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Attribute reduction in interval-valued fuzzy ordered decision tables via evidence theory

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Jia Zhang[1], *Xiaoyan Zhang*[2] ✉, *Weihua Xu*[2]
[1]*School of Science, Chongqing University of Technology, Chongqing 400054, People's Republic of China*
[2]*School of Mathematics and Statistics, Southwest University, Chongqing, 400715, People's Republic of China*
✉ *E-mail: zxy19790915@163.com*

Abstract: There are two different theory methods that are rough set theory and evidence theory, but these two theories can both handle some incomplete and uncertain information. In this study, these two models are combined in the interval-valued fuzzy ordered information system (IVFOIS). Belief functions and plausibility functions are proposed based on dominance relations in IVFOISs. The belief and plausibility reducts are defined in interval-valued fuzzy ordered decision tables (IVFODTs) and the attribute reduction of IVFODTs based on evidence theory is established. Finally, the authors use an instance to verify the above argument.

1 Introduction

Rough set theory (RST) [1] which was pioneered by the scientist Pawlak in 1982 is an effective mathematical paradigm to tackle the imprecise, uncertain and tremendous information in intelligent systems [2]. The key benefit of the rough set method is that no extra information is needed, rather than other theories, such as membership of fuzzy set theory, the probability distribution in probability theory or basic probability distribution in Dempster–Shafer evidence theory [3], all of which require their respective appendages. The starting point of the theory is to observe the indiscernible objects with the same description in the available information [4, 5]. It can analyse and reason data, and extract hidden knowledge from data, and reveal its potential rules. It is also an objective and effective data mining method. RST is based on the classification mechanism. It links knowledge with classification, and considers that knowledge is the ability to classify objects, and knowledge base is a set of classification methods. Its main idea is to approximate inaccurate or uncertain knowledge by using knowledge known in the knowledge base. Ever since the advent of RST, it has been triumphantly applied in lots of fields especially in expert systems, granular computing and machine learning [6]. Nevertheless, the classical rough set cannot find and handle the inconsistencies from the standard considerations, that is, the order of preference attributes, such as project investment, market share, test scores and debt ratio. In order to handle this problem, Greco and others put forward that the rough set method with preference is used to sort attributes under the dominance criterion, that is, the extension of the traditional RST. It is referred as the dominance-based rough set approach (DRSA) [7–9]. Moreover, the DRSA also considers the relationship between classification based on condition criteria and object set in different information systems, and studies its related contents [10–15]. Since the establishment of the DRSA, it has been extended to the environment of various information systems to cope with different types of rule extraction, knowledge acquisition, attribute reduction and so on.

Another important way to handle the uncertainty of information and data is the Dempster–Shafer evidence theory, which is an extension of Bayesian theory of subjective judgment. Belief structure is the basic representative structure in evidence theory, and it is composed of a subset of elements called focus elements. In addition, a belief structure is also the sum of associated individual positive weights. Belief and plausibility functions [16] are basic numeric measures in the belief structure. Combining the traditional rough set with the Dempster–Shafer evidence theory [17, 18], the mass measure of a partition of the universe is described by using the approximate values of the rough membership of the belief and the plausibility functions [19]. The non-numerical aspects of a set of available information are described by the lower and the upper approximation operators. The numeric aspects about uncertain information of the same set are represented by the belief and plausibility measures. Different belief structures have been shown to be related to different approximation spaces. The two different pairs of lower and upper approximation can make sense to explain corresponding the belief and the plausibility functions [20–24]. Therefore, we can use the above-mentioned two functions to characterise attribute reduct.

Many studies on attribute reduction concerned in the certainly single-valued or ordered information system, there are few concerns about IVFOISs. For example, Qian *et al.* [25] investigated, respectively, a pre-ordered relation in interval-value information systems and decision tables with interval-value and established a rough set method via dominance relation to the interval-valued decision analysis. Yang *et al.* [26, 27] put forward the contents and related properties of dominance relations in the interval-valued information system. Wang and Shi [28] studied on the basis of evidence theory the basic notions and related properties of attribute reduction and obtained the characterisation and knowledge reduction methods via this theory. Du and Hu [29] introduced knowledge reduct in an ordered decision table on the basis of evidence theory. The belief reduction and plausibility reduction were proposed based on belief and plausibility functions. Moreover, their relationships with the traditional reduction were discussed. In our paper, combine the evidence theory with the rough set in the fuzzy decision table, and explore the belief and plausibility function based on pre-ordered relations, and combine the attribute measure to get the relative reduction.

For the sake of discussion, we give some preliminaries on an IVFOIS and fundamental introduction to evidence theory in Section 2. In Section 3, we use the basic set allocation function to establish the probability measure space under the relation with preference in an IVFOIS. Then, in this context, two important functions of evidence theory are defined and their attribute reduction is further studied. At the same time, in Section 4, the decision attribute is added and the content of Section 3 is further investigated. Furthermore, in Section 3, combining interior and exterior significance measures of attribute sets, we establish a general method for finding some type of reduction that we are concerned with, which is applied to the case in Section 4 as well. Section 5 summarises the related research work of knowledge reduction in IVFODTs, and puts forward suggestions for further study.

2 Preliminary

In the following, we first review the basic knowledge, which is divided into two parts: interval-valued fuzzy ordered information system (IVFOIS) and evidence theory.

2.1 Interval-valued fuzzy ordered information systems

An interval-valued fuzzy information system (IVFIS) is a quadruple $I = (U, AT, V, F)$, where U is not a empty finite universe, AT is a finite non-empty attribute set, $V = \bigcup_{a \in A} V_a$ is the domain of all criteria a, where V_a is an interval-valued fuzzy set of universe based on attribute a and $F = \{f : U \to V\}$ are mapping sets of object attribute value, in which $f : U \times AT \to V$ is a function that satisfies $f(x, a) \in V_a$ for each $a \in AT$. That is $f(x, a) = [a^L, a^R]$ for $a \in AT$, where $a^L(x) : U \to [0, 1]$ and $a^R(x) : U \to [0, 1]$ and satisfy $a^L(x) \le a^R(x)$ for each $x \in U$. In particular, when $a^L(x) = a^R(x)$, $f(x, a)$ degenerates into a real number.

Let $I = (U, AT, V, F)$ be an IVFIS. $\forall a \in AT$, the attribute values are compared in an IVFIS. We define

$$\begin{aligned} f(x_i, a) \le f(x_j, a) &\Leftrightarrow a^L(x_i) \le a^L(x_j), \quad a^R(x_i) \le a^R(x_j), \\ f(x_i, a) \ge f(x_j, a) &\Leftrightarrow a^L(x_i) \ge a^L(x_j), \quad a^R(x_i) \ge a^R(x_j), \end{aligned} \tag{1}$$

where '$\le$' and '$\ge$' are, respectively, represented as an decreasing preference and an increasing preference. An attribute is a criterion if the domain of criteria is preference according to an decreasing order and an increasing order.

Definition 1: Let $I = (U, AT, V, F)$ be an IVFIS. If all attributes are criteria, I is referred as an IVFOIS [28, 29].

We suppose that the domain of criterion $a \in AT$ is preference with the relation $\ge_a$. The statement $x_s \ge_a x_t$ indicates that x_s is at least as good as x_t based on the criterion a. We say that $A \subseteq AT$ are criteria. Then $x_s \ge_A x_t \Leftrightarrow x_s \ge_a x_t (\forall a \in A)$. The dominance relation with regard to A can be defined as (see (2)) . Based on dominance relation $R_A^{\ge}$, dominance classes can be defined as

$$\begin{aligned} [x_s]_A^{\ge} &= \{x_t \in U | (x_s, x_t) \in R_A^{\ge}\} \\ &= \{x_t \in U | (\forall a \in A) a^L(x_t) \le a^L(x_j), \quad a^R(x_s) \le a^R(x_t)\} \end{aligned} \tag{3}$$

Let $U/R_A^{\ge} = \{[x]_A^{\ge} | x \in U\}$ represent all of interval-valued dominance classes based on the dominance relation $R_A^{\ge}$. Generally, dominance classes $[x]_A^{\ge}$ are not a partition of the universe U but just a cover of the universe U.

There is an interval-valued fuzzy ordered decision table (IVFODT) $I = (U, AT \cup \{d\}, V, F)$, where $d (d \notin AT)$ is only decision criterion and makes a division of universe U. Let $DC = \{D_s, s \in N\}, N = \{1, 2, \ldots, n\}$ be the set of these ordered classes. It means that for all $s, r \in T$ we have the objects which from D_s are preferred to the objects from D_r, if $r \le s$. The approximated sets based on the decision attribute are upward unions $D_s^{\ge}$ and downward unions $D_s^{\le}$, which, respectively, are defined as $D_s^{\ge} = \bigcup_{r \ge s} D_r, D_s^{\le} = \bigcup_{r \le s} D_r (s \in N)$. So it is easy to get that $D_1^{\ge} = D_n^{\le} = U, D_n^{\ge} = D_n, D_1^{\le} = D_1$. Then $D_s^{\ge}, D_{s-1}^{\le}$ are complementary to each other for all $s \in T$.

Definition 2: Let $I = (U, AT, V, F)$ be an IVFOIS and $A \subseteq AT$. Then the lower and upper approximations with regard to $D_s^{\ge}$ are defined as

$$\begin{aligned} \underline{R_A^{\ge}}(D_s^{\ge}) &= \{x \| [x]_A^{\ge} \subseteq D_s^{\ge}, \quad x \in U\}, \\ \overline{R_A^{\ge}}(D_s^{\ge}) &= \{x \| [x]_A^{\ge} \cap D_s^{\ge} \ne \varnothing, \quad x \in U\}, \end{aligned} \tag{4}$$

$\underline{R_A^{\ge}}(D_s^{\ge})$ definitely belongs to sets of objects of $D_s^{\ge}$. $\overline{R_A^{\ge}}(D_s^{\ge})$ possibly belongs to sets of objects of $D_s^{\ge}$. We denote $Bn_A^{\ge}(D_s^{\ge}) = \overline{R_A^{\ge}}(D_s^{\ge}) - \underline{R_A^{\ge}}(D_s^{\ge})$, where $Bn_A^{\ge}(D_s^{\ge})$ is called the boundary region with respect to $D_s^{\ge}$.

Proposition 1: Let $I = (U, AT, V, F)$ be an IVFOIS and $A \subseteq AT$. Then for $D_s^{\ge} (1 \le s \le n)$, rough approximations $\overline{R_A^{\ge}}(D_s^{\ge}), \underline{R_A^{\ge}}(D_s^{\ge})$ meet the following conclusions:

(i) $\underline{R_A^{\ge}}(D_s^{\ge}) \subseteq D_s^{\ge} \subseteq \overline{R_A^{\ge}}(D_s^{\ge})$.
(ii) $\underline{R_A^{\ge}}(\varnothing) = \varnothing, \overline{R_A^{\ge}}(\varnothing) = \varnothing$.
(iii) $\underline{R_A^{\ge}}(U) = U, \overline{R_A^{\ge}}(U) = U$.
(iv)
$\underline{R_A^{\ge}}(D_s^{\ge} \cap D_r^{\ge}) = \underline{R_A^{\ge}}(D_s^{\ge}) \cap \underline{R_A^{\ge}}(D_r^{\ge}), \overline{R_A^{\ge}}(D_s^{\ge} \cup D_r^{\ge}) = \overline{R_A^{\ge}}(D_s^{\ge}) \cup \overline{R_A^{\ge}}(D_r^{\ge})$.
(v)
$\underline{R_A^{\ge}}(D_s^{\ge} \cup D_r^{\ge}) \supseteq \underline{R_A^{\ge}}(D_s^{\ge}) \cup \underline{R_A^{\ge}}(D_r^{\ge}), \overline{R_A^{\ge}}(D_s^{\ge} \cap D_r^{\ge}) \subseteq \overline{R_A^{\ge}}(D_s^{\ge}) \cap \overline{R_A^{\ge}}(D_r^{\ge})$.

2.2 Dempster–Shafer evidence theory

Next, we mainly review the belief and plausibility functions of evidence theory. First, let us first introduce the mass function that is closely related to the two functions mentioned above.

Definition 3: Let a function $m : P(U) \to [0, 1]$ be to satisfy the following two conditions [30].

(i) $m(\varnothing) = 0$.
(ii) $\sum_{X \subseteq U} m(X) = 1$.

So, the function m can be referred as a mass function.

Combine the belief structure (M, m), the belief and plausibility functions are given in the following.

Definition 4: Let a function $\mathrm{Bel} : P(U) \to [0, 1]$ be a belief function if for $X \in P(U)$, the function satisfies the following equation where (M, m) is a belief structure on universe U [30, 31]:

$$\mathrm{Bel}(X) = \sum_{\substack{T \subseteq X \\ T \in M}} m(T). \tag{5}$$

Let a function $\mathrm{Pl} : P(U) \to [0, 1]$ be a plausibility function on universe U if the function satisfies the following condition for $X \in P(U)$:

$$\mathrm{Pl}(X) = \sum_{\substack{T \cap X \ne 0 \\ T \in M}} m(T). \tag{6}$$

The belief and the plausibility functions correspond to the lower approximation and the upper approximation, which are used to indicate the certainty and possibility of the support for the set. They satisfy the following property:

(i) $\mathrm{Bel}(\varnothing) = 0, \quad \mathrm{Pl}(\varnothing) = 0$.
(ii) $\mathrm{Bel}(U) = 1, \quad \mathrm{Pl}(U) = 1$.
(iii) For $\forall X_i \in P(U)$

$$\begin{aligned} R_A^{\ge} &= \{(x_s, x_t) \in U \times U | x_t \ge_a x_s, \quad \forall a \in A\} \\ &= \{(x_s, x_t) \in U \times U | (\forall a \in A) a^L(x_s) \le a^L(x_t), \quad a^R(x_s) \le a^R(x_t)\}. \end{aligned} \tag{2}$$

$$\mathrm{Bel}\left(\bigcup_{i=1}^{n} X_i\right) \geq \sum_{\phi \neq K \subseteq \{1,2,\ldots,n\}} (-1)^{|K|+1}\mathrm{Bel}\left(\bigcap_{k=K} X_i\right),$$
$$\mathrm{Pl}\left(\bigcup_{i=1}^{n} X_i\right) \leq \sum_{\phi \neq K \subseteq \{1,2,\ldots,n\}} (-1)^{|K|+1}\mathrm{Bel}\left(\bigcap_{k=K} X_i\right). \quad (7)$$

3 Attribute reduction in IVFOISs

Reduction is the minimal subset of attributes in an information system. In an IVFOIS, attribute reduction is to remove some unnecessary criteria from the information system according to the dominance relation. In this part, the evidence theory is extended to the relation with preference, and the attribute reduct method of IVFOISs in view of evidence theory is established.

Definition 5: Let $I = (U, AT, V, F)$ be an IVFOIS. For $X \in P(U), A \subseteq AT$. We denote

$$h_A(X) = \{x : [x]_A^{\gtrsim} = X, \quad x \in U\}. \quad (8)$$

Apparently, $y \in h_A(X) \Leftrightarrow [y]_A^{\gtrsim} = X$. Meanwhile, $h_A(X)$ satisfies the following two properties:

(i) $\bigcup_{X \subseteq U} h_A(X) = U$,
(ii) If $X_s \neq X_t$, then $h_A(X_s) \cap h_A(X_t) = \varnothing$.

Obviously, the function $h_A(X)$ is the universe partition into equivalence classes. So a mass function m_A based on attribute set A can be defined as according to the function $h_A(X)$

$$m_A(X) = \frac{|h_A(X)|}{|U|}. \quad (9)$$

It is easy to know that numeric measure m_A stems from ratio of cardinalities of sets corresponding h_A and the universe U.

Definition 6: Let $I = (U, AT, V, F)$ be an IVFOIS, $X \in P(U)$ and $A \subseteq AT$. m_A is a mass function with regard to A. We have a belief and a plausibility function on U in the following [24, 29]:

$$\mathrm{Bel}_A^{\gtrsim}(X) = \sum_{T \subseteq X, T \in U/R_A^{\gtrsim}} m_A(T),$$
$$\mathrm{Pl}_A^{\gtrsim}(X) = \sum_{T \cap X \neq \varnothing, T \in U/R_A^{\gtrsim}} m_A(T). \quad (10)$$

It is apparent to know that $0 \leq \mathrm{Bel}_A^{\gtrsim}(X) \leq 1, 0 \leq \mathrm{Pl}_A^{\gtrsim}(X) \leq 1$. In the same time, the above two functions meet the following two properties:

(i) $\mathrm{Pl}_A^{\gtrsim}(X) = 1 - \mathrm{Bel}_A^{\gtrsim}(X^C)$.
(ii) $\mathrm{Bel}_A^{\gtrsim}(X) \leq \mathrm{Pl}_A^{\gtrsim}(X)$, for every $X \in P(U)$.
(iii) $\mathrm{Bel}_A^{\gtrsim}(X) \leq \mathrm{Bel}_A^{\gtrsim}(X) \leq \mathrm{Pl}_A^{\gtrsim}(X) \leq \mathrm{Pl}_A^{\gtrsim}(X)$, if $B \subseteq A \subseteq AT$.

By the definition of belief function $\mathrm{Bel}_{\gtrsim}^{A}(X)$ and mass function $m_A(X)$ it follows that (see (11)) . Finally, we can find that $\mathrm{Bel}_A^{\gtrsim}(X) = \left(\left|\underline{R_A^{\gtrsim}}(X)\right|/|U|\right)$.

In the same way, it is easy to find $\mathrm{Pl}_A^{\gtrsim}(X) = \left(\left|\overline{R_A^{\gtrsim}}(X)\right|/|U|\right)$. Thus, we get that $\mathrm{Bel}_A^{\gtrsim}(X) \leq (|X|/|U|) \leq \mathrm{Pl}_A^{\gtrsim}(X)$, for every $X \in P(U)$. Especially, when $X \in U/R_A^{\gtrsim}$, the equation $\mathrm{Bel}_A^{\gtrsim}(X) = |X|/|U|$ holds. Due to $\exists\, y \in U$, such that $R_A^{\gtrsim}(y) = X$, we get that

$$\begin{aligned}\mathrm{Bel}_A^{\gtrsim}(X) &= \frac{\left|\{x \in U : [x]_A^{\gtrsim} \subseteq X\}\right|}{|U|} \\ &= \frac{\left|\{x \in U : [x]_A^{\gtrsim} \subseteq R_A^{\gtrsim}(y)\}\right|}{|U|} \\ &= \frac{\left|\{x \in U : x \in R_A^{\gtrsim}(y)\}\right|}{|U|} = \frac{\left|R_A^{\gtrsim}(y)\right|}{|U|} = \frac{|X|}{|U|}.\end{aligned} \quad (12)$$

Therefore, there is property in the following.

Theorem 1: Let $I = (U, AT, V, F)$ be an IVFOIS and $A \subseteq AT$. Then

(i)
$\mathrm{Bel}_A^{\gtrsim}(X) = \left(\left|\underline{R_A^{\gtrsim}}(X)\right|/|U|\right)$, especially $X \in U/R_A^{\gtrsim}$, we have $\mathrm{Bel}_A^{\gtrsim}(X) = |X|/|U|$.
(ii) $\mathrm{Pl}_A^{\gtrsim}(X) = \left(\left|\overline{R_A^{\gtrsim}}(X)\right|/|U|\right)$.

Definition 7: Let $I = (U, AT, V, F)$ be an IVFOIS and $B \subseteq A \subseteq AT$, then

(i) A is called a consistent set if $R_A^{\gtrsim}(X) = R_{AT}^{\gtrsim}(X)$ holds. Moreover, if $R_A^{\gtrsim}(X) = R_{AT}^{\gtrsim}(X)$ and $R_B^{\gtrsim}(X) \neq R_{AT}^{\gtrsim}(X)$, then A is called a reduct.
(ii) A is called a belief consistent set if $\mathrm{Bel}_A^{\gtrsim}(X) = \mathrm{Bel}_{AT}^{\gtrsim}(X)$ holds. Moreover, $\forall X \in U/R_A^{\gtrsim}$, if $\mathrm{Bel}_A^{\gtrsim}(X) = \mathrm{Bel}_{AT}^{\gtrsim}(X)$ and $\mathrm{Bel}_B^{\gtrsim}(X) \neq \mathrm{Bel}_{AT}^{\gtrsim}(X)$, then A is called a belief reduct.
(iii) A is called a plausibility consistent set if $\mathrm{Pl}_A^{\gtrsim}(X) = \mathrm{Pl}_{AT}^{\gtrsim}(X)$ holds. Moreover, $\forall X \in U/R_A^{\gtrsim}$, if $\mathrm{Pl}_A^{\gtrsim}(X) = \mathrm{Pl}_{AT}^{\gtrsim}(X)$ and $\mathrm{Pl}_B^{\gtrsim}(X) \neq \mathrm{Pl}_{AT}^{\gtrsim}(X)$, then A is called a plausibility reduct.

Theorem 2: Let $I = (U, AT, V, F)$ be an IVFOIS, $X \in U/R_{AT}^{\gtrsim}$ and $A \subseteq AT$. Then we have

(i) A is called a belief consistent set $\Leftrightarrow \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_A^{\gtrsim}(X) = H_{\mathrm{Bel}}$.
(ii) A is called a belief reduction set $\Leftrightarrow \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_A^{\gtrsim}(X) = H_{\mathrm{Bel}}$, and for any set of non-empty subsets $B\varnothing A$, $\sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) < H_{\mathrm{Bel}}$.
(iii) A is called a belief reduct set $\Leftrightarrow \sum_{X \in U/R_{AT}^{\gtrsim}} (\mathrm{Bel}_A^{\gtrsim}(X))/|X| = 1$, and for any set of non-empty subset $B\varnothing A$, $\sum_{X \in U/R_{AT}^{\gtrsim}} (\mathrm{Bel}_B^{\gtrsim}(X))/|X| < 1$..

where H_{Bel} is represented as the belief sum $\sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_{AT}^{\gtrsim}(X) = H_{\mathrm{Bel}}$.

Proof:

(i) A is called a belief consistent set

$$\begin{aligned}\mathrm{Bel}_A^{\gtrsim}(X) &= \sum_{T \subseteq X, T \in U/R_A^{\gtrsim}} m_A(T) = \sum_{T \subseteq X, T \in U/R_A^{\gtrsim}} \frac{|h_A(T)|}{|U|} \\ &= \sum_{T \subseteq X, T \in U/R_A^{\gtrsim}} \frac{\left|\{x \in U : [x]_A^{\gtrsim} = T\}\right|}{|U|} = \frac{\left|\{x \in U : [x]_A^{\gtrsim} \subseteq X\}\right|}{|U|} \\ &= \frac{\left|\underline{R_A^{\gtrsim}}(X)\right|}{|U|}.\end{aligned} \quad (11)$$

$$\Leftrightarrow \mathrm{Bel}_A^{\geq}(X) = \mathrm{Bel}_{AT}^{\geq}(X) \Leftrightarrow \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_{AT}^{\geq}(X) = \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_A^{\geq}(X) = H_{\mathrm{Bel}}. \tag{13}$$

(ii) According to the property $\mathrm{Bel}_B^{\geq}(X) \leq \mathrm{Bel}_A^{\geq}(X)$ if $B \subseteq A$, we can obtain the conclusion that A is called a belief reduction set. $\Leftrightarrow \mathrm{Bel}_B^{\geq} \leq \mathrm{Bel}_A^{\geq}$.

$$\Leftrightarrow \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_B^{\geq}(X) < \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_A^{\geq}(X) = H_{Bel}. \tag{14}$$

(iii) Based on Theorem 1, it is apparent to know that $\mathrm{Bel}_A^{\geq}(X) = |X|/|U|$ when $X \in U/R_A^{\geq}$. So

$$\sum_{X \in U/R_{AT}^{\geq}} \frac{\mathrm{Bel}_A^{\geq}(X)}{|X|} = \sum_{\substack{x \in U \\ X = R_A^{\geq}(x)}} \frac{\mathrm{Bel}_A^{\geq}(X)}{|X|} = \sum_{x \in U} \frac{1}{|X|} \frac{|X|}{|U|} = 1. \tag{15}$$

□

Now, we know that A satisfies the condition of a belief consistent set, and combine Theorem 2 (ii) $\sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_B^{\geq}(X) < \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_A^{\geq}(X) = H_{\mathrm{Bel}}$. Furthermore

$$\sum_{X \in U/R_{AT}^{\geq}} \frac{\mathrm{Bel}_B^{\geq}(X)}{|X|} < 1. \tag{16}$$

In the following, based on belief function and plausibility function, the concept of interior and exterior significance measures is introduced. By studying the importance measure of each attribute, we can achieve the belief reduction and the plausibility reduction.

For $B \subseteq AT$, if the attribute a belongs to attribute set B, the interior significance measure of attribute a indicates the importance of attributes in the attribute set B. So it is to determine whether the attribute a is necessary to an attribute set B. The interior significance measure of the attribute is defined by deleting this attribute a to compare the changes in the belief function.

Definition 8: Let $I = (U, AT, V, F)$ be an IVFOIS, $B \subseteq AT$ and $a \in B$.

Then the interior significance measure of criterion a with regard to B is defined as

$$\mathrm{sig}_{\mathrm{interior}}^{\geq}(a, B) = \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_B^{\geq}(X) - \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_{B-\{a\}}^{\geq}(X). \tag{17}$$

Obviously, it is apparent to know that $\mathrm{sig}_{\mathrm{interior}}^{\geq}(a, B) \geq 0$. When $\mathrm{sig}_{\mathrm{iinterior}}^{\geq}(a, B) > 0$, it is represented as a to be indispensable in B. Otherwise, the attribute a is redundant.

Definition 9: Let $I = (U, AT, V, F)$ be an IVFOIS, $B \subseteq AT$ and $a \notin B$.

Then the exterior significance measure of attribute a with regard to B is defined as

$$\mathrm{sig}_{\mathrm{exterior}}^{\geq}(a, B) = \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_{B \cup \{a\}}^{\geq}(X) - \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_B^{\geq}(X). \tag{18}$$

Similarly, for $B \subseteq AT$, if the attribute a does not belong to attribute set B, the exterior significance measure means that the attribute set B does not have this attribute a and that is whether produces the change of the belief function in the attribute set when the attribute a is added. Thus, the exterior importance of the attribute to the attribute set is judged.

Certainly, the belief core is essential attribute for the attribute set AT. According to the above, we can obtain the belief core $\mathrm{Core} = \{a \in AT: \mathrm{sig}_{\mathrm{interior}}^{\geq}(a, AT) > 0\}$. The core is the only one and the intersection of all the belief reductions. So, when we have the core, compare the exterior significance measure of attributes which are out of core and expand from a belief core to a belief reduction B by adding a number of attributes to satisfy the condition

$$\sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_B^{\geq}(X) = \sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_{AT}^{\geq}(X) \tag{19}$$

Next, we use an example to better illustrate relative belief reduction and relative plausibility reduction.

Example 1: Venture investment has become a more and more important source of funding for a lot of firms and plays a vital role in the process of entrepreneurship, especially in emerging cutting-edge technologies and markets. For investors and policy makers, before investing, you have to choose a better project based on some potential capital projects, or find some direction according to your existing successful investment cases. Considering the investment problem of VC firm, there are now eight investors who can evaluate them from the risk factors. There are four kinds of risk factors: market, technology, management and production. Consider the IVFOISs and the IVFOIS are given in Table 1, where $U = \{x_1, x_2, \ldots, x_8\}$ and $AT = \{\text{market}, \text{technology}, \text{production}, \text{management}\} = \{a_1, a_2, a_3, a_4\}$

By calculation, we have $U/R_A^{\geq} = \{[x]_A^{\geq} \in P(U) | x \in U\}$, where

$$\begin{aligned} &[x_1]_{AT}^{\geq} = \{x_1, x_3, x_7\}, \quad [x_2]_{AT}^{\geq} = \{x_2, x_7, x_8\}, \quad [x_3]_{AT}^{\geq} = \{x_3, x_7\}. \\ &[x_4]_{AT}^{\geq} = \{x_3, x_4, x_7\}, \quad [x_5]_{AT}^{\geq} = [x_8]_{AT}^{\geq}\{x_3, x_5, x_7, x_8\}, \\ &[x_6]_{AT}^{\geq} = \{x_1, x_3, x_6, x_7, x_8\}, \quad [x_7]_{AT}^{\geq} = \{x_7\}. \end{aligned} \tag{20}$$

According to Definition 5 that we have

$$h_{AT}(X) = \begin{cases} \{x\}, & X = [x]_{AT}^{\geq} \\ \varnothing, & \text{otherwise} \end{cases} \tag{21}$$

Therefore, $[x]_{AT}^{\geq} = \{y \in U: y = h_{AT}(X), X \in P(U)\}$, thus

$$m_{AT}(X) = \begin{cases} \frac{1}{8}, & X = [x]_{AT}^{\geq} \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

Table 1 Interval-valued fuzzy ordered information system

U	a_1	a_2	a_3	a_4
x_1	[0.12,0.23]	[0.72,0.79]	[0.45,0.63]	[0.45,0.63]
x_2	[0.72,0.38]	[0,07,0.21]	[0.45,0.63]	[0.23,0.38]
x_3	[0.47,0.60]	[0,77,0.81]	[0.72,0.81]	[0.57,0.69]
x_4	[0.42,0.58]	[0,72,0.79]	[0.45,0.63]	[0.14,0.32]
x_5	[0.47,0.60]	[0.14,0.24]	[0.72,0.81]	[0,49,0.62]
x_6	[0.07,0.21]	[0.42,0.49]	[0.21,0.38]	[0.41,0.57]
x_7	[0.82,0.94]	[0.82,0.83]	[0.79,0.85]	[0.76,0.89]
x_8	[0.74,0.88]	[0.47,0.53]	[0.72,0.81]	[0.51,0.64]

Then (see (23)) .

(i) Let $B = \{a_1, a_2, a_3\}$, then $U/R_B^{\gtrsim} = \{[x]_B^{\gtrsim} \in P(U): x \in U\}$, where

$$\begin{aligned} &[x_1]_B^{\gtrsim} = \{x_1, x_3, x_4, x_7\}, \quad [x_2]_B^{\gtrsim} = \{x_2, x_7, x_8\}, \\ &[x_3]_B^{\gtrsim} = \{x_3, x_7\}, \quad [x_4]_B^{\gtrsim} = \{x_3, x_4, x_7\}, \\ &[x_5]_B^{\gtrsim} = \{x_3, x_5, x_7, x_8\}, \quad [x_6]_B^{\gtrsim} = \{x_1, x_3, x_4, x_6, x_7, x_8\}, \\ &[x_7]_B^{\gtrsim} = \{x_7\}, \quad [x_8]_B^{\gtrsim} = \{x_7, x_8\}. \end{aligned} \tag{24}$$

Then, we have

$$\sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) = \frac{2}{8} \times 3 + \frac{1}{8} + \frac{3}{8} \times 3 + \frac{4}{8} = \frac{20}{8}. \tag{25}$$

Therefore

$$\begin{aligned} \mathrm{sig}_{\mathrm{interior}}^{\gtrsim}(a_4, AT) &= \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_{AT}^{\gtrsim}(X) - \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) \\ &= \frac{23}{8} - \frac{20}{8} > 0. \end{aligned} \tag{26}$$

(ii) Let $B = \{a_1, a_2, a_4\}$, then $U/R_B^{\gtrsim} = \{[x]_B^{\gtrsim} \in P(U)\}$ where

$$\begin{aligned} &[x_1]_B^{\gtrsim} = \{x_1, x_3, x_7\}, \quad [x_2]_B^{\gtrsim} = \{x_2, x_7, x_8\}, \quad [x_3]_B^{\gtrsim} = \{x_3, x_7\}, \\ &[x_4]_B^{\gtrsim} = \{x_3, x_4, x_7\}, \quad [x_5]_B^{\gtrsim} = \{x_3, x_5, x_7, x_8\}, \\ &[x_6]_B^{\gtrsim} = \{x_1, x_3, x_6, x_7, x_8\}, \quad [x_7]_B^{\gtrsim} = \{x_7\}, \quad [x_8]_B^{\gtrsim} = \{x_7, x_8\}. \end{aligned} \tag{27}$$

Then, we have

$$\sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) = \frac{2}{8} \times 3 + \frac{5}{8} + \frac{2}{8} \times 2 + \frac{4}{8} + \frac{1}{8} = \frac{23}{8}. \tag{28}$$

Therefore

$$\begin{aligned} \mathrm{sig}_{\mathrm{interior}}^{\gtrsim}(a_3, AT) &= \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_{AT}^{\gtrsim}(X) - \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) \\ &= \frac{23}{8} - \frac{23}{8} = 0. \end{aligned} \tag{29}$$

(iii) Let $B = \{a_1, a_3, a_4\}$, then $U/R_B^{\gtrsim} = \{[x]_B^{\gtrsim} \in P(U): x \in U\}$ where

$$\begin{aligned} &[x_1]_B^{\gtrsim} = \{x_1, x_3, x_5, x_7, x_8\}, [x_2]_B^{\gtrsim} = \{x_2, x_7, x_8\}, \\ &[x_3]_B^{\gtrsim} = \{x_3, x_5, x_7, x_8\}, [x_4]_B^{\gtrsim} = \{x_2, x_3, x_4, x_5, x_7, x_8\}, \\ &[x_5]_B^{\gtrsim} = \{x_3, x_5, x_7, x_8\}, [x_6]_B^{\gtrsim} = \{x_1, x_3, x_5, x_6, x_7, x_8\}, \\ &[x_7]_B^{\gtrsim} = \{x_7\}, [x_8]_B^{\gtrsim} = \{x_7, x_8\}. \end{aligned} \tag{30}$$

Then, we have

$$\sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) = \frac{1}{8} \times 4 + \frac{3}{8} + \frac{2}{8} \times 2 + \frac{4}{8} = \frac{15}{8}, \tag{31}$$

Therefore

$$\begin{aligned} \mathrm{sig}_{\mathrm{interior}}^{\gtrsim}(a_2, AT) &= \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_{AT}^{\gtrsim}(X) - \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) \\ &= \frac{23}{8} - \frac{15}{8} > 0. \end{aligned} \tag{32}$$

(iv) Let $B = \{a_2, a_3, a_4\}$, then $U/R_B^{\gtrsim} = \{[x]_B^{\gtrsim} \in P(U): x \in U\}$ where

$$\begin{aligned} &[x_1]_B^{\gtrsim} = \{x_1, x_3, x_7\}, [x_2]_B^{\gtrsim} = \{x_1, x_2, x_3, x_4, x_5, x_7, x_8\}, \\ &[x_3]_B^{\gtrsim} = \{x_3, x_7\}, [x_4]_B^{\gtrsim} = \{x_1, x_3, x_4, x_7\}, \\ &[x_5]_B^{\gtrsim} = \{x_3, x_5, x_7, x_8\}, [x_6]_B^{\gtrsim} = \{x_1, x_3, x_6, x_7, x_8\}, \\ &[x_7]_B^{\gtrsim} = \{x_7\}, [x_8]_B^{\gtrsim} = \{x_3, x_7, x_8\}. \end{aligned} \tag{33}$$

Then, we have

$$\sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) = \frac{1}{8} \times 3 + \frac{3}{8} + \frac{2}{8} \times 2 + \frac{4}{8} + \frac{5}{8} = \frac{19}{8}, \tag{34}$$

Therefore

$$\begin{aligned} \mathrm{sig}_{\mathrm{interior}}^{\gtrsim}(a_1, AT) &= \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_{AT}^{\gtrsim}(X) - \sum_{X \in U/R_{AT}^{\gtrsim}} \mathrm{Bel}_B^{\gtrsim}(X) \\ &= \frac{23}{8} - \frac{19}{8} > 0 \end{aligned} \tag{35}$$

Thus, we know a_3 is redundant, the only one belief core and the belief reduct both are $\{a_1, a_2, a_4\}$. It means that production is not the risk of investment for eight objects.

4 Attribute reductions in IVFODTs

In an decision table, we have to add the decision makers' opinions. Therefore, attribute reduction needs to delete redundant attributes with respect to decision classification in information systems with decision. Next, we introduce the concepts of relative belief reduction and relative plausibility reduction with respect to decision attributes and discuss the relation between them in IVFODTs.

Definition 10: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS and $B \subseteq A \subseteq AT$. Then

(i) A is called a relative lower consistent set of I if $\underline{R_A^{\gtrsim}}(D_s^{\gtrsim}) = \underline{R_{AT}^{\gtrsim}}(D_s^{\gtrsim})$, for all $D_s^{\gtrsim}$. If $\underline{R_A^{\gtrsim}}(D_s^{\gtrsim}) = \underline{R_{AT}^{\gtrsim}}(D_s^{\gtrsim})$ and $\underline{R_B^{\gtrsim}}(D_s^{\gtrsim}) \neq \underline{R_{AT}^{\gtrsim}}(D_s^{\gtrsim})$, then A is called a relative lower reduction of I.
(ii) A is called a relative upper consistent set of I if $\overline{R_A^{\gtrsim}}(D_s^{\gtrsim}) = \overline{R_{AT}^{\gtrsim}}(D_s^{\gtrsim})$, for all $D_s^{\gtrsim}$. If $\overline{R_A^{\gtrsim}}(D_s^{\gtrsim}) = \overline{R_{AT}^{\gtrsim}}(D_s^{\gtrsim})$ and $\overline{R_B^{\gtrsim}}(D_s^{\gtrsim}) \neq \overline{R_{AT}^{\gtrsim}}(D_s^{\gtrsim})$, then A is called a relative upper reduction of I.

Definition 11: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS and $B \subseteq A \subseteq AT$. Then

(i) A is called a relative boundary consistent set if $Bn_A(D_s^{\gtrsim}) = Bn_{AT}(D_s^{\gtrsim})$, for all $D_s^{\gtrsim}$. If $Bn_A(D_s^{\gtrsim}) = Bn_{AT}(D_s^{\gtrsim})$ and $Bn_B(D_s^{\gtrsim}) \neq Bn_{AT}(D_s^{\gtrsim})$, then A is called a relative boundary reduction.
(ii) A is called a relative belief consistent set if $\mathrm{Bel}_A^{\gtrsim}(D_s^{\gtrsim}) = \mathrm{Bel}_{AT}^{\gtrsim}(D_s^{\gtrsim})$, for all $D_s^{\gtrsim}$. If $\mathrm{Bel}_A^{\gtrsim}(D_s^{\gtrsim}) = \mathrm{Bel}_{AT}^{\gtrsim}(D_s^{\gtrsim})$ and $\mathrm{Bel}_B^{\gtrsim}(D_s^{\gtrsim}) \neq \mathrm{Bel}_{AT}^{\gtrsim}(D_s^{\gtrsim})$, then A is called a relative belief reduction.
(iii) A is called a relative plausibility consistent set if $\mathrm{Pl}_A^{\gtrsim}(D_s^{\gtrsim}) = \mathrm{Pl}_{AT}^{\gtrsim}(D_s^{\gtrsim})$, for all $D_s^{\gtrsim}$. If $\mathrm{Pl}_A^{\gtrsim}(D_s^{\gtrsim}) = \mathrm{Pl}_{AT}^{\gtrsim}(D_s^{\gtrsim})$ and $\mathrm{Pl}_B^{\gtrsim}(D_s^{\gtrsim}) \neq \mathrm{Pl}_{AT}^{\gtrsim}(D_s^{\gtrsim})$, then A is called a relative plausibility reduction.

Certain rules can be obtained from the lower approximations. Relative to the rules of possible decision, it is supported by the upper approximations. Due to decision makers who have different preferences for risk, we need different relative reductions such as a relative belief and a relative plausibility reductions.

Proposition 2: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS and $A \subseteq AT$. Then

(i) $\mathrm{Bel}_A^{\gtrsim}(D_s^{\gtrsim}) = \left(\left|\underline{R_A^{\gtrsim}}(D_s^{\gtrsim})\right|/|U|\right)$.
(ii) $\mathrm{Pl}_A^{\gtrsim}(D_s^{\gtrsim}) = \left(\left|\overline{R_A^{\gtrsim}}(D_s^{\gtrsim})\right|/|U|\right)$.

Proof: On the basis of Theorem 1, we have

$$\sum_{X \in U/R_{AT}^{\geq}} \mathrm{Bel}_{AT}^{\geq}(X) = (3/8) \times 3 + (5/8) + (2/8) \times 2 + (4/8) = (23/8) \tag{23}$$

Table 2 Interval-valued fuzzy ordered decision table

U	a_1	a_2	a_3	a_4	d
x_1	[0.12,0.23]	[0.72,0.79]	[0.45,0.63]	[0.45,0.63]	1
x_2	[0.72,0.38]	[0,07,0.21]	[0.45,0.63]	[0.23,0.38]	1
x_3	[0.47,0.60]	[0,77,0.81]	[0.72,0.81]	[0.57,0.69]	2
x_4	[0.42,0.58]	[0,72,0.79]	[0.45,0.63]	[0.14,0.32]	3
x_5	[0.47,0.60]	[0.14,0.24]	[0.72,0.81]	[0,49,0.62]	3
x_6	[0.07,0.21]	[0.42,0.49]	[0.21,0.38]	[0.41,0.57]	2
x_7	[0.82,0.94]	[0.82,0.83]	[0.79,0.85]	[0.76,0.89]	3
x_8	[0.74,0.88]	[0.47,0.53]	[0.72,0.81]	[0.51,0.64]	2

$$\mathrm{Bel}_A^{\geq}(X) = \frac{\left|\{x \in U : [x]_A^{\geq} \subseteq X\}\right|}{|U|}, \qquad \mathrm{Pl}_A^{\geq}(X) = \frac{\left|\{x \in U : [x]_A^{\geq} \cap X \neq \varnothing\}\right|}{|U|}. \tag{36}$$

Combine the definitions of $\underline{R_A^{\geq}}(D_s^{\geq})$ and $\overline{R_A^{\geq}}(D_s^{\geq})$. So we have $Bel_A^{\geq}(D_s^{\geq}) = \left(\left|\underline{R_A^{\geq}}(D_s^{\geq})\right|/|U|\right), Pl_A^{\geq}(D_s^{\geq}) = \left(\left|\overline{R_A^{\geq}}(D_s^{\geq})\right|/|U|\right)$. □

Theorem 3: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS and $A \subseteq AT$. Then

A is a relative belief consistent set(reduct)

$\Leftrightarrow$ A isa relative lower consistent set(reduct)

Proof:

A is a relative belief consistent set

$\Leftrightarrow \mathrm{Bel}_A^{\geq}(D_s^{\geq}) = \mathrm{Bel}_{AT}^{\geq}(D_s^{\geq})$ for all $D_s^{\geq}$.

$\Leftrightarrow \left|\underline{R_A^{\geq}}(D_s^{\geq})\right| = \left|\underline{R_{AT}^{\geq}}(D_s^{\geq})\right|$ for all $D_s^{\geq}$.

$\Leftrightarrow \underline{R_A^{\geq}}(D_s^{\geq}) = \underline{R_{AT}^{\geq}}(D_s^{\geq})$ for all $D_s^{\geq}$.

$\Leftrightarrow A$ is a relative lower consistent set

The relationship between various reductions in IVFODTs will be specified in the following study. □

Definition 12: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS and the decision criterion d. Then the pre-ordered relation with regard to d can be defined

$$R_d^{\geq} = \{(x, y) \in U \times U | f(x, d) \geq f(y, d)\}. \tag{37}$$

An information system is consistent if $\forall (x, y) \in R_{AT}^{\geq}$ such that $(x, y) \in R_d^{\geq}$. That means it satisfies $R_{AT}^{\geq} \subseteq R_d^{\geq}$, then it is called a consistent information system.

For a inconsistent IVFOIS, the following characterisations hold.

Proposition 3: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS. Then

$$I \text{ is consistent} \Leftrightarrow \underline{R_{AT}^{\geq}}(D_s^{\geq}) = D_s^{\geq} = \overline{R_{AT}^{\geq}}(D_s^{\geq}),\ \forall s \in \{1, \ldots, n\} \tag{38}$$

Proof: $' \Rightarrow '$ Obviously, it is that $\underline{R_{AT}^{\geq}}(D_s^{\geq}) \subseteq D_s^{\geq}$. From other perspective, it follows that $R_d^{\geq}(x) \subseteq D_s^{\geq}$ for all $x \in D_s^{\geq}$. According to the consistency of I, it is clear that $R_{AT}^{\geq}(x) \subseteq R_d^{\geq}(x)$. So we have $x \in \underline{R_{AT}^{\geq}}(D_s^{\geq})$. Thus

$$\underline{R_{AT}^{\geq}}(D_s^{\geq}) = D_s^{\geq}. \tag{39}$$

First, we have $D_s^{\geq} \subseteq \overline{R_{AT}^{\geq}}(D_s^{\geq})$. And $\forall x \in \overline{R_{AT}^{\geq}}(D_s^{\geq}), \exists y \in D_s^{\geq}$, such that $x \in R_{AT}^{\geq}(y)$. As I is consistent, it is that $R_{AT}^{\geq}(y) \subseteq R_d^{\geq}(y)$. By considering $\overline{R_d^{\geq}}(y) \subseteq D_s^{\geq}$. Then we have $x \in D_s^{\geq}$. Hence, $\overline{R_{AT}^{\geq}}(D_s^{\geq}) \subseteq D_s^{\geq}$. Finally, we get that $\overline{R_{AT}^{\geq}}(D_s^{\geq}) = D_s^{\geq}$.

$' \Leftarrow '$ If there exist an $x \in U$ such that $R_{AT}^{\geq}(x) \not\subset R_d^{\geq}(x)$ then there exists a y of universe U that satisfies $y \in R_{AT}^{\geq}(x)$ not yet $y \notin R_d^{\geq}(x)$. By $f(x, d) = s$, then $x \in D_s^{\geq}$ and $y \notin D_s^{\geq}$. According to the assumption $\underline{R_{AT}^{\geq}}(D_s^{\geq}) = D_s^{\geq}$, we have $x \in D_s^{\geq}$ then it is that $R_{AT}^{\geq}(x) \subseteq D_s^{\geq}$. Thus, $y \in D_s^{\geq}$. It is a contradiction. □

Definition 13: Let $I = (U, AT \cup \{d\}, V, F)$ be an IVFOIS, $B \subseteq AT$. Then the interior and exterior significance measure of the criterion a with regard to decision criterion d when, respectively, $a \in B$ and $a \notin B$ is defined as

$$\mathrm{sig}_{\mathrm{interior}}^{\geq}(a, B, d) = \sum_{s \in T} \mathrm{Bel}_B^{\geq}(D_s^{\geq}) - \sum_{s \in T} \mathrm{Bel}_{B - \{a\}}^{\geq}(D_s^{\geq}) \quad (a \in B). \tag{40}$$

$$\mathrm{sig}_{\mathrm{exterior}}^{\geq}(a, B, d) = \sum_{s \in T} \mathrm{Bel}_{B \cup \{a\}}^{\geq}(D_s^{\geq}) - \sum_{s \in T} \mathrm{Bel}_B^{\geq}(D_s^{\geq}) \quad (a \notin B). \tag{41}$$

Based on IVFODTs, the following example is given to verify the above theory.

Example 2: Continue from Example 1. Let us add the decision criterion d, which represents the degree of risk taking (Table 2).

So we can get

$$U/d = DC = \{D_1, D_2, D_3\}, \tag{42}$$

where $D_1 = \{x_1, x_2\}, D_2 = \{x_4, x_7, x_8\}, D_3 = \{x_3, x_5, x_6\}$.

Thus, $D_1^{\geq} = \{x_1, \ldots, x_8\}$, $D_2^{\geq} = \{x_3, \ldots, x_8\}$, $D_3^{\geq} = \{x_3, x_5, x_6\}$.

Let

$$\sum_{s \in T} \mathrm{Bel}_{AT}^{\geq}(D_s^{\geq}) = \mathrm{Bel}_{AT}^{\geq}(D_1^{\geq}) + \mathrm{Bel}_{AT}^{\geq}(D_2^{\geq}) + \mathrm{Bel}_{AT}^{\geq}(D_3^{\geq}) = \frac{8}{8} + \frac{5}{8} + \frac{0}{8} = \frac{13}{8}. \tag{43}$$

(i) Let $B = \{a_1, a_2, a_3\}$, then

$$\sum_{s \in T} \mathrm{Bel}_{AT}^{\geq}(D_s^{\geq}) = (8/8) + (5/8) + (0/8) = (13/8). \tag{44}$$

Thus,

$$\mathrm{sig}^{\geq}_{\mathrm{exterior}}(a_3, C, d) = \mathrm{sig}^{\geq}_{\mathrm{exterior}}(a_4, C, d) = \max_{b \in AT - B} \mathrm{sig}^{\geq}_{\mathrm{exterior}}(b, C, d) \neq 0 \quad (55)$$

$$\begin{aligned}\mathrm{sig}^{\geq}_{\mathrm{interior}}(a_4, AT, d) &= \sum_{s \in T} \mathrm{Bel}^{\geq}_{AT}(D^{\geq}_s) - \sum_{s \in T} \mathrm{Bel}^{\geq}_{B}(D^{\geq}_s) \\ &= \frac{13}{8} - \frac{13}{8} = 0.\end{aligned} \quad (45)$$

(ii) Let $B = \{a_1, a_2, a_4\}$, then

$$\sum_{s \in T} \mathrm{Bel}^{\geq}_{AT}(D^{\geq}_s) = (8/8) + (5/8) + (0/8) = (13/8) \quad (46)$$

Thus,

$$\begin{aligned}\mathrm{sig}^{\geq}_{\mathrm{interior}}(a_3, AT, d) &= \sum_{s \in T} \mathrm{Bel}^{\geq}_{AT}(D^{\geq}_s) - \sum_{s \in T} \mathrm{Bel}^{\geq}_{B}(D^{\geq}_s) \\ &= \frac{13}{8} - \frac{13}{8} = 0.\end{aligned} \quad (47)$$

(iii) Let $B = \{a_1, a_3, a_4\}$, then

$$\sum_{s \in T} \mathrm{Bel}^{\geq}_{AT}(D^{\geq}_s) = (8/8) + (4/8) + (0/8) = (12/8) \quad (48)$$

Thus,

$$\begin{aligned}\mathrm{sig}^{\geq}_{\mathrm{interior}}(a_2, AT, d) &= \sum_{s \in T} \mathrm{Bel}^{\geq}_{AT}(D^{\geq}_s) - \sum_{s \in T} \mathrm{Bel}^{\geq}_{B}(D^{\geq}_s) \\ &= \frac{13}{8} - \frac{12}{8} > 0\end{aligned} \quad (49)$$

(iv) Let $B = \{a_2, a_3, a_4\}$, then

$$\sum_{s \in T} Bel^{\geq}_{AT}(D^{\geq}_s) = (8/8) + (4/8) + (0/8) = (12/8) \quad (50)$$

Thus,

$$\begin{aligned}\mathrm{sig}^{\geq}_{\mathrm{interior}}(a_1, AT, d) &= \sum_{s \in T} Bel^{\geq}_{AT}(D^{\geq}_s) - \sum_{s \in T} Bel^{\geq}_{B}(D^{\geq}_s) \\ &= \frac{13}{8} - \frac{12}{8} > 0\end{aligned} \quad (51)$$

Thus, it can be seen that combining the decision attribute the relative belief core $\mathrm{Core}_r = \{a_1, a_2\}$ and

$$\sum_{s \in T} Bel^{\geq}_{\mathrm{Core}_r}(D^{\geq}_s) = (8/8) + (4/8) + (0/8) = (12/8) \neq (13/8) \quad (52)$$

Moreover, it is calculated by adding the attribute a_3 or a_4.

- Let $A = \{a_1, a_2, a_3\}$, then $\sum_{s \in T} \mathrm{Bel}^{\geq}_{A}(D^{\geq}_s) = (13/8)$,

$$\begin{aligned}&\mathrm{sig}^{\geq}_{\mathrm{exterior}r}(a_3, C, d) = \sum_{s \in T} \mathrm{Bel}^{\geq}_{A}(D^{\geq}_s) - \sum_{s \in T} \mathrm{Bel}^{\geq}_{\mathrm{Core}_r}(D^{\geq}_s) \\ &= \frac{1}{8}.\end{aligned} \quad (53)$$

- Let $A = \{a_1, a_2, a_4\}$, then $\sum_{s \in T} \mathrm{Bel}^{\geq}_{A}(D^{\geq}_s) = (13/8)$,

$$\begin{aligned}&\mathrm{sig}^{\geq}_{\mathrm{exterior}}(a_4, C, d) = \sum_{s \in T} \mathrm{Bel}^{\geq}_{A}(D^{\geq}_s) - \sum_{s \in T} \mathrm{Bel}^{\geq}_{\mathrm{Core}_r}(D^{\geq}_s) \\ &= \frac{1}{8}.\end{aligned} \quad (54)$$

We get (see (55)) . So, the relative belief reducts are $\{a_1, a_2, a_3\}$ and $\{a_1, a_2, a_4\}$.

5 Conclusion

The method of rough set based on pre-ordered relation is an extension of classical RST. Attribute reduction is to delete redundant information without affecting the final classification result, so as to better find effective information and analyse potential rules. In terms of real-life applications, it takes into account the preferences of users. Taking into account the two aspects of theoretical and practical, it is very effective to combine the preference RST with evidence theory to establish a new model. First of all, belief and plausibility functions in an IVFOIS are introduced by mass function. Through the interior significance measure, core elements can be found in IVFOIS. Next, reduction is obtained through the exterior significance measure, which is including core elements. On this basis, we have that attribute reduction is studied under the background of the IVFOIS. At the end of this paper, some examples confirm the effectiveness and performance of our proposed method.

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