Information Sciences 378 (2017) 264–281

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Double-quantitative rough fuzzy set based decisions: A logical operations method

Bingjiao Fan[a], Eric C. C. Tsang[a,*], Weihua Xu[b], Jianhang Yu[b]

[a] Faculty of Information Technology, Macau University of Science and Technology, Taipa, Macau
[b] School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, PR China

ARTICLE INFO

Article history:
Received 28 August 2015
Revised 2 May 2016
Accepted 23 May 2016
Available online 27 May 2016

Keywords:
Decision-theoretic rough set
Double quantification
Fuzzy concept
Graded rough set
Logical conjunction and disjunction

ABSTRACT

As two important expanded quantification rough set models, the probabilistic rough set (PRS) model and the graded rough set (GRS) model are used to measure relative quantitative information and absolute quantitative information between the equivalence classes and a basic concept, respectively. The decision-theoretic rough set (DTRS) model is a special case of PRS model which mainly utilizes the conditional probability to express relative quantification. Since the fuzzy concept is more general than classical concept in real life, how to make decision for a fuzzy concept using relative and absolute quantitative information is becoming a hot topic. In this paper, a couple of double-quantitative decision-theoretic rough fuzzy set (Dq-DTRFS) models based on logical conjunction and logical disjunction operation are proposed. Furthermore, we discuss decision rules and the inner relationship between these two models. Then, an experiment in the medical diagnosis is studied to support the theories. Finally, to apply our methods to solve a pattern recognition problem in big data, experiments on data sets downloaded from UCI are conducted to test the proposed models. In addition, we also offer a comparative analysis using two non-rough set based methods. From the results obtained, one finds that the proposed method is efficient for dealing with practical issues.

1. Introduction

Rough set theory was first proposed by Pawlak [26]. It is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. It is currently one of the most promising research directions in artificial intelligence. The classical Pawlak rough set model defines a pair of lower and upper approximations, or equivalently three pair-wise disjoint positive, negative, and boundary regions, of a given set by using the set-inclusion relation and the set non-empty overlapping condition [27]. The three regions can be interpreted as three-way decisions consisting of acceptance, rejection and non-commitment [37–39]. The qualitative formulation ensures that both positive and negative regions are error free, namely, there is no incorrect acceptance error, nor incorrect rejection error. Whenever there is any doubt, a decision of non-commitment is made. Such a stringent formulation, although making an analysis of a rough set model easier, may unnecessarily restrict its applicability and flexibility. In many situations, we are willing to allow some degree of errors in order to make an acceptance or a rejection decision for more

* Corresponding author. Tel.: +85388972136.
E-mail addresses: fanbingjiao890617@126.com (B. Fan), cctsang@must.edu.mo (E. C. C. Tsang).

http://dx.doi.org/10.1016/j.ins.2016.05.035
0020-0255/

objects. To solve the limitations that the relationship between equivalence classes and the basic set are strict and there are no fault tolerance mechanisms, several proposals of generalized quantitative rough set models have been made by using a graded set inclusion. The PRS model [40] and GRS model [41] are two important expanded models to measure relative and absolute quantitative information between the equivalence class and a basic concept, respectively.

In 1987, Wong and Ziarko [34,35] introduced probabilistic approximation space to rough set theory and then presented the concept of PRS model. Subsequently, Yao et al. [42,43] proposed a more concrete PRS model called DTRS model. This perspective was given to deal with the degree of overlapping of an equivalence class with the set to be approximated, and an approach was presented to select the needed parameters in lower and upper approximations. As far as the probabilistic approach to rough set theory is concerned, Pawlak and Skowron [28], proposed a method to characterize a rough set by a single membership function. By the definition of a rough membership function, elements in the same equivalence class have the same degree of membership. The rough membership may be interpreted as the probability of any element belonging to a set, given that the element belongs to an equivalence class. This interpretation led to proposing PRS model [40,42]. Greco et al. [3] introduced a new generalization of the original definition of rough set and variable precision rough set models, named the parameterized rough set model. They aimed at modeling data relationships expressed in terms of frequency distribution rather than in terms of a full inclusion relation, which is used in the classical definition of rough set model. PRS model extends the applied range of classical rough set model. The major change is the consideration regarding the probability of an element being in a set to determine inclusion in approximation regions. Two probabilistic thresholds are used to determine the division between the boundary-positive region and boundary-negative region.

Decision making is an important issue in our daily life. Pedrycz [29] proposed the collaborative and linguistic models of decision making based on granular computing. Chen and Zhao [1] discussed the local reduction of decision system with fuzzy rough set. Yao et al. [43] proposed the DTRS model by using the Bayesian decision theory. Herbet et al. studied the game-theoretic rough set [5,6]. The DTRS model as a concrete PRS model, has aroused interests of many scholars and obtained great achievements in recent years. Herbert and Yao [4,5] studied the combination of the DTRS model and the game-theoretic rough set model. Qian et al. studied the multi-granulation decision-theoretic rough set model [30]. Li et al. proposed the multi-granulation decision-theoretic rough set model in an ordered information system [22]. Li and Zhou [11,12] presented a multi-perspective explanation of the DTRS model and discussed attribute reduction and its application for the DTRS model. Jia et al. [7] also discussed the attribute reduction problem for the DTRS theory. Liu et al. [13–15] discussed multiple-category classification with DTRS model and its applications in management science. In [9,10,16], both Li and Lingras used the DTRS theory to discuss the clustering analysis. Yang [44] studied the multi-agent DTRS model. Li et al. and Liang et al. [17,18] discussed information retrieval and filtering by using the DTRS theory. With the aid of multi-attribute group decision making, Liang et al. proposed three-way decisions by extending DTRS model to the qualitative environment [19]. Greco and Slowinski [4] combined the DTRS model with the dominance-based rough set model and then gave a new generalized rough set model. Based on the basic idea of the DTRS model, Zhou [52] presented a new description of this model. Ma and Sun [23,24] studied the DTRS theory over two universes based on the idea of the classical DTRS theory. Ju et al. proposed a moderate attribute reduction approach in DTRS [8]. Sun et al. investigated the decision-theoretic rough fuzzy set (DTRFS) model and application [32]. Zhang and Min applied three-way decision to recommender systems [46].

The GRS model based on graded modal logics exploring the relationships between rough set and modal logics was proposed by Yao and Lin [41]. The GRS model primarily considered the absolute quantitative information regarding the basic concepts and knowledge granules and was a generalization of the Pawlak model. The regions of the GRS model were extensions of grade approximations. Since the inclusion relation of the grade approximations did not hold any longer, positive and negative regions, upper and lower boundary regions were naturally proposed. Obviously, regions of the GRS model also extended the corresponding notions of the classical rough set model. Also Yao and Lin studied the graded rough set approximations based on nested neighborhood systems. Xu et al. investigated the GRS model based on rough membership function [36]. Liu et al. researched the GRS model based on two universes and its properties were discussed [20]. They classified the universe more precisely and had their own logical meanings related to the grade quantitative index. In addition, GRS model also considered absolute quantitative information between equivalence classes and the basic concept [49]. By defining the upper approximation number, Wang et al. constructed the quantitative analysis for covering-based rough set model [33]. Zhang and Miao constructed the double-quantitative approximation space, and then they proposed two double-quantitative rough set models [47,48]. Later, they investigated the attribute reduction of the proposed rough set models [50,51].

As two useful expanded rough set models, DTRS and GRS can respectively reflect relative and absolute quantitative information about the degree of overlapping between equivalence classes and a basic set. The relative and absolute quantitative information are two distinctive objective sides that describe approximate space, and each has its own virtues and application environments, so none can be neglected. Relative quantitative information and absolute quantitative information are two kinds of quantification methodologies in certain applications. Usually, most researchers prefer to use the relative quantitative information [11,13,14,23–25,35,47]. However, the absolute quantitative information is more important than or as important as the relative quantitative information in some specific fields or special cases. Many related examples can be found in practice. Zhang investigated the double-quantitative rough set model of precision and grade using granular computing [48]. Recently, Zhang and Miao did researches on an expanded double-quantitative model regarding probabilities and grades and its hierarchical double-quantitative attribute [50,51]. Li discussed the double-quantitative decision-theoretic rough set model based on assembling the lower and upper approximations of DTRS and GRS models [21]. However, there is no research on incorporating fuzzy concept in double-quantitative models.

So far, none of these proposed double-quantitative rough set models could deal with problems with fuzzy concept. For example, when dealing with the cold patients' medical diagnosis, we find that some patients have a serious cold. So they should be treated as cold patients. However, other patients may have a mild fever or headache, that is to say, they are not necessarily catching a cold. They may need to do other examinations to determine whether they have other diseases or not. The motivation of this study is to combine DTRS model with GRS model by using logical operators and incorporating a fuzzy concept. We introduce the absolute quantitative information to DTRS model in order to significantly improve the results obtained by DTRS model. We construct the rough set model which considers the relative and absolute quantitative information about the degree of overlapping between equivalence classes and a concept set. Two new double-quantitative decision-theoretic rough fuzzy set models ($\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS) based on logical conjunction and logical disjunction operation are proposed, respectively. Moreover, some important properties of these models are investigated thoroughly. After further studies to discuss decision rules and the inner relationship between these two models, we introduce an illustrative case study in the medical diagnosis to interpret and support the theories. Experiments on real-life big data sets are proposed to demonstrate our model could deal with practical problems.

The rest of this paper is organized as follows. In Section 2, some basic concepts of rough set theory, GRS and DTRS models, and some related necessary preliminaries are briefly introduced. We propose two kinds of Dq-DTRFS models, and some important properties of these models are investigated in Section 3. In Section 4, comparisons and analyses of $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS are done, we also discuss the basic relation among these two models and the classical rough set model. In Section 5, an illustrative example is presented. Then experiments about real-life big data sets are conducted and analyzed in Section 6. Finally, the paper ends with conclusions.

2. Preliminaries

In this section, some basic preliminaries and necessary concepts are briefly introduced. More details can be found in [2,26,28,31,32,40–42,45].

Throughout this paper, we assume that the universe U is a non-empty finite set, and the class of all subsets of U is denoted by $\mathcal{P}(U)$, the class of all fuzzy subsets of U is denoted by $\mathcal{F}(U)$, the complementary set of X is denoted by X^c.

2.1. Pawlak rough set

For a non-empty set U, we call it the universe of discourse. The class of all subsets of U is denoted by $\mathcal{P}(U)$. For $X \in \mathcal{P}(U)$, the equivalence relation R in a Pawlak approximation space (U, R) partitions the universe U into disjoint subsets. Such a partition of the universe is a quotient set of U and is denoted by $U/R = \{[x]_R | x \in U\}$, where $[x]_R = \{y \in U | (x, y) \in R\}$ is the equivalence class containing x with respect to R. In the view of granular computing, equivalence classes are the basic building blocks for the representation and approximation of any subset of the universe of discourse. Each equivalence class may be viewed as a granule consisting of indistinguishable elements. In the basic concept $X \in \mathcal{P}(U)$, one can characterize X by a pair of upper and lower approximations which are

$$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\} = \cup\{[x]_R \mid [x]_R \cap X \neq \emptyset\};$$
$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\} = \cup\{[x]_R \mid [x]_R \subseteq X\}.$$

Here, $pos(X) = \underline{R}(X), neg(X) = (\overline{R}(X))^c, bn(X) = \overline{R}(X) - \underline{R}(X)$ are called the positive region, negative region, and boundary region of X, respectively [26].

2.2. Fuzzy set

Zadeh introduced the fuzzy set [45] in which a fuzzy subset $\widetilde{A}$ of U is defined as a function assigning to each element x of U. The value $\widetilde{A}(x) \in [0, 1]$ and $\widetilde{A}(x)$ is referred to as the membership degree of x to the fuzzy set $\widetilde{A}$. Let $\mathcal{F}(U)$ denotes all fuzzy subsets of U. For any fuzzy concept $\widetilde{A}, \widetilde{B} \in \mathcal{F}(U)$, we say that $\widetilde{A}$ is contained in $\widetilde{B}$, denoted by $\widetilde{A} \subseteq \widetilde{B}$, if $\widetilde{A}(x) \leq \widetilde{B}(x)$ for all $x \in U$, we say that $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A} \subseteq \widetilde{B}$ and $\widetilde{A} \supseteq \widetilde{B}$, given that $\widetilde{A}, \widetilde{B} \in \mathcal{F}(U)$ and $\forall x \in U$. The basic computing rules of fuzzy set are described as follows.

$$(\widetilde{A} \cup \widetilde{B})(x) = max\{\widetilde{A}(x), \widetilde{B}(x)\} = \widetilde{A}(x) \vee \widetilde{B}(x);$$
$$(\widetilde{A} \cap \widetilde{B})(x) = min\{\widetilde{A}(x), \widetilde{B}(x)\} = \widetilde{A}(x) \wedge \widetilde{B}(x);$$
$$\widetilde{A}^c(x) = 1 - \widetilde{A}(x).$$

Here "$\vee$" and "$\wedge$" are the maximum operation and minimum operation, respectively. The $\widetilde{A}^c$ is the complementary set of $\widetilde{A}$. In [31], Sarkar proposed a rough-fuzzy membership function for any two fuzzy sets($\widetilde{A}$ and $\widetilde{B}$) of the universe of discourse as:

$$\mu_{\widetilde{A}} = \frac{|\widetilde{A} \cup \widetilde{B}|}{|\widetilde{B}|}, \quad x \in U.$$

Here $|\widetilde{A}| = \Sigma\widetilde{A}(x), x \in U$. It is useful to construct the decision-theoretic rough fuzzy set model which is very important in our study.

2.3. Graded rough set

Yao and Lin [41] explored the relationships between rough set theory and modal logics and proposed the GRS model based on graded modal logics. The GRS model primarily considers the absolute quantitative information regarding the basic concept and knowledge granules and is a generalization of the Pawlak rough set model. In the GRS model, $k \in \mathbf{N}$ is a non-negative integer called "grade" and the following approximations are made:

$$\overline{R_k}(X) = \{x \in U \mid |[x]_R \cap X| > k\} = \cup\{[x]_R \mid |[x]_R \cap X| > k\};$$
$$\underline{R_k}(X) = \{x \in U \mid |[x]_R| - |[x]_R \cap X| \le k\} = \cup\{[x]_R \mid |[x]_R| - |[x]_R \cap X| \le k\} = \cup\{[x]_R \mid |[x]_R \cap X^c| \leqslant k\}.$$

These two approximations are called grade k upper and lower approximations of X. Where $|\cdot|$ stands for the cardinality of the objects in set, and X^c stands for the complementary set of X. If $\overline{R_k}(X) = \underline{R_k}(X)$, then X is called a definable set by grade k; otherwise, X is called a rough set by grade k. $\overline{R_k}$ and $\underline{R_k}$ are called grade k upper and lower approximation operators, respectively. Here, $\overline{R_k}(X)$ is the union of the equivalence classes which satisfy the cardinality of the intersection of X and equivalence classes to exceed parameter k. $\underline{R_k}\ (X)$ is the union of the equivalence classes which satisfy the cardinality of the intersection of the complementary set of X and equivalence classes not to exceed parameter k.

Especially, if $k = 0$, then $\overline{R_k}(X) = \overline{R}(X)$, $\underline{R_k}(X) = \underline{R}(X)$. Therefore, the classical rough set model is a special case of GRS model. It must be pointed out that the lower approximation included in the upper approximation does not hold usually. So, the boundary region could be defined as union of lower and upper boundary regions. Accordingly, we can get the following regions.

$$pos_k(X) = \overline{R_k}(X) \cap \underline{R_k}(X);$$
$$neg_k(X) = (\overline{R_k}(X) \cup \underline{R_k}(X))^c;$$
$$Ubn_k(X) = \overline{R_k}(X) - \underline{R_k}(X);$$
$$Lbn_k(X) = \underline{R_k}(X) - \overline{R_k}(X);$$
$$bn_k(X) = Ubn_k(X) \cup Lbn_k(X) = \overline{R_k}(X) \triangle \underline{R_k}(X).$$

Here $pos_k(X)$, $neg_k(X)$, $Ubn_k(X)$, $Lbn_k(X)$ and $bn_k(X)$ are called grade k positive region, negative region, upper boundary region, lower boundary region, and boundary region of X, respectively. The $\triangle$ is the symmetric difference of the upper and lower approximation sets.

Moreover, if the set X is generalized to a fuzzy set $\widetilde{A} \in \mathcal{F}(U)$, the GRS model will be generalized to graded rough fuzzy set (GRFS) model. The following definition can be got.

$$\overline{R_k}(\widetilde{A}) = \{x \in U \mid \ \Sigma_{y\in[x]_R}\widetilde{A}(y) > k\} = \cup\{[x]_R \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > k\};$$
$$\underline{R_k}(\widetilde{A}) = \{x \in U \mid \ \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) \le k\} = \cup\{[x]_R \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) \le k\}.$$

According to the above definition, the rough regions can be calculated similar to previous one.

2.4. Decision-theoretic rough set

In [28] Pawlak and Skowron suggested using a rough membership function to redefine the two approximations and the rough membership function μ_R is defined by:

$$\mu_R(x) = P(X|[x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|}.$$

Bayesian decision procedure mainly deals with making decisions have minimum risk or cost under probabilistic uncertainty. The following processes can be found in [2]. In the Bayesian decision procedure, a finite set of states can be written as $\Omega = \{\omega_1, \omega_2, \ldots, \omega_s\}$, and a finite set of r possible actions can be denoted by $A = \{a_1, a_2, \ldots, a_r\}$. Let $P(\omega_j|x)$ be the conditional probability of an object x being in state ω_j given that the object is described by x. Let $\lambda(a_i|\omega_j)$ denote the loss or the cost for taking action a_i when the state is ω_j. The expected loss function associated with taking action a_i is given by

$$R(a_i|x) = \sum_{j=1}^{s} \lambda(a_i|\omega_j) P(\omega_j|x).$$

With respect to the membership of an object in X, we have a set of two states and a set of three actions for each state. The set of states is given by $\Omega = \{X, X^c\}$ indicating that an element is in X and not in X, respectively. The set of actions with respect to a state is given by $A = \{a_P, a_B, a_N\}$, where P, B and N represent the three actions in deciding $x \in pos(X)$, $x \in bn(X)$, and $x \in neg(X)$, respectively. The loss function regarding the risk or the cost of actions in different states is given in the following:

Table 1
The loss function.

	$X(P)$	$X^c(N)$
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

In Table 1, λ_{PP}, λ_{BP} and λ_{NP} denote the losses incurred for taking actions a_P, a_B and a_N, respectively, when an object belongs to X. And λ_{PN}, λ_{BN} and λ_{NN} denote the losses incurred for taking the same actions when the object does not belong to X. The expected loss $R(a_i|[x]_R)$ associated with taking the individual actions can be expressed as [42].

$$R(a_P|[x]_R) = \lambda_{PP}P(X|[x]_R) + \lambda_{PN}P(X^C|[x]_R);$$

$$R(a_B|[x]_R) = \lambda_{BP}P(X|[x]_R) + \lambda_{BN}P(X^C|[x]_R);$$

$$R(a_N|[x]_R) = \lambda_{NP}P(X|[x]_R) + \lambda_{NN}P(X^C|[x]_R).$$

When $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$, the Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) If $P(X|[x]_R) \ge \gamma$ and $P(X|[x]_R) \ge \alpha$, decide $pos(X)$;

(N) If $P(X|[x]_R) \le \beta$ and $P(X|[x]_R) \le \gamma$, decide $neg(X)$;

(B) If $\beta \le P(X|[x]_R) \le \alpha$, decide $bn(X)$.

Where the parameters α, β and γ are defined as:

$$\alpha = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})};$$
$$\beta = \frac{\lambda_{NN} - \lambda_{BN}}{(\lambda_{NN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{NP})};$$
$$\gamma = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}.$$

If a loss function further satisfies the condition: $(\lambda_{PN} - \lambda_{NN})(\lambda_{BP} - \lambda_{NP}) \ge (\lambda_{NN} - \lambda_{BN})(\lambda_{NP} - \lambda_{PP})$, then we can get $\alpha \ge \gamma \ge \beta$.

When $\alpha > \beta$, we have $\alpha > \gamma > \beta$. The DTRS has the decision rules:

(P) If $P(X|[x]_R) \ge \alpha$, decide $pos(X)$;

(N) If $P(X|[x]_R) \le \beta$, decide $neg(X)$;

(B) If $\beta < P(X|[x]_R) < \alpha$, decide $bn(X)$.

Using these three decision rules, we get the probabilistic approximations, namely the upper and lower approximations of the DTRS model:

$$\overline{R_{(\alpha,\beta)}}(X) = \{x \in U \mid P(X|[x]_R) > \beta\} = \cup\{[x]_R \mid P(X|[x]_R) > \beta\};$$
$$\underline{R_{(\alpha,\beta)}}(X) = \{x \in U \mid P(X|[x]_R) \ge \alpha\} = \cup\{[x]_R \mid P(X|[x]_R) \ge \alpha\}.$$

If $\underline{R_{(\alpha,\beta)}}(X) = \overline{R_{(\alpha,\beta)}}(X)$, then X is a definable set, otherwise X is a rough set. Here, $pos_{(\alpha,\beta)}(X) = \underline{R_{(\alpha,\beta)}}(X)$, $neg_{(\alpha,\beta)}(X) = (\overline{R_{(\alpha,\beta)}}(X))^c$, $bn_{(\alpha,\beta)}(X) = \overline{R_{(\alpha,\beta)}}(X) - \underline{R_{(\alpha,\beta)}}(X)$ are the positive region, negative region and boundary region, respectively.

Sun et al. introduced the PRS model and its extensions in [32]. Let U be a non-empty finite universe and R be an equivalence relation of U and P be the probabilistic measure. For any $\widetilde{A} \in \mathcal{F}(U)$, $P(\widetilde{A}|[x]_R)$ is called the conditional probability of fuzzy event $\widetilde{A}$ given the description $[x]_R$. The $P(\widetilde{A}|[x]_R)$ is defined as follows:

$$P(\widetilde{A}|[x]_R) = \frac{\Sigma_{y\in[x]_R}\widetilde{A}(y)}{|[x]_R|}, x \in U.$$

The $P(\widetilde{A}|[x]_R)$ can also be explained as the probability that a randomly selected object $x \in U$ belongs to the fuzzy concept $\widetilde{A}$ given the description $[x]_R$. Based on the above conditional probability of fuzzy event $\widetilde{A}$, the upper and lower approximations of fuzzy set $\widetilde{A}$ with respect to α and β are defined as follows:

$$\overline{R_{(\alpha,\beta)}}(\widetilde{A}) = \{x \in U|P(\widetilde{A}|[x]_R) > \beta\} = \cup\{[x]_R \mid P(\widetilde{A}|[x]_R) > \beta\};$$
$$\underline{R_{(\alpha,\beta)}}(\widetilde{A}) = \{x \in U|P(\widetilde{A}|[x]_R) \ge \alpha\} = \cup\{[x]_R \mid P(\widetilde{A}|[x]_R) \ge \alpha\}.$$

According to the definitions, it is easy to get that the upper and lower approximations are the binary operators from $\mathcal{F}(U) \to \mathcal{P}(U)$, where the $\mathcal{F}(U)$ stands for all fuzzy set of U and $\mathcal{P}(U)$ means the power set of U. At this time, the DTRS model will be generalized to the DTRFS model.

DTRS model based on Bayesian decision principle was initially proposed by Yao [42]. The conditional probability in this model is determined by the rough membership functions $P(X|[x]_R) = |[x]_R \cap X|/|[x]_R|$, which implies the relative quantitative information. The decision-theoretic approximations are made with $0 \leq \beta < \alpha \leq 1$. The parameters α and β were obtained from the losses of the Bayesian decision procedure, which are related to the relative quantitative information. The loss function can be considered as the standard threshold values, which are not abstract notions. However, they have an intuitive interpretation. One can easily interpret and measure the loss or the cost in a real application.

3. Double-quantitative decision-theoretic rough fuzzy set models

Based on previous introduction, here we will discuss a fuzzy concept which comprehensively describes relative and absolute quantitative information. We will further discuss the composite study of DTRFS and GRFS models based on logical operation. Based on the knowledge that logical conjunction and logical disjunction operation are a pair of symmetric operators, we will construct two expanded Dq-DTRFS models by using logical conjunction and logical disjunction operation, respectively.

3.1. Logical conjunction double-quantitative decision-theoretic rough fuzzy set ($\bigwedge$-Dq-DTRFS) model

In many scientific fields, not only the relative quantitative information should be considered, but also the absolute quantitative information should be researched by considering the upper and lower approximations at the same time. By taking the absolute quantitative information into consideration in the Bayesian decision procedure in the DTRFS model, we can get a kind of Dq-DTRFS model based on logical conjunction.

Definition 3.1. Let $I = (U, R)$ be an approximation space, and $U = \{x_1, x_2, \ldots, x_n\}$ be a universe. For any $\widetilde{A} \in \mathcal{F}(U)$, $0 \leq \beta < \alpha < 1$, $k \in \mathbf{N}$ and $x \in U$, the logical conjunction double-quantitative upper and lower approximations of fuzzy set $\widetilde{A}$ based on the relation R are defined as:

$$\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) = \{x \in U \mid P(\widetilde{A}|[x]_R) > \beta,\ \Sigma_{y\in[x]_R}\widetilde{A}(y) > k\};$$
$$\underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) = \{x \in U \mid P(\widetilde{A}|[x]_R) \geq \alpha,\ \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) \leq k\}.$$

Based on these operators, we determine a rough set model called the logical conjunction double-quantitative decision-theoretic rough fuzzy set ($\bigwedge$-Dq-DTRFS) model, which is also denoted by $(U, \overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}), \underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}))$. The positive region, negative region, upper boundary region, lower boundary region and boundary region are presented as follows:

$$pos_k^{\wedge}(\widetilde{A}) = \overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) \cap \underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A});$$
$$neg_k^{\wedge}(\widetilde{A}) = (\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) \cup \underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}))^c;$$
$$Ubn_k^{\wedge}(\widetilde{A}) = \overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) - \underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A});$$
$$Lbn_k^{\wedge}(\widetilde{A}) = \underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) - \overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A});$$
$$bn_k^{\wedge}(\widetilde{A}) = Ubn_k^{\wedge}(\widetilde{A}) \cup Lbn_k^{\wedge}(\widetilde{A}) = \overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) \triangle \underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}).$$

According to the definition of $P(\widetilde{A}|[x]_R)$ and the above definition, one can get the following theorem.

Theorem 3.1. *The logical conjunction double-quantitative upper and lower approximations of fuzzy set $\widetilde{A}$ based on the relation R can also be defined as:*

$$\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > max(k, \beta|[x]_R|)\};$$
$$\underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > max(|[x]_R| - k, \alpha|[x]_R|)\}.$$

Proof. It can be proved directly from the definition of $P(\widetilde{A}|[x]_R)$ and Definition 3.1. □

Corollary 3.1. *If $\widetilde{A}$ is degenerated into a classical set $A \subseteq U$, then,*

$$\overline{R_{(\alpha,\beta)\wedge k}}(A) = \{x \in U \mid P(A|[x]_R) > \beta, |[x]_R \cap A| > k\} = \overline{R_{(\alpha,\beta)}}(A) \cap \overline{R_k}(A);$$
$$\underline{R_{(\alpha,\beta)\wedge k}}(A) = \{x \in U \mid P(A|[x]_R) \geq \alpha, |[x]_R| - |[x]_R \cap A| \leq k\} = \underline{R_{(\alpha,\beta)}}(A) \cap \underline{R_k}(A).$$

Furthermore, these parameters in their special case with $\alpha = 1$, $\beta = 0$ and $k = 0$, the $\bigwedge$-Dq-DTRFS model will be degenerated into Pawlak rough set $\overline{R_{(\alpha,\beta)\wedge k}}(A) = \overline{R}(A)$, and $\underline{R_{(\alpha,\beta)\wedge k}}(A) = \underline{R}(A)$. It means the model is a directional expansion of the Pawlak model.

The relative information similarly complements the absolute description and can be used to improve the GRS model. The sharp contrast between the relative and grade environments is typical of double quantification applications. For example, if the relative quantification varies over a small range while the grade changes significantly, then the double quantification can play a significant role. Based on the descriptions of the regions, the following decision rules can be obtained.

$(P^\wedge)$ If $k \le \beta|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge |[x]_R| - k$, decide $pos_k^\wedge(\widetilde{A})$;
$(P^\wedge)$ If $\beta|[x]_R| < k < \alpha|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge \alpha|[x]_R|$, decide $pos_k^\wedge(\widetilde{A})$;
$(P^\wedge)$ If $\alpha|[x]_R| \le k$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) > k$, decide $pos_k^\wedge(\widetilde{A})$;
$(N^\wedge)$ If $k \le \beta|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \le \beta|[x]_R|$, decide $neg_k^\wedge(\widetilde{A})$;
$(N^\wedge)$ If $\beta|[x]_R| < k < \alpha|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \le k$, decide $neg_k^\wedge(\widetilde{A})$;
$(N^\wedge)$ If $\alpha|[x]_R| \le k$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha|[x]_R|$, decide $neg_k^\wedge(\widetilde{A})$;
$(UB^\wedge)$ If $k \le \beta|[x]_R|$, $\beta|[x]_R| < \Sigma_{y\in[x]_R}\widetilde{A}(y) < |[x]_R| - k$, decide $Ubn_k^\wedge(\widetilde{A})$;
$(UB^\wedge)$ If $\beta|[x]_R| < k < \alpha|[x]_R|$, $k < \Sigma_{y\in[x]_R} < \alpha|[x]_R|$, decide $Ubn_k^\wedge(\widetilde{A})$;
$(LB^\wedge)$ If $\alpha|[x]_R| \le k$, $\alpha|[x]_R| \le \Sigma_{y\in[x]_R}\widetilde{A}(y) \le k$, decide $Lbn_k^\wedge(\widetilde{A})$.

With these rules, one can make decisions based on the following positive, upper boundary, lower boundary and negative rules. For $\bigwedge$-Dq-DTRFS model, we have the following decisions:

$Des([x]_R) \to Des_{P_k^\wedge}(\widetilde{A})$, for $x \in pos_k^\wedge(\widetilde{A})$;
$Des([x]_R) \to Des_{N_k^\wedge}(\widetilde{A})$, for $x \in neg_k^\wedge(\widetilde{A})$;
$Des([x]_R) \to Des_{UB_k^\wedge}(\widetilde{A})$, for $x \in Ubn_k^\wedge(\widetilde{A})$;
$Des([x]_R) \to Des_{LB_k^\wedge}(\widetilde{A})$, for $x \in Lbn_k^\wedge(\widetilde{A})$.

3.2. Logical disjunction double-quantitative decision-theoretic rough fuzzy set ($\bigvee$-Dq-DTRFS) model

Based on the previous discussion, we find that sometimes the relative quantitative information and the absolute quantitative information should not be considered at the same time. We may just need to satisfy at least one of these conditions.

Definition 3.2. Let $I = (U, R)$ be an approximation space, and $U = \{x_1, x_2, \ldots, x_n\}$ be a universe. For any $\widetilde{A} \in \mathcal{F}(U)$, $0 \le \beta < \alpha < 1$ and $x \in U$, the logical disjunction double-quantitative upper and lower approximations of fuzzy set $\widetilde{A}$ based on the relation R are defined as:

$$\overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) = \{x \in U \mid P(\widetilde{A}|[x]_R) > \beta,\ \text{or}\ \Sigma_{y\in[x]_R}\widetilde{A}(y) > k\};$$
$$\underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) = \{x \in U \mid P(\widetilde{A}|[x]_R) \ge \alpha,\ \text{or}\ \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) \le k\}.$$

Similar to the previous rough set model, the regions of $\bigvee$-Dq-DTRFS model can be described in the same way.

$$pos_k^\vee(\widetilde{A}) = \overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) \cap \underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A});$$
$$neg_k^\vee(\widetilde{A}) = (\overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) \cup \underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}))^c;$$
$$Ubn_k^\vee(\widetilde{A}) = \overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) - \underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A});$$
$$Lbn_k^\vee(\widetilde{A}) = \underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) - \overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A});$$
$$bn_k^\vee(\widetilde{A}) = Ubn_k^\vee(\widetilde{A}) \cup Lbn_k^\vee(\widetilde{A}) = \overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) \triangle \underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}).$$

According to the definition of $P(\widetilde{A}|[x]_R)$ and the above definition, one can get the following theorem.

Theorem 3.2. *The logical disjunction double-quantitative upper and lower approximations of fuzzy set $\widetilde{A}$ based on the relation R can also be defined as:*

$$\overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > min(k, \beta|[x]_R|)\};$$
$$\underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}) = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > min(|[x]_R| - k, \alpha|[x]_R|)\}.$$

Proof. It can easily be verified by the Definition 3.2 and the definition of $P(\widetilde{A}|[x]_R)$. □

Here, the second type of Dq-DTRFS model was established based on the logical disjunction. In this model the relative and absolute information are considered at the same time. The lower and upper approximations in Definition 3.2 must contain at least one quantification.

Corollary 3.2. *Similar to the first model, if $\widetilde{A}$ is degenerated into a classical set $A \subseteq U$, then,*

$$\overline{R_{(\alpha,\beta)\vee k}}(A) = \{x \in U \mid P(A|[x]_R) > \beta,\ \text{or}\ |[x]_R \cap A| > k\} = \overline{R_{(\alpha,\beta)}}(A) \cup \overline{R}_k(A);$$
$$\underline{R_{(\alpha,\beta)\vee k}}(A) = \{x \in U \mid P(A|[x]_R) \ge \alpha,\ \text{or}\ |[x]_R| - |[x]_R \cap A| \le k\} = \underline{R_{(\alpha,\beta)}}(A) \cup \underline{R_k}(A).$$

Furthermore, these parameters in their special case with $\alpha = 1$, $\beta = 0$ and $k = 0$, then the $\bigvee$-Dq-DTRFS model will be degenerated into Pawlak rough set $\overline{R_{(\alpha,\beta)\vee k}}(A) = \overline{R}(A)$, and $\underline{R_{(\alpha,\beta)\vee k}}(A) = \underline{R}(A)$. It means the model is a directional expansion of the Pawlak model, too.

Table 2
Comparison decision rules of $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models.

Value	$\bigwedge$-*Dq-DTRFS*	$\bigvee$-*Dq-DTRFS*	Decision
$\alpha\lvert[x]_R\rvert \le k$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) > k$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) > \beta\lvert[x]_R\rvert$	$pos_k(\widetilde{A})$
$\alpha\lvert[x]_R\rvert \le k$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) < \lvert[x]_R\rvert - k$	$neg_k(\widetilde{A})$
$\alpha\lvert[x]_R\rvert \le k$	$\alpha\lvert[x]_R\rvert \le \Sigma_{y\in[x]_R}\widetilde{A}(y) \le k$	$\lvert[x]_R\rvert - k \le \Sigma_{y\in[x]_R}\widetilde{A}(y) \le \beta\lvert[x]_R\rvert$	$Lbn_k(\widetilde{A})$
$k \le \beta\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge \lvert[x]_R\rvert - k$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge \alpha\lvert[x]_R\rvert$	$pos_k(\widetilde{A})$
$k \le \beta\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \le \beta\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \le k$	$neg_k(\widetilde{A})$
$k \le \beta\lvert[x]_R\rvert$	$\beta\lvert[x]_R\rvert < \Sigma_{y\in[x]_R}\widetilde{A}(y) < \lvert[x]_R\rvert - k$	$k < \Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha\lvert[x]_R\rvert$	$Ubn_k(\widetilde{A})$
$\beta\lvert[x]_R\rvert < k < \alpha\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge \alpha\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge \lvert[x]_R\rvert - k$	$pos_k(\widetilde{A})$
$\beta\lvert[x]_R\rvert < k < \alpha\lvert[x]_R\rvert$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \le k$	$\Sigma_{y\in[x]_R}\widetilde{A}(y) \le \beta\lvert[x]_R\rvert$	$neg_k(\widetilde{A})$
$\beta\lvert[x]_R\rvert < k < \alpha\lvert[x]_R\rvert$	$k < \Sigma_{y\in[x]_R} < \alpha\lvert[x]_R\rvert$	$\beta\lvert[x]_R\rvert < \Sigma_{y\in[x]_R} < \lvert[x]_R\rvert - k$	$Ubn_k(\widetilde{A})$

According to the constructed model $\bigvee$-Dq-DTRFS, the following decision rules can be obtained.
($P^\vee$) If $k \le \beta|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge \alpha|[x]_R|$, decide $pos_k^\vee(\widetilde{A})$;
($P^\vee$) If $\beta|[x]_R| < k < \alpha|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \ge |[x]_R| - k$, decide $pos_k^\vee(\widetilde{A})$;
($P^\vee$) If $\alpha|[x]_R| \le k$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) > \beta|[x]_R|$, decide $pos_k^\vee(\widetilde{A})$;
($N^\vee$) If $k \le \beta|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \le k$, decide $neg_k^\vee(\widetilde{A})$;
($N^\vee$) If $\beta|[x]_R| < k < \alpha|[x]_R|$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) \le \beta|[x]_R|$, decide $neg_k^\vee(\widetilde{A})$;
($N^\vee$) If $\alpha|[x]_R| \le k$, $\Sigma_{y\in[x]_R}\widetilde{A}(y) < |[x]_R| - k$, decide $neg_k^\vee(\widetilde{A})$;
($UB^\vee$) If $k \le \beta|[x]_R|$, $k < \Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha|[x]_R|$, decide $Ubn_k^\vee(\widetilde{A})$;
($UB^\vee$) If $\beta|[x]_R| < k < \alpha|[x]_R|$, $\beta|[x]_R| < \Sigma_{y\in[x]_R} < |[x]_R| - k$, decide $Ubn_k^\vee(\widetilde{A})$;
($LB^\vee$) If $\alpha|[x]_R| \le k$, $|[x]_R| - k \le \Sigma_{y\in[x]_R}\widetilde{A}(y) \le \beta|[x]_R|$, decide $Lbn_k^\vee(\widetilde{A})$.
With these decision rules, one can make decisions based on the following positive, upper boundary, lower boundary and negative rules. For $\bigvee$-Dq-DTRFS model, we have the following decisions:

$$Des([x]_R) \to Des_{P_k^\vee}(\widetilde{A}),\ \text{for } x \in pos_k^\vee(\widetilde{A});$$
$$Des([x]_R) \to Des_{N_k^\vee}(\widetilde{A}),\ \text{for } x \in neg_k^\vee(\widetilde{A});$$
$$Des([x]_R) \to Des_{UB_k^\vee}(\widetilde{A}),\ \text{for } x \in Ubn_k^\vee(\widetilde{A});$$
$$Des([x]_R) \to Des_{LB_k^\vee}(\widetilde{A}),\ \text{for } x \in Lbn_k^\vee(\widetilde{A}).$$

4. Comparisons and analyses of $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models

To complete our previous work, in this paper we build two Dq-DTRFS models and give their basic description. In this section, we will discuss the relationship between $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS and more analyses will be provided. According to the definitions of these two Dq-DTRFS models, we have shown that the expressions are similar. They are constructed on different logical conjunction and disjunction. The related theorems are held. In the last section, the decision rules of these two Dq-DTRFS models are presented and their comparisons are shown in Table 2.

There are serval parameters in models where α, β are conditional probability and k is a grade. The conditional probability and the grade represent the levels and grade of tolerance in making incorrect decisions. When the conditional probability is too low for acceptance (below α) and at the same time too high for rejection (above β), we will discuss the influence on decisions based on different parameter values.

Case 1. $\alpha + \beta = 1$.

Theorem 4.1. *For any $\widetilde{A} \in \mathcal{F}(U)$, $0 \le \beta < \alpha < 1$, $k \in \mathbf{N}$ and $x \in U$, if $\alpha + \beta = 1$ so, $\alpha = 1 - \beta$ then, the following theorem holds between $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models.*

$$\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}^c) = (\underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}))^c;$$
$$\underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}^c) = (\overline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}))^c.$$

Proof. According to the Theorem 3.1, one can get the left hand side of first part that $\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}^c) = \overline{R_{(\alpha,\beta)}}(\widetilde{A}^c) \cap \overline{R_k}(\widetilde{A}^c) = \{x \in U \mid P(\widetilde{A}^c|[x]_R) > \beta\} \cap \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}^c(y) > k\} = \{x \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > \beta|[x]_R|\} \cap \{x \in U \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > k\} = \{x \in U \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > max(\beta|[x]_R|, k)\}$; for the right hand side, one can get $(\underline{R_{(\alpha,\beta)\vee k}}(\widetilde{A}))^c = [\underline{R_{(\alpha,\beta)}}(\widetilde{A}) \cup \underline{R_k}(\widetilde{A})]^c = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha|[x]_R|\} \cap \{x \in U \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > k\} = \{x \in U \mid |[x]_R| - \Sigma_{y\in[x]_R}\widetilde{A}(y) > |[x]_R| - \alpha|[x]_R|\} \cap \{x \in U \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > k\} = \{x \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > \beta|[x]_R|\} \cap \{x \in U \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > k\} = \{x \in U \mid \Sigma_{y\in[x]_R}(1 - \widetilde{A}(y)) > max(\beta|[x]_R|, k)\}$. So, the first part has been proved and the second part of the theorem can be verified similarly. □

If a loss function satisfies $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$, $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$ and $(\lambda_{PN} - \lambda_{NN})(\lambda_{BP} - \lambda_{NP}) > (\lambda_{NN} - \lambda_{BN})(\lambda_{NP} - \lambda_{PP})$, we have $\alpha > \beta$. Thus, $\beta < 0.5$ and $\alpha > 0.5$ hold, from the fact that $\alpha = 1 - \beta$ and $\beta = 1 - \alpha$. At the same time, for the k, it

Table 3
Initial medical data.

Patient	*Fever*	*Headache*	$\widetilde{d}$	Patient	*Fever*	*Headache*	$\widetilde{d}$
x_1	0	0	0.1	x_{19}	0	0	0
x_2	1	1	0.4	x_{20}	1	2	0.8
x_3	0	2	0.6	x_{21}	2	0	0.8
x_4	2	1	0.8	x_{22}	0	0	0
x_5	1	0	0.3	x_{23}	2	1	0.6
x_6	2	2	1.0	x_{24}	1	2	0.7
x_7	0	0	0.1	x_{25}	0	2	0.5
x_8	1	2	0.7	x_{26}	2	2	1.0
x_9	2	2	0.9	x_{27}	1	1	0.2
x_{10}	1	1	0.5	x_{28}	2	0	0.4
x_{11}	1	2	1.0	x_{29}	2	1	0.5
x_{12}	2	0	0.5	x_{30}	0	0	0
x_{13}	0	0	0	x_{31}	1	2	0.7
x_{14}	2	1	0.8	x_{32}	0	1	0.1
x_{15}	0	1	0.3	x_{33}	2	1	0.7
x_{16}	1	1	0.5	x_{34}	1	1	0.6
x_{17}	0	2	0.5	x_{35}	0	0	0.1
x_{18}	2	1	0.8	x_{36}	2	0	0.5

follows that for $\alpha + \beta = 1$, a loss function must satisfy $\lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$, $\lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$ and the condition:

$$\frac{(\lambda_{PN} - \lambda_{NN})}{(\lambda_{NP} - \lambda_{PP})} = \frac{(\lambda_{BP} - \lambda_{NP})}{(\lambda_{NN} - \lambda_{BN})}.$$

The classical rough set model is obtained by letting $\alpha = 1$, $\beta = 0$ and $k = 0$, which satisfies the condition $\alpha + \beta = 1$. Thus the classical rough set model is a special case of the Case 1. It should be pointed out that the two models proposed by Zhang et al. in [38] is also a special case of Case 1 in this section. The reason is that when $\alpha + \beta = 1$, the DTRS model can be degenerated into variable precision rough set model, which is investigated in [48].

Case 2. $\alpha + \beta < 1$.

The condition $\alpha > \beta, \beta < 0.5$ holds for Case 2. A loss function must satisfy the condition:

$$\frac{(\lambda_{NP} - \lambda_{PP})}{(\lambda_{PN} - \lambda_{NN})} < \frac{(\lambda_{BP} - \lambda_{NP})}{(\lambda_{NN} - \lambda_{BN})}.$$

For Case 2, we choose an accepted region $pos(\widetilde{A})^{\vee}$ in $\bigvee$-Dq-DTRFS model which is described as $\{x \mid k/\alpha < |[x]_R| < k/\beta, \Sigma_{y\in[x]_R}\widetilde{A}(y) \geq |[x]_R| - k\}$ namely, $\{x \mid k/\alpha < |[x]_R| < k/\beta, \Sigma_{y\in[x]_R}\widetilde{A}^c(y) < |[x]_R| - k\}$ then, we have $\{x \mid k/\alpha < |[x]_R| < k/\beta, \Sigma_{y\in[x]_R}\widetilde{A}(y) > k\}$. That means it is part of $Ubn^{\wedge}$ in $\bigwedge$-Dq-DTRFS model. That is to say, from the accepted region of one class in the kind of $\bigvee$-Dq-DTRFS model, we can not decide where to include it in $\bigwedge$-Dq-DTRFS model. It is just possible to include it in accepted region $\bigwedge$-Dq-DTRFS model.

Case 3. $\alpha + \beta > 1$.

The condition $\alpha > \beta, \alpha > 0.5$ holds for Case 3. A loss function must satisfy the condition:

$$\frac{(\lambda_{PN} - \lambda_{NN})}{(\lambda_{NP} - \lambda_{PP})} > \frac{(\lambda_{NN} - \lambda_{BN})}{(\lambda_{BP} - \lambda_{NP})}.$$

For case 3, we choose any x in rejection of $\bigwedge$-Dq-DTRFS model as $\{x \mid \alpha|[x]_R| \leq k, \Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha|[x]_R|\}$ to be equivalent to $\{x \mid \alpha|[x]_R| \leq k, \Sigma_{y\in[x]_R}\widetilde{A}^c(y) \geq \alpha|[x]_R|\}$ namely, $\{x \mid \alpha|[x]_R| \leq k, |[x]_R| - \Sigma_{y\in[x]_R}\widetilde{A}(y) \geq \alpha|[x]_R|\}$ then, we have $\{x \mid \alpha|[x]_R| \leq k, \Sigma_{y\in[x]_R}\widetilde{A}(y) \leq (1-\alpha)|[x]_R|\}$. Because $\alpha > 0.5 \Rightarrow 1 - \alpha < 0.5 < \alpha$ then, $\{x \mid \alpha|[x]_R| \leq k, \Sigma_{y\in[x]_R}\widetilde{A}(y) < \alpha|[x]_R|\}$. That is to say, from the acceptance of one class in a kind of $\bigwedge$-Dq-DTRFS model, we can compute the acceptance class in the $\bigvee$-Dq-DTRFS model. That means the acceptance condition is more strict than the first model.

5. Case study

In real life, patients go to a doctor when they have a headache or a fever. The doctor may decide the degree of coldness (flu) of the patients based on the degree of fever and the degree of headache of the patients. In this section, the medical example [21,49] is introduced to illustrate the utilization of our two $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models by comparing with DTRFS and GRFS models. Let $I = (U, C \cup \widetilde{d})$ be a decision fuzzy table, where U is composed of 36 patients, and the condition and decision attributes are *fever*, *headache* and *heavy cold*, respectively. Let R denote the equivalence relation on the condition attributes. Based on the measured medical data in Table 3, we provide the statistical results of the patient classes in Table 4, where (i, j) $(i, j \in [0, 2])$ denotes the rank of condition attributes and X denotes the cold patient set.

Based on the condition attributes *Fever* and *Headache*, the universe is classified into nine classes. From Table 3, the fuzzy decision attribute is represented as $\widetilde{d} = (0.1, 0.4, 0.6, \cdots, 0.6, 0.1, 0.5)$. In the following, we will discuss the limitations of DTRFS and GRFS models as well as the advantages of our two models proposed in the paper.

Table 4
Statistical results of the patient classes.

(i, j)	$[x]_R$	$\|[x]_R\|$	$\Sigma_{y\in[x]_R}\widetilde{d}(y)$	$P(\widetilde{d}\|[x]_R)$	$\Sigma_{y\in[x]_R}(1-\widetilde{d}(y))$
(0, 0)	$x_{1,\ 7,\ 13,\ 19,\ 22,\ 30,\ 35}$	7	0.3	0.04	6.7
(0, 1)	$x_{15,\ 32}$	2	0.4	0.20	1.6
(0, 2)	$x_{3,\ 17,\ 25}$	3	1.6	0.53	1.4
(1, 0)	x_5	1	0.3	0.30	0.7
(1, 1)	$x_{2,\ 10,\ 16,\ 27,\ 34}$	5	2.2	0.44	2.8
(1, 2)	$x_{8,\ 11,\ 20,\ 24,\ 31}$	5	3.9	0.78	1.1
(2, 0)	$x_{12,\ 21,\ 28,\ 36}$	4	2.2	0.55	1.8
(2, 1)	$x_{4,\ 14,\ 18,\ 23,\ 29,\ 33}$	6	4.2	0.70	1.8
(2, 2)	$x_{6,\ 9,\ 26}$	3	2.9	0.97	0.1

Table 5
Regions of GRFS model.

$Region\backslash k$	0	1	2	3
$\overline{R_k}(\widetilde{d})$	U	$U-([x_1]_R, [x_5]_R, [x_{15}]_R)$	$[x_2]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_4]_R$, $[x_8]_R$
$\underline{R_k}(\widetilde{d})$	$\emptyset$	$[x_5]_R$, $[x_6]_R$	$U-([x_1]_R, [x_2]_R)$	$U-[x_1]_R$
$pos_k(\widetilde{d})$	$\emptyset$	$[x_6]_R$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_4]_R$, $[x_8]_R$
$neg_k(\widetilde{d})$	$\emptyset$	$[x_1]_R$, $[x_{15}]_R$	$[x_1]_R$	$[x_1]_R$
$Ubn_k(\widetilde{d})$	U	$[x_2]_R$, $[x_3]_R$, $[x_4]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_2]_R$	$\emptyset$
$Lbn_k(\widetilde{d})$	$\emptyset$	$[x_5]_R$	$[x_3]_R$, $[x_5]_R$, $[x_{15}]_R$	$U-([x_1]_R, [x_4]_R, [x_8]_R)$

From Table 4, by utilizing the GRFS model, we can get the upper and lower approximations of $\widetilde{d}$ with different grades $k = 0, 1, 2, 3$ in Table 5. And then, we could calculate all the positive regions, negative regions, upper boundary regions and lower boundary regions of $\widetilde{d}$ with $k = 0, 1, 2, 3$.

From Tables 4 and 5, it is easy to see that $\Sigma_{y\in[x_1]_R}\widetilde{d}(y) = \Sigma_{y\in[x_5]_R}\widetilde{d}(y) = 0.3$. $[x_1]_R$ and $[x_5]_R$ are indiscernible and equal in the GRFS model. However, $[x_1]_R$ belongs to the negative region and $[x_5]_R$ belongs to the lower boundary region with the grades $k = 1, 2, 3$. Since $\Sigma_{y\in[x_4]_R}(1-\widetilde{d}(y)) = \Sigma_{y\in[x_{12}]_R}(1-\widetilde{d}(y)) = 1.8$, $[x_4]_R$ and $[x_{12}]_R$ should be indiscernible and equal in the GRFS model. While $[x_4]_R$ belongs to the positive region and $[x_{12}]_R$ belongs to the lower boundary region with the grade $k = 3$. For grade $k = 2$, $[x_4]_R$ and $[x_6]_R$ belong to the positive region and they should be indiscernible and equal in the GRFS model. However, $P(\widetilde{d}|[x_4]_R) = 0.70 \neq 0.97 = P(\widetilde{d}|[x_6]_R)$. So the GRFS model has some shortcomings sometimes. Therefore the GRFS model can not discern a valuable description in some circumstances.

In the Bayesian decision procedure, from the losses, one can give the values λ_{i1}, λ_{i2}, and $i = 1, 2, 3$. We make some changes to the loss function defined in [21], and the parameters can be calculated as follows.

Case 1. $\alpha + \beta = 1$. Consider the following loss function:

$$\lambda_{PP} = 0, \quad \lambda_{PN} = 18,$$
$$\lambda_{BP} = 9, \quad \lambda_{BN} = 2,$$
$$\lambda_{NP} = 12, \quad \lambda_{NN} = 0.$$

Then we can get $\alpha = 0.6$, $\beta = 0.4 \Rightarrow \alpha + \beta = 1$. We can obtain the decision-theoretic upper and lower approximations.

$$\overline{R_{(0.6,0.4)}}(\widetilde{d}) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$\underline{R_{(0.6,0.4)}}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R.$$

Accordingly, one can get the positive region, negative region and boundary region presented as follows:

$$pos_{(0.6,0.4)}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$$
$$neg_{(0.6,0.4)}(\widetilde{d}) = [x_1]_R \cup [x_5]_R \cup [x_{15}]_R;$$
$$bn_{(0.6,0.4)}(\widetilde{d}) = [x_2]_R \cup [x_3]_R \cup [x_{12}]_R.$$

Case 2. $\alpha + \beta < 1$. Consider the following loss function:

$$\lambda_{PP} = 0, \quad \lambda_{PN} = 19,$$
$$\lambda_{BP} = 12, \quad \lambda_{BN} = 3,$$
$$\lambda_{NP} = 19, \quad \lambda_{NN} = 0.$$

Then we can get $\alpha = 0.5$, $\beta = 0.3 \Rightarrow \alpha + \beta < 1$. We can obtain the decision-theoretic upper and lower approximations.

$$\overline{R_{(0.5,0.3)}}(\widetilde{d}) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$\underline{R_{(0.5,0.3)}}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R.$$

Table 6
Regions of DTRFS model.

$Region\backslash(\alpha,\beta)$	(0.5, 0.3)	(0.6, 0.4)	(0.7, 0.5)
$\overline{R}(\widetilde{d})$	$U-([x_1]_R,[x_5]_R,[x_{15}]_R)$	$U-([x_1]_R,[x_5]_R,[x_{15}]_R)$	$[x_3]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$
$\underline{R}(\widetilde{d})$	$[x_3]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$
$pos(\widetilde{d})$	$[x_3]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$
$neg(\widetilde{d})$	$[x_1]_R$, $[x_5]_R$, $[x_{15}]_R$	$[x_1]_R$, $[x_5]_R$, $[x_{15}]_R$	$[x_1]_R$, $[x_2]_R$, $[x_5]_R$, $[x_{15}]_R$
$bn(\widetilde{d})$	$[x_2]_R$	$[x_2]_R$, $[x_3]_R$, $[x_{12}]_R$	$[x_3]_R$, $[x_{12}]_R$

Table 7
Regions of $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models.

	Model	Pos. region	Neg. region	Ubn. region	Lbn. region
Case 1	$\bigwedge$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$	$[x_1]_R$, $[x_3]_R$, $[x_5]_R$, $[x_{15}]_R$	$[x_2]_R$, $[x_{12}]_R$	$\emptyset$
	$\bigvee$	$[x_3]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_1]_R$	$[x_2]_R$	$[x_5]_R$, $[x_{15}]_R$
Case 2	$\bigwedge$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_1]_R$, $[x_5]_R$, $[x_{15}]_R$	$[x_2]_R$	$[x_3]_R$
	$\bigvee$	$[x_3]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_1]_R$	$[x_2]_R$	$[x_5]_R$, $[x_{15}]_R$
Case 3	$\bigwedge$	$[x_4]_R$, $[x_6]_R$, $[x_8]_R$	$[x_1]_R$, $[x_2]_R$, $[x_3]_R$, $[x_5]_R$, $[x_{12}]_R$, $[x_{15}]_R$	$\emptyset$	$\emptyset$
	$\bigvee$	$[x_3]_R$, $[x_4]_R$, $[x_6]_R$, $[x_8]_R$, $[x_{12}]_R$	$[x_1]_R$	$[x_2]_R$	$[x_5]_R$, $[x_{15}]_R$

Accordingly, one can get the positive region, negative region and boundary region presented as follows:

$$pos_{(0.5,0.3)}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$neg_{(0.5,0.3)}(\widetilde{d}) = [x_1]_R \cup [x_5]_R \cup [x_{15}]_R;$$
$$bn_{(0.5,0.3)}(\widetilde{d}) = [x_2]_R.$$

Case 3. $\alpha + \beta > 1$. Consider the following loss function:

$$\lambda_{PP} = 0, \quad \lambda_{PN} = 21,$$
$$\lambda_{BP} = 7, \quad \lambda_{BN} = 2,$$
$$\lambda_{NP} = 9, \quad \lambda_{NN} = 0.$$

Then we can get $\alpha = 0.7,\ \beta = 0.5 \Rightarrow \alpha + \beta > 1$. We can obtain the decision-theoretic upper and lower approximations.

$$\overline{R_{(0.7,0.5)}}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$\underline{R_{(0.7,0.5)}}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R.$$

Accordingly, one can get the positive region, negative region and boundary region presented as follows:

$$pos_{(0.7,0.5)}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$$
$$neg_{(0.7,0.5)}(\widetilde{d}) = [x_1]_R \cup [x_2]_R \cup [x_5]_R \cup [x_{15}]_R;$$
$$bn_{(0.7,0.5)}(\widetilde{d}) = [x_3]_R \cup [x_{12}]_R.$$

All the regions of DTRFS model with three groups parameters (α, β) are shown in Table 6.

From Tables 4 and 6, it is easy to see that $\Sigma_{y\in[x_2]_R}\widetilde{d}(y) = \Sigma_{y\in[x_{12}]_R}\widetilde{d}(y) = 2.2$. $[x_2]_R$ and $[x_{12}]_R$ are indiscernible and equal in the DTRFS model. However, $[x_2]_R$ belongs to the boundary region and $[x_{12}]_R$ belongs to the positive region with the threshold values $\alpha = 0.5, \beta = 0.3$. And $[x_2]_R$ belongs to the negative region and $[x_{12}]_R$ belongs to the boundary region with the threshold values $\alpha = 0.7, \beta = 0.5$. For $\Sigma_{y\in[x_4]_R}(1-\widetilde{d}(y)) = \Sigma_{y\in[x_{12}]_R}(1-\widetilde{d}(y)) = 1.8$, $[x_4]_R$ and $[x_{12}]_R$ should be indiscernible and equal in the DTRFS model. While $[x_4]_R$ belongs to the positive region and $[x_{12}]_R$ belongs to the boundary region with the threshold values $\alpha = 0.6, \beta = 0.4$ or $\alpha = 0.7, \beta = 0.5$. Especially, $[x_1]_R$, $[x_5]_R$ and $[x_{15}]_R$ all belong to the boundary region, and one can not know whether these patients need to be treated as a flu. This is a serious deficiency of the DTRFS model. So the DTRFS model can not discern a complete and valuable description in these circumstances.

Therefore, neither DTRFS nor GRFS can discern a complete and valuable description in some circumstances.

The description of $\bigwedge$-Dq-DTRFS and $\bigvee$**-Dq-DTRFS models:**

According to the results which are computed previously, we will calculate the proposed models for Case 1, Case 2 and Case 3 (see Table 7) in the following. Here, we choose the grade $k = 2$ for convenience.

Case 1. The upper and lower approximations of $\bigwedge$-Dq-DTRFS model are

$$\overline{R_{(0.6,0.4)\wedge 2}}(\widetilde{d}) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$\underline{R_{(0.6,0.4)\wedge 2}}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of $\bigwedge$-Dq-DTRFS model:

$pos_2^{\wedge}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$

$neg_2^{\wedge}(\widetilde{d}) = [x_1]_R \cup [x_3]_R \cup [x_5]_R \cup [x_{15}]_R;$

$Ubn_2^{\wedge}(\widetilde{d}) = [x_2]_R \cup [x_{12}]_R;$

$Lbn_2^{\wedge}(\widetilde{d}) = \emptyset.$

The upper and lower approximations of $\bigvee$-Dq-DTRFS model are

$\overline{R_{(0.6,0.4)\vee 2}}(\widetilde{d}) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$

$\underline{R_{(0.6,0.4)\vee 2}}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R.$

We can also get the positive region, negative region, upper boundary region and lower boundary region of $\bigvee$-Dq-DTRFS model:

$pos_2^{\vee}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$

$neg_2^{\vee}(\widetilde{d}) = [x_1]_R;$

$Ubn_2^{\vee}(\widetilde{d}) = [x_2]_R;$

$Lbn_2^{\vee}(\widetilde{d}) = [x_5]_R \cup [x_{15}]_R.$

For $\alpha = 0.6,\ \beta = 0.4,\ k = 2$, these $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models have their own quantitative semantics for the relative and absolute degree quantification. In $\bigwedge$-Dq-DTRFS model, $pos_2^{\wedge}(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$ denotes the relative degree of the patients belonging to cold patient set to exceed 0.6 and the external grade with respect to the heavy cold patient set not to exceed 2. In $\bigvee$-Dq-DTRFS model, $pos_2^{\vee}(X) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R$ denotes the relative degree of the patients belonging to cold patient set to be at least 0.4 and the internal grade with respect to the cold patient set to exceed 2. The same analysis result can be obtained for the negative region, upper boundary region and lower boundary region in both two models with the thresholds $\alpha = 0.6,\ \beta = 0.4$, and the grade $k = 2$.

Case 2. The upper and lower approximations of $\bigwedge$-Dq-DTRFS model are

$\overline{R_{(0.5,0.3)\wedge 2}}(\widetilde{d}) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$

$\underline{R_{(0.5,0.3)\wedge 2}}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R.$

We can also get the positive region, negative region, upper boundary region and lower boundary region of $\bigwedge$-Dq-DTRFS model:

$pos_2^{\wedge}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$

$neg_2^{\wedge}(\widetilde{d}) = [x_1]_R \cup [x_5]_R \cup [x_{15}]_R;$

$Ubn_2^{\wedge}(\widetilde{d}) = [x_2]_R;$

$Lbn_2^{\wedge}(\widetilde{d}) = [x_3]_R.$

The upper and lower approximations of $\bigvee$-Dq-DTRFS model are

$\overline{R_{(0.5,0.3)\vee 2}}(\widetilde{d}) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$

$\underline{R_{(0.5,0.3)\vee 2}}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R.$

We can also get the positive region, negative region, upper boundary region and lower boundary region of $\bigvee$-Dq-DTRFS model:

$pos_2^{\vee}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$

$neg_2^{\vee}(\widetilde{d}) = [x_1]_R;$

$Ubn_2^{\vee}(\widetilde{d}) = [x_2]_R;$

$Lbn_2^{\vee}(\widetilde{d}) = [x_5]_R \cup [x_{15}]_R.$

For $\alpha = 0.5,\ \beta = 0.3,\ k = 2$, these two models have their own quantitative semantics for the relative and absolute degree quantification. In $\bigwedge$-Dq-DTRFS model, $pos_2^{\wedge}(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R$ denotes the relative degree of the patients belonging to heavy cold patient set to exceed 0.5 and the external grade with respect to the cold patient set not to exceed 2. In $\bigvee$-Dq-DTRFS model, $pos_2^{\vee}(X) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R$ denotes the relative degree of the patients belonging to heavy cold patient set to be at least 0.3 and the internal grade with respect to the cold patient set to exceed 2. The same analysis result can be obtained for the negative region, upper boundary region and lower boundary region in both two models with the thresholds $\alpha = 0.5,\ \beta = 0.3$, and the grade $k = 2$.

Case 3. The upper and lower approximations of $\bigwedge$-Dq-DTRFS model are

$$\overline{R_{(0.7,0.5)\wedge 2}}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$$

$$\underline{R_{(0.7,0.5)\wedge 2}}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of $\bigwedge$-Dq-DTRFS model:

$$pos_2^{\wedge}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$$

$$neg_2^{\wedge}(\widetilde{d}) = [x_1]_R \cup [x_2]_R \cup [x_3]_R \cup [x_5]_R \cup [x_{12}]_R \cup [x_{15}]_R;$$

$$Ubn_2^{\wedge}(\widetilde{d}) = \emptyset;$$

$$Lbn_2^{\wedge}(\widetilde{d}) = \emptyset.$$

The upper and lower approximations of $\bigvee$-Dq-DTRFS model are:

$$\overline{R_{(0.7,0.5)\vee 2}}(\widetilde{d}) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$

$$\underline{R_{(0.7,0.5)\vee 2}}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of $\bigvee$-Dq-DTRFS model:

$$pos_2^{\vee}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$

$$neg_2^{\vee}(\widetilde{d}) = [x_1]_R;$$

$$Ubn_2^{\vee}(\widetilde{d}) = [x_2]_R;$$

$$Lbn_2^{\vee}(\widetilde{d}) = [x_5]_R \cup [x_{15}]_R.$$

For $\alpha = 0.7,\ \beta = 0.5,\ k = 2,$ in $\bigwedge$-Dq-DTRFS model, $pos_2^{\wedge}(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$ denotes the relative degree of the patients belonging to cold patient set to exceed 0.7 and the external grade with respect to the cold patient set not to exceed 2. In $\bigvee$-Dq-DTRFS model, $pos_2^{\vee}(X) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R$ denotes the relative degree of the patients belonging to cold patient set to be at least 0.5 and the internal grade with respect to the cold patient set to exceed 2. The same analysis result can be obtained for the negative region, upper boundary region and lower boundary region in both two models with the thresholds $\alpha = 0.7,\ \beta = 0.5,$ and the grade $k = 2.$

From Tables 4 and 5, it is easy to see that $\Sigma_{y\in[x_1]_R}\widetilde{d}(y) = \Sigma_{y\in[x_5]_R}\widetilde{d}(y) = 0.3$. So $[x_1]_R$ and $[x_5]_R$ are indiscernible and equal in the GRFS model. But we find that $[x_1]_R$ and $[x_5]_R$ belong to different regions. While in Table 7, $[x_1]_R$ and $[x_5]_R$ belong to the negative region based on the $\bigwedge$-Dq-DTRFS model in cases 1, 2, and 3. This shows that $[x_1]_R$ and $[x_5]_R$ are indiscernible to a certain conditions.

From Table 5, one can see the positive region of $\widetilde{d}$ is $pos_k(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R$ when $k = 2$. That means these four classes are accepted, and they are indiscernible in this grade. Meanwhile, according to Table 6, it is easy to get that for the threshold $\alpha = 0.6, \beta = 0.4$ the positive region is $pos_{(0.6,0.4)}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$. For the threshold $\alpha = 0.5, \beta = 0.3$ the positive region is $pos_{(0.5,0.3)}(\widetilde{d}) = [x_3]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R$. For the threshold $\alpha = 0.7, \beta = 0.5$ the positive region is $pos_{(0.7,0.5)}(\widetilde{d}) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$. Based on these analyses, one can combine the regions of $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models as shown in Table 7. The regions which are $[x_4]_R \cup [x_6]_R \cup [x_8]_R$ are always in positive region, namely, the patients who have heavy cold under the given loss condition. When the relative information is considered, the $[x_{12}]_R$ could be changed into lower boundary or upper boundary of $\bigwedge$-Dq-DRTFS model. Also, For the $\bigvee$-Dq-DTRFS models of cases 1, 2 and 3, $[x_1]_R$ always belongs to the negative region, $[x_5]_R$ and $[x_{15}]_R$ always belong to the lower boundary region. Most importantly, each patients can know whether they need to be treated as a flu with the minimum risk according to the regions in Table 7 based on the Dq-DTRFS models. Therefore, the double-quantitative of the relative and absolute information provides a valuable description in decision analysis fields.

6. Numerical experiments

In last section, we mainly introduced a medical example to illustrate the utilization of our $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models by comparing our models with DTRFS and GRFS models. In this section, we apply our two methods to solve a pattern recognition problem in big data. In the era of big data, decision making about big data is a very important issue. In order to make better decisions, it is imperative to handle the big data. To apply our $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models to big data, the big data should satisfy the condition that all the attribute values should be known and the decision attribute values should be in the range [0, 1]. That is to say, the information system should be a complete information system (with known attribute values) and decision attribute should be a fuzzy set. So if the big data satisfy these two conditions, one can use our $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models to deal with the big data. And if the big data set is a complete information system and the decision attribute values are not in the range [0, 1], then our models can be applied after the decision attribute

Table 8
Characteristics of testing date sets.

Date set	*Samples*	*Attributes*
Wine quality – red	1599	12
Wine quality – white	4898	12
Statlog (Landsat Satellite)	6435	37
Statlog (Shuttle)	30, 000	10

Table 9
Wine quality-red: The number of elements in each pattern with respect to different attribute sets.

Patterns	*methods*	1	2	3	4	5	6	7	8	9	10	11
Pos.region	$\bigwedge$-Dq-DTRFS	219	295	75	56	52	52	52	52	52	52	52
	$\bigvee$-Dq-DTRFS	1598	1573	1571	1571	1571	1571	1571	1571	1571	1571	1571
Excellent	*k-means*	156	126	138	224	226	547	516	516	516	516	516
	Hierarchical Clustering	876	19	1	1	1	2	1	1	1	1	1
Ubn.region	$\bigwedge$-Dq-DTRFS	1357	0	0	0	0	0	0	0	0	0	0
	$\bigvee$-Dq-DTRFS	0	0	0	0	0	0	0	0	0	0	0
Good	*k-means*	428	315	360	526	525	329	716	716	716	716	716
	Hierarchical Clustering	1	38	1	1	1	2	1	1	1	1	2
Lbn.region	$\bigwedge$-Dq-DTRFS	21	1229	1444	1462	1466	1466	1466	1466	1466	1466	1466
	$\bigvee$-Dq-DTRFS	1	24	26	26	26	26	26	26	26	26	26
General	*k-means*	362	584	530	76	76	613	265	265	265	265	265
	Hierarchical Clustering	721	1532	1587	1596	1595	1592	1596	1596	1596	1596	1595
Neg.region	$\bigwedge$-Dq-DTRFS	2	75	80	81	81	81	81	81	81	81	81
	$\bigvee$-Dq-DTRFS	0	2	2	2	2	2	2	2	2	2	2
Bad	*k-means*	653	574	571	773	772	110	102	102	102	102	102
	Hierarchical Clustering	1	10	10	1	2	3	1	1	1	1	1

Table 10
Wine quality-white: The number of elements in each pattern with respect to different attribute sets.

Patterns	*methods*	1	2	3	4	5	6	7	8	9	10	11
Pos.region	$\bigwedge$-Dq-DTRFS	29	2802	665	358	352	352	347	347	347	347	343
	$\bigvee$-Dq-DTRFS	4890	4754	4704	4698	4698	4698	4698	4698	4698	4698	4698
Excellent	*k-means*	1107	1769	1872	2114	2114	1872	1720	1720	1720	1720	1719
	Hierarchical Clustering	2111	259	1	1	1	1	1	2	2	1	1
Ubn.region	$\bigwedge$-Dq-DTRFS	4844	1027	12	0	0	0	0	0	0	0	0
	$\bigvee$-Dq-DTRFS	6	70	16	0	0	0	0	0	0	0	0
Good	*k-means*	1316	1107	763	871	871	1255	992	992	992	992	978
	Hierarchical Clustering	1	2146	1	1	1	2	1	1	2	2	1
Lbn.region	$\bigwedge$-Dq-DTRFS	20	896	3860	4177	4183	4183	4188	4188	4188	4188	4192
	$\bigvee$-Dq-DTRFS	2	37	105	124	124	124	124	124	124	124	124
General	*k-means*	2001	1474	1715	1373	1373	1346	1449	1449	1449	1449	1444
	Hierarchical Clustering	2785	2023	4893	4892	4895	4894	4894	4894	4892	4894	4895
Neg.region	$\bigwedge$-Dq-DTRFS	5	173	361	363	363	363	363	363	363	363	363
	$\bigvee$-Dq-DTRFS	0	37	73	76	76	76	76	76	76	76	76
Bad	*k-means*	474	548	548	540	540	425	737	737	737	737	757
	Hierarchical Clustering	1	470	3	4	1	1	2	1	2	1	1

values are normalized to the range [0, 1]. In this section, we will classify some real-life big data sets based on our $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models, using four real-life data sets available from the UCI databases. First we normalized the decision attribute values to the range [0, 1]. Among the four data sets, the Statlog (Landsat Satellite) data have 6435 samples and 37 attributes, the attribute set is relatively large. The Statlog(Shuttle) data have 30,000 samples and 10 attributes, the sample set is relatively large. The characteristics of the data sets are summarized in Table 8.

From the view of pattern recognition, each data set is divided into four patterns. Tables 9–12 show the number of elements in each pattern with respect to different attribute sets using our two $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models and k-means($k = 4$) and hierarchical clustering methods. Our two $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models divide each data set into *positive region*, *upper boundary region*, *lower boundary region* and *negative region* four patterns. While k-means and hierarchical clustering methods divide each data set into *excellent*, *good*, *general* and *bad* four patterns. Figs. 1–3 are obtained based on Tables 9–12.

In Fig. 1a,b,c,d show the number of elements in positive region, upper boundary region, lower boundary region and negative region with respect to different sizes of attribute sets using our $\bigwedge$-Dq-DTRFS model. Similarly, e,f,g,h show the number of elements in each region with respect to different sizes of attribute sets using our $\bigvee$-Dq-DTRFS model. Figs. 2 and 3 show the number of elements in each class with respect to different sizes of attribute sets using k-means and hierarchical

Table 11
Statlog (Landsat Satellite): The number of elements in each pattern with respect to different attribute sets.

Patterns	methods	4	8	12	16	20	24	28	32	36
Pos.region	$\bigwedge$-Dq-DTRFS	864	18	0	0	0	0	0	0	0
	$\bigvee$-Dq-DTRFS	2793	2855	2841	2841	2841	2841	2841	2841	2841
Excellent	*k-means*	1782	1807	1841	1853	1817	1875	1787	1848	1838
	Hierarchical Clustering	1	1	2	1	1	1	1	1	1
Ubn.region	$\bigwedge$-Dq-DTRFS	368	0	0	0	0	0	0	0	0
	$\bigvee$-Dq-DTRFS	521	0	0	0	0	0	0	0	0
Good	*k-means*	1783	1798	1779	1794	1800	1772	1805	1779	1774
	Hierarchical Clustering	1	1	1	1	1	1	1	1	1
Lbn.region	$\bigwedge$-Dq-DTRFS	1443	2209	2215	2215	2215	2215	2215	2215	2215
	$\bigvee$-Dq-DTRFS	2093	3526	3584	3592	3594	3594	3594	3594	3594
General	*k-means*	2276	2236	2209	2184	2213	2184	2236	2199	2216
	Hierarchical Clustering	6432	6432	6431	6432	6432	6432	6432	6432	6432
Neg.region	$\bigwedge$-Dq-DTRFS	3760	4208	4220	4220	4220	4220	4220	4220	4220
	$\bigvee$-Dq-DTRFS	1028	54	10	2	0	0	0	0	0
Bad	*k-means*	594	594	606	604	605	604	607	609	607
	Hierarchical Clustering	1	1	1	1	1	1	1	1	1

Table 12
Statlog (Shuttle): The number of elements in each pattern with respect to different attribute sets.

Patterns	methods	1	2	3	4	5	6	7	8	9
Pos.region	$\bigwedge$-Dq-DTRFS	1702	1531	1629	1635	1834	12	0	0	0
	$\bigvee$-Dq-DTRFS	4775	6203	6270	6116	6376	6376	6376	6376	6376
Excellent	*k-means*	1697	1677	1677	1687	2724	5655	4337	5625	4177
	Hierarchical Clustering	14, 903	23, 566	1	1	1	1	1	1	1
Ubn.region	$\bigwedge$-Dq-DTRFS	6009	5610	4632	3551	1653	0	0	0	0
	$\bigvee$-Dq-DTRFS	16, 747	1633	692	739	0	0	0	0	0
Good	*k-means*	9082	9534	5298	8910	5797	8	8	8	8
	Hierarchical Clustering	1	1	1	1	1	1	1	1	5
Lbn.region	$\bigwedge$-Dq-DTRFS	4	211	482	1465	2828	6303	6315	6315	6315
	$\bigvee$-Dq-DTRFS	6	166	1545	5221	7216	19, 693	22, 455	23, 224	23, 624
General	*k-means*	8355	7731	9348	5265	6041	3	3	3	3
	Hierarchical Clustering	15095	1749	29, 987	29, 997	29, 997	29, 997	29, 997	29, 996	29, 990
Neg.region	$\bigwedge$-Dq-DTRFS	22, 285	22, 648	23, 257	23, 349	23, 685	23, 685	23, 685	23, 685	23, 685
	$\bigvee$-Dq-DTRFS	8472	21, 998	21, 493	17, 924	16, 408	3931	1169	400	0
Bad	*k-means*	10, 866	11, 058	13, 677	14, 138	15, 438	24, 334	25, 652	24, 364	25, 812
	Hierarchical Clustering	1	4684	11	1	1	1	1	2	4

clustering methods respectively. The *X*-axis represents the size of attribute set, from one attribute to all attributes. The *Y*-axis represents the number of elements in each region versus attribute set. Each figure has four lines.

From Fig. 1, we find that the changes on the number of elements in each region are irregular in both $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models. It is because the definitions of upper approximations and lower approximations are irregular in these two models. For example, in the $\bigwedge$-Dq-DTRFS model, the logical conjunction double-quantitative upper and lower approximations of fuzzy set $\widetilde{A}$ based on the relation R is defined as: $\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > max(k, \beta|[x]_R|)\}$, and $\underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A}) = \{x \in U \mid \Sigma_{y\in[x]_R}\widetilde{A}(y) > max(|[x]_R| - k, \alpha|[x]_R|)$. It is obvious that the equivalence classes get larger when the size of attribute sets gets larger. That is to say $[x]_R$ gets larger when the attribute set increases from one attribute to more attributes, $\Sigma_{y\in[x]_R}\widetilde{A}(y)$ and $\beta|[x]_R|$ get larger at the same time. So the size of $\Sigma_{y\in[x]_R}\widetilde{A}(y)$ and $\beta|[x]_R|$ can not be determined even though the value of k is always equal to 2. And then the size of $\overline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A})$ and $\underline{R_{(\alpha,\beta)\wedge k}}(\widetilde{A})$ also can not be determined. So the positive region, negative region, upper boundary region and lower boundary region also can not be determined based on the definition of each region. Since *a*, *b*, *c*, *d* show the number of elements in positive region, upper boundary region, lower boundary region and negative region with respect to different sizes of attribute sets using our $\bigwedge$-Dq-DTRFS model, the changes on the number of elements in each region are irregular in *a*, *b*, *c*, *d*. While *e*, *f*, *g*, *h* show the number of elements in each region with respect to different sizes of attribute sets using our $\bigvee$-Dq-DTRFS model, the changes on the number of elements in each region are also irregular in *e*, *f*, *g*, *h*. Moreover, all the data sets can be classified into different regions based on our $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS models.

For these four methods, *k*-means method randomly chooses *k* points as the initial points, which poses a problem. The problem is that there are different classification results of the same data set and which one to use is difficult to decide. This is the biggest disadvantage of this approach. Hierarchical clustering mainly take into account of class distance and object distance. One should take into account the risk theory in the decision-making process though there are some risks when one makes decisions. While both of these two methods do not consider the factor of the cost or the risk. Our two $\bigwedge$-Dq-

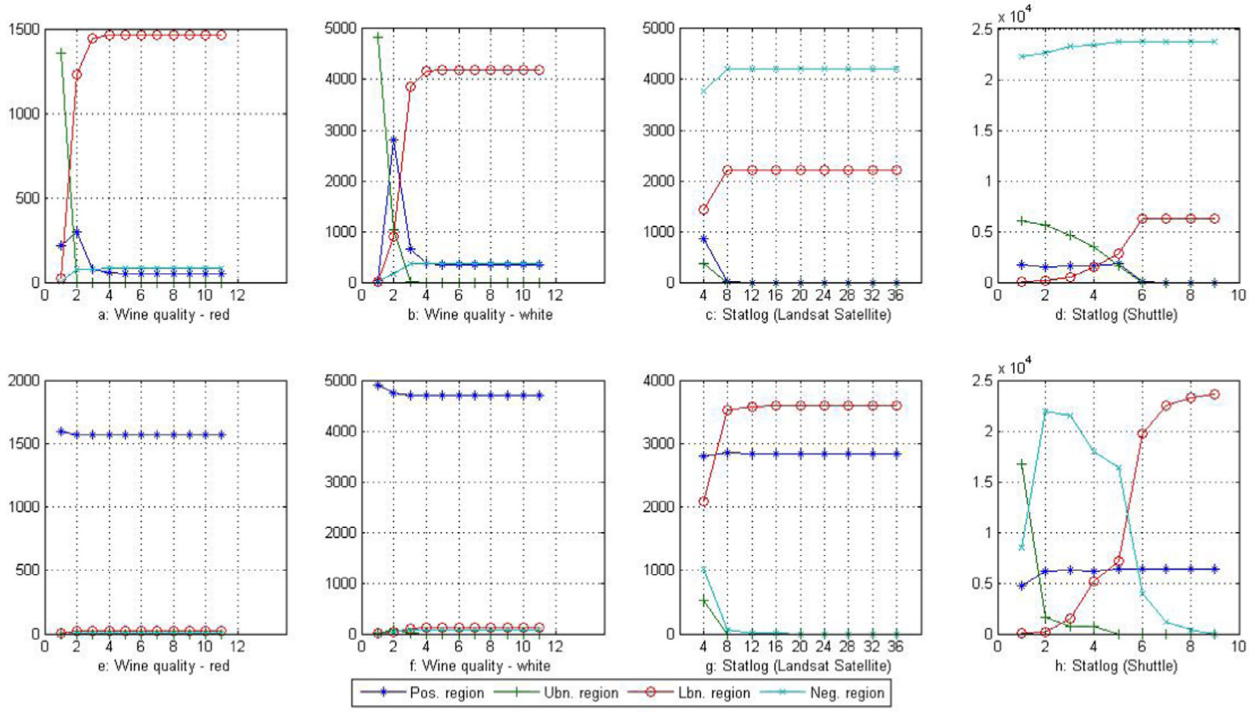

Fig. 1. Dq-DTRFS: The number of elements in each region with respect to different attribute sets.

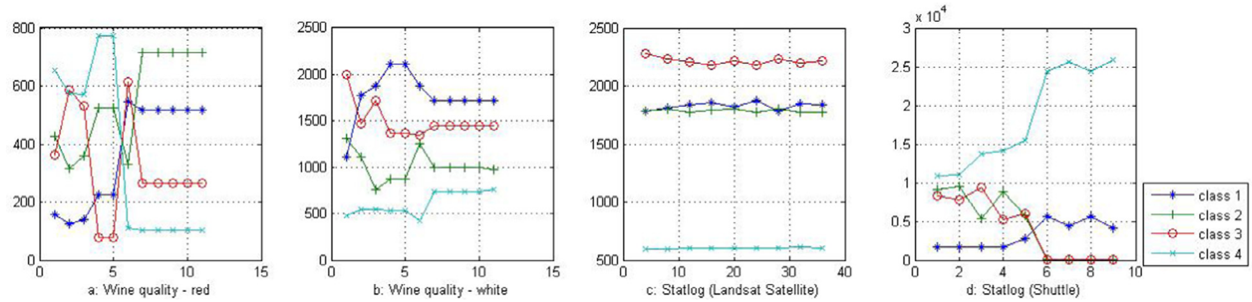

Fig. 2. *k*-means: The number of elements in each class with respect to different attribute sets.

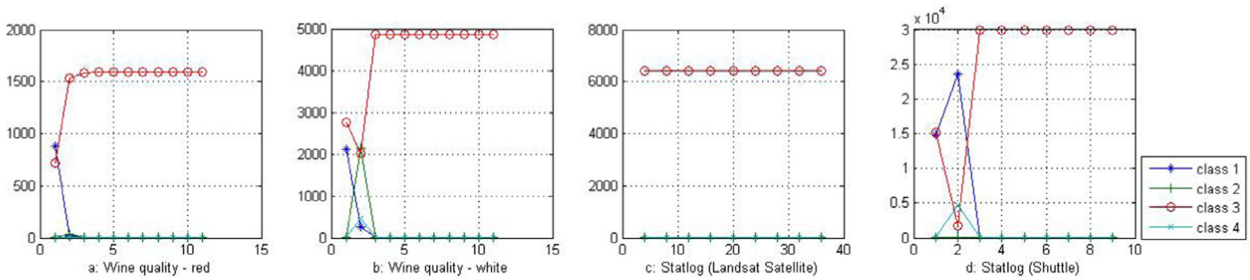

Fig. 3. Hierarchical Clustering: The number of elements in each class with respect to different attribute sets.

DTRFS and $\bigvee$-Dq-DTRFS models not only can divide data set into four patterns, but also can take into account of the risk factor.

7. Conclusions

The relative and absolute quantitative information of the approximate space are two fundamental quantitative indexes, which represent two distinct objective descriptors. The double quantification formed by adding the absolute quantitative information can improve the descriptive abilities of DTRFS model and expand their range of applicability. The proposed models, $\bigwedge$-Dq-DTRFS and $\bigvee$-Dq-DTRFS, perform a basic double quantification of the relative information and absolute information based on logical operation. These new models are directional expansions of Pawlak rough set model, satisfying the quantitative completeness properties, and exhibiting strong double fault tolerance capabilities. This paper mainly investigates double quantification, namely the relative and absolute information by combining DTRFS and GRFS models together with a fuzzy concept. Moreover, after proposing the decision rules containing both relative quantification and absolute quantification in two types of models, the inner relationships between these two models are studied. In this article, we provide an example on medical diagnosis and experiment data sets downloaded from UCI to illustrate and support our proposed models. In future work, several aspects of these two models will be investigated and studied, which include the uncertainty measures and other properties of these models with respect to the concepts and parameters setting.

Acknowledgments

This work is supported by the Macau Science and Technology Development Fund (no. 100/2013/A3 and no. 081/2015/A3), the Natural Science Foundation of China (no. 61472463 and no. 61402064).

References

[1] D.G. Chen, S.Y. Zhao, Local reduction of decision system with fuzzy rough sets, Fuzzy Sets Syst. 161 (13) (2010) 1871–1883.
[2] R.O. Duda, P.E. Hart, Pattern classification and scene analysis, New York: Wiley, 1973.
[3] S. Greco, B. Matarazzo, R. Slowinski, Parameterized rough set model using rough membership and bayesian confirmation measures, Int. J. Approx. Reason. 49 (2) (2008) 285–300.
[4] S. Greco, R. Slowinski, Y.Y. Yao, Bayesian decision theory for dominance-based rough set approach, Rough Sets Knowledge Technology, 2007, pp. 134–141.
[5] J.P. Herbert, J.T. Yao, Game-theoretic rough set, Fundam. Inform. 108 (3-4) (2011) 267–286.
[6] J.P. Herbert, J.T. Yao, Analysis of data-driven parameters in game-theoretic rough sets, Rough Sets Knowledge Technology, 2011, pp. 447–456.
[7] X.Y. Jia, W.W. Li, L. Shang, J.J. Chen, An optimization viewpoint of decision-theoretic rough set model, Rough Sets Knowledge Technology, 2011, pp. 457–465.
[8] H.R. Ju, X.B. Yang, P. Yang, et al., A moderate attribute reduction approach in decision-theoretic rough set, in: 15th International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing (RSFDGrC), vol. 9437, 2015, pp. 376–388.
[9] P. Lingras, M. Chen, D.Q. Miao, Rough multi-category decision theoretic framework, Rough Sets Knowledge Technology, 2008, pp. 676–683.
[10] P. Lingras, M. Chen, D.Q. Miao, Rough cluster quality index based on decision theory, IEEE Trans. Knowl. Data Eng. 21 (7) (2009) 1014–1026.
[11] H.X. Li, X.Z. Zhou, Risk decision making based on decision-theoretic rough set: a three-way view decision model, Int. J. Comput. Intell. Syst. 4 (1) (2011) 1–11.
[12] H.X. Li, X.Z. Zhou, J.B. Zhao, D. Liu, Attribute reduction in decision-theoretic rough set model: a further investigation, Rough Sets Knowledge Technology, 2011, pp. 466–475.
[13] D. Liu, H.X. Li, X.Z. Zhou, Two decades' research on decision-theoretic rough sets, IEEE International Conference on Cognitive Informatics, 2010, pp. 968–973.
[14] D. Liu, T.R. Li, D. Ruan, Probabilistic model criteria with decision-theoretic rough sets, Inf. Sci. 181 (17) (2011) 3709–3722.
[15] D. Liu, Y.Y. Yao, T.R. Li, Three-way investment decisions with decision-theoretic rough sets, Int. J. Comput. Intell. Syst. 4 (1) (2011) 66–74.
[16] W. Li, D.Q. Miao, W.L. Wang, N. Zhang, Hierarchical rough decision theoretic framework for text classification, IEEE International Conference on Cognitive Informatics, 2010, pp. 484–489.
[17] Y.F. Li, C.Q. Zhang, J.R. Swan, Rough set based decision model in information retrieval and filtering, International Conference on Information Systems Analysis and Synthesis, 5, 1999, pp. 398–403.
[18] D.C. Liang, D. Liu, W. Pedrycz, P. Hu, Triangular fuzzy decision-theoretic rough sets, Int. J. Approx. Reason. 54 (8) (2013) 1087–1106.
[19] D.C. Liang, W. Pedrycz, D. Liu, P. Hu, Three-way decisions based on decision-theoretic rough sets under linguistic assessment with the aid of group decision making, Appl. Soft Comput. 29 (2015) 256–269.
[20] C.H. Liu, D.Q. Miao, N. Zhang, Graded rough set model based on two universes and its properties, Knowl. Based Syst. 33 (2012) 65–72.
[21] W.T. Li, W.H. Xu, Double-quantitative decision-theoretic rough set, Inf. Sci. 316 (2015) 54–67.
[22] W.T. Li, W.H. Xu, Multigranulation decision-theoretic rough set in ordered information system, Fundam. Inform. 139 (2015) 67–89.
[23] W.M. Ma, B.Z. Sun, Probabilistic rough set over two universes and rough entropy, Int. J. Approx. Reason. 53 (4) (2012) 608–619.
[24] W.M. Ma, B.Z. Sun, On relationship between probabilistic rough set and bayesian risk decision over two universes, Int. J. Gen. Syst. 41 (3) (2012) 225–245.
[25] J.S. Mi, W.Z. Wu, W.X. Zhang, Approaches to knowledge reduction based on variable precision rough set model, Inf. Sci. 159 (3) (2004) 255–272.
[26] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[27] Z. Pawlak, Rough sets: Theoretical aspects of resoning about data, Kluwer Academic, Boston, 1991.
[28] Z. Pawlak, A. Skowron, Rough membership functions. Advances in the Dempster-Shafer Theory of Evidence, John Wiley and Sons, New York, 1994, pp. 251–271.
[29] W. Pedrycz, Granular computing: Analysis and design of intelligent systems, CRC Press Taylor & Francis Group, 2013.
[30] Y.H. Qian, H. Zhang, Y.L. Sang, J.Y. Liang, Multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 55 (1) (2014) 225–237.
[31] M. Sarkar, Rough-fuzzy functions in classification, Fuzzy Sets Syst. 132 (3) (2002) 353–369.
[32] B.Z. Sun, W.M. Ma, H.Y. Zhao, Decision-theoretic rough fuzzy set model and application, Inf. Sci. 283 (2014) 180–196.
[33] S.P. Wang, Q.X. Zhu, W. Zhu, F. Min, Quantitative analysis for covering-based rough sets through the upper approximation number, Inf. Sci. 220 (2013) 483–491.
[34] S.K.M. Wong, W. Ziarko, INFER: an adaptative decision support system based on the probabilistic approximate classification, International Workshop on Expert Systems and Their Applications, 1987, pp. 713–726.
[35] S.K.M. Wong, W. Ziarko, Comparison of the probabilistic approximate classification and the fuzzy set model, Fuzzy Sets Syst. 21 (3) (1987) 357–362.
[36] W.H. Xu, S.H. Liu, Q.R. Wang, W.X. Zhang, The first type of graded rough set based on rough membership function, Fuzzy Syst. Knowl. Discov. 4 (2010) 1922–1926.
[37] Y.Y. Yao, Three-way decisions with probabilistic rough sets, Inf. Sci. 180 (3) (2010) 341–353.
[38] Y.Y. Yao, The superiority of three-way decisions in probabilistic rough set models, Inf. Sci. 181 (6) (2011) 1080–1096.
[39] Y.Y. Yao, An outline of a theory of three-way decisions, Rough Sets Curr. Trends Comput. 7413 (2012) 1–17.
[40] Y.Y. Yao, Probabilistic approaches to rough sets, Expert Syst. 20 (5) (2003) 287–297.
[41] Y.Y. Yao, T.Y. Lin, Generalization of rough sets using modal logic, Intell. Autom. Soft Comput. 2 (2) (1996) 103–119.
[42] Y.Y. Yao, S.K.M. Wong, A decision theoretic framework for approximating concepts, Int. J. Man Mach. Stud. 37 (6) (1992) 793–809.

[43] Y.Y. Yao, S.K.M. Wong, P.J. Lingras, A decision-theoretic rough set model, International Conference on Methodologies for Intelligent Systems, 5, 1990, pp. 17–24.
[44] X.P. Yang, J.T. Yao, A multi-agent decision-theoretic rough set model, Rough Sets Knowledge Technology, 2010, pp. 711–718.
[45] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (3) (1965) 338–353.
[46] H.R. Zhang, F. Min, Three-way recommender systems based on random forests, Knowl. Based Syst. 91 (2016) 275–286.
[47] X.Y. Zhang, D.Q. Miao, Quantitative information architecture, granular computing and rough set models in the double-quantitative approximation space of precision and grade, Inf. Sci. 268 (2014) 147–168.
[48] X.Y. Zhang, D.Q. Miao, Two basic double-quantitative rough set models of precision and grade and their investigation using granular computing, Int. J. Approx. Reason. 54 (8) (2013) 1130–1148.
[49] X.Y. Zhang, Z.W. Mo, F. Xiong, W. Cheng, Comparative study of variable precision rough set model and graded rough set model, Int. J. Approx. Reason. 53 (1) (2012) 104–116.
[50] X.Y. Zhang, D.Q. Miao, An expanded double-quantitative model regarding probabilities and grades and its hierarchical double-quantitative attribute reduction, Inf. Sci. 299 (2015) 312–336.
[51] X.Y. Zhang, D.Q. Miao, Region-based quantitative and hierarchical attribute reduction in the two-category decision theoretic rough set model, Knowl. Based Syst. 71 (2014) 146–161.
[52] B. Zhou, A new formulation of multi-category decision-theoretic rough sets, Rough Sets Knowledge Technology, 2011, pp. 514–522.