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The Graded Multi-granulation Rough Set Based on Interval-valued Fuzzy Information System

Derong Shi*, Binin Sang* and Weihua Xu*†
*School of Science, Chongqing University of Technology, Chongqing, 400054, P.R. China
†Corresponding author is W.H. Xu (E-mail: chxuwh@gmail.com; Tel: +86-23-15998921583).

***Abstract*—Currently, the representative and extended rough set models are the graded rough set (GRS) and multi-granulation rough set (MGRS). And both of them have been applied in many fields. The main work of this paper is to combine the GRS with the MRS in the interval-valued fuzzy information system (IVFIS) and explore its mechanism. First, in the frame of IVFIS, we construct three new types of multi-granulation rough sets, i.e., graded optimistic MGRS, graded pessimistic MGRS and graded generalized MGRS, respectively. Then, their corresponding basic structure and some properties are studied as well. Finally, based on the above analysis, an illustrative example about class evaluation is introduced to further verify the validity of the model.**

***Index Terms*—Graded Rough Set; Multi-granulation Rough set; Interval-valued Fuzzy Information System; Graded Generalized MRS**

I. Introduction

The classical rough set theory (RST) [1-2] was first proposed by Pawlak in early 1980s and now it is still a powerful mathematical tool used in the data analysis. At present, it has been widely used in pattern recognition, machine learning, decision analysis, knowledge acquisition and data mining and some other fields.

However, Pawlak rough set model also has some defects that the relationship between the equivalence classes of knowledge and the concept set is too strict. To a certain extent, they do not have subset relationship. At the same time, there is a lack of quantitative information about the intersection of equivalence classes and concept sets. Therefore, the conclusion of small scale objects can not be applied to large scale objects. To describe quantitative information, Y.Y. Yao and T.Y. Lin proposed the graded rough set (GRS) based on the modal logics in the literature [3]. A lot of researches [4-9] have been completed on GRS. For example, [10-13] studied the establishment and properties of GRS. And the reduction of GRS was investigated in article [14-15] and so on. It is important to note that the inclusion between the upper approximation and the lower approximation does not hold in most cases.

From the point of view of Granular Computing, these rough sets are essentially defined in a single granular space. The upper (lower) approximation of the target concept is represented by the information granules in the granular space which induced by a single binary relation. Qian et al believe that the relationship among multiple decision makers may be independent in the problem of decision analysis. For this reason, he raised the concept of multi-granulation rough set (MGRS) model [16-18].The optimistic (pessimistic) multi-granulation rough set model is further defined. Moreover, the classical multi- granulation rough set is still built on account of equivalence relation and one of the limitations is that it must be completely correct or certain. So, the rough set is systematically studied by Xu in an ordered information system [19]. The main innovation of these works is that the dominance relation takes the place of equivalence relation. Some scholars combine the GRS with the MGRS to reach the graded multi-granulation rough set. Wu et al. [20] constructed a multi-granulation rough set. Wang et al. discussed graded multi-granulation rough set based on weighting granulations and dominance relation. Shen et al. [21] extended the graded multi-granulation rough set to the graded rough set model based on tolerance relation. Further, this paper proposed the graded optimistic (pessimistic) rough set model based on limited tolerance relation. In addition, multi-granulation with different grades rough set in ordered information system was considered by Yu [22].

Based on the above elaboration, the motivation of this study is to explore the graded multi-granulation rough set models in view of IVFIS. Then, in accordance with the types of the GRS and MGRS, three kinds of decision-theoretic rough set models are constructed.

The reminder of this paper is structured as follows: Section 2 some preliminary recalls some necessary basic concepts briefly. Three new types of graded multi-granulation rough set are constructed in IVFIS and their properties are investigated in Section 3. In Section 4, an illustrative case about class evaluation is introduced to further verify the validity of the models. The paper ends with conclusions and further research topic in section 5.

II. Preliminaries

A. The Rough Set of Interval-valued Fuzzy Information System

Let a quadruple $I^{\succeq} = (U, AT, V, F)$ be an interval-valued fuzzy information system, where the universe of discourse $U = \{x_1, x_2, \cdots, x_n\}$ is a nonempty finite set of objects, AT is a nonempty finite set of attributes, $V = \underset{a\in A}{\cup} V_a$, and $F = \{U \to V_a, a \in AT\}$ is the relationship set from U to AT , and V_a is the finite domain of a. for each $f \in F, a \in AT$ and $x_i \in U$, $f(x_i, a) = [a^L(x_i), a^U(x_i)]$. Especially, $a^L(x_i), a^U(x_i) \in [0,1]$ and $a^L(x_i) \leq a^U(x_i)$.

$f(x_i, a)$ denotes the value of object x_i under the attribute a . When $a^L(x) = a^U(x)$, $f(x, a)$ degenerates into a real number.

For an IVFIS, if there is an ordered relation on V_a, then the attribute a is termed as a criterion. $\forall x, y \in U$, $x \succeq_a y$ represents that x with respect to the criterion a is superior to y, and $x \succeq_a y \Leftrightarrow f(x, a) \succeq_a f(y, a)$. For $\forall B \subseteq AT, x \succeq_B y$ represents that x with respect to the criterion set B is superior to y and $x \succeq_B y \Leftrightarrow$ it is true that $f(x, a) \succeq_a f(y, a)$ for any $a \in B$.

Definition 2.1 Suppose an interval $f(x_i, a) = [a^L(x_i), a^U(x_i)$], for $\forall \lambda \in [0.1]$,

$$G_\lambda(f(x_i, a)) = (a^L(x_i))^{1-\lambda}(a^U(x_i))^\lambda$$

is called λ geometric average ranking function of interval number $f(x_i, a)$.

Evidently, for any two intervals $f(x_i, a)$, $f(x_j, a)$, if $G_\lambda(f(x_j, a)) \geq G_\lambda(f(x_i, a))$, we decide that $f(x_j, a)$ precedes $f(x_i, a)$, denoted by $f(x_j, a) \succeq f(x_i, a)$.

Definition 2.2 [23] Let $I^\succeq = (U, AT, V, F)$ be an IVFIS. For $\forall A \subseteq AT$, , the IVF dominance relation $R_A^\succeq$ is defined as $R_A^\succeq = \{(x_i, x_j) \in U \times U : \forall a \in A, f(x_j, a) \succeq f(x_i, a)\}$. The IVF dominance class of x introduced by the dominance relation $R_A^\succeq$ is defined as $[x_i]_A^\succeq = \{x_j \in U : (x, y) \in R_A^\succeq\} = \{x_j \in U | \forall a \in A, f(x_j, a) \succeq f(x_i, a)\}$.

In an IVFIS, for all $X \subseteq U$, the upper and lower approximations are

$$\overline{R_A^\succeq}(X) = \{x \in U | [x]_A^\succeq \cap X \neq \emptyset\} = \cup\{[x]_A^\succeq | [x]_A^\succeq \cap X \neq \emptyset\},$$
$$\underline{R_A^\succeq}(X) = \{x \in U | [x]_A^\succeq \subseteq X\} = \cup\{[x]_A^\succeq | [x]_A^\succeq \subseteq X\}.$$

If $\overline{R_A^\succeq}(X) \neq \underline{R_A^\succeq}(X)$, then X with regard to $R_A^\succeq$ is rough; if $\overline{R_A^\succeq}(X) = \underline{R_A^\succeq}(X)$, then X concerning $R_A^\succeq$ is precise.

B. Graded Rough Set Theory

Let $\overline{R}_k(X)$ and $\underline{R}_k(X)$ denote the upper and lower approximation sets of X when the grade is a non-negative integer k. $\overline{R}_k(X)$ and $\underline{R}_k(X)$ can be defined as $\overline{R}_k(X) = \{x \in U| \ |[x]_R \cap X| > k\}$ and $\underline{R}_k(X) = \{x \in U| \ |[x]_R| - |[x]_R \cap X| \leq k\}$. If $\overline{R}_k(X) \neq \underline{R}_k(X)$, then X is rough while the grade is k ; otherwise X is precise. The upper approximation $\overline{R}_k(X)$ expresses a collection of equivalence classes which must satisfy $|[x]_R \cap X| > k$, and the lower approximation $\underline{R}_k(X)$ represents the union of some objects whose equivalence classes satisfy $|[x]_R| - |[x]_R \cap X| \leq k$. Accordingly, the positive, negative, the upper, the lower and boundary region of are exposed as $pos(X) = \overline{R}_k(X) \cap \underline{R}_k(X)$, $neg(X) = \sim (\overline{R}_k(X) \cup \underline{R}_k(X))$, $Ubn(X) = \overline{R}_k(X) - \underline{R}_k(X)$, $Lbn(X) = \underline{R}_k(X) - \overline{R}_k(X)$, $bn(X) = Ubn_k(X) \cup Lbn_k(X)$, respectively. When $k = 0$, the graded rough set becomes the classic rough set. Consequently, the inclusion between $\overline{R}_k(X)$ and $\underline{R}_k(X)$ does not hold in most cases.

C. Multi-Granulation Rough Set

This section will introduce the models of multi- granulation rough set which are caused by multiple dominance relations and the specific details can be found in reference [19].

Let $I = (U, AT, V, F)$ be an ordered information system, $A_1, \cdots, A_s \subseteq AT$ are attribute sets, $R_{A_1}^\succeq, \cdots, R_{A_s}^\succeq$ are dominance relations. $X \subseteq U$, the optimistic lower and upper approximations are defined as follows.

$$\underline{OM_{\sum_{i=1}^s A_i}^\succeq}(X) = \{x | \bigvee_{i=1}^{s} ([x]_{A_i}^\succeq \subseteq X\},$$
$$\overline{OM_{\sum_{i=1}^s A_i}^\succeq}(X) = \{x | \bigwedge_{i=1}^{s} ([x]_{A_i}^\succeq \cap X \neq \varnothing\}.$$

The pessimistic lower and upper approximations can be similarly defined as the following way.

$$\underline{PM_{\sum_{i=1}^s A_i}^\succeq}(X) = \{x | \bigwedge_{i=1}^{s} ([x]_{A_i}^\succeq \subseteq X\},$$
$$\overline{PM_{\sum_{i=1}^s A_i}^\succeq}(X) = \{x | \bigvee_{i=1}^{s} ([x]_{A_i}^\succeq \cap X \neq \varnothing\}.$$

Moreover, if $\underline{OM_{\sum_{i=1}^s A_i}^\succeq}(X) \neq \overline{OM_{\sum_{i=1}^s A_i}^\succeq}(X)$, $(\underline{PM_{\sum_{i=1}^s A_i}^\succeq}(X) \neq \overline{PM_{\sum_{i=1}^s A_i}^\succeq}(X))$, then we think that X is optimistic(pessimistic) rough set under multiple dominance relations. If not, the X is optimistic(pessimistic) precise.

Now, we look at the more general model in order information system. The generalized multi-granulation rough set is defined in the following manner.

$$\underline{GM_{\sum_{i=1}^s A_i}^\succeq}(X)_\beta = \{x | (\sum_{i=1}^{s} S_X^{A_i}(x))/s \geq \beta\},$$
$$\overline{GM_{\sum_{i=1}^s A_i}^\succeq}(X)_\beta = \{x | (\sum_{i=1}^{s} (1 - S_{\sim X}^{A_i}(x))/s > 1 - \beta\}.$$

Where $S_X^{A_i}(x)$ is the support characteristic function of x with respect to concept X under A_i; if $[x]_{A_i}^\succeq \subseteq X$, then $S_X^{A_i}(x) = 1$, else $S_X^{A_i}(x) = 0$. X is referred to as a rough set concerning $\sum_{i=1}^s A_i$ if and only if $\underline{GM_{\sum_{i=1}^s A_i}^\succeq}(X)_\beta \neq \overline{GM_{\sum_{i=1}^s A_i}^\succeq}(X)_\beta$; otherwise, X is termed a definable set with regard to $\sum_{i=1}^s A_i$. β is called the level of information about $\sum_{i=1}^s A_i$.

On the basis of above definitions of optimistic, pessimistic and generalized multi-granulation rough set in ordered information system, the following conclusions are established.

$$(1)\underline{PM_{\sum_{i=1}^s A_i}^\succeq}(X) \subseteq \underline{GM_{\sum_{i=1}^s A_i}^\succeq}(X)_\beta \subseteq \underline{OM_{\sum_{i=1}^s A_i}^\succeq}(X),$$
$$(2)\overline{OM_{\sum_{i=1}^s A_i}^\succeq}(X) \subseteq \overline{GM_{\sum_{i=1}^s A_i}^\succeq}(X)_\beta \subseteq \overline{PM_{\sum_{i=1}^s A_i}^\succeq}(X).$$

III. Graded Multi-Granulation Rough Set Based on IVFIS

In this section, three new kinds of multi-granulation rough sets with different grades on the basis of IVFIS are explored. Namely, the graded optimistic multi-granulation rough set, the graded pessimistic multi-granulation rough set and the graded generalized multi-granulation rough set.

A. The Graded Optimistic Multi-Granulation Rough Set in IVFIS

This part combines optimistic multi-granulation with degree rough set in IVFIS.

Definition 3.1 Let $I^{\succeq}=(U,AT,V,F)$ be an interval-valued fuzzy information system, $A_1,\cdots,A_s\subseteq AT$, $R^{\succeq}$ is the dominance relation of $I^{\succeq}$, for any $X\subseteq U$,$k_i\in n$, then

$$\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)=\{x|\bigvee_{i=1}^{s}(|[x]^{\succeq}_{A_i}|-|X\cap[x]^{\succeq}_{A_i}|)\le k_i\},$$
$$\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)=\{x|\bigwedge_{i=1}^{s}|[x]^{\succeq}_{A_i}\cap X|>k_i\}.$$

is called the graded optimistic multi-granulation lower and upper approximation of in view of IVFIS.

Due to the graded rough set, the graded optimistic multi-granulation upper and lower approximations are no longer a strict inclusion relation.

Furthermore, the graded optimistic multi-granulation positive , negative and boundary region of in IVFIS are defined as the following way :

(1) $Pos(X)^{OM}_{\sum_{i=1}^{s}A_i}=\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\cap\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$;

(2) $Neg(X)^{OM}_{\sum_{i=1}^{s}A_i}=\sim(\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\cup\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X))$;

(3) $Lbn(X)^{OM}_{\sum_{i=1}^{s}A_i}=\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)-\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$;

(4) $Ubn(X)^{OM}_{\sum_{i=1}^{s}A_i}=\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)-\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$;

(5) $Bn(X)^{OM}_{\sum_{i=1}^{s}A_i}=Lbn(X)^{OM}_{\sum_{i=1}^{s}A_i}\cup Ubn(X)^{OM}_{\sum_{i=1}^{s}A_i}$.

Theorem 3.2 Given an IVFIS $I^{\succeq}=(U,AT,V,F)$, $A_1,\cdots,A_s\subseteq AT$ are subsets of AT, for any $X\subseteq U$,$k_i\in n$, then we can get the relationship between the graded optimistic multi-granulation lower approximation of $\sim X$ and the graded optimistic multi-granulation upper approximation of X.

$$\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)=\sim\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(\sim X)$$

Proof. In light of definition 3.1, we can obtain

$$\begin{aligned}&\sim\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(\sim X)=\sim\{x|(|[x]^{\succeq}_{A_1}|-|\sim X\cap[x]^{\succeq}_{A_1}|)\\&\le k_1\vee(|[x]^{\succeq}_{A_2}|-|\sim X\cap[x]^{\succeq}_{A_2}|)\le k_2\vee\cdots\vee k_s\}\\&=\{x|(|[x]^{\succeq}_{A_1}|-|\sim X\cap[x]^{\succeq}_{A_1}|)>k_1\wedge(|[x]^{\succeq}_{A_2}|-|\sim X\\&\cap[x]^{\succeq}_{A_2}|)>k_2\wedge\cdots\wedge(|[x]^{\succeq}_{A_s}|-|\sim X\cap[x]^{\succeq}_{A_s}|)>k_s\}\\&=\{(|[x]^{\succeq}_{A_i}|-|\sim X\cap[x]^{\succeq}_{A_i}|)>k_i,\forall i=1,2,\cdots,s\}\\&=\{(X\cap[x]^{\succeq}_{A_i}|)>k_i,\forall i=1,2,\cdots,s\}\\&=\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\end{aligned}$$

Therefore, the theorem is completely proved. □

Theorem 3.3 For an IVFIS $I^{\succeq}=(U,AT,V,F)$, $A_1,\cdots,A_s\subseteq AT$ are subsets of AT, $X\subseteq U$,$k_i\in n$. Then there are a series of properties holding.

(1) $\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(U)=U,\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(\varnothing)=\varnothing$;

(2) $X\subseteq Y\Rightarrow\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\subseteq\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)$;

(3) $X\subseteq Y\Rightarrow\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\subseteq\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)$;

(4) $\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\cap\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)\supseteq\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X\cap Y)$;

(5) $\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\cap\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)\supseteq\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X\cap Y)$;

(6) $\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\cup\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)\subseteq\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X\cup Y)$;

(7) $\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\cup\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)\subseteq\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X\cup Y)$.

Proof: According to the definition 3.1, the property (1) is clearly true for U and $\emptyset$.

(2) If $X\subseteq Y$, then it means that $(|X\cap[x]^{\succeq}_{A_i}|)\subseteq(|Y\cap[x]^{\succeq}_{A_i}|)$ for $\forall i$. Thus, $(|[x]^{\succeq}_{A_i}|-|X\cap[x]^{\succeq}_{A_i}|)\ge(|[x]^{\succeq}_{A_i}|-|Y\cap[x]^{\succeq}_{A_i}|)$. For arbitrary $x\in\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$, there is always i, s.t. $(|[x]^{\succeq}_{A_i}|-|X\cap[x]^{\succeq}_{A_i}|)\le k_i$, i.e. $|[x]^{\succeq}_{A_i}|\le k_i+|X\cap[x]^{\succeq}_{A_i}|$, therefore, $(|[x]^{\succeq}_{A_i}|-|Y\cap[x]^{\succeq}_{A_i}|)\le k_i$, that is $x\in\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)$. So we have $\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\subseteq\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)$.

(3) For $\forall x\in\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$, there exists i such that $|[x]^{\succeq}_{A_i}\cap X|>k_i$, hence, from (2) it can get $|[x]^{\succeq}_{A_i}\cap Y|>k_i$. So, $x\in\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(Y)$. This property has been proved.

(4) Through the operation properties of sets, we can obtain $X\cap Y\subseteq X,Y$, then the right $''\supseteq''$ is illustrated. The remaining (5), (6) and (7) are similar to certify. □

Theorem 3.4 Suppose an IVFIS $I^{\succeq}=(U,AT,V,F)$, $A_1,\cdots,A_s\subseteq AT$, for any $X\subseteq U$, $k_i\in n$, we have:

$$\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}(X)\subseteq\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X);$$
$$\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)\subseteq\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}(X).$$

In particular, $\forall k_i=0,i=1,2,\cdots,s$, then $\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}(X)=\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{o}(X)$, $\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{o}(X)=\overline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}(X)$. This shows that the graded optimistic multi-granulation rough set model is the extension of optimistic multi-granulation rough set in ordered information system.

Proof: For $\forall x\in\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}(X)$, there is always a i satisfying $[x]^{\succeq}_{A_i}\subseteq X$. It is also bound to meet the formula $(|[x]^{\succeq}_{A_i}|-|X\cap[x]^{\succeq}_{A_i}|)\le k_i$, further, $x\in\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$. Accordingly, $\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}(X)\subseteq\underline{OM^{\succeq}_{\sum_{i=1}^{s}A_i}}^{k_i}(X)$. The following expression can be proved in the same way. □

B. The Graded Pessimistic Multi-Granulation Rough Set in IVFIS

This section is similar to the graded optimistic multi-granulation rough set. We will explore some properties and applications of graded pessimistic multi-granulation rough set on account of IVFIS.

Definition 3.5 Let $I^{\succeq} = (U, AT, V, F)$ be an interval-valued fuzzy information system, $A_1, \cdots, A_s \subseteq AT$, $R^{\succeq}$ is the interval-valued fuzzy dominance relation of $I^{\succeq}$, for any $X \subseteq U$, $k_i \in N$, then

$$\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) = \{x | \bigwedge_{i=1}^{s} (|[x]^{\succeq}_{A_i}| - |X \cap [x]^{\succeq}_{A_i}|) \leq k_i\};$$

$$\overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) = \{x | \bigvee_{i=1}^{s} |[x]^{\succeq}_{A_i} \cap X| > k_i\}.$$

is referred to as the graded pessimistic multi-granulation lower and upper approximation of X based on IVFIS.

Additionally, if $\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \neq \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X)$, we call that graded pessimistic multi-granulation rough set about the multiple granular structure. On the contrary, one can say that is graded pessimistic multi-granulation precise set in IVFIS. Based upon the above defined upper and lower approximation operators, the pessimistic rough regions can also be defined as the optimistic case.

Theorem 3.6 There is an IVFIS $I^{\succeq} = (U, AT, V, F)$, $A_1, \cdots, A_s \subseteq AT$ are subsets of X, for any $X \subseteq U$, $k_i \in N$. The graded pessimistic multi-granulation lower approximation of $\sim X$ and the graded optimistic multi-granulation upper approximation of X satisfy the following relationship.

$$\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) = \sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X).$$

Factually, it is also true that

$$\sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) = \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X).$$

Proof: By definition 3.5, we can get

$$\begin{aligned}
&\sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X) = \sim \{x | (|\sim X \cap [x]^{\succeq}_{A_1}|) > k_1 \\
&\vee (|\sim X \cap [x]^{\succeq}_{A_2}|) > k_2 \vee \cdots \vee k_s\} \\
&= \{x | (|\sim X \cap [x]^{\succeq}_{A_1}|) \leq k_1 \wedge (|\sim X \cap [x]^{\succeq}_{A_2}|) \leq k_2 \\
&\wedge \cdots \wedge (|\sim X \cap [x]^{\succeq}_{A_s}|) \leq k_s\} \\
&= \{(|\sim X \cap [x]^{\succeq}_{A_i}|) \leq k_i, \forall i = 1, 2, \cdots, s\} \\
&= \{(|[x]^{\succeq}_{A_i}| - |X \cap [x]^{\succeq}_{A_i}|) \leq k_i, \forall i = 1, 2, \cdots, s\} \\
&= \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X).
\end{aligned}$$

Consequently, $\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) = \sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X)$. By proving the previous equation, we have

$$\begin{aligned}
&\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) = \sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X) \\
&\Rightarrow \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X) = \sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim (\sim X)) \\
&\Leftrightarrow \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\sim X) = \sim \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X)
\end{aligned}$$

So far, both proofs have been finished. □

Theorem 3.7 For an IVFIS $I^{\succeq} = (U, AT, V, F)$, $A_1, \cdots, A_s \subseteq AT$, for any $X, Y \subseteq U$, $k_i \in N$. Then the following properties are valid.

(1) $\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(U) = U$, $\overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(\varnothing) = \varnothing$;

(2) $X \subseteq Y \Rightarrow \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \subseteq \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y)$;

(3) $X \subseteq Y \Rightarrow \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \subseteq \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y)$;

(4) $\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \cap \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y) \supseteq \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X \cap Y)$;

(5) $\overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \cap \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y) \supseteq \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X \cap Y)$;

(6) $\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \cup \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y) \subseteq \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X \cup Y)$;

(7) $\overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \cup \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y) \subseteq \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X \cup Y)$.

Proof: The properties can be proved in similar theorem 3.3. □

Theorem 3.8 Let $I^{\succeq} = (U, AT, V, F)$ be an IVFIS, $A_1, \cdots, A_s \subseteq AT$, for any $X \subseteq U$, $k_i \in N$, we have:

$$\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X) \subseteq \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X),$$

$$\overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \subseteq \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X).$$

Particularly, $\forall k_i = 0, i = 1, 2, \cdots, s$, then $\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X) = \underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{o}(X)$, $\overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{o}(X) = \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)$. The conclusions demonstrate that the graded pessimistic multi-granulation rough set model is the prolongation of pessimistic multi-granulation rough set in ordered information system.

Proof: This conclusion can be easily obtained. □

C. The Graded Generalized Multi-Granulation Rough Set in IVFIS

In the above two sections, we have discussed two special types of graded multi-granulation rough sets based on IVFIS. However, it is notable that the requirements of approximation operators are either too strict or too loose. In other words, the description of approximation operators does not have a restrictive standard condition. In reality, a qualified, feasible and credible principle is that the minority should be subordinated to the majority. In the basic model of multi-granulation rough sets, the graded optimism and pessimism multi-granulation rough sets do not take into account the case of the minority comply with the majority. Therefore, in this section, a graded generalized multi-granulation rough set on the basis of IVFIS is proposed. In addition, this model is more logical and consistent with the characterization of the real world.

Definition 3.9 Let $I^{\succeq} = (U, AT, V, F)$ be an interval-valued fuzzy information system, $A_i \subseteq AT, i = 1, 2 \cdots, s$, for any

$X \subseteq U$, $k_i \in N$, the graded generalized multi-granulation lower and upper approximations of X with respect to $\sum_{i=1}^{s} A_i$ can be defined as

$$\underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} = \{x \in U | (\sum_{i=1}^{s} LS_X^{A_i}(x))/s \geq \beta\},$$

$$\overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} = \{x \in U | (\sum_{i=1}^{s} US_X^{A_i}(x))/s > 1 - \beta\}.$$

respectively. Where $LS_X^{A_i}(x)$ is the lower support characteristic function of $x \in U$ under A_i,

$$LS_X^{A_i}(x) = \begin{cases} 1, & \text{if } |[x]^{\succeq}_{A_i}| - |X \cap [x]^{\succeq}_{A_i}| \leq k_i; \\ 0, & \text{other.} \end{cases}$$

And $US_X^{A_i}(x)$ is the upper support characteristic function of $x \in U$ under A_i,

$$US_X^{A_i}(x) = \begin{cases} 1, & \text{if } |X \cap [x]^{\succeq}_{A_i}| > k_i; \\ 0, & \text{other.} \end{cases}$$

X is referred to as a rough set with regard to $\sum_{i=1}^{s} A_i$ if and only if $\underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \neq \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta}$; otherwise, X is termed a definable set with regard to $\sum_{i=1}^{s} A_i$. β is called the level of information concerning $\sum_{i=1}^{s} A_i$. This model is framed as graded generalized multi-granulation rough set on account of IVFIS.

By the graded generalized multi-granulation lower and upper approximations, the positive region, negative region, and boundary region of X are represented as:

(1) $Pos(X)^{GM}_{\sum_{i=1}^{s} A_i} = \underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \cap \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta}$;

(2) $Neg(X)^{GM}_{\sum_{i=1}^{s} A_i} =\sim (\underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \cup \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta})$;

(3) $Lbn(X)^{GM}_{\sum_{i=1}^{s} A_i} = \underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} - \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta}$;

(4) $Ubn(X)^{GM}_{\sum_{i=1}^{s} A_i} = \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} - \underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta}$;

(5) $Bn(X)^{GM}_{\sum_{i=1}^{s} A_i} = Lbn(X)^{GM}_{\sum_{i=1}^{s} A_i} \cup Ubn(X)^{GM}_{\sum_{i=1}^{s} A_i}$.

The graded generalized multi-granulation rough set on IVFIS is a generalization of the graded optimistic and pessimistic multi-granulation rough sets. According to definition 3.9, we can get the following relations among the graded generalized, the graded optimistic and the graded pessimistic multi-granulation rough sets.

Theorem 3.10 Assume an IVFIS $I^{\succeq} = (U, AT, V, F)$, $A_i \subseteq AT, i = 1, 2 \cdots, s$, $\beta \in (0.5, 1]$, for any $X \subseteq U$, $k_i \in N$, We have the following conclusions established.

$$\underline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \subseteq \underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \subseteq \underline{OM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X);$$

$$\overline{OM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X) \subseteq \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \subseteq \overline{PM^{\succeq}_{\sum_{i=1}^{s} A_i}}^{k_i}(X).$$

Proof: The conclusions are obvious. □

Theorem 3.11 For an IVFIS $I^{\succeq} = (U, AT, V, F)$, $A_i \subseteq AT, i = 1, 2 \cdots, s$, for any $X, Y \subseteq U$, $k_i \in N$. Then there are the following properties.

(1) $\underline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(U)^{k_i}_{\beta} = U$, $\overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(\varnothing)^{k_i}_{\beta} = \varnothing$;

(2) $\underline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(\sim X)^{k_i}_{\beta} =\sim \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta}$;

(3) $\sim \underline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} = \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(\sim X)^{k_i}_{\beta}$;

(4) $X \subseteq Y \Rightarrow \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \subseteq \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y)^{k_i}_{\beta}$;

(5) $\overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \cap \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y)^{k_i}_{\beta} \supseteq \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X \cap Y)^{k_i}_{\beta}$;

(6) $\overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} \cup \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}^{k_i}(Y)^{k_i}_{\beta} \subseteq \overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(X \cup Y)^{k_i}_{\beta}$.

Proof: (1) By definition 3.9, $LS_U^{A_i}(x) = 1$, $US_{\varnothing}^{A_i}(x) = 0$. Then $\underline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(U)^{k_i}_{\beta} = \{x | (\sum_{i=1}^{s} LS_U^{A_i}(x))/s = (\sum_{i=1}^{s} 1)/s = 1 \geq \beta\} = U$, $\overline{GM^{\geqq}_{\sum_{i=1}^{s} A_i}}(\varnothing)^{k_i}_{\beta} = \{x \in U | (\sum_{i=1}^{s} US_{\varnothing}^{A_i}(x))/s = 0 > 1 - \beta\} = \varnothing$.

(2) For the upper support characteristic function, we have

$$US_X^{A_i}(x) = \begin{cases} 1, & \text{if } |X \cap [x]^{\succeq}_{A_i}| > k_i; \\ 0, & \text{other.} \end{cases}$$

$$\Rightarrow US_{\sim X}^{A_i}(x) = \begin{cases} 1, & \text{if } |\sim X \cap [x]^{\succeq}_{A_i}| > k_i; \\ 0, & \text{other.} \end{cases}$$

$$\Rightarrow US_{\sim X}^{A_i}(x) = \begin{cases} 1, & \text{if } |[x]^{\succeq}_{A_i}| - |X \cap [x]^{\succeq}_{A_i}| > k_i; \\ 0, & \text{other.} \end{cases}$$

$$\Rightarrow US_{\sim X}^{A_i}(x) = \begin{cases} 1, & \text{if } |[x]^{\succeq}_{A_i}| - |X \cap [x]^{\succeq}_{A_i}| \leq k_i; \\ 0, & \text{other.} \end{cases}$$

$$\Leftrightarrow LS_X^{A_i}(x).$$

Hence, $\sum_{i=1}^{s} LS_{\sim X}^{A_i}(x) =\sim \sum_{i=1}^{s} US_X^{A_i}(x) = s - \sum_{i=1}^{s} US_X^{A_i}(x)$. So

$$\underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(\sim X)^{k_i}_{\beta} = \{x \in U | (\sum_{i=1}^{s} LS_{\sim X}^{A_i}(x))/s \geq \beta\}$$

$$= \{x \in U | (s - \sum_{i=1}^{s} US_X^{A_i}(x))/s \geq \beta\}$$

$$= \{x \in U | (1 - (\sum_{i=1}^{s} US_X^{A_i}(x))/s) \geq \beta\}$$

$$= \{x \in U | (\sum_{i=1}^{s} US_X^{A_i}(x))/s \leq 1 - \beta\}$$

$$=\sim \overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta}.$$

The proof has been finished. (3)The method analogous to that used above. (4), (5), (6) are easy to prove. □

IV. CASE STUDY

Let $I^{\succeq} = (U, AT, V, F)$ be an IVFIS, where $U = \{x_1, \cdots, x_{10}\}$ is composed of 10 classes. $AT = \{a_1, a_2, a_3, a_4, a_5\}$ is the attribute set that stands for learning atmosphere, the construction of class atmosphere, class activities, safety and discipline, and the work of class head-teachers. The school evaluates the ten classes under the 5 indexes respectively. And the detailed evaluation data are shown in Table 1.

At first, we randomly select a part of the classes as the research object, namely $X = \{x_1, x_2, x_3, x_4, x_5\}$. And two granules are $A_1 = \{a_1, a_2, a_3\}$ and $A_2 = \{a_2, a_4, a_5\}$, the grades are $k_1 = 1$ and $k_2 = 2$. Now given $\lambda = 0.5$, we can calculate the number of sorting on 10 classes. The calculated data are shown in table 2.

TABLE I
Classes evaluation statistics

U	a_1	a_2	a_3	a_4	a_5
x_1	[0.75,0.84]	[0.90,0.92]	[0.81,0.94]	[0.93,0.96]	[0.90,0.91]
x_2	[0.90,0.95]	[0.89,0.93]	[0.85,0.95]	[0.80,0.94]	[0.94,0.97]
x_3	[0.80,0.91]	[0.84,0.86]	[0.81,0.84]	[0.81,0.84]	[0.86,0.89]
x_4	[0.86,0.89]	[0.89,0.92]	[0.90,0.95]	[0.90,0.93]	[0.91,0.94]
x_5	[0.86,0.95]	[0.91,0.93]	[0.91,0.95]	[0.90,0.94]	[0.92,0.97]
x_6	[0.89,0.95]	[0.92,0.93]	[0.91,0.95]	[0.87,0.90]	[0.91,0.95]
x_7	[0.90,0.97]	[0.85,0.93]	[0.87,0.97]	[0.83,0.93]	[0.94,0.95]
x_8	[0.77,0.83]	[0.90,0.93]	[0.77,0.83]	[0.82,0.88]	[0.83,0.90]
x_9	[0.86,0.89]	[0.90,0.93]	[0.85,0.89]	[0.83,0.89]	[0.90,0.95]
x_{10}	[0.84,0.89]	[0.84,0.85]	[0.82,0.87]	[0.86,0.89]	[0.88,0.92]

TABLE II
The numbers of sorting on 10 classes

U	a_1	a_2	a_3	a_4	a_5
x_1	0.7937	0.9099	0.8726	0.9449	0.9050
x_2	0.9247	0.9098	0.8986	0.8672	0.9549
x_3	0.8532	0.8499	0.8249	0.8249	0.8749
x_4	0.8749	0.9049	0.9247	0.9149	0.9249
x_5	0.9039	0.9199	0.9298	0.9198	0.9447
x_6	0.9195	0.9250	0.9298	0.8849	0.9298
x_7	0.9343	0.8891	0.9186	0.8786	0.9450
x_8	0.7994	0.9149	0.7994	0.8495	0.8643
x_9	0.8749	0.9149	0.8698	0.8595	0.9247
x_{10}	0.8646	0.8450	0.8446	0.8749	0.8998

Then the dominance classes for every object with regard to granule A_1 and A_2 can be calculated. The results of classes under granular structures are shown in Table 3 and 4.

Case 1. The values for graded optimistic multi-granulation rough set can be calculated as follows:

$$\underline{OM^{\succeq}_{\sum_{i=1}^{2} A_i}}^{k_i}(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_9\};$$

$$\overline{OM^{\succeq}_{\sum_{i=1}^{2} A_i}}^{k_i}(X) = \{x_3, x_{10}\}.$$

On account of the definition of the graded optimistic multi-granulation, rough regions of X can be obtained:

$$Pos(X)^{OM}_{\sum_{i=1}^{s} A_i} = \{\varnothing\}; Neg(X)^{OM}_{\sum_{i=1}^{s} A_i} = \{x_8\};$$
$$Lbn(X)^{OM}_{\sum_{i=1}^{s} A_i} = \{x_1, x_2, x_4, x_5, x_6, x_7, x_9\};$$
$$Ubn(X)^{OM}_{\sum_{i=1}^{s} A_i} = \{x_3, x_{10}\};$$
$$Bn(X)^{OM}_{\sum_{i=1}^{s} A_i} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}\}.$$

Case 2. The results for graded pessimistic multi-granulation rough set can be computed as follows:

$$\underline{PM^{\succeq}_{\sum_{i=1}^{2} A_i}}^{k_i}(X) = \{x_1, x_2, x_4, x_5, x_6\};$$

$$\overline{PM^{\succeq}_{\sum_{i=1}^{2} A_i}}^{k_i}(X) = \{x_1, x_3, x_4, x_8, x_9, x_{10}\}.$$

On the basis of the definition of the graded pessimistic multi-granulation, rough regions of X can be got:

$$Pos(X)^{PM}_{\sum_{i=1}^{s} A_i} = \{x_1, x_4\}; Neg(X)^{PM}_{\sum_{i=1}^{s} A_i} = \{x_7\};$$
$$Lbn(X)^{PM}_{\sum_{i=1}^{s} A_i} = \{x_2, x_5, x_6\};$$
$$Ubn(X)^{PM}_{\sum_{i=1}^{s} A_i} = \{x_3, x_8, x_9, x_{10}\};$$
$$Bn(X)^{PM}_{\sum_{i=1}^{s} A_i} = \{x_2, x_3, x_5, x_6, x_8, x_9, x_{10}\}.$$

Case 3. Here, let the information level$\beta = 0.7$. Then the results for graded generalized multi-granulation rough set can be computed as follows:

$$\underline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} = \{x_1, x_2, x_4, x_5, x_6, x_7\};$$

$$\overline{GM^{\succeq}_{\sum_{i=1}^{s} A_i}}(X)^{k_i}_{\beta} = \{x_1, x_3, x_4, x_8, x_9, x_{10}\}.$$

TABLE III
Statistical data of classes under granular structure A_1

x_i	$[x]^{\succeq}_{A_1}$	$\lvert [x]^{\succeq}_{A_1} \rvert$	$\lvert [x]^{\succeq}_{A_1} \cap X \rvert$	$\lvert [x]^{\succeq}_{A_1} \rvert - \lvert [x]^{\succeq}_{A_1} \cap X \rvert$
x_1	$x_{1,5,6}$	3	2	1
x_2	x_2	1	1	0
x_3	$x_{2,3,4,5,6,7,9}$	7	4	3
x_4	$x_{4,5,6}$	3	2	1
x_5	$x_{5,6}$	2	1	1
x_6	x_6	1	0	1
x_7	x_7	1	0	1
x_8	$x_{5,6,8,9}$	4	1	3
x_9	$x_{5,6,9}$	3	1	2
x_{10}	$x_{2,4,5,6,7,9,10}$	7	3	4

TABLE IV
Statistical data of classes under granular structure A_2

x_i	$[x]^{\succeq}_{A_2}$	$\lvert [x]^{\succeq}_{A_2} \rvert$	$\lvert [x]^{\succeq}_{A_2} \cap X \rvert$	$\lvert [x]^{\succeq}_{A_2} \rvert - \lvert [x]^{\succeq}_{A_2} \cap X \rvert$
x_1	$x_{1,2}$	2	2	0
x_2	$x_{1,2}$	2	2	0
x_3	$x_{3,4,5,6,7,9}$	6	3	3
x_4	$x_{4,5}$	2	2	0
x_5	x_5	1	1	0
x_6	x_6	1	0	1
x_7	x_7	1	0	1
x_8	$x_{5,6,8,9}$	4	1	3
x_9	$x_{5,6,9}$	3	1	2
x_{10}	$x_{1,4,5,6,7,10}$	6	3	3

Based on the definition of the graded generalized multi-granulation, rough regions of X can be obtained:

$$Pos(X)^{GM}_{\sum_{i=1}^{s} A_i} = \{x_1, x_4\}; Neg(X)^{GM}_{\sum_{i=1}^{s} A_i} = \{\varnothing\};$$
$$Lbn(X)^{GM}_{\sum_{i=1}^{s} A_i} = \{x_2, x_5, x_6, x_7\};$$
$$Ubn(X)^{GM}_{\sum_{i=1}^{s} A_i} = \{x_3, x_8, x_9, x_{10}\};$$
$$Bn(X)^{GM}_{\sum_{i=1}^{s} A_i} = \{x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}.$$

Obviously, calculation results of three methods are different. That is to say, during the selection of excellent classes, the results of three models are different in depicting the concept X. However, case 3 can use the information level to control objects selectively. Through this parameter β, the objects can be positively described in most classifications. Therefore, different models should be chosen according to different situations in practice.

V. Conclusions

In real life, the interval-valued fuzzy phenomenon can be seen everywhere. In this paper, three different kinds of graded multi-granulation rough sets are firstly constructed based on interval-valued fuzzy information systems. Further, this paper defines the upper and lower approximation operators of the graded multi-granulation under multiple dominance relations. Moreover, some properties of the three types of models have been discussed. Finally, an instance is presented to verify the effectiveness of the models.

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