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ORIGINAL ARTICLE

Attributes reduction and rules acquisition in an lattice-valued information system with fuzzy decision

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Abstract In this paper, we aim to investigate lattice-valued information systems with fuzzy decision (LvISFD), where the domain of every condition attribute is a finite lattice. Firstly, we propose the concept of LvISFD by combining dominance relation and lattice structure. Meanwhile, we establish a rough set approach and give a ranking method for all objects in this complex system. Secondly, we address approximation reductions and rules acquisition in LvISFD. Furthermore, an algorithm of the presented reduction approach is constructed. Finally, an illustrative example is given to show the effectiveness of the proposed method, and experiment evaluation is performed by four datasets from UCI. These results of this study will be more valuable to solve practical issues.

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1 Introduction

Rough set theory was proposed by Pawlak in the early 1980s [15, 16], which is based upon the classification mechanism. From granular computing viewpoint, a classification can be viewed as knowledge granules induced by a corresponding equivalence relation. It is a new mathematical approach to deal with uncertain and vague problems. And rough set theory plays an important role in many fields, such as data mining, knowledge discovery, and so on. Lower and upper approximations are two essential operators in rough set theory, because any subset of the given universe can be approximated by using these two operators. It is well-known that not all conditional attributes are necessary to depict the decision attribute in an information system. So, attribute reduction is one of the most important problems in rough set theory. For an information system with discrete attribute values, this can be done by reducing redundant attributes and finding a subset of the original attributes. However, an information system may usually have more than one reduction. This means rules set deriving from attribute reduction is not unique. In practice, it is always hoped to obtain the most concise rules set. Therefore, people have been attempting to find the minimal reduction of an information system, which means that the number of attributes contained in the reduction is minimal. Unfortunately, it has been proven that finding the minimal reduction of an information system is an NP-hard problem. However, many types of attribute reduction have been proposed in the area of rough sets [7, 12, 18, 20, 24]. Possible rules and possible reductions have been proposed as different ways to deal with inconsistence in an inconsistent decision table [8].

Approximation rules [21] are also used as an alternative approach to possible rules. Uncertainty processing plays a key role in learning rules discovery [28, 29]. It is found that, in a knowledge information system, the modeling of fuzziness and roughness can significantly improves the system performance [27, 30, 31].

In real life, due to the existing of uncertainty and complexity, some problems would not be perfectly solved by original rough set theory. Therefore, it is vital to generalize the classical rough set model. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful models in recent years by proposing other binary relations, such as tolerance relations [19], neighborhood operators [35], and so on [9, 16, 17, 23, 25, 26, 32, 34–36, 38, 39]. However, original rough set theory does not consider attributes with preference ordered domain. We often face some problems in which order of properties of the considered attributes plays a crucial role. For this reason, Greco, Matarazzo, and Slowinski [4–6] proposed an extension rough set theory, the dominance-based rough set approach, to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In dominance-based rough set approach condition attributes are criteria, and classes are preference ordered [11]. The approximated knowledge is a collection of upward and downward unions of classes, and dominance classes are sets of objects defined by using a dominance relation [3, 10]. In recent years, several studies have been achieved about properties and algorithmic implementations of dominance-based rough set approach [1, 2, 22, 34, 35]. Nevertheless, only a limited number of methods have been proposed by using dominance-based rough set approach to acquire knowledge from the inconsistent ordered information systems. Pioneering work on inconsistent ordered information systems with the dominance-based rough set approach has been proposed by Greco, Matarazzo, and Slowinski [4–6], but they did not clearly point out the semantic explanation of unknown values. Shao and Zhang [19] further proposed an extension of the dominance relation in an inconsistent ordered information systems.

The purpose of this paper is to address a complex information system. We not only consider dominance relation of condition attributes, but also take account into the lattice structure of attribute values in an information system with fuzzy decision. This system is called lattice-valued information system (LvIS) with fuzzy decision in the study. By discussing some important properties in an LvISFD, methods for attribute reductions and rules acquisition are constructed carefully. The rest of this paper is organized as follows. Some preliminary concepts required in our work are briefly recalled in Sect. 2. In Sect. 3, the definition of LvISFD is proposed and some properties are discussed. In Sect. 4, a rank approach with dominance classes is considered by proposing dominance degree in an LvISFD. In Sect. 5, approaches to approximation reduction are constructed in an LvISFD, and dominance rules acquisition are also discussed in this system. In Sect. 6, the algorithm analysis to attribute reduction is developed in an LvISFD, and a small dataset about the fund investment by persons and four real-life datasets from the UCI are calculated by the C++ computer program. Through the fund investment issue, it is illustrated that how to make a decision by using the proposed approach. Finally, we conclude the paper with a summary.

2 Preliminaries

In this section, we make a brief overview of some necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in the source papers [3–6, 10].

Let U be a finite and non-empty set called the universe. A fuzzy set X is a mapping from U into the interval [0, 1]: $\mu_X : U \mapsto [0,1]$, where for each $u \in U$ we call the membership degree of u in X [37]. The fuzzy power set, i.e., the set of all fuzzy sets in the universe U is denoted $F(U)$.

An information system is a quadruple $\mathcal{I} = (U, AT, V, f)$, where U is a non-empty finite set with n objects, $\{u_1, u_2, \ldots, u_n\}$, called the universe of discourse; $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with m attributes; $V = \bigcup_{a \in AT} V_a$ and V_a is the domain of attribute a; $f : U \times AT \longrightarrow V$ is an information function such that $f(u, a) \in V_a$ for any $u \in U$. An information system with fuzzy decision is a special case of an information system $\mathcal{I} = (U, AT \cup \{d\}, V, f)$, where A is called conditional attribute set, and d is a fuzzy decision of $\mathcal{I}$. In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criteria.

Definition 2.1 [4–6] An information system is called an ordered information system if all condition attributes are criteria.

Assumed that the domain of the criteria $a \in AT$ is completely pre-ordered by an outranking relation $\succcurlyeq_a$, then $u \succcurlyeq_a v$ means that u is at least as good as (outranks) v with respect to a, and we say that u dominates v or v is dominated by u. Being of type gain, that is, $u \succcurlyeq_a v \Longleftrightarrow f(u, a) \geq f(v, a)$ (according to increasing preference) or $u \succcurlyeq_a v \Longleftrightarrow f(u, a) \leq f(v, a)$ (according to decreasing preference). Without any loss of generality, we only consider attributes with increasing preference [3, 10].

Definition 2.2 Let $\mathcal{I}_d = (U, AT \bigcup \{d\}, V, f)$ be an information system with fuzzy decision and $A \subseteq AT$. Given

$$R_A^{\succcurlyeq} = \{(u,v) \mid f(u,a) \geq f(v,a)\ \forall u,v \in U, a \in A)\}. \quad (1)$$

then $R_A^{\succcurlyeq}$ is called the dominance relation with respect to condition attributes set A, and in which case the information system $\mathcal{I}_d = (U, AT \bigcup \{d\}, V, f)$ is called an ordered information system with fuzzy decision and denoted by $\mathcal{I}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$, or $\mathcal{I}_d^{\succcurlyeq}$ for simplicity. Let us denote

$$[u_i]_A^{\succcurlyeq} = \{u_j \mid f(u_j,a) \geq f(u_i,a), \forall a \in A\},$$
$$U/R_A^{\succcurlyeq} = \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \ldots, [u_n]_A^{\succcurlyeq}\},$$

where $i \in \{1,2,\ldots,n\}$, then $[u_i]_A^{\succcurlyeq}$ will be called the dominance class of $u_i \in U$ and $U/R_A^{\succcurlyeq}$ be cover of U with respect to condition attributes set A, respectively.

Proposition 2.1 [4–6] *Let $\mathcal{I}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an ordered information system with fuzzy decision and $B, A \subseteq AT$, then we have that*

1. *$R_{AT}^{\succcurlyeq}$ is reflective, transitive, but not symmetric, so it is not an equivalence relation;*
2. If *$B \subseteq A \subseteq AT$, then $R_{AT}^{\succcurlyeq} \subseteq R_A^{\succcurlyeq} \subseteq R_B^{\succcurlyeq}$.*

Similarly, for the dominance class induced by dominance relation $R_A^{\succcurlyeq}$, the following properties are still true.

Proposition 2.2 [4–6] Let *$\mathcal{I}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an ordered information system with fuzzy decision and $B, A \subseteq AT$, then we have that*

1. If *$B \subseteq A \subseteq AT$, then $[u]_{AT}^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq} \subseteq [u]_B^{\succcurlyeq}$ for any $u \in U$;*
2. *If $v \in [u]_A^{\succcurlyeq}$, then $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ and $[u]_A^{\succcurlyeq} = \bigcup\{[v]_A^{\succcurlyeq} \mid v \in [u]_A^{\succcurlyeq}\}$;*
3. *$[u]_{AT}^{\succcurlyeq} = [v]_{AT}^{\succcurlyeq}$ if and only if $f(u,a) = f(v,a)$* for any *$a \in AT$;*
4. *$|[u]_{AT}^{\succcurlyeq}| \geq 1$ for any $u \in U$,*

where |X| denotes the cardinality of the set X.

For any subset *$X \subseteq U$ and $A \subseteq AT$ in $\mathcal{I}^{\succcurlyeq}$, if we denote*

$$\underline{R_A^{\succcurlyeq}}(X) = \{u_i \mid [u_i]_A^{\succcurlyeq} \subseteq X, u_i \in U\}, \quad (2)$$

$$\overline{R_A^{\succcurlyeq}}(X) = \{u_i \mid [u_i]_A^{\succcurlyeq} \cap X \neq \emptyset, u_i \in U\}, \quad (3)$$

then $\underline{R_A^{\succcurlyeq}}(X)$ and $\overline{R_A^{\succcurlyeq}}(X)$ are the lower and upper approximation of X with respect to $R_A^{\succcurlyeq}$, respectively.

3 Lattice-valued information system with fuzzy decision (LvISFD)

In this section, we first propose the concept of lattice-valued information system with fuzzy decision (LvISFD), which is an extension of lattice-valued information system [13, 14, 33] . Moreover, lower/upper approximations and some important properties are considered in lattice-valued information system with fuzzy decision.

Definition 3.1 An LvISFD is an information system $\mathcal{L}_d = (U, AT \bigcup \{d\}, V, f)$, where

1. $U = \{u_1, u_2, \ldots, u_n\}$ is a non-empty finite set with n objects, called the universe of discourse;
2. $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with m condition attributes and $\{d\}$ is the decision attributes set;
3. $V = \bigcup_{a \in AT} V_a$ and V_a is the domain of attribute a such that $(V_a, \succcurlyeq_a)$ is a finite lattice;
4. $d = \{(u, d(u)) \mid u \in U, d(u) \in I\ ([0,1])\}$ is fuzzy decision attribute. The decision can be divided into many classes as follows

$$D_i = \{u | a_i \leq f(x) \leq a_{i+1}, a_i \in [0,1]\};$$

5. $f : U \times AT \longrightarrow V$ is an information function such that for any $u \in U, f(u,a) \in V_a$ when $a \in AT$.

From above definition, we can find that the domain of every condition attribute can be ordered according to a decreasing or increasing preference, that is, every attribute is a criteria. Thus, LvISFD is a kind of ordered information system. In general, it can be denoted by $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$, or $\mathcal{L}_d^{\succcurlyeq}$ for simplicity.

In the following, just like the description of dominance relation in Sect. 2, the dominance relation in lattice-valued information systems with fuzzy decision can de redefined as follows.

Definition 3.2 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD and $A \subseteq AT$. Given

$$\mathcal{R}_A^{\succcurlyeq} = \{(u,v) \mid f(u,a) \succcurlyeq_a f(v,a),\ \forall a \in A\} \quad (4)$$

then $\mathcal{R}_A^{\succcurlyeq}$ is called the dominance relations with respect to A.

Let us denote

$$[u_i]_A^{\succcurlyeq} = \{u_j \mid f(u_j,a) \succcurlyeq_a f(u_i,a),\ \forall a \in A\},$$
$$U/\mathcal{R}_A^{\succcurlyeq} = \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \ldots, [u_n]_A^{\succcurlyeq}\},$$

where $i \in \{1,2,\ldots,n\}$, then $[u_i]_A^{\succcurlyeq}$ will be called a dominance class and $U/\mathcal{R}_A^{\succcurlyeq}$ be a cover of U with respect to A in LvISFD. We use $\mathcal{R}_a^{\succcurlyeq}$ instead of $\mathcal{R}_{\{a\}}^{\succcurlyeq}$ and $[u_i]_a^{\succcurlyeq}$ instead of $[u_i]_{\{a\}}^{\succcurlyeq}$ for any $a \in AT$.

Example 3.1 Consider an lattice-valued information system with fuzzy decision in Table 1, where $U = \{u_1, u_2, \ldots, u_6\}$ and $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

According to above expression, we can find $V_{a_1} = \{1, 2, 3\}$ is a finite lattice with real numbers, where the

Table 1 A LvISFD

U	a_1	a_2	a_3	a_4	a_5	d
u_1	2	0.6	[0.3, 0.8]	$\{0,1,2\}$	(0.8, 0.1)	0.9
u_2	3	0.7	[0.1, 0.5]	$\{0,1,2\}$	(0.0, 1.0)	0.2
u_3	2	0.6	[0.3, 0.8]	$\{0\}$	(0.5, 0.3)	0.7
u_4	2	0.7	[0.3, 0.8]	$\{0\}$	(0.3, 0.6)	1
u_5	1	0.6	[0.4, 0.9]	$\{0,1,2\}$	(0.5, 0.3)	0.2
u_6	1	0.6	[0.2, 0.6]	$\{0,1\}$	(0.3, 0.6)	0.3

partial order relation on V_{a_1} is "$\geq$" between two real numbers. So a dominance relation on U according to attribute a_1 can be defined as

$$\mathcal{R}^{\succcurlyeq}_{a_1} = \{(u,v) \mid f(u,a_1) \geq f(v,a_1)\}.$$

The domain $V_{a_2} = \{0.6, 0.7\}$ is a finite lattice with fuzzy elements where the partial order relation on V_{a_2} is "$\geq$" between two fuzzy elements. And a dominance relation on U according to attribute a_2 can be defined as

$$\mathcal{R}^{\succcurlyeq}_{a_2} = \{(u,v) \mid f(u,a_2) \geq f(v,a_2)\}.$$

The domain $V_{a_3} = \{[0.1,0.5],[0.3,0.8],[0.2,0.6],[0.4,0.9]\}$ is a finite lattice with interval-valued elements, and a dominance relation on it can be defined as

$$\mathcal{R}^{\succcurlyeq}_{a_3} = \{(u,v) \mid f^{\pm}(u,a_3) \geq f^{\pm}(v,a_3)\},$$

where $f^{\pm}(u,a_3) \geq f^{\pm}(v,a_3)$ if and only if $f^{+}(u,a_3) \geq f^{+}(v,a_3)$ and $f^{-}(u,a_3) \geq f^{-}(v,a_3)$, $f^{+}(u,a_3)$ is the right endpoint of $f(u,a_3)$ and $f^{-}(u,a_3)$ is the left endpoint of $f(u,a_3)$, to name a couple for explanation.

The domain $V_{a_4} = \{\{0\},\{0,1\},\{0,1,2\}\}$ is a finite lattice with set-valued elements, where the partial order relation on V_{a_4} is "$\supseteq$" between two sets. Thus a dominance relation on U according to attribute a_4 can be defined as

$$\mathcal{R}^{\succcurlyeq}_{a_4} = \{(u,v) \mid f(u,a_4) \supseteq f(v,a_4)\}.$$

The domain $V_{a_5} = \{(0,1),(0.3,0.6),(0.5,0.3),(0.8,0.1)\}$ is a finite lattice, every element of which is a classical intuitionistic fuzzy set. Thus a dominance relation on U according to attribute a_5 can be defined as

$$\mathcal{R}^{\succcurlyeq}_{a_5} = \{(u,v) \mid \mu_{a_5}(u) \geq \mu_{a_5}(v) \text{ and } \nu_{a_5}(\mathrm{u}) \leq \nu_{a_5}(\mathrm{v})\}.$$

The decision $d = \{0.9, 0.2, 0.7, 1, 0.2, 0.3\}$ is a fuzzy set. If the membership degree $d(u_i) \geq 0.7$, then the decision is called high decision. Otherwise, it is called lower decision. So, the decision can be divided into two decision classes "High" and "low", where $D = \{\mathrm{High}, \mathrm{Low}\} = \{\{u_1,u_3,u_4\},\{u_2,u_5,u_6\}\}$. We can also get the dominance relation classes as following:

$$[u_1]^{\succcurlyeq}_{AT} = \{u_1\},\ [u_2]^{\succcurlyeq}_{AT} = \{u_2\},\ [u_3]^{\succcurlyeq}_{AT} = \{u_1,u_3\},$$
$$[u_4]^{\succcurlyeq}_{AT} = \{u_4\},\ [u_5]^{\succcurlyeq}_{AT} = \{u_5\},\ [u_6]^{\succcurlyeq}_{AT} = \{u_1,u_5,u_6\}.$$

Definition 3.3 Let $\mathcal{L}^{\succcurlyeq}_d = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $B, A \subseteq AT$.

1. If $[u]^{\succcurlyeq}_B = [u]^{\succcurlyeq}_A$ for all $u \in U$, then we have that cover $U/\mathcal{R}^{\succcurlyeq}_B$ is equal to $U/\mathcal{R}^{\succcurlyeq}_A$, denoted by $U/\mathcal{R}^{\succcurlyeq}_B = U/\mathcal{R}^{\succcurlyeq}_A$.
2. If $[u]^{\succcurlyeq}_B \subseteq [u]^{\succcurlyeq}_A$ for all $u \in U$, then we have that cover $U/\mathcal{R}^{\succcurlyeq}_B$ is finer than $U/\mathcal{R}^{\succcurlyeq}_A$, denoted by $U/\mathcal{R}^{\succcurlyeq}_B \subseteq U/\mathcal{R}^{\succcurlyeq}_A$.
3. If $[u]^{\succcurlyeq}_B \subseteq [u]^{\succcurlyeq}_A$ for all $u \in U$ and $[v]^{\succcurlyeq}_B \neq [v]^{\succcurlyeq}_A$ for some $v \in U$, then we have that cover $U/\mathcal{R}^{\succcurlyeq}_B$ is proper finer than $U/\mathcal{R}^{\succcurlyeq}_A$, denoted by $U/\mathcal{R}^{\succcurlyeq}_B \subset U/\mathcal{R}^{\succcurlyeq}_A$.

From the definition of $\mathcal{R}^{\succcurlyeq}_A$ and $[u]^{\succcurlyeq}_A$, the following properties can be obtained directly.

Proposition 3.1 *Let $\mathcal{L}^{\succcurlyeq}_d = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $B, A \subseteq AT$, then we can get*

1. $\mathcal{R}^{\succcurlyeq}_A = \bigcap_{a \in A} \mathcal{R}^{\succcurlyeq}_a$;
2. *$\mathcal{R}^{\succcurlyeq}_A$ is reflective, transitive, but not symmetric, so it is not an equivalence relation;*
3. *If $B \subseteq A \subseteq AT$, then $\mathcal{R}^{\succcurlyeq}_{AT} \subseteq \mathcal{R}^{\succcurlyeq}_A \subseteq \mathcal{R}^{\succcurlyeq}_B$.*

Proposition 3.2 *Let $\mathcal{L}^{\succcurlyeq}_d = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $B, A \subseteq AT$, then we have that*

1. *If $B \subseteq A \subseteq AT$, then $[u]^{\succcurlyeq}_{AT} \subseteq [u]^{\succcurlyeq}_A \subseteq [u]^{\succcurlyeq}_B$ for all $u \in U$.*
2. *If $u \in [v]^{\succcurlyeq}_A$, then $[u]^{\succcurlyeq}_A \subseteq [v]^{\succcurlyeq}_A$ and $[v]^{\succcurlyeq}_A = \bigcup\{[u]^{\succcurlyeq}_A \mid u \in [v]^{\succcurlyeq}_A\}$.*
3. *$[u]^{\succcurlyeq}_{AT} = [v]^{\succcurlyeq}_{AT}$ if and only if $f(u,a) = f(v,a)$ for all $a \in AT$.*
4. *$|[u]^{\succcurlyeq}_{AT}| \geq 1$ for all $u \in U$.*

In the following, we will investigate the problem of approximation operators with respect to $\mathcal{R}^{\succcurlyeq}_A$ in lattice-valued information systems with fuzzy decision.

Definition 3.4 Let $\mathcal{L}^{\succcurlyeq}_d = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD and $A \subseteq AT$. The lower and upper approximation operators of $\{d\}$ with respect to A is denoted by $\underline{A^{\succcurlyeq}_d}$ and $\overline{A^{\succcurlyeq}_d}$, respectively. And their membership functions are defined by

$$\underline{A^{\succcurlyeq}_d}(u) = \bigwedge\{f(v,d) \mid v \in [u]^{\succcurlyeq}_A\}, \tag{5}$$

$$\overline{A^{\succcurlyeq}_d}(u) = \bigvee\{f(v,d) \mid v \in [u]^{\succcurlyeq}_A\}. \tag{6}$$

From above definition, one can easily obtain the following results.

Proposition 3.3 *Let* $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup\{d\}, V, f)$ *be an LvISFD, and* $B, A \subseteq AT$, *then we have that*

1. *If* $B \subseteq A$, *then* $\underline{B_d^{\succcurlyeq}} \subseteq \underline{A_d^{\succcurlyeq}}$ *and* $\overline{A_d^{\succcurlyeq}} \subseteq \overline{B_d^{\succcurlyeq}}$.
2. *If* $\mathcal{R}_A^{\succcurlyeq} = \mathcal{R}_B^{\succcurlyeq}$, *then* $\underline{B_d^{\succcurlyeq}} = \underline{A_d^{\succcurlyeq}}$ *and* $\overline{A_d^{\succcurlyeq}} = \overline{B_d^{\succcurlyeq}}$.

These properties can be illustrated through the following example.

Example 3.2 (Continued from Example 3.1) Suppose $A = \{a_1, a_2, a_3\}$, we can obtain

$[u_1]_A^{\succcurlyeq} = \{u_1, u_3, u_4\}$,
$[u_2]_A^{\succcurlyeq} = \{u_2\}$,
$[u_3]_A^{\succcurlyeq} = \{u_1, u_3, u_4\}$,
$[u_4]_A^{\succcurlyeq} = \{u_4\}$,
$[u_5]_A^{\succcurlyeq} = \{u_5\}$,
$[u_6]_A^{\succcurlyeq} = \{u_1, u_3, u_4, u_5, u_6\}$.

Obviously, $U/\mathcal{R}_A^{\succcurlyeq}$ is a covering of U and $U/\mathcal{R}_C^{\succcurlyeq} \subset U/\mathcal{R}_A^{\succcurlyeq}$.

Moreover, one can obtain that

$$\underline{A_d^{\succcurlyeq}} = \frac{0.7}{u_1} + \frac{0.2}{u_2} + \frac{0.7}{u_3} + \frac{1}{u_4} + \frac{0.2}{u_5} + \frac{0.2}{u_6},$$
$$\overline{A_d^{\succcurlyeq}} = \frac{1}{u_1} + \frac{0.2}{u_2} + \frac{1}{u_3} + \frac{1}{u_4} + \frac{0.2}{u_5} + \frac{1}{u_6};$$

and

$$\underline{AT_d^{\succcurlyeq}} = \frac{0.9}{u_1} + \frac{0.2}{u_2} + \frac{0.7}{u_3} + \frac{1}{u_4} + \frac{0.2}{u_5} + \frac{0.2}{u_6},$$
$$\overline{AT_d^{\succcurlyeq}} = \frac{0.9}{u_1} + \frac{0.2}{u_2} + \frac{0.9}{u_3} + \frac{1}{u_4} + \frac{0.2}{u_5} + \frac{0.9}{u_6}.$$

Thus, we have that $\underline{A_d^{\succcurlyeq}} \subseteq \underline{C_d^{\succcurlyeq}}$ and $\overline{C_d^{\succcurlyeq}} \subseteq \overline{A_d^{\succcurlyeq}}$.

4 Ranking for objects in LvISFD

In general, there are two classes of problems in intelligent decision-making. One is to find satisfactory results through ranking with information aggregation. And the other is to find dominance rules through relations. In this section, we mainly investigate that how to rank all objects by the dominance relation in lattice-valued information systems.

Definition 4.1 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup\{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$. Dominance degree between two objects $u_i, u_j \in U$ with respect to the dominance relation $R_A^{\succeq}$ is defined as

$$d_A(u_i, u_j) = 1 - \frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq})|}{|U|}. \tag{7}$$

We say that dominance degree of u_i to u_j is $d_A(u_i, u_j)$.

From the definition, the dominance degree $d_A(u_i, u_j)$ depicts the proportion of some objects which are as least as good as u_j in dominance class $[u_i]_A^{\succcurlyeq}$. Moreover, we can obtain the following properties.

Proposition 4.1 *Let* $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup\{d\}, V, f)$ *be an LvISFD,* $A \subseteq AT$ *and dominance degree between two objects* u_j *and* u_i *be* $d_A(u_i, u_j)$ *with respect to the dominance relation* $R_A^{\succcurlyeq}$, *then the following hold.*

1. $0 \le d_A(u_i, u_j) \le 1$ *and* $d_A(u_i, u_i) = 1$.
2. *If* $u_i \in [u_j]_A^{\succcurlyeq}$, *then* $d_A(u_i, u_j) = 1$.
3. *If* $u_j \in [u_k]_A^{\succcurlyeq}$, *then* $d_A(u_i, u_j) \le d_A(u_i, u_k)$.
4. *If* $u_j \in [u_k]_A^{\succcurlyeq}$ *and* $u_k \in [u_i]_A^{\succcurlyeq}$ $d_A(u_i, u_j) \le d_A(u_k, u_j)$ *and* $d_A(u_i, u_j) \le d_A(u_i, u_k)$.

Proof

1. is directly obtained from the definition.
2. Since $u_i \in [u_j]_A^{\succcurlyeq}$, one can have $[u_i]_A^{\succcurlyeq} \subseteq [u_j]_A^{\succcurlyeq}$ by Proposition 3.2. So, we have $[u_i]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq}) = \emptyset$. That is to say
$$d_A(u_i, u_j) = 1 - \frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq})|}{|U|} = 1.$$
3. If $u_j \in [u_k]_A^{\succcurlyeq}$, then we can obtain $[u_j]_A^{\succcurlyeq} \subseteq [u_k]_A^{\succcurlyeq}$. So we have $(\sim [u_j]_A^{\succcurlyeq}) \supseteq (\sim [u_k]_A^{\succcurlyeq})$. Thus
$$\frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq})|}{|U|} \ge \frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_k]_A^{\succcurlyeq})|}{|U|}.$$
Then we can get $d_A(u_i, u_j) \le d_A(u_i, u_k)$.
4. If $u_j \in [u_k]_A^{\succcurlyeq}$ and $u_k \in [u_i]_A^{\succcurlyeq}$, then we can obtain $[u_j]_A^{\succcurlyeq} \subseteq [u_k]_A^{\succcurlyeq} \subseteq [u_i]_A^{\succcurlyeq}$. That is $(\sim [u_j]_A^{\succcurlyeq}) \supseteq (\sim [u_k]_A^{\succcurlyeq}) \supseteq (\sim [u_i]_A^{\succcurlyeq})$ hold. So we have
$$\frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq})|}{|U|} \ge \frac{|[u_k]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq})|}{|U|},$$
and
$$\frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_j]_A^{\succcurlyeq})|}{|U|} \ge \frac{|[u_i]_A^{\succcurlyeq} \cap (\sim [u_k]_A^{\succcurlyeq})|}{|U|}.$$

Then we can get $d_A(u_i, u_j) \le d_A(u_k, u_j)$, $d_A(u_i, u_j) \le d_A(u_i, u_k)$. □

Definition 4.2 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup\{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$. Denote

$$M_A^{\succcurlyeq} = (r_{ij})_{|U| \times |U|}, \text{ where } \mathrm{r_{ij}} = \mathrm{d_A(u_i, u_j)}. \tag{8}$$

Then, we call the matrix $M_A^{\succcurlyeq}$ to be a dominance matrix with respect to A induced by the intuitionistic fuzzy dominance relation $R_A^{\succcurlyeq}$. Moreover, if denote

$$d_A(u_i) = \frac{1}{|U|}\sum_{u_j \in U} d_A(u_i, u_j), \tag{9}$$

then we call $d_A(u_i)$ to be dominance degree of u_i with respect to relation $R_A^{\succcurlyeq}$, for every $u_i \in U$.

By definition of dominance matrix and dominance degree of the object with respect to relation $R_A^{\succcurlyeq}$, we can directly receive the following properties. For all $u_i \in U$, the degree can be calculated according to the following formula

$$d_A(u_i) = \frac{1}{|U|}\sum_{j=1}^{|U|} r_{ij}. \tag{10}$$

As a result of the above discussions, we come to the following corollary.

Corollary 4.1 *Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$. If $R_A^{\succcurlyeq} = R_{AT}^{\succcurlyeq}$, then $d_A(u_i, u_j) = d_{AT}(u_i, u_j)$, $d_A(u_i) = d_{AT}(u_i)$ and $M_A^{\succcurlyeq} = M_{AT}^{\succcurlyeq}$, for $u_i, u_j \in U$.*

From the dominance degree of each object on the universe, we can rank all objects according to the number of d_A. A larger number implies a better object. This idea can be understood by the following example.

Example 4.1 (Continued From Example 3.1) Rank all objects in U according to the dominance relation $R_{AT}^{\succcurlyeq}$ in the system of Example 3.1. By Example 3.1, we can easily obtain the dominance degree of two objects and dominance matrix in the system as follows:

$$M_{AT}^{\succcurlyeq} = \begin{bmatrix} 1 & 5/6 & 1 & 5/6 & 5/6 & 1 \\ 5/6 & 1 & 5/6 & 5/6 & 5/6 & 5/6 \\ 5/6 & 4/6 & 1 & 4/6 & 4/6 & 5/6 \\ 5/6 & 5/6 & 5/6 & 1 & 5/6 & 5/6 \\ 5/6 & 5/6 & 5/6 & 5/6 & 1 & 1 \\ 4/6 & 3/6 & 4/6 & 3/6 & 4/6 & 1 \end{bmatrix}.$$

So, we can have

$$d_{AT}(u_1) = 0.92, \quad d_{AT}(u_2) = 0.86, \quad d_{AT}(u_3) = 0.78,$$
$$d_{AT}(u_4) = 0.86, \quad d_{AT}(u_5) = 0.89, \quad d_{AT}(u_6) = 0.67.$$

Therefore, we rank all objects in the following:

$$u_1 \succeq u_5 \succeq u_2 = u_4 \succeq u_3 \succeq u_6.$$

5 Approximation reductions and rules acquisition in an LvISFD

The approximation reduction is an important attribute reduction, which can be used to simplify an inconsistent classical decision table, and extract more briefer rules. However, there is not any practical approach to attribute reduction in lattice-valued information systems with fuzzy decision. In this section, we present the notions of lower approximation reduction and upper approximation reductions in lattice-valued information systems with fuzzy decision, and then we develop the method based on discernibility matrix to compute all approximation approximation reductions. Moreover, we investigate rules acquisition of lattice-valued information systems with fuzzy decision.

Definition 5.1 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$.

1. If $\overline{A_d^{\succcurlyeq}} = \overline{AT_d^{\succcurlyeq}}$, then A is referred to as an upper approximation consistent set of $\mathcal{L}_d^{\succcurlyeq}$. Moreover, if A is an upper approximation consistent set and $\overline{B_d^{\succcurlyeq}} \neq \overline{AT_d^{\succcurlyeq}}$ for any $B \subset A$, then A is referred to as an upper approximation reduction of $\mathcal{L}_d^{\succcurlyeq}$.
2. If $\underline{A_d^{\succcurlyeq}} = \underline{AT_d^{\succcurlyeq}}$, then A is referred to as a lower approximation consistent set of $\mathcal{L}_d^{\succcurlyeq}$. Moreover, if A is a lower approximation consistent set and $\underline{B_d^{\succcurlyeq}} \neq \underline{AT_d^{\succcurlyeq}}$ for any $B \subset A$, then A is referred to as a lower approximation reduction of $\mathcal{L}_d^{\succcurlyeq}$.

Proposition 5.1 *Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$. Then A is an upper approximation consistent set if and only if there exists $a_k \in A$ such that $f(u, a_k) \not\leq f(v, a_k)$ when $\overline{AT_d^{\succcurlyeq}}(u) < \overline{AT_d^{\succcurlyeq}}(v)$ for $u, v \in U$.*

Proof "$\Longrightarrow$": Suppose $f(u, a_k) \leq f(v, a_k)$ for any $a_k \in A$ when $\overline{AT_d^{\succcurlyeq}}(v) < \overline{AT_d^{\succcurlyeq}}(u)$, then we have that $v \in [u]_A^{\succcurlyeq}$. According to the Proposition 3.2, one can get that $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$. By Definition 3.4, $\overline{A_d^{\succcurlyeq}}(v) \leq \overline{A_d^{\succcurlyeq}}(u)$ is true. With the condition that A is an upper approximation consistent set, we have that $\overline{A_d^{\succcurlyeq}}(u) = \overline{AT_d^{\succcurlyeq}}(u)$ and $\overline{A_d^{\succcurlyeq}}(v) = \overline{AT_d^{\succcurlyeq}}(v)$. That is, $\overline{AT_d^{\succcurlyeq}}(u) \geq \overline{AT_d^{\succcurlyeq}}(v)$, which is a contradiction with $\overline{AT_d^{\succcurlyeq}}(u) < \overline{AT_d^{\succcurlyeq}}(v)$. Therefore, there exists $a_k \in A$ s.t. $f(u, a_k) \not\leq f(v, a_k)$ when $\overline{AT_d^{\succcurlyeq}}(u) < \overline{AT_d^{\succcurlyeq}}(v)$ for any $u, v \in U$.

"$\Longleftarrow$": Suppose A is not an upper approximation consistent set, that is, $\overline{A_d^{\succcurlyeq}} \neq \overline{AT_d^{\succcurlyeq}}$. Thus, there exists $u_0 \in U$ s.t. $\overline{A_d^{\succcurlyeq}}(u_0) \neq \overline{AT_d^{\succcurlyeq}}(u_0)$. By Definition 3.4 and Proposition 3.3, we have that $\overline{A_d^{\succcurlyeq}}(u_0) > \overline{AT_d^{\succcurlyeq}}(u_0)$. Moreover, let $v_0 \in [u_0]_A^{\succcurlyeq}$ s.t. $\overline{A_d^{\succcurlyeq}}(u_0) = f(v_0, d)$. And we have that $v_0 \in [v_0]_C^{\succcurlyeq}$, so one can obtain that $\max\{f(u,d) \mid u \in [v_0]_C^{\succcurlyeq}\} \geq f(v_0, d)$. Hence, $\overline{AT_d^{\succcurlyeq}}(v_0) > \overline{AT_d^{\succcurlyeq}}(u_0)$. So, there exists $a_k \in A$ s.t. $f(u_0, a_k) \not\leq f(v_0, a_k)$, that is, $v_0 \notin [u_0]_A^{\succcurlyeq}$. And this is a contradiction with $v_0 \in [u_0]_A^{\succcurlyeq}$. $\square$

Proposition 5.2 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$. Then A is a lower approximation consistent set if and only if there exists $a_k \in A$ s.t. $f(u, a_k) \not\geq f(v, a_k)$ when $\underline{AT_d^{\succcurlyeq}}(u) < \underline{AT_d^{\succcurlyeq}}(v)$ for any $u, v \in U$.

Proof It is similar to the proof of Proposition 5.1. □

From above, one can get that Propositions 5.1 and 5.2 are just an equivalent description of the upper and lower approximation consistent sets respectively. To realize the purpose of obtaining the approximation reduction in an LvISFD, the notion of discernibility matrix will be proposed and, then, the detailed methods for researching upper and lower approximation reductions are constructed.

Let us take

$$\mathcal{D}_f^+ = \{(u, v) \mid \overline{AT_d^{\succcurlyeq}}(u) < \overline{AT_d^{\succcurlyeq}}(v),\ u, v \in U\}, \tag{11}$$

$$\mathcal{D}_f^- = \{(u, v) \mid \underline{AT_d^{\succcurlyeq}}(u) < \underline{AT_d^{\succcurlyeq}}(v),\ u, v \in U\}, \tag{12}$$

and then the notion of discernibility matrix can be defined as follows.

Definition 5.2 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD. For any $u, v \in U$, if we denote

$$\mathcal{D}_f^+(u, v) = \begin{cases} \{a \mid f(u, a) \not\leq f(v, a),\ a \in AT\} & (u, v) \in \mathcal{D}_f^+ \\ \emptyset & (u, v) \notin \mathcal{D}_f^+ \end{cases} \tag{13}$$

and

$$\mathcal{D}_f^-(u, v) = \begin{cases} \{a \mid f(u, a) \not\geq f(v, a),\ a \in AT\} & (u, v) \in \mathcal{D}_f^- \\ \emptyset & (u, v) \notin \mathcal{D}_f^- \end{cases}, \tag{14}$$

then we call $\mathcal{D}_f^+(u, v)$ an upper approximation discernibility attributes set and $\mathcal{D}_f^-(u, v)$ a lower approximation discernibility attributes set between u and v, respectively.

The matrix $\mathcal{M}^+ = (\mathcal{D}_f^+(u, v))_{|U| \times |U|}$ and $\mathcal{M}^- = (\mathcal{D}_f^-(u, v))_{|U| \times |U|}$ are called upper approximation discernibility matrix and lower approximation discernibility matrix, respectively.

Proposition 5.3 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD, and $A \subseteq AT$.

1. A is an upper approximation consistent set if and only if $A \bigcap \mathcal{D}_f^+(u, v) \neq \emptyset$ for all $(u, v) \in \mathcal{D}_f^+$.
2. A is a lower approximation consistent set if and only if $A \bigcap \mathcal{D}_f^-(u, v) \neq \emptyset$ for all $(u, v) \in \mathcal{D}_f^-$.

Proof

1. "$\Longrightarrow$": By Proposition 5.1 we have that there exists $a \in A$ s.t. $f(u, a) \not\leq f(v, a)$ for any $(u, v) \in \mathcal{D}_f^+$. It is easy to obtain that $a \in \mathcal{D}_f^+(u, v)$. Therefore, we have that $A \bigcap \mathcal{D}_f^+(u, v) \neq \emptyset$. "$\Longleftarrow$": If $A \bigcap \mathcal{D}_f^+(u, v) \neq \emptyset$ for all $(u, v) \in \mathcal{D}_f^+$, then there exists $a \in A \bigcap \mathcal{D}_f^+(u, v)$, i.e., $a \in \mathcal{D}_f^+(u, v)$. With the definition of $\mathcal{D}_f^+(u, v)$, we have that $f(u, a) \not\leq f(v, a)$. According to Proposition 5.1, one can get that A is an upper approximation consistent set.
2. The proof of (2) is similar to (1).

Definition 5.3 Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD. If we denote

$$\mathcal{F}^+ = \bigwedge_{i,j=1}^{n} \left(\bigvee \mathcal{D}_f^+(u_i, u_j) \right) \tag{15}$$

$$\mathcal{F}^- = \bigwedge_{i,j=1}^{n} \left(\bigvee \mathcal{D}_f^-(u_i, u_j) \right), \tag{16}$$

then $\mathcal{F}^+$ is referred to as an upper approximation discernibility formula and $\mathcal{F}^-$ is referred to as a lower approximation discernibility formula.

Moreover, if $\mathcal{F}^+$ and $\mathcal{F}^-$ can be expressed as

$$\mathcal{F}^+ = \bigvee_{k=1}^{p} \left(\bigwedge_{l=1}^{q_k} a_{i_l} \right) = \bigvee_{k=1}^{p} \mathcal{B}_k \tag{17}$$

$$\mathcal{F}^- = \bigvee_{k=1}^{t} \left(\bigwedge_{l=1}^{s_k} a_{j_l} \right) = \bigvee_{k=1}^{t} \mathcal{A}_k, \tag{18}$$

then $\mathcal{F}^+$ is referred to as an upper approximation minimal disjunctive normal form of the discernibility formula and $\mathcal{F}^-$ is referred to as a lower approximation minimal disjunctive normal form if $|\mathcal{B}_k| = a_k$ and $|\mathcal{A}_k| = s_k$.

Proposition 5.4 *Let $\mathcal{L}_d^{\succcurlyeq} = (U, AT \bigcup \{d\}, V, f)$ be an LvISFD. Then we have that*

1. *$\mathcal{B}_k$ is an upper approximation reduction and $\mathcal{F}^+ = \bigvee_{k=1}^{p} \mathcal{B}_k$ contains all upper approximation reductions.*
2. *$\mathcal{A}_k$ is a lower approximation reduction and $\mathcal{F}^- = \bigvee_{k=1}^{p} \mathcal{A}_k$ contains all lower approximation reductions.*

Proof The proof is easy by Definition 5.1. □

Example 5.1 (Continued from Examples 3.1 and 3.2) By computing we can get $\mathcal{M}^+$ as follows (Table 2)

According to Definition 5.1, one can get that

$$\begin{aligned} \mathcal{F}^+ &= (a_1 \vee a_2 \vee a_4) \wedge (a_1 \vee a_2) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_4) \\ &\quad \wedge (a_3 \vee a_4) \wedge (a_3) \wedge (a_4) \wedge (a_5) \\ &= (a_1 \vee a_2) \wedge (a_3) \wedge (a_4) \wedge (a_5) \\ &= (a_1 \wedge a_3 \wedge a_4 \wedge a_5) \vee (a_2 \wedge a_3 \wedge a_4 \wedge a_5) \end{aligned}$$

Table 2 Upper approximation discernibility matrix of Table 1

u_i/u_j	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_4a_5\}$	$\emptyset$	$\emptyset$
u_2	$\{a_1a_2\}$	$\emptyset$	$\{a_1a_2a_4\}$	a_1a_4	$\emptyset$	$\{a_1a_2a_4\}$
u_3	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_5\}$	$\emptyset$	$\emptyset$
u_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_5	$\{a_3\}$	$\emptyset$	$\{a_3a_4\}$	$\{a_3a_4a_5\}$	$\emptyset$	$\{a_3a_4a_5\}$
u_6	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_4\}$	$\emptyset$	$\emptyset$

Table 3 Lower approximation discernibility matrix of Table 1

u_i/u_j	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_2\}$	$\emptyset$	$\emptyset$
u_2	$\{a_3a_5\}$	$\emptyset$	$\{a_3a_5\}$	a_3a_5	$\emptyset$	$\emptyset$
u_3	$\{a_4a_5\}$	$\emptyset$	$\emptyset$	$\{a_2\}$	$\emptyset$	$\emptyset$
u_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_5	$\{a_1a_5\}$	$\emptyset$	$\{a_1\}$	$\{a_1a_2\}$	$\emptyset$	$\emptyset$
u_6	$\{a_1a_3a_4a_5\}$	$\emptyset$	$\{a_1a_3a_5\}$	$\{a_1a_2a_3\}$	$\emptyset$	$\emptyset$

Similarly, one can get that $\mathcal{M}^-$ as follows (Table 3)

And by Definition 5.1, we have that

$$\begin{aligned}\mathcal{F}^- &= (a_1 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3) \\ &\quad \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_1 \vee a_2) \wedge (a_1 \vee a_5) \\ &\quad \wedge (a_3 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1) \wedge (a_2) \\ &= (a_3 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1) \wedge (a_2) \\ &= (a_1 \wedge a_2 \wedge a_3 \wedge a_4) \vee (a_1 \wedge a_2 \wedge a_5).\end{aligned}$$

So, $\{a_2, a_3\}$ is the only one lower approximation reduction of this system.

Thus, we can conclude that $\{a_1, a_3, a_4, a_5\}$ and $\{a_2, a_3, a_4, a_5\}$ are all the upper approximation reductions, and $\{a_1, a_2, a_3, a_4\}$ and $\{a_1, a_2, a_5\}$ are all the lower approximation reductions of LvISFD, which accord with the result of Example 3.1.

In an ordered information system, an atomic expression over a single attribute a is defined as $(a, \geq)$ in an ordered information system. For any $A \subseteq AT$, an expression over A in ordered information systems is defined by $\bigwedge_{a \in A} e(a)$, where $e(a)$ is an atomic expression over a. Given $a \in AT$, $v_1 \in V_a$, an atomic formula over a single attribute a is defined as $(a, \geq)$. For any $A \in AT$, a formula over A in ordered information system is denoted by $M(A)$. Let the formulas $\phi \in M(A)$, $||\phi||$ denotes the set of objects satisfying formula ϕ. For example, $(a, \geq, v_1)$, is atomic formula, then

$$||(a, \geq, v_1)|| = \{u \in U | f(x, a) \geq v_1\}. \tag{19}$$

However, in an LvISFD, we modify the definition of a formula over a according to the dominance relation $R_A^{\succcurlyeq}$ as follows

$$||(a, \succcurlyeq, v_1)|| = \{u \in U | f(x, a) \succcurlyeq v_1\}. \tag{20}$$

Now we consider an LvISFD $\mathcal{I}^{\succcurlyeq} = (U, AT \cup \{d\}, V, f)$ and a subset of attributes $A \subseteq AT$. For formulas $\phi \in M(A)$, a decision rule, denoted by $\phi \rightarrow \varphi$, is read "if ϕ then φ." The formula ϕ is called the rule's antecedent, and the formula φ is called the rules consequent. We say that an object supports a decision rule if it matches both the condition and the decision parts of the rule. On the other hand, an object is covered by a decision rule if it matches the condition parts of the rule.

In order to obtain more meaningful rules, the lower approximation $\underline{A_d^{\succcurlyeq}}$ and upper approximation $\overline{A_d^{\succcurlyeq}}$ also can be divided into many classes as follows

$$(\underline{A_d^{\succcurlyeq}})_i = \{u |\ a_i \leq \underline{A_d^{\succcurlyeq}}(u) \leq a_{i+1}, a_i \in [0, 1]\}, \tag{21}$$

$$(\overline{A_d^{\succcurlyeq}})_i = \{u |\ a_i \leq \overline{A_d^{\succcurlyeq}}(u) \leq a_{i+1}, a_i \in [0, 1]\}. \tag{22}$$

There are two types of dominance rules to be considered as follows.

1. Certain dominance rules with the following syntax: if $(f(u, a_1) \succcurlyeq v_{a_1}) \wedge (f(u, a_2) \succcurlyeq v_{a_2}) \wedge \cdots \wedge (f(u, a_k) \succcurlyeq v_{a_k})$, then $u \in D_i$, the syntax is equivalent to "If $u \in (\underline{A_d^{\succcurlyeq}})_i$, then $u \in D_i$";
2. Possible dominance rules with the following syntax: if $(f(u, a_1) \succcurlyeq v_{a_1}) \wedge (f(x, a_2) \succcurlyeq v_{a_2}) \wedge \cdots \wedge (f(u, a_k) \succcurlyeq v_{a_k})$, then u possible belong to D_i, the syntax is equivalent to "If $u \in (\overline{A_d^{\succcurlyeq}})_i$, then u possible belong to D_i".

Now we employ an example to illustrate dominance rules acquisition of lattice-valued ordered information system with fuzzy decision.

Example 5.2 (Continued from Examples 3.1 and 5.1) Consider an LvISFD in Table 1, we can easy obtain that

$$(\underline{A_d^{\succcurlyeq}})_1 = \{x |\ \underline{A_d^{\succcurlyeq}}(u) \geq 0.7\} = \{u_1, u_3, u_4\},$$
$$(\overline{A_d^{\succcurlyeq}})_1 = \{x |\ \overline{A_d^{\succcurlyeq}}(u) \geq 0.7\} = \{u_1, u_3, u_4, u_6\}.$$

So, the following set of dominance rules from the Table 1 are:

1. Certain dominance rules with the following syntax: r_1 : $(a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.6) \wedge (a_3 \succcurlyeq [0.3, 0.8]) \wedge (a_4 \succcurlyeq 0) \wedge (a_5 \succcurlyeq (0.5, 0.3)) \rightarrow (d =$ High) supported by objects u_1, u_3; r_2 : $(a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.7) \wedge (a_3 \succcurlyeq [0.3, 0.8]) \wedge$ $(a_4 \succcurlyeq 0) \wedge (a_5 \succcurlyeq (0.3, 0.6)) \rightarrow (d =$ High) supported by objects u_4.
2. Possible dominance rules with the following syntax: r_3 : $(a_1 \succcurlyeq 1) \wedge (a_2 \succcurlyeq 0.6) \wedge (a_3 \succcurlyeq [0.2, 0.4]) \wedge (a_4 \succcurlyeq \{0,$

1\}) $\wedge(a_5 \succcurlyeq (0.3, 0.6)) \rightarrow (d = \text{High}) \vee (d = \text{Low})$ supported by objects u_6; Where r_1, r_2 are certain dominance rules, r_3 is possible dominance rules. According to Example 5.1, we know the attribute a_3, a_4, a_5 are indispensable to extract certain dominance rules, and the attribute a_1, a_2 are indispensable to extract possible dominance rules. Through a upper and lower approximation reduction, one can obtain more briefer dominance rules. For example, by taking the upper approximation reduction $\{a_1, a_3, a_4, a_5\}$ and lower approximation reduction $\{a_1, a_2, a_5\}$.The three dominance rules in Example 5.2 can be simply represented as follows.

3. Certain dominance rules with the following syntax: $r_1 : (a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.6) \wedge (a_5 \succcurlyeq (0.5, 0.3)) \rightarrow (d = \text{High})$ supported by objects u_1, u_3; $r_2 : (a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.7) \wedge (a_5 \succcurlyeq (0.3, 0.6)) \rightarrow (d = \text{High})$ supported by objects u_4.
4. Possible dominance rules with the following syntax: $r_3 : (a_1 \succcurlyeq 1) \wedge (a_3 \succcurlyeq [0.2, 0.4]) \wedge (a_4 \succcurlyeq \{0, 1\}) \wedge (a_5 \succcurlyeq (0.3, 0.6)) \rightarrow (d = \text{High}) \vee (d = \text{Low})$ supported by u_6;

where r_1, r_2 are certain dominance rules, r_3 is possible dominance rule.

6 Case study and experiments

In this section, we present the algorithm of the attribute reduction in an LvISFD (see Algorithm 1) and a small artificial case study (see Table 4), then give experimental evaluation by four datasets from the UCI dataset. Firstly, we construct an lattice-valued dataset with fuzzy decision about the fund investment by persons. Fund has become an increasingly important source of financing. For a decision maker, he may need to adopt a better one from some possible fund projects or find some directions from existing successful fund projects before investing. The purpose of this section is, through a fund investment issue, to illustrate how to make a decision by using the approaches proposed in this paper.

We consider ten projects u_i $(i = 1, 2, \ldots, 10)$. They can be evaluated from the view of profit factors. Profit factors are classified into five factors, which are market, technology, management, environment and production. These five factors are all increasing preference and the value of each project under each factor is given by an evaluation expert through an intuitionistic number. Table 4 is an evaluation table about fund investment given by an expert, where $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$, $AT = \{$Market, Technology, Management, Environment, Production$\}$ and $d = \{$Venture$\}$, For convenience, in the sequel, a_1, a_2, a_3, a_4, a_5 and P_f will stand for Market, Technology, Management, Environment, Production, and Profit, respectively.

From Table 4, we have that $U/R^{\succcurlyeq}_{AT} = \{[u_1]^{\succcurlyeq}_{AT}, [u_2]^{\succcurlyeq}_{AT}, [u_3]^{\succcurlyeq}_{AT}, [u_4]^{\succcurlyeq}_{AT}, [u_5]^{\succcurlyeq}_{AT}, [u_6]^{\succcurlyeq}_{AT}, [u_7]^{\succcurlyeq}_{AT}, [u_8]^{\succcurlyeq}_{AT}, [u_9]^{\succcurlyeq}_{AT}, [u_{10}]^{\succcurlyeq}_{AT}\}$. And the dominance classes are

$[u_1]^{\succcurlyeq}_{AT} = \{u_1, u_5, u_7, u_8\}$; $[u_2]^{\succcurlyeq}_{AT} = \{u_1, u_2, u_3, u_5, u_6, u_7, u_8, u_9, u_{10}\}$;

$[u_3]^{\succcurlyeq}_{AT} = \{u_1, u_3, u_5, u_6, u_7, u_8\}$; $[u_4]^{\succcurlyeq}_{AT} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$; $[u_5]^{\succcurlyeq}_{AT} = \{u_5\}$;

$[u_6]^{\succcurlyeq}_{AT} = \{u_5, u_6, u_8\}$; $[u_7]^{\succcurlyeq}_{AT} = \{u_5, u_7, u_8\}$; $[u_8]^{\succcurlyeq}_{AT} = \{u_8\}$; $[u_9]^{\succcurlyeq}_{AT} = \{u_9\}$; $[u_{10}]^{\succcurlyeq}_{AT} = \{u_{10}\}$.

From the definition of dominance degree, we can get the dominance matrix of this system with respect to $U/R^{\succcurlyeq}_{AT}$ as

$$M^{\succcurlyeq}_{AT} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0.7 & 0.8 & 0.9 & 0.7 & 0.6 & 0.6 \\ 0.5 & 1 & 0.7 & 1 & 0.2 & 0.4 & 0.4 & 0.2 & 0.2 & 0.2 \\ 0.8 & 1 & 1 & 1 & 0.5 & 0.7 & 0.7 & 0.5 & 0.4 & 0.4 \\ 0.4 & 0.9 & 0.6 & 1 & 0.1 & 0.3 & 0.3 & 0.1 & 0.1 & 0.1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.9 & 0.9 & 0.9 \\ 0.9 & 1 & 1 & 1 & 0.8 & 1 & 0.9 & 0.8 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 0.8 & 0.9 & 1 & 0.8 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 0.9 & 1 & 1 & 1 & 0.9 & 0.9 \\ 0.9 & 1 & 0.9 & 1 & 0.9 & 0.9 & 0.9 & 0.9 & 1 & 0.9 \\ 0.9 & 1 & 0.9 & 1 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 1 \end{bmatrix}.$$

Therefore, one can obtain that

$D_{AT}(u_1) = 0.83$, $D_{AT}(u_2) = 0.48$, $D_{AT}(u_3) = 0.7$, $D_{AT}(u_4) = 0.39$, $D_{AT}(u_5) = 0.97$, $D_{AT}(u_6) = 0.88$, $D_{AT}(u_7) = 0.89$, $D_{AT}(u_8) = 0.97$, $D_{AT}(u_9) = 0.93$, $D_{AT}(u_{10}) = 0.93$.

We rank these five projects according to the number of $D_{AT}(u_i)$. A project with whole dominance degree implies that it has higher investment venture.

$$u_5 = u_8 \succcurlyeq_{AT} u_9 = u_{10} \succcurlyeq_{AT} u_7 \succcurlyeq_{AT} u_6 \succcurlyeq_{AT} u_1 \succcurlyeq_{AT} u_3 \succcurlyeq_{AT} u_2 \succcurlyeq_{AT} u_4.$$

Thus, the investment profit of project u_5 and u_8 are highest and that of project u_4 is lowest. The decision maker should select the project u_5 and u_8 to invest.

From Table 4, it is easy to see that $d = \{0.8, 0.4, 0.3, 0.5, 1.0, 0.7, 0.6, 0.9, 0.7, 0.5\}$.

From Definition 3.1, we have that

$$\underline{AT^{\succcurlyeq}_d} = \{0.6, 0.3, 0.3, 0.3, 1.0, 0.7, 0.6, 0.9, 0.7, 0.5\},$$
$$\overline{AT^{\succcurlyeq}_d} = \{1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0, 0.9, 0.7, 0.5\}.$$

Take the threshold $\alpha = 0.7$, then we have

$$(\underline{AT^{\succcurlyeq}_d})_1 = \{u_5, u_6, u_8, u_9\},$$
$$(\overline{AT^{\succcurlyeq}_d})_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}.$$

Table 4 An LvISFD about fund investment

U	Market	Technology	Management	Environment	Production	Profit
u_1	3	0.6	[0.5, 0.8]	$\{2,3\}$	(0.5, 0.4)	0.8
u_2	2	0.1	[0.4, 0.5]	$\emptyset$	(0.2, 0.8)	0.4
u_3	2	0.1	[0.4, 0.5]	$\{2,3\}$	(0.2, 0.8)	0.3
u_4	1	0.1	[0.1, 0.2]	$\emptyset$	(0.2, 0.8)	0.5
u_5	8	0.8	[0.8, 0.9]	$\{1,2,3,4\}$	(0.7, 0.1)	1.0
u_6	4	0.8	[0.6, 0.7]	$\{1,2,3,4\}$	(0.7, 0.1)	0.7
u_7	3	0.7	[0.5, 0.9]	$\{2,3\}$	(0.6, 0.2)	0.6
u_8	7	0.8	[0.7, 0.9]	$\{1,2,3,4,5\}$	(0.7, 0.1)	0.9
u_9	7	0.9	[0.7, 0.9]	$\{2,3,4\}$	(0.9, 0.0)	0.7
u_{10}	8	0.9	[0.8, 0.9]	$\{2\}$	(0.9, 0.0)	0.5

Table 5 Upper approximation discernibility matrix of Table 4

u_i/u_j	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_8	AT	AT	AT	AT	$\{a_4\}$	$\{a_1a_3a_4\}$	AT	$\emptyset$	$\emptyset$	$\emptyset$
u_9	AT	AT	AT	AT	$\{a_2a_5\}$	$\{a_1a_2a_3a_5\}$	AT	$\{a_2a_5\}$	$\emptyset$	$\emptyset$
u_{10}	$\{a_1a_2a_3a_5\}$	AT	$\{a_1a_2a_3a_5\}$	AT	$\{a_2\}$	$\{a_1a_2a_3a_5\}$	$\{a_1a_2a_3a_5\}$	$\{a_1a_2a_3a_5\}$	$\{a_1a_3\}$	$\emptyset$

Table 6 Lower approximation discernibility matrix of Table 4

u_i/u_j	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AT	AT	$\emptyset$	AT	AT	$\emptyset$
u_2	AT	$\emptyset$	$\emptyset$	$\emptyset$	AT	AT	AT	AT	AT	AT
u_3	$\{a_1a_2a_3a_5\}$	$\emptyset$	$\emptyset$	$\emptyset$	AT	AT	$\{a_1a_2a_3a_5\}$	AT	AT	$\{a_1a_2a_3a_5\}$
u_4	AT	$\emptyset$	$\emptyset$	$\emptyset$	AT	AT	AT	AT	AT	AT
u_5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_1a_3\}$	$\emptyset$	$\emptyset$	AT	AT	$\emptyset$
u_7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AT	AT	$\emptyset$	AT	AT	$\emptyset$
u_8	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_1a_3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_9	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_1a_3a_4\}$	$\emptyset$	$\emptyset$	a_4	$\emptyset$	$\emptyset$
u_{10}	$\{a_4\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{a_4\}$	$\{a_4\}$	$\{a_4\}$	$\{a_4\}$	$\emptyset$

Therefore, we can obtain the following set of dominance rules from the considered lattice-valued information systems with fuzzy decision:

1. Certain dominance rules with the following syntax:

r'_1 : $(a_1 \succcurlyeq 4) \wedge (a_2 \succcurlyeq 0.8) \wedge (a_3 \succcurlyeq [0.6, 0.7]) \wedge (a_4 \succcurlyeq \{1, 2, 3, 4\}) \wedge (a_5 \succcurlyeq (0.7, 0.1)) \rightarrow (P_f = \text{High})$ // supported by objects u_5, u_6, u_8;

r'_2 : $(a_1 \succcurlyeq 7) \wedge (a_2 \succcurlyeq 0.9) \wedge (a_3 \succcurlyeq [0.7, 0.9]) \wedge (a_4 \succcurlyeq \{2, 3, 4\}) \wedge (a_5 \succcurlyeq (0.9, 0.0)) \rightarrow (P_f = \text{High})$ // supported by u_9.

2. Possible dominance rules with the following syntax:

r'_3 : $(a_1 \succcurlyeq 3) \wedge (a_2 \succcurlyeq 0.6) \wedge (a_3 \succcurlyeq [0.5, 0.8]) \wedge (a_4 \succcurlyeq \{2, 3\}) \wedge (a_5 \succcurlyeq (0.5, 0.4)) \rightarrow (P_f = \text{High})$ // supported by u_1;

r'_4 : $(a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.1) \wedge (a_3 \succcurlyeq [0.4, 0.5]) \wedge (a_4 \succcurlyeq \emptyset) \wedge (a_5 \succcurlyeq (0.2, 0.8)) \rightarrow (P_f = \text{High})$ // supported by u_2;

r_5' : $(a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.1) \wedge (a_3 \succcurlyeq [0.4, 0.5]) \wedge (a_4 \succcurlyeq \{2, 3\}) \wedge (a_5 \succcurlyeq (0.2, 0.8)) \rightarrow (P_f = \text{High})$ // supported by u_3;

r_6' : $(a_1 \succcurlyeq 1) \wedge (a_2 \succcurlyeq 0.1) \wedge (a_3 \succcurlyeq [0.1, 0.2]) \wedge (a_4 \succcurlyeq \emptyset) \wedge (a_5 \succcurlyeq (0.2, 0.8)) \rightarrow (P_f = \text{High})$ // supported by u_4;

r_7' : $(a_1 \succcurlyeq 3) \wedge (a_2 \succcurlyeq 0.7) \wedge (a_3 \succcurlyeq [0.5, 0.9]) \wedge (a_4 \succcurlyeq \{2, 3\}) \wedge (a_5 \succcurlyeq (0.6, 0.2)) \rightarrow (P_f = \text{High})$ // supported by u_7.

To extract much simpler dominance rules, we compute the lower and upper approximation reductions of this decision system. The lower and upper approximation reductions of this decision system can be obtained by the proposed approach in Sect. 5. Tables 5 and 6 are Upper and lower approximation discernibility matrix of the system in Table 4.

From Tables 5 and 6, one can caculate that

$$
\begin{aligned}
\mathcal{F}^+ &= (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \\
&\quad \wedge (a_1 \vee a_3 \vee a_4) \wedge (a_1 \vee a_3) \\
&\quad \wedge (a_2 \vee a_5) \wedge (a_2) \wedge (a_4) \\
&= (a_1 \vee a_3) \wedge (a_2) \wedge (a_4) \\
&= (a_1 \wedge a_2 \wedge a_4) \vee (a_2 \wedge a_3 \wedge a_4). \\
\mathcal{F}^- &= (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \\
&\quad \wedge (a_1 \vee a_3 \vee a_4) \wedge (a_1 \vee a_3) \wedge (a_4) \\
&= (a_1 \vee a_3) \wedge (a_4) \\
&= (a_1 \wedge a_4) \vee (a_3 \wedge a_4).
\end{aligned}
$$

Hence, there are two lower approximation approximation reductions in this information system about fund investment, which is {Market, Environment} and {Management, Environment}. Lower approximation reduction is keeps certain dominance rules invariant. Through a lower approximation reduction, one can obtain more briefer certain dominance rules. For example, by taking the lower approximation reduction {Market, Environment}, the two certain dominance rules in above part can be simply represented as follows.

3. Certain dominance rules with the following syntax:

r_1'' : $(a_1 \succcurlyeq 4) \wedge (a_4 \succcurlyeq \{1, 2, 3, 4\}) \rightarrow (P_f = \text{High})$ // supported by objects u_5, u_6, u_8;

r_2'' : $(a_1 \succcurlyeq 7) \wedge (a_4 \succcurlyeq \{2, 3, 4\}) \rightarrow (P_f = \text{High})$ // supported by objects u_9.

There are two upper approximation approximation reductions in this information system about fund investment, which is {Market, Technology, Environment} and {Technology, Management, Environment}. Upper approximation reduction is keeps possible dominance rules invariant. Through an upper approximation reduction, one can obtain more briefer possible dominance rules. For example, by taking the upper approximation reduction {Market, Technology, Environment}, the five possible dominance rules in above part can be simply represented as follows.

4. possible dominance rules with the following syntax:

r_3'' : $(a_1 \succcurlyeq 3) \wedge (a_2 \succcurlyeq 0.6) \wedge (a_4 \succcurlyeq \{2, 3\}) \rightarrow (P_f = \text{High})$ // supported by objects u_1;

r_4'' : $(a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.1) \wedge (a_4 \succcurlyeq \emptyset) \rightarrow (P_f = \text{High})$ // supported by objects u_2;

r_5'' : $(a_1 \succcurlyeq 2) \wedge (a_2 \succcurlyeq 0.1) \wedge (a_4 \succcurlyeq \{2, 3\}) \rightarrow (P_f = \text{High})$ // supported by objects u_3;

r_6'' : $(a_1 \succcurlyeq 1) \wedge (a_2 \succcurlyeq 0.1) \wedge (a_4 \succcurlyeq \emptyset) \rightarrow (P_f = \text{High})$ // supported by objects u_4;

r_7'' : $(a_1 \succcurlyeq 3) \wedge (a_2 \succcurlyeq 0.7) \wedge (a_4 \succcurlyeq \{2, 3\}) \rightarrow (P_f = \text{High})$ // supported by objects u_7.

Where r_1'' and r_1'' are two certain dominance rules and $r_3'', r_4'', r_5'', r_6'', r_7''$ are possible dominance rules.

Algorithm 1: An algorithm for attributes reduction in LvISFD

```
Input  : The UCI experimental data where include n objects and m attributes.
Output : The lower and upper approximation reductions, respectively
begin
    load    : The UCI dataset saved as I = (U, AT).        // U include n objects and m attributes.
    let     : D_f^- = ∅, D_f^+ = ∅, F^+ = ∅, F^- = ∅.
    for each u ∈ U do
        compute: [u]_AT^≽.                    // According to the compute ways in the Example 3.1.
    end
    for each u ∈ U do
        for each v ∈ [u]_AT^≽ do
            lower(AT_d^≽)(u) = min f(v, d).        // According to the Definition 3.4.
            upper(AT_d^≽)(u) = max f(v, d).
        end
    end
    for each u ∈ U do
        for each v ∈ U do
            if lower(AT_d^≽)(u) < lower(AT_d^≽)(v) then
                D_f^- = D_f^- ∪ {(u, v)}.
            end
            if upper(AT_d^≽)(u) < upper(AT_d^≽)(v) then
                D_f^+ = D_f^+ ∪ {(u, v)}.
            end
            for each a ∈ AT do
                if (u, v) ∈ D_f^+ and f(u, a) ⋠ f(v, a) then
                    D_f^+(u, v) = D_f^+(u, v) ∪ a.        // According the Definition 5.2.
                end
                if (u, v) ∈ D_f^- and f(u, a) ⋡ f(v, a) then
                    D_f^-(u, v) = D_f^-(u, v) ∪ a.
                end
            end
        end
    end
    for i = 1 : n do
        for j = 1 : n do
            F^+ = ∧{∨{D_f^+(u_i, u_j)}}.        // According to the Definition 5.3.
            F^- = ∧{∨{D_f^-(u_i, u_j)}}.
        end
    end
    return: The lower and upper approximation reductions F^-, F^+.
end
```

Table 7 The basic information datasets

	Dataset	Objects	Attributes	Decision classes
1	Energy efficiency	768	8	3
2	Airfoil self-noise	1503	6	5
3	Wine quality-red	1599	11	6
4	Wine quality-white	4898	11	7

In the following, we will test approaches to attribute reductions in an LvISFD on some real-life datasets. Datasets used in the experiments are underlined in Table 7, which are downloaded from University of California - Irvine (UCI) Repository of Machine Learning datasets. In these UCI datasets, most of the initial data are single-valued. what means one has to construct the lattice-valued datasets with fuzzy decision. When a dataset is selected from the UCI, we need to preprocess these data on the selected datasets. The means of preprocess to obtain the target data are shown as follows:

1. The real elements: Do not need preprocess.
2. The fuzzy elements: $|A(x_i) - E(A)|/|m(A) - n(A)|$. Where $A(x_i)$ denotes the value of x_i with respect to condition attribute A. $E(A)$, $m(A)$ and $n(A)$ denote the average value of A, the maximum value of A and the minimum value of A, respectively.
3. The interval-valued elements: $[A(x_i) \times (1 - \alpha), A(x_i) \times (1 + \alpha)]$. Where $A(x_i)$ denotes the value of x_i with respect to condition attribute A, and α denotes an any number in the interval [0, 1].
4. The set-valued elements: $A(x_i) \times \beta$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$, and $\beta_i \in [0.1, 1]$.
5. The classical intuitionistic fuzzy set elements: $(sin^2(A(x_i)), cos^2(A(x_i)) - \gamma)$, $\gamma \in [0, 0.05]$.

This experimental computing program is running on a personal computer with the following hardware and software configuration.

Names	Model	Parameters
CPU	Intel Core i3-2350M	2.3GHz
Memory	Samsung DDR3 SDRAM	2×2GB 1333MHz
Hard disk	West Data	500GB
System	Windows 7	32bit
Platform	C++	Leasehold

By using of Algorithm 1 and proposed methods for attribute reductions in an LvISFD, we can get the results of experiments. The number of lower/upper approximation reductions are shown in Table 8.

Table 8 The number of lower/upper approximation reductions

	Dataset	Low-red	Up-red
1	Energy efficiency	52	45
2	Airfoil self-noise	13	18
3	Wine quality-red	10	9
4	Wine quality-white	37	49

7 Conclusions

Rough set theory is a new mathematical tool to deal with vagueness and uncertainty. Development of a rough computational method is one of the most important research tasks. However, an LvISFD confines the applications of classical rough set theory. In this article, we mainly considered some important concepts and properties in this system. We defined two approximation operators and established the rough set approach to an LvISFD. For dominance rules acquisition, we have discussed dominance rules acquisition in this kind of decision information system. In order to extract much simpler dominance rules, based on the discernibility matrices, we have proposed approximation reductions of an LvISFD, and presented method of the corresponding reduction respectively. The approaches show how to find much simpler dominance rules directly from an LvISFD. In order to hillustrate our methods of attribute reduction, we constructed an artificial example about the fund investment, and tested the reduction algorithm in the UCI datasets.

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