance-based double-quantitative rough fuzzy set (Db-DqI-RFS) model is denoted as

$$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} > \beta \wedge |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\},$$

$$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} \ge \alpha \wedge (|[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}|) \le k\}.$$

Based on these two operators, we determine the Db-DqI-RFS model, which is also denoted by $(U, \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}), \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}))$.

In Db-DqI-RFS model, the conjunction operator is applied to reflect both the relative quantitative information and absolute quantitative information in upper and lower approximations. Each element, in the upper approximation and the lower approximation, exhibits both relative quantification and absolute quantification at the same time with the conjunction operator.

Theorem 4.1. *Db-DqI-RFS can be also defined as*

$$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid |[x]_{\widetilde{R}} \cap \widetilde{X}| > max(\beta \cdot |[x]_{\widetilde{R}}|, k)\},$$

$$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq max(\alpha \cdot |[x]_{\widetilde{R}}|, |[x]_{\widetilde{R}}| - k)\}.$$

Proof. For the upper approximation, $\frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} > \beta \wedge |[x]_{\widetilde{R}} \cap \widetilde{X}| > k \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}| \wedge |[x]_{\widetilde{R}} \cap \widetilde{X}| > k \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| > max(\beta \cdot |[x]_{\widetilde{R}}|, k)$. For the lower approximation, $\frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} \geq \alpha \wedge (|[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}|) \leq k \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq \alpha \cdot |[x]_{\widetilde{R}}| \wedge |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq (|[x]_{\widetilde{R}}| - k) \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq max(\alpha \cdot |[x]_{\widetilde{R}}|, |[x]_{\widetilde{R}}| - k)$. □

From the Definition 4.3 about the upper approximation and lower approximation in Db-DqI-RFS model, we obtain $\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cap \overline{\widetilde{R}_k}(\widetilde{X})$ and $\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cap \underline{\widetilde{R}_k}(\widetilde{X})$.

The positive region, negative region, upper boundary region, lower boundary region are given as

$$Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) \cap \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X});$$

$$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \sim (\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) \cup \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}));$$

$$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) - \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X});$$

$$LBn^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) - \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}).$$

The boundary region is the union of upper boundary region and lower boundary region, which means $Bn^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X}) \cup LBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

If the upper approximation must contain two kinds of quantitative information, and the lower approximation contains at least one kind of quantitative information, the following model in Definition 4.4 can be applied.

Definition 4.4. Suppose $I = (U, A, V, f)$ is an information system, $R \subseteq A$. For any $\widetilde{X} \in \mathcal{F}(U)$, $0 \leq \beta < \alpha \leq 1$, $k \in \mathbf{R}$, $0 \leq k \leq |U|$ and $x \in U$. The second kind of distance-based double-quantitative rough fuzzy set (Db-DqII-RFS) model is denoted as

$$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} > \beta \wedge |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\},$$

$$\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} \geq \alpha \vee (|[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}|) \leq k\}.$$

Based on these two operators, we can determine the Db-DqII-RFS model, which is also denoted by $(U, \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}), \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}))$.

In Db-DqII-RFS model, the conjunction operator is applied to reflect the relative quantitative information and absolute quantitative information in upper approximation, and the disjunction operator is applied to reflect the relative quantitative information and absolute quantitative information in lower approximation. Each element, in upper approximation, exhibits both relative quantification and absolute quantification at the same time with the conjunction operator; as to each element in lower approximation, it contains relative quantification or absolute quantification with the disjunction operator.

Theorem 4.2. *Db-DqII-RFS can be also defined as*

$$\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > max(\beta \cdot |[x]_{\widetilde{R}}|, k)\},$$
$$\underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq min(\alpha \cdot |[x]_{\widetilde{R}}|, |[x]_{\widetilde{R}}| - k)\}.$$

Proof. The proof of upper approximation is the same to the proof in Theorem 4.1. As to the proof of lower approximation, $\frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} \geq \alpha \vee (|[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}|) \leq k \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq \alpha \cdot |[x]_{\widetilde{R}}| \vee |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq (|[x]_{\widetilde{R}}| - k) \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq min(\alpha \cdot |[x]_{\widetilde{R}}|, |[x]_{\widetilde{R}}| - k)$. □

From Definition 4.4 dealing with the upper approximation and lower approximation in Db-DqII-RFS model, we obtain $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cap \overline{\widetilde{R}_k}(\widetilde{X})$ and $\underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cup \underline{\widetilde{R}_k}(\widetilde{X})$.

The positive region, negative region, upper boundary region, lower boundary region are presented as follows.

$$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) \cap \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X});$$
$$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \sim (\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) \cup \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}));$$
$$UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) - \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X});$$
$$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) - \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}).$$

The boundary region is the union of upper boundary region and lower boundary region, which means $Bn^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X}) \cup LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

If the upper approximation contains at least one kind of quantitative information, and the lower approximation also contains at least one kind of quantitative information, the following model in Definition 4.5 can be applied.

Definition 4.5. Suppose $I = (U, A, V, f)$ is an information system, $R \subseteq A$. For any $\widetilde{X} \in \mathcal{F}(U)$, $0 \leq \beta < \alpha \leq 1$, $k \in \mathbf{R}$, $0 \leq k \leq |U|$ and $x \in U$. The third kind of distance-based double-quantitative rough fuzzy set (Db-DqIII-RFS) model is denoted as

$$\overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} > \beta \vee |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\},$$
$$\underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} \geq \alpha \vee (|[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}|) \leq k\}.$$

Based on these two operators, we can determine the Db-DqIII-RFS model, which is also denoted by $(U, \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}), \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}))$.

In Db-DqIII-RFS model, the disjunction operator is applied to reflect the relative quantitative information and absolute quantitative information in upper and lower approximations. With the disjunction operator, the elements in upper approximation and lower approximation exhibit relative quantification or absolute quantification at the same time.

Theorem 4.3. *Db-DqIII-RFS can be also defined as*

$$\overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > min(\beta \cdot |[x]_{\widetilde{R}}|, k)\},$$
$$\underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| \geq min(\alpha \cdot |[x]_{\widetilde{R}}|, |[x]_{\widetilde{R}}| - k)\}.$$

Proof. For the upper approximation, $\frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} > \beta \vee |[x]_{\widetilde{R}} \cap \widetilde{X}| > k \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}| \vee |[x]_{\widetilde{R}} \cap \widetilde{X}| > k \Leftrightarrow |[x]_{\widetilde{R}} \cap \widetilde{X}| > min(\beta \cdot |[x]_{\widetilde{R}}|, k)$. The proof of lower approximation is the same to the proof in Theorem 4.2. □

From the Definition 4.5 about the upper approximation and lower approximation in Db-DqIII-RFS model, we obtain $\overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cup \overline{\widetilde{R}_k}(\widetilde{X})$ and $\underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cup \underline{\widetilde{R}_k}(\widetilde{X})$.

The positive region, negative region, upper boundary region, lower boundary region are presented as follows.

$$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) \cap \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X});$$

$$Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \sim (\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) \cup \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}));$$

$$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) - \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X});$$

$$LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) - \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}).$$

The boundary region is the union of upper boundary region and lower boundary region, which means $Bn^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X}) \cup LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

If the upper approximation contains at least one kind of quantitative information, and the lower approximation must contain two kinds of quantitative information, the following model in Definition 4.6 can be applied.

Definition 4.6. Suppose $I = (U, A, V, f)$ is an information system, $R \subseteq A$. For any $\widetilde{X} \in \mathcal{F}(U)$, $0 \le \beta < \alpha \le 1$, $k \in \mathbf{R}$, $0 \le k \le |U|$ and $x \in U$. The fourth kind of distance-based double-quantitative rough fuzzy set (Db-DqIV-RFS) model is denoted as

$$\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} > \beta \vee |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\},$$

$$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid \frac{|[x]_{\widetilde{R}} \cap \widetilde{X}|}{|[x]_{\widetilde{R}}|} \ge \alpha \wedge (|[x]_{\widetilde{R}}| - |[x]_{\widetilde{R}} \cap \widetilde{X}|) \le k\}.$$

Based on these two operators, we determine the Db-DqIV-RFS model, which is also denoted by $(U, \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}), \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}))$.

In Db-DqIV-RFS model, the disjunction operator is applied to reflect the relative quantitative information and absolute quantitative information in upper approximation, and the conjunction operator is applied to reflect the relative quantitative information and absolute quantitative information in lower approximation. Each element, in upper approximation, exhibits relative quantification or absolute quantification with the disjunction operator; as to each element in lower approximation, it contains both relative quantification and absolute quantification at the same time with the conjunction operator.

Theorem 4.4. *Db-DqIV-RFS can be also defined as*

$$\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid |[x]_{\widetilde{R}} \cap \widetilde{X}| > min(\beta \cdot |[x]_{\widetilde{R}}|, k)\},$$

$$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \mid |[x]_{\widetilde{R}} \cap \widetilde{X}| \ge max(\alpha \cdot |[x]_{\widetilde{R}}|, |[x]_{\widetilde{R}}| - k)\}.$$

Proof. The proof of upper approximation and lower approximation is the same as the proofs of Theorem 4.3 and Theorem 4.1. □

From the Definition 4.6 about the upper approximation and lower approximation in Db-DqIV-RFS model, we obtain $\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \overline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cup \overline{\widetilde{R}_k}(\widetilde{X})$ and $\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \underline{\widetilde{R}_{(\alpha,\beta)}}(\widetilde{X}) \cap \underline{\widetilde{R}_k}(\widetilde{X})$.

The positive region, negative region, upper boundary region, lower boundary region are presented as follows.

$$Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) \cap \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X});$$

$$Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \sim (\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) \cup \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}));$$

$$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) - \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X});$$

$$LBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) - \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}).$$

The boundary region is the union of upper boundary region and lower boundary region, which means $Bn^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) \cup LBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

Example 4.3 *(Continuation of Example 3.1).* We also consider the values of the parameters $\alpha = 0.60$, $\beta = 0.45$ and grade $k = 1.6$. The upper and lower approximations of Db-DqI-RFS are shown below.

$$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_4, x_6, x_8\};$$
$$\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_1, x_4\}.$$

Accordingly, the elements of the positive region, negative region, upper boundary region, and lower boundary region are as follows.

$$Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_4\};$$
$$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_2, x_3, x_5, x_7\};$$
$$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_6, x_8\};$$
$$LBn^{I}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_1\}.$$

The patient x_4 belongs to the positive region means that x_4 really needs to receive treatment; x_2, x_3, x_5 and x_7 belong to the negative region means that they do not need to get treatments; x_6 and x_8 belong to the upper boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely not to receive treatment; x_1 belongs to the lower boundary region means that although x_1 needs to be observed to make decisions about whether he (or she) needs to get treatment, he (or she) is more likely to receive treatment.

The upper and lower approximations of Db-DqII-RFS are shown below.

$$\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_4, x_6, x_8\};$$
$$\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_1, x_2, x_4, x_7\}.$$

The elements of the positive region, negative region, upper boundary region, and lower boundary region are as follows.

$$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_4\};$$
$$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_3, x_5\};$$
$$UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_6, x_8\};$$
$$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_1, x_2, x_7\}.$$

The patient x_4 belongs to the positive region means that x_4 really needs to receive treatment; x_3 and x_5 belong to the negative region means that they do not need to get treatments; x_6 and x_8 belong to the upper boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely not to receive treatment; x_1, x_2 and x_7 belong to the lower boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely to receive treatment.

The upper and lower approximations of Db-DqIII-RFS are shown below.

$$\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\};$$
$$\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_1, x_2, x_4, x_7\}.$$

The elements of the positive region, negative region, upper boundary region, and lower boundary region are as follows.

$$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_1, x_2, x_4, x_7\};$$
$$Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_5\};$$
$$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_3, x_6, x_8\};$$
$$LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X}) = \emptyset.$$

The patients x_1, x_2, x_4 and x_7 belong to the positive region means that they really need to receive treatment; x_5 belongs to the negative region means that he (or she) does not need to get treatments; x_3, x_6 and x_8 belong to the upper boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely not to receive treatment.

The upper and lower approximations of Db-DqIV-RFS are shown below.

$$\overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\};$$

$$\underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x_1, x_4\}.$$

The elements of the positive region, negative region, upper boundary region, and lower boundary region are as follows.

$$Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_1, x_4\};$$

$$Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_5\};$$

$$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \{x_2, x_3, x_6, x_7, x_8\};$$

$$LBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X}) = \emptyset.$$

Patients x_1 and x_4 belong to the positive region means that they really need to receive treatment; x_5 belongs to the negative region means that he (or she) does not need to get treatments; x_2, x_3, x_6, x_7 and x_8 belong to the upper boundary region means that although they need to be observed to make decisions about whether they need to get treatment, they are more likely not to receive treatment.

5. Decision rules with parameters variation for four kinds of Db-Dq-RFS models

The circumstance of double quantitative information included in both upper and lower approximations can form four kinds of Db-Dq-RFS models, and the decision rules in each Db-Dq-RFS model is much more complex than Dq-DTRS model [26]. We consider all the elements of U separately to analyze the decision rules with parameters variation in each Db-Dq-RFS model. According to the conditions α and β meeting $0 \le \beta < \alpha \le 1$, it is easy to see that $\beta \cdot |[x]_{\widetilde{R}}| < \alpha \cdot |[x]_{\widetilde{R}}|$. There are three cases of the value of k between $\beta \cdot |[x]_{\widetilde{R}}|$ and $\alpha \cdot |[x]_{\widetilde{R}}|$. The following Fig. 5.1 is about the different cases of the grade k, the items (1), (2) and (3) (in Fig. 5.1) represent the Case 1 (for $k \le \beta \cdot |[x]_{\widetilde{R}}|$), Case 2 (for $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$) and Case 3 (for $k > \alpha \cdot |[x]_{\widetilde{R}}|$), respectively.

Here, we analyze the decision rules in detail with the parameters changing of the proposed four kinds of Db-Dq-RFS models, respectively. For each $x \in U$ and a given fuzzy set $\widetilde{X}$, we can easily get the values of $\alpha \cdot |[x]_{\widetilde{R}}|$, $\beta \cdot |[x]_{\widetilde{R}}|$, $|[x]_{\widetilde{R}} \cap \widetilde{X}|$ and $|[x]_{\widetilde{R}}| - k$. Because $\beta \cdot |[x]_{\widetilde{R}}| < \alpha \cdot |[x]_{\widetilde{R}}|$ always holds for each object $x \in U$, we should first compare the values between k and $\beta \cdot |[x]_{\widetilde{R}}|$, and compare the values between k and $\alpha \cdot |[x]_{\widetilde{R}}|$, and then judge which item in Fig. 5.1 it is satisfied.

5.1. Effect of parameters on decision rules in Db-DqI-RFS

We discuss the values of k step by step ranging from small to large values. First, we study Case 1 where $k \le \beta \cdot |[x]_{\widetilde{R}}|$.

Case 1: (See item (1) in Fig. 5.1.) For $k \le \beta \cdot |[x]_{\widetilde{R}}|$, which means that $max(\beta \cdot |[x]_{\widetilde{R}}|, k) = \beta \cdot |[x]_{\widetilde{R}}|$, we can get $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U | |[x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}|\}$.

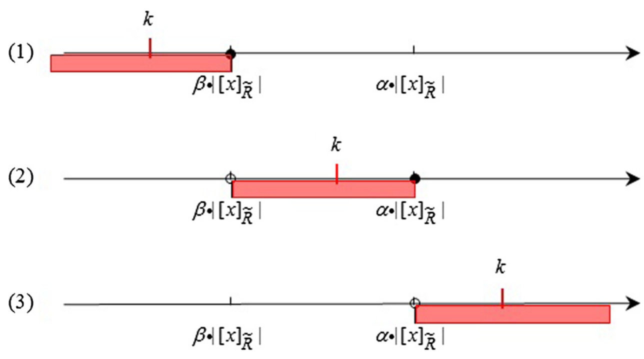

Fig. 5.1. Different cases of the grade k.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we have $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\beta \cdot |[x]_{R}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we get $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

After discussing Case 1 of $k \le \beta \cdot |[x]_{\widetilde{R}}|$, we then study Case 2 of $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$ about decision rules in the following.

Case 2: (See (2) in Fig. 5.1.) For $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $max(\beta \cdot |[x]_{\widetilde{R}}|, k) = k$, we have $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

The following Case 3 concerns $k > \alpha \cdot |[x]_{\widetilde{R}}|$, we analyze the corresponding decision rules.

Case 3: (See (3) in Fig. 5.1.) For $k > \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $max(\beta \cdot |[x]_{\widetilde{R}}|, k) = k$, so $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\alpha \cdot |[x]_{R}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > k$, then one has $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$.

Through the three cases discussed above, we have the variation of values of the parameters to the decision rules in Db-DqI-RFS model.

5.2. *Effect of parameters on decision rules in Db-DqII-RFS*

We discuss the value of k step by step moving from small to large, and first study the Case 1 about $k \le \beta \cdot |[x]_{\widetilde{R}}|$ in the following.

Case 1: (See (1) in Fig. 5.1.) For $k \le \beta \cdot |[x]_{\widetilde{R}}|$, it means that $max(\beta \cdot |[x]_{\widetilde{R}}|, k) = \beta \cdot |[x]_{\widetilde{R}}|$, we can get $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}|\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

After discussing Case 1 of $k \le \beta \cdot |[x]_{\widetilde{R}}|$, we then study Case 2 of $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$ about decision rules.

Case 2: (See (2) in Fig. 5.1.) For $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $max(\beta \cdot |[x]_{\widetilde{R}}|, k) = k$, so $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

Case 3 concerns $k > \alpha \cdot |[x]_{\widetilde{R}}|$, here we analyze the corresponding decision rules.

Case 3: (See (3) in Fig. 5.1.) For $k > \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $max(\beta \cdot |[x]_{\widetilde{R}}|, k) = k$, so $\overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\alpha \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \sim \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > k$, then we can get $x \in \overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$.

By studying the three cases discussed above, we obtain a clear correspondence of the variation of values and their impact on the values variation of parameters to the decision rules in Db-DqII-RFS model.

5.3. Effect of parameters on decision rules in Db-DqIII-RFS

We discuss the value of k step by step starting from small values, and first study Case 1 where $k \le \beta \cdot |[x]_{\widetilde{R}}|$.

Case 1: (See (1) in Fig. 5.1.) For $k \le \beta \cdot |[x]_{\widetilde{R}}|$, it means that $min(\beta \cdot |[x]_{\widetilde{R}}|, k) = k$, we can get $\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > k\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \sim \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

After discussing Case 1 with $k \le \beta \cdot |[x]_{\widetilde{R}}|$, we then study Case 2 with $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$ with regard to decision rules.

Case 2: (See (2) in Fig. 5.1.) For $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $min(\beta \cdot |[x]_{\widetilde{R}}|, k) = \beta \cdot |[x]_{\widetilde{R}}|$, we can get $\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}|\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

Case 3 is about $k > \alpha \cdot |[x]_{\widetilde{R}}|$. We can analyze the corresponding decision rules.

Case 3: (See (3) in Fig. 5.1.) For $k > \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $min(\beta \cdot |[x]_{\widetilde{R}}|, k) = \beta \cdot |[x]_{\widetilde{R}}|$, we can get $\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U || [x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}|\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\alpha \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > k$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$.

Through the three cases discussed above, we form a clear correspondence between the variation of values of the parameters and their impact on the decision rules in the Db-DqIII-RFS model.

5.4. Effect of parameters on decision rules in Db-DqIV-RFS

We discuss the value of k step by step, we first study the Case 1 when $k \le \beta \cdot |[x]_{\widetilde{R}}|$ in the following.

Case 1: (See (1) in Fig. 5.1.) For $k \le \beta \cdot |[x]_{\widetilde{R}}|$, it means that $min(\beta \cdot |[x]_{\widetilde{R}}|, k) = k$, we can get $\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U || [x]_{\widetilde{R}} \cap \widetilde{X}| > k\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \sim \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$,

(2) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\beta \cdot |[x]_{R}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

After discussing Case 1 with $k \le \beta \cdot |[x]_{\widetilde{R}}|$, we then study Case 2 when $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$ about decision rules.

Case 2: (See (2) in Fig. 5.1.) For $\beta \cdot |[x]_{\widetilde{R}}| < k \le \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $min(\beta \cdot |[x]_{\widetilde{R}}|, k) = \beta \cdot |[x]_{\widetilde{R}}|$, we can get $\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U || [x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}|\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \le \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$,

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le k$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $k < |[x]_{\widetilde{R}} \cap \widetilde{X}| \le \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$,
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \ge |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

Table 5.1
Effect of parameters on decision rules for Case 1 in different models.

$\vert[x_i]_{\widetilde{R}} \cap \widetilde{X}\vert$		Db-DqI-RFS	Db-DqII-RFS	Db-DqIII-RFS	Db-DqIV-RFS
$[0,k]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$		$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(k, \beta\vert[x_i]_{\widetilde{R}}\vert]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$		$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(\beta\vert[x_i]_{\widetilde{R}}\vert, \alpha\vert[x_i]_{\widetilde{R}}\vert]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$		$UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(\alpha\vert[x_i]_{\widetilde{R}}\vert, \vert U\vert]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$	$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$			$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$

Table 5.2
Effect of parameters on decision rules for Case 2 in different models.

$\vert[x_i]_{\widetilde{R}} \cap \widetilde{X}\vert$		Db-DqI-RFS	Db-DqII-RFS	Db-DqIII-RFS	Db-DqIV-RFS
$[0, \beta\cdot\vert[x_i]_{\widetilde{R}}\vert]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$		$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(\beta\cdot\vert[x_i]_{\widetilde{R}}\vert, k]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$		$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(k, \alpha\vert[x_i]_{\widetilde{R}}\vert]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$		$UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(\alpha\vert[x_i]_{\widetilde{R}}\vert, \vert U\vert]$	$\geq \vert[x_i]_{\widetilde{R}}\vert - k$	$Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$< \vert[x_i]_{\widetilde{R}}\vert - k$	$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$			$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$

The following Case 3 is about $k > \alpha \cdot |[x]_{\widetilde{R}}|$, we can also analyze the corresponding decision rules.

Case 3: (See (3) in Fig. 5.1.) For $k > \alpha \cdot |[x]_{\widetilde{R}}|$, which means that $min(\beta \cdot |[x]_{\widetilde{R}}|, k) = \beta \cdot |[x]_{\widetilde{R}}|$, we can get $\overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X}) = \{x \in U \,|\, |[x]_{\widetilde{R}} \cap \widetilde{X}| > \beta \cdot |[x]_{\widetilde{R}}|\}$.

(1) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| \leq \beta \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \sim \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$,

(2) If $\beta \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \leq \alpha \cdot |[x]_{\widetilde{R}}|$, then we can get $x \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) If $\alpha \cdot |[x]_{\widetilde{R}}| < |[x]_{\widetilde{R}} \cap \widetilde{X}| \leq k$, then we can get $x \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \geq |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) If $|[x]_{\widetilde{R}} \cap \widetilde{X}| > k$, then we can get $x \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$,

- $|[x]_{\widetilde{R}} \cap \widetilde{X}| \geq |[x]_{\widetilde{R}}| - k$, then $x \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$;
- $|[x]_{\widetilde{R}} \cap \widetilde{X}| < |[x]_{\widetilde{R}}| - k$, then $x \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

Through the three cases discussed above, we can get a clearly corresponding of the values variation of parameters to the decision rules in Db-DqIV-RFS model.

It is easy to see that the maximum and minimum values of $|[x_i]_{\widetilde{R}} \cap \widetilde{X}|$ are 0 and $|U|$, respectively. Tables 5.1–5.3 are comparisons of the decision rules with parameters variation for four kinds of Db-Dq-RFS models.

Table 5.3
Effect of parameters on decision rules for Case 3 in different models.

$\lvert[x_i]_{\widetilde{R}}\cap\widetilde{X}\rvert$		Db-DqI-RFS	Db-DqII-RFS	Db-DqIII-RFS	Db-DqIV-RFS
$[0,\beta\cdot\lvert[x_i]_{\widetilde{R}}\rvert]$	$\geq\lvert[x_i]_{\widetilde{R}}\rvert-k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$<\lvert[x_i]_{\widetilde{R}}\rvert-k$		$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(\beta\cdot\lvert[x_i]_{\widetilde{R}}\rvert,\alpha\cdot\lvert[x_i]_{\widetilde{R}}\rvert]$	$\geq\lvert[x_i]_{\widetilde{R}}\rvert-k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$<\lvert[x_i]_{\widetilde{R}}\rvert-k$		$Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	
$(\alpha\lvert[x_i]_{\widetilde{R}}\rvert,k]$	$\geq\lvert[x_i]_{\widetilde{R}}\rvert-k$	$LBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$<\lvert[x_i]_{\widetilde{R}}\rvert-k$	$Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$			$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
$(k,\lvert U\rvert]$	$\geq\lvert[x_i]_{\widetilde{R}}\rvert-k$	$Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$	$Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$
	$<\lvert[x_i]_{\widetilde{R}}\rvert-k$	$UBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$			$UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$

Table 5.4
Statistical results.

	$\lvert[x_i]_{\widetilde{R}}\rvert$	$\lvert[x_i]_{\widetilde{R}}\cap\widetilde{X}\rvert$	$\lvert[x_i]_{\widetilde{R}}\rvert-\lvert[x_i]_{\widetilde{R}}\cap\widetilde{X}\rvert$	$\lvert[x_i]_{\widetilde{R}}\cap\widetilde{X}\rvert/\lvert[x_i]_{\widetilde{R}}\rvert$	$\beta\cdot\lvert[x_i]_{\widetilde{R}}\rvert$	$\alpha\cdot\lvert[x_i]_{\widetilde{R}}\rvert$	$\lvert[x_i]_{\widetilde{R}}\rvert-k$
x_1	2.36	1.46	0.90	0.62	1.06	1.42	0.76
x_2	2.92	1.45	1.47	0.50	1.31	1.75	1.32
x_3	3.98	1.70	2.28	0.43	1.79	2.39	2.38
x_4	3.27	1.96	1.31	0.60	1.47	1.96	1.67
x_5	3.26	1.42	1.84	0.44	1.47	1.96	1.66
x_6	3.80	1.74	2.06	0.46	1.71	2.28	2.20
x_7	3.14	1.57	1.57	0.50	1.41	1.88	1.54
x_8	3.64	1.67	1.97	0.46	1.64	2.18	2.04

5.5. An illustrative example

Example 5.1 *(Continuation of Example 3.1).* We consider the parameters $\alpha=0.60$, $\beta=0.45$ and grade $k=1.6$. Based on the normalized information system shown in Table 3.2, Table 5.4 provides the statistical results for each object in U. In what follows, we present the analysis of decision rules of each elements in U for given parameters α, β and grade k to verify the decision rule analysis completed in Subsections 5.1–5.4.

From the statistical results presented in Table 5.4, we can accurately classify each object into four disjoint regions, namely positive region, negative region, upper boundary region and lower boundary region, and then make corresponding decisions based on the four different regions in different Db-Dq-RFS models.

(1) For the object x_1, $\alpha\cdot\lvert[x_1]_{\widetilde{R}}\rvert=1.42$, $\beta\cdot\lvert[x_1]_{\widetilde{R}}\rvert=1.06$, and $\lvert[x_1]_{\widetilde{R}}\cap\widetilde{X}\rvert=1.46$. So $k>\alpha\cdot\lvert[x_1]_{\widetilde{R}}\rvert$, $\alpha\cdot\lvert[x_1]_{\widetilde{R}}\rvert<\lvert[x_1]_{\widetilde{R}}\cap\widetilde{X}\rvert<k$ and $\lvert[x_1]_{\widetilde{R}}\cap\widetilde{X}\rvert>\lvert[x_1]_{\widetilde{R}}\rvert-k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 3** (because $k>\alpha\cdot\lvert[x_1]_{\widetilde{R}}\rvert$), and in Case 3, $\alpha\cdot\lvert[x_1]_{\widetilde{R}}\rvert<\lvert[x_1]_{\widetilde{R}}\cap\widetilde{X}\rvert<k$. In Db-DqI-RFS model, $x_1\in\sim\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_1\in\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_1\in\sim\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_1\in\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_1\in\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_1\in\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_1\in\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_1\in\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(2) For the object x_2, $\alpha\cdot\lvert[x_2]_{\widetilde{R}}\rvert=1.75$, $\beta\cdot\lvert[x_2]_{\widetilde{R}}\rvert=1.31$, and $\lvert[x_2]_{\widetilde{R}}\cap\widetilde{X}\rvert=1.45$. So $\beta\cdot\lvert[x_2]_{\widetilde{R}}\rvert<k<\alpha\cdot\lvert[x_2]_{\widetilde{R}}\rvert$, $\beta\cdot\lvert[x_2]_{\widetilde{R}}\rvert<\lvert[x_2]_{\widetilde{R}}\cap\widetilde{X}\rvert<k$ and $\lvert[x_2]_{\widetilde{R}}\cap\widetilde{X}\rvert>\lvert[x_2]_{\widetilde{R}}\rvert-k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 2**, and in Case 2, $\beta\cdot\lvert[x_2]_{\widetilde{R}}\rvert<\lvert[x_2]_{\widetilde{R}}\cap\widetilde{X}\rvert<k$. In Db-DqI-RFS model, $x_2\in\sim\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_2\in\sim\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_2\in\sim\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_2\in\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_2\in\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_2\in\underline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_2\in\overline{\widetilde{R}^{\vee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_2\in\sim\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(3) For the object x_3, $\alpha\cdot\lvert[x_3]_{\widetilde{R}}\rvert=2.39$, $\beta\cdot\lvert[x_3]_{\widetilde{R}}\rvert=1.79$, and $\lvert[x_3]_{\widetilde{R}}\cap\widetilde{X}\rvert=1.70$. So $k<\beta\cdot\lvert[x_3]_{\widetilde{R}}\rvert$, $k<\lvert[x_3]_{\widetilde{R}}\cap\widetilde{X}\rvert<\beta\cdot\lvert[x_3]_{\widetilde{R}}\rvert$ and $\lvert[x_3]_{\widetilde{R}}\cap\widetilde{X}\rvert<\lvert[x_3]_{\widetilde{R}}\rvert-k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 1**, and in Case 1, $k<\lvert[x_3]_{\widetilde{R}}\cap\widetilde{X}\rvert<\beta\cdot\lvert[x_3]_{\widetilde{R}}\rvert$. In Db-DqI-RFS model, $x_3\in\sim\overline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_3\in\sim\underline{\widetilde{R}^{\wedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model,

$x_3 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_3 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_3 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_3 \in \sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_3 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_2 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(4) For the object x_4, $\alpha \cdot |[x_4]_{\widetilde{R}}| = 1.96$, $\beta \cdot |[x_4]_{\widetilde{R}}| = 1.47$, and $|[x_4]_{\widetilde{R}} \cap \widetilde{X}| = 1.96$. So $\beta \cdot |[x_4]_{\widetilde{R}}| < k < \alpha \cdot |[x_4]_{\widetilde{R}}|$, $|[x_4]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x_4]_{\widetilde{R}}|$ and $|[x_4]_{\widetilde{R}} \cap \widetilde{X}| > |[x_4]_{\widetilde{R}}| - k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 2**, and in Case 2, $|[x_2]_{\widetilde{R}} \cap \widetilde{X}| > \alpha \cdot |[x_4]_{\widetilde{R}}|$ and $|[x_4]_{\widetilde{R}} \cap \widetilde{X}| > |[x_4]_{\widetilde{R}}| - k$. In Db-DqI-RFS model, $x_4 \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_4 \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_4 \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_4 \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_4 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_4 \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_4 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_4 \in \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(5) For the object x_5, $\alpha \cdot |[x_5]_{\widetilde{R}}| = 1.96$, $\beta \cdot |[x_5]_{\widetilde{R}}| = 1.47$, and $|[x_5]_{\widetilde{R}} \cap \widetilde{X}| = 1.42$. So $\beta \cdot |[x_5]_{\widetilde{R}}| < k < \alpha \cdot |[x_5]_{\widetilde{R}}|$, $|[x_5]_{\widetilde{R}} \cap \widetilde{X}| < \beta \cdot |[x_5]_{\widetilde{R}}|$ and $|[x_5]_{\widetilde{R}} \cap \widetilde{X}| < |[x_5]_{\widetilde{R}}| - k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 2**, and in Case 2, $|[x_5]_{\widetilde{R}} \cap \widetilde{X}| < \beta \cdot |[x_5]_{\widetilde{R}}|$ and $|[x_5]_{\widetilde{R}} \cap \widetilde{X}| < |[x_5]_{\widetilde{R}}| - k$. In Db-DqI-RFS model, $x_5 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_5 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_5 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_5 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_5 \in\sim \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_5 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_5 \in\sim \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_5 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(6) For the object x_6, $\alpha \cdot |[x_6]_{\widetilde{R}}| = 2.28$, $\beta \cdot |[x_6]_{\widetilde{R}}| = 1.71$, and $|[x_6]_{\widetilde{R}} \cap \widetilde{X}| = 1.74$. So $k < \beta \cdot |[x_6]_{\widetilde{R}}|$, $\beta \cdot |[x_6]_{\widetilde{R}}| < |[x_6]_{\widetilde{R}} \cap \widetilde{X}| < \alpha \cdot |[x_6]_{\widetilde{R}}|$ and $|[x_6]_{\widetilde{R}} \cap \widetilde{X}| < |[x_6]_{\widetilde{R}}| - k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 1**, and in Case 1, $\beta \cdot |[x_6]_{\widetilde{R}}| < |[x_6]_{\widetilde{R}} \cap \widetilde{X}| < \alpha \cdot |[x_6]_{\widetilde{R}}|$ and $|[x_6]_{\widetilde{R}} \cap \widetilde{X}| < |[x_6]_{\widetilde{R}}| - k$. In Db-DqI-RFS model, $x_6 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_6 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_6 \in \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_6 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_6 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_6 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_6 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_6 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(7) For the object x_7, $\alpha \cdot |[x_7]_{\widetilde{R}}| = 1.88$, $\beta \cdot |[x_7]_{\widetilde{R}}| = 1.41$, and $|[x_7]_{\widetilde{R}} \cap \widetilde{X}| = 1.57$. So $\beta \cdot |[x_7]_{\widetilde{R}}| < k < \alpha \cdot |[x_7]_{\widetilde{R}}|$, $\beta \cdot |[x_7]_{\widetilde{R}}| < |[x_7]_{\widetilde{R}} \cap \widetilde{X}| < k$ and $|[x_7]_{\widetilde{R}} \cap \widetilde{X}| > |[x_7]_{\widetilde{R}}| - k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 2**, and in Case 2, $\beta \cdot |[x_7]_{\widetilde{R}}| < |[x_7]_{\widetilde{R}} \cap \widetilde{X}| < k$. In Db-DqI-RFS model, $x_7 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_7 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_7 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_7 \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $LBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_7 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_7 \in \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Pos^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_7 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_7 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

(8) For the object x_8, $\alpha \cdot |[x_8]_{\widetilde{R}}| = 2.18$, $\beta \cdot |[x_8]_{\widetilde{R}}| = 1.64$, and $|[x_8]_{\widetilde{R}} \cap \widetilde{X}| = 1.67$. So $k < \beta \cdot |[x_8]_{\widetilde{R}}|$, $\beta \cdot |[x_8]_{\widetilde{R}}| < |[x_8]_{\widetilde{R}} \cap \widetilde{X}| < \alpha \cdot |[x_8]_{\widetilde{R}}|$ and $|[x_8]_{\widetilde{R}} \cap \widetilde{X}| < |[x_8]_{\widetilde{R}}| - k$. In the four kinds of Db-Dq-RFS models, it takes the **Case 1**, and in Case 1, $\beta \cdot |[x_8]_{\widetilde{R}}| < |[x_8]_{\widetilde{R}} \cap \widetilde{X}| < \alpha \cdot |[x_8]_{\widetilde{R}}|$ and $|[x_8]_{\widetilde{R}} \cap \widetilde{X}| < |[x_8]_{\widetilde{R}}| - k$. In Db-DqI-RFS model, $x_8 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_8 \in\sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $Neg^{I}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqII-RFS model, $x_8 \in\sim \overline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_8 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{II}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIII-RFS model, $x_8 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_8 \in\sim \underline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{III}_{(\alpha,\beta,k)}(\widetilde{X})$; in Db-DqIV-RFS model, $x_8 \in \overline{\widetilde{R}^{\bigvee}_{(\alpha,\beta,k)}}(\widetilde{X})$ and $x_8 \in \sim \underline{\widetilde{R}^{\bigwedge}_{(\alpha,\beta,k)}}(\widetilde{X})$, decide $UBn^{IV}_{(\alpha,\beta,k)}(\widetilde{X})$.

Table 5.5
Disjoint regions of four kinds of Db-Dq-RFS models.

Model	Positive region	Negative region	Boundary region	
			Upper	Lower
Db-DqI-RFS	x_4	x_2, x_3, x_5, x_7	x_6, x_8	x_1
Db-DqII-RFS	x_4	x_3, x_5	x_6, x_8	x_1, x_2, x_7
Db-DqIII-RFS	x_1, x_2, x_4, x_7	x_5	x_3, x_6, x_8	$\emptyset$
Db-DqIV-RFS	x_1, x_4	x_5	x_2, x_3, x_6, x_7, x_8	$\emptyset$

It is easy to verify that the above analysis from the view of parameters on the decision rules of the whole elements in U is consistent with the decision regions obtained from the upper and lower approximations in each Db-Dq-RFS model (see Table 5.5).

6. Experimental studies

The models presented in [11] provide us with an approach capable of dealing with specified numeric information system, where the data in the information system are fuzzy and the values of each data are located in the range $[0, 1]$. When the information system is not fuzzy, some preprocessing is required if we want to use the methods in [11]. This constitutes a serious limitation when we apply the methods to practical problems. As it has mentioned in the Introduction, we do not need to preprocess the data for any numeric information system when we use the models presented in this paper. We can directly use the proposed four kinds of Db-Dq-RFS models to handle raw data. Compared with the rough set method which needs preprocessing, the presented methods reduce the loss of information in the preprocessing phase. In this section, to introduce and interpret the effectiveness of dealing with real-life information system of the proposed four kinds of Db-Dq-RFS models, we conduct some experiments on a PC with Windows 7, Inter (R) Core (TM) i7-4770M CPU 3.40 GHZ and 16 GB Memory. Publicly available datasets come from the UCI data repository are shown in Table 6.1.

The elements in the decision regions are closely related with the values of the tested parameters α, β and grade k. Generally, different parameters and different grades may result in different decision regions for each dataset. In the previous work [26], we discussed the decision rules on three cases regard to the most representative three pairs of parameters α and β, specifically as follows. $\alpha + \beta < 1$, $\alpha + \beta = 1$ and $\alpha + \beta > 1$. Here, we also consider three pairs of parameters $\alpha,\ \beta$, which are $\alpha = 0.6,\ \beta = 0.2$ (satisfies $\alpha + \beta < 1$), $\alpha = 0.7,\ \beta = 0.3$ (satisfies $\alpha + \beta = 1$), and $\alpha = 0.8,\ \beta = 0.4$ (satisfies $\alpha + \beta > 1$). As expressed in Table 6.1, we randomly select the grade $k = 4$ for dataset "SHU", $k = 340$ for dataset "VEH", $k = 215$ for dataset "WCU", $k = 700$ for dataset "WRE" and $k = 2330$ for dataset "WWH", respectively. As to the target fuzzy concept $\widetilde{X}$, we use the computer program to randomly generate a number between 0 and 1 for each $x \in U$ to make sure that it is randomly selected, then we get a series of numbers, which represent the membership degree for elements in the fuzzy set $\widetilde{X}$. The following Tables 6.2–6.6 indicate the number of elements of decision regions for five tested datasets using different methods. For the convenience of comparison, we use "M1" and "M2" to represent the two models in reference

Table 6.1
Dataset description.

Datasets	Number of data	Features	Grade
Shuttle Landing Control (SHU)	15	7	4
Vehicle (VEH)	846	18	340
Wholesale customers (WCU)	440	6	215
Winequality-Red (WRE)	1599	12	700
Winequality-White (WWH)	4898	12	2330

Table 6.2
Number of elements located in decision regions (SHU).

		$\alpha = 0.6,\ \beta = 0.2$				$\alpha = 0.7,\ \beta = 0.3$				$\alpha = 0.8,\ \beta = 0.4$			
		POS	NEG	UBn	LBn	POS	NEG	UBn	LBn	POS	NEG	UBn	LBn
SHU	M1	0	0	0	15	0	0	0	15	0	0	0	15
	M2	15	0	0	0	15	0	0	0	15	0	0	0
	I	2	9	2	2	0	11	4	0	0	11	4	0
	II	4	0	1	10	4	0	1	10	4	0	1	10
	III	9	3	2	1	9	3	2	1	9	3	2	1
	IV	4	0	9	2	0	0	15	0	0	0	15	0

Table 6.3
Number of elements located in decision regions (VEH).

		$\alpha = 0.6,\ \beta = 0.2$				$\alpha = 0.7,\ \beta = 0.3$				$\alpha = 0.8,\ \beta = 0.4$			
		POS	NEG	UBn	LBn	POS	NEG	UBn	LBn	POS	NEG	UBn	LBn
VEH	M1	0	0	0	846	0	0	0	846	0	0	0	846
	M2	661	0	0	185	604	0	0	242	526	0	0	320
	I	656	0	60	130	22	87	634	103	0	189	656	1
	II	656	0	60	130	656	0	60	130	656	0	60	130
	III	746	0	53	47	746	0	53	47	746	0	53	47
	IV	746	0	53	47	125	0	721	0	1	0	845	0

Table 6.4
Number of elements located in decision regions (WCU).

		$\alpha=0.6,\ \beta=0.2$				$\alpha=0.7,\ \beta=0.3$				$\alpha=0.8,\ \beta=0.4$			
		POS	NEG	UBn	LBn	POS	NEG	UBn	LBn	POS	NEG	UBn	LBn
WCU	M1	0	440	0	0	0	440	0	0	0	440	0	0
	M2	339	0	0	101	311	0	0	129	266	0	0	174
	I	0	29	354	57	0	77	354	9	0	82	354	4
	II	354	0	0	86	354	0	0	86	354	0	0	86
	III	440	0	0	0	440	0	0	0	440	0	0	0
	IV	57	0	383	0	9	0	431	0	4	0	436	0

Table 6.5
Number of elements located in decision regions (WRE).

		$\alpha=0.6,\ \beta=0.2$				$\alpha=0.7,\ \beta=0.3$				$\alpha=0.8,\ \beta=0.4$			
		POS	NEG	UBn	LBn	POS	NEG	UBn	LBn	POS	NEG	UBn	LBn
WRE	M1	0	63	0	1536	0	744	0	855	0	1382	0	217
	M2	1304	0	0	295	1176	0	0	423	961	0	0	638
	I	1364	0	0	235	0	152	1364	83	0	232	1364	3
	II	1364	0	0	235	1364	0	0	235	1364	0	0	235
	III	1599	0	0	0	1599	0	0	0	1599	0	0	0
	IV	1599	0	0	0	83	0	1516	0	3	0	1596	0

Table 6.6
Number of elements located in decision regions (WWH).

		$\alpha=0.6,\ \beta=0.2$				$\alpha=0.7,\ \beta=0.3$				$\alpha=0.8,\ \beta=0.4$			
		POS	NEG	UBn	LBn	POS	NEG	UBn	LBn	POS	NEG	UBn	LBn
WWH	M1	0	1640	0	3258	0	3838	0	1060	0	4718	0	180
	M2	4171	0	0	727	3654	0	0	1244	3114	0	0	1784
	I	110	7	166	4615	0	4587	276	35	0	4620	276	2
	II	276	0	0	4622	276	0	0	4622	276	0	0	4622
	III	4898	0	0	0	4898	0	0	0	4898	0	0	0
	IV	4725	0	173	0	35	0	4863	0	2	0	4896	0

[11], where symbols M1 and M2 stand for conjunction and disjunction double-quantitative decision-theoretic rough fuzzy set model, respectively. The results obtained from M1 and M2 follow the preprocessing process [11].

For example, we take the data VEH to analyze the results of M1 and M2 shown in Tables 6.2–6.6; the analysis of other datasets can be obtained in a similar manner. For dataset VEH, M1 classifies all objects into the lower boundary region; M2 classifies 611 objects into positive region and 185 objects into lower boundary region when $\alpha=0.6,\ \beta=0.2$, 604 objects into positive region and 242 objects into lower boundary region when $\alpha=0.7,\ \beta=0.3$, 526 objects into positive region and 320 objects into lower boundary region when $\alpha=0.8,\ \beta=0.4$. From Tables 6.2–6.6, we can easily obtain that these two methods M1 and M2 divide each dataset into at most two classes. Moreover, objects can only be divided into one class in many cases. Such classification methods are not practical in real-life applications. It is necessary to propose new models to overcome defects of M1 and M2.

When $\alpha=0.6,\ \beta=0.2$: Db-DqI-RFS classifies 656 objects into positive region, 60 objects into upper boundary region and 130 objects into lower boundary region; Db-DqII-RFS classifies 656 objects into positive region, 60 objects into upper boundary region and 130 objects into lower boundary region; Db-DqIII-RFS classifies 746 objects into positive region, 53 objects into upper boundary region and 47 objects into lower boundary region; Db-DqIV-RFS classifies 746 objects into positive region, 53 objects into upper boundary region and 47 objects into lower boundary region. When $\alpha=0.7,\ \beta=0.3$: Db-DqI-RFS classifies 22 objects into positive region, 87 objects into negative region, 634 objects into upper boundary region and 103 objects into lower boundary region; Db-DqII-RFS classifies 656 objects into positive region, 60 objects into upper boundary region and 130 objects into lower boundary region; Db-DqIII-RFS classifies 746 objects into positive region, 53 objects into upper boundary region and 47 objects into lower boundary region; Db-DqIV-RFS classifies 125 objects into positive region, 721 objects into upper boundary region. When $\alpha=0.8,\ \beta=0.4$: Db-DqI-RFS classifies 189 objects into negative region, 656 objects into upper boundary region and 1 object into lower boundary region; Db-DqII-RFS classifies 656 objects into positive region, 60 objects into upper boundary region and 130 objects into lower boundary region; Db-DqIII-RFS classifies 746 objects into positive region, 53 objects into upper boundary region and 47 objects into lower boundary region; Db-DqIV-RFS classifies 1 object into positive region, 845 objects into upper boundary region.

From the above analysis, we can draw some conclusions. (1) Db-Dq-RFS models presented in this paper can partition the universe of discourse more accurately, generate less losses of information, and are applicable to a much wider environment than M1 and M2. (2) The decision regions in each dataset for Db-Dq-RFS models vary with the change of the parameters α and β when we keep the k constant. The change shows a certain regularity for four kinds of Db-Dq-RFS models: the number of elements in positive region becomes smaller or remains the same when the parameters α and β become larger; the number of elements in negative region becomes larger or remains the same when the parameters α and β become larger; the number of elements in upper boundary region becomes larger or remains the same when the parameters α and β become larger; the number of elements in lower boundary region becomes smaller or remains the same when the parameters α and β assume higher values. In other words, given two pairs of (α, β) for each dataset, (α_1, β_1) and (α_2, β_2) which satisfy the condition $\alpha_1 \le \alpha_2,\ \beta_1 \le \beta_2$, we obtain that $POS^{\blacklozenge}_{(\alpha_1,\beta_1,k)}(\widetilde{X}) \supseteq POS^{\blacklozenge}_{(\alpha_2,\beta_2,k)}(\widetilde{X})$; $NEG^{\blacklozenge}_{(\alpha_1,\beta_1,k)}(\widetilde{X}) \subseteq NEG^{\blacklozenge}_{(\alpha_2,\beta_2,k)}(\widetilde{X})$; $UBn^{\blacklozenge}_{(\alpha_1,\beta_1,k)}(\widetilde{X}) \subseteq UBn^{\blacklozenge}_{(\alpha_2,\beta_2,k)}(\widetilde{X})$; $LBn^{\blacklozenge}_{(\alpha_1,\beta_1,k)}(\widetilde{X}) \supseteq LBn^{\blacklozenge}_{(\alpha_2,\beta_2,k)}(\widetilde{X})$, where $\blacklozenge \in \{I, II, III, IV\}$.

7. Conclusions

When we use classical rough sets to tackle continuous data, a discretization should be implemented, this preprocessing will reduce the classification accuracy and result in a certain loss of contained information. We need to develop novel rough set models to deal with continuous data, not to discretize the continuous data. Meanwhile, PRS and GRS are two fundamental rough set model expansions that achieve strong fault tolerance capabilities by utilizing single quantitative descriptions. If both relative quantitative information and absolute quantitative information are contained in upper and lower approximations of the generalized rough set model, then we call the generalized rough set as the double-quantitative rough set model. The relative measure and absolute measure reflect relative and absolute accuracy or fault tolerance from two different quantitative viewpoints. Thus, both of the two quantification exhibit a close and supplementary relationship, and each one actually has its own representation virtues and application environments.

The proposed Db-Dq-RFS models perform a developed double quantification of the relative quantitative information and absolute quantitative information. These new models are directional expansions of Pawlak rough set model and satisfy the quantitative completeness properties, exhibit much stronger double fault tolerance capabilities to compare with the existing double-quantitative rough set models. This paper introduces a distance matrix to measure the real-valued data, and investigates distance-based fuzzy similarity relation to formulate the Dq-Dq-RFS models. Moreover, after presenting the four kinds of Db-Dq-RFS models, we study the decision rules with parameters variation for proposed Db-Dq-RFS models. Among this article, we use examples to interpret and analyze the concepts we investigated. We also test the decision rules of proposed models with parameters variation from five UCI datasets, and make a comparison with two existing models from the resulting number of elements in joint decision regions. This paper gives a framework of Db-Dq-RFS model, in the future work, several aspects of these four models are worth investigating, including the parameters selection method, the uncertainty measures and underlying properties of the models with respect to the concept and decision rules.

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