Granular Computing Approach to Two-Way Learning Based on Formal Concept Analysis in Fuzzy Datasets

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Abstract—**The main task of granular computing (GrC) is about representing, constructing, and processing information granules. Information granules are formalized in many different approaches. Different formal approaches emphasize the same fundamental facet in different ways. In this paper, we propose a novel GrC method of machine learning by using formal concept description of information granules. Based on information granules, the model and mechanism of two-way learning system is constructed in fuzzy datasets. It is addressed about how to train arbitrary fuzzy information granules to become necessary, sufficient, and necessary and sufficient fuzzy information granules. Moreover, an algorithm of the presented approach is established, and the complexity of the algorithm is analyzed carefully. Finally, to interpret and help understand the theories and algorithm, a real-life case study is considered and experimental evaluation is performed by five datasets from the University of California—Irvine, which is valuable for applying these theories to deal with practical issues.**

Index Terms—**Formal concept analysis (FCA), fuzzy datasets, fuzzy information granules, granular computing (GrC), machine learning.**

I. Introduction

NOWADAYS, machine learning is becoming an essential method for handling the problems of knowledge acquisition. Machine learning, a branch of artificial intelligence, concerns the construction and study of systems that can learn from data. Machine learning usually concerned with the question of how to construct computer programs that automatically improve with experience. In 1959, A. Samuel defined machine learning as a "field of study that gives computers the ability to learn without being explicitly programmed" [21]. The research of machine learning is based on the understanding of human learning mechanism, establishes the learning processes of calculation model and cognitive model [25]. For machine learning, what we need to do is to develop a variety of machine learning theory and machine-learning methods [13], excogitate the general machine-learning algorithm and do the theoretical analysis, and set up the task with a specific application system.

In real-life, we can perceive a world filling different shapes, colors, sounds, smells, and so on. All these modalities are not in merely chaotic disorder. Through our learning abilities, we may find kinds of senses over the world. We can identify objects, and distinguish them from one to another. We also can manipulate different things and changing relationships between them, which help us to cognitive the world. For a newborn baby, almost everything is empty, overtime, more strong learning ability, more things one knows. This is a never ending learning process. Similarly, learning is not accompanied by the advent of machine. The machine, without implanting any system, naturally has problems in solving mathematical equations. To make the learning abilities, namely the learning process, visualization, and theorization, we want to build a new mathematical model of machine learning by use of formal concept analysis (FCA) from the view of GrC in this paper.

Zadeh [33] first explored the concept of GrC between 1996 and 1997. Zadeh coined an informal yet highly descriptive notion of an information granule. In a general sense, by information granule, one regards a collection of elements drawn together by their closeness (resemblance, proximity, functionality, etc.), articulated in terms of some useful spatial, temporal, or functional relationships. GrC is about representing, constructing, and processing information granules [17]. Dick *et al*. [6] established a novel architecture for a granular neural network and Zhang and Miao [34] investigated two basic double-quantitative rough set models of precision and grade and their investigation using GrC. Bargiela and Pedrycz [1], [2] researched the roots of GrC and expanded the theory of GrC for human-centered information processing. Yao [27], [28], [30] discussed integrative levels of granularity and theory of GrC. Many excellent achievements can be seen in [9], [12], [23], and [26].

It is worth stressing that information granules, as encountered in natural language, are implicit in their nature, and information granules permeate human endeavors [18], [19], [29]. No matter which problem is taken into consideration, we usually set it up in a certain conceptual framework composed of some generic and conceptually meaningful entities-information granules, which we regard to be of relevance to the problem formulation, further problem solving, and a way in which the findings are communicated to the community.

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Information granules realize a framework in which we formulate generic concepts by adopting a certain level of abstraction.

Information granules are formalized in many different ways. Clustering delivers a natural mechanism to construct information granules in the presence of numeric data. As a matter of fact, a main agenda of clustering is to reveal a structure of data, namely, form a collection of clusters-information granules [17]. There is a genuine diversity of clustering algorithms. Depending upon the method used, the results arise as information granules expressed in terms of sets, fuzzy sets [33], rough sets [15], [16], [34], formal concepts [25], and so forth. Sets stress the notion of dichotomy. Fuzzy sets depart from the dichotomy by emphasizing the idea of partial membership and in this way offer a possibility to deal with concepts where the binary view of the underlying concept is not suitable or overly restrictive. Rough sets bring another conceptual and algorithmic perspective by admitting an inability for a full description of concepts in the presence of a limited collection of information granules. Roughness comes as a manifestation of the limited descriptive capabilities of the vocabulary used, with which the description is realized. Each formal concept is the pair (objects and attributes), which consists of two parts: 1) the extension (objects covered by the concept) and 2) intension (attributes describing the concept) in FCA theory. FCA was proposed by Wille [22] in 1982, and the key of this theory is based on mathematical expressions of the formal concept. Concept lattice can be depicted by a Hassen diagram, where each node expresses a formal concept. A concept lattice is an ordered hierarchical structure of formal concepts that are defined by a binary relation between a set of objects and a set of attributes.

In addition, Zadeh [32] first proposed and discussed the issue of fuzzy information granulation in 1979. Burusco [4] presented the first paper on FCA in a fuzzy setting. Later, Pollandt [20], Belohlavek [3], Yahia and Jaoua [31], and Krajci [10] proposed the L-fuzzy context to combine fuzzy logic with FCA. Pang *et al*. [14] and Xu *et al*. [24] developed the knowledge reduction in lattice-valued information systems with interval-valued intuitionist fuzzy decision is the further study about the fuzzy concept lattice. Kumar [11] introduced the fuzzy clustering-based FCA for association rules mining, it is an application of the fuzzy formal concept.

The choice of a suitable formal framework of information granules is problem oriented and has to be done according to the requirements of the problem, ways of acquisition of information granules (both on the basis of domain knowledge and numerical evidence), and related processing and optimization mechanisms available in light of the assumed formalism of information granularity.

In fact, there are two types of fuzzy information granules, one is that intension and extension are fuzzy, the other one is that the intension is fuzzy while the extension is classic. In this paper, we mainly consider the fuzzy concept which the intension is fuzzy and the extension is classic. We will investigate the issues that are fuzzy intension and fuzzy extension in the future work. We describe concepts and form models using formal concept description of information granules. These constructs become of particular interest when information granules have to capture a multifaceted nature of the problem. To realize a novel machine-learning process, we will establish a special two-way learning model in the view of GrC, and discuss fuzzy information granules in the fuzzy datasets. The fuzzy concept is described as a necessary and sufficient fuzzy information granule. The aim of two-way learning methods is to obtain the fuzzy concepts from general fuzzy information granule.

This paper is organized as follows. Some preliminary concepts of fuzzy set theory and FCA that are required in this paper are briefly reviewed in Section II. In Section III, we propose a novel model of two-way learning system in fuzzy datasets. Notions of fuzzy information granules, such as necessary fuzzy information granules, sufficient fuzzy information granules as well as necessary and sufficient fuzzy information granules, are presented. Moreover, the relationship between an object and its fuzzy attributes is discussed in view of information granules in this two-way learning system. In Section IV, we suggest how to learn necessary fuzzy information granules, sufficient fuzzy information granules, and necessary and sufficient fuzzy information granules from any fuzzy information granule. Furthermore, we present a real-life case study and give experimental evaluation by five datasets from the University of California—Irvine (UCI) dataset. Finally, we conclude our contribution with a summary and an outlook for the further research.

II. Related Work and Foundations

As we study a novel-learning system in fuzzy datasets, it is necessary to introduce the basic concepts of fuzzy set and FCA. In this section, the involved notions of fuzzy set theory and FCA are introduced briefly. Detailed description of them can be found in corresponding references. Throughout this paper, the power set of U is denoted by $P(U)$.

A. Fuzzy Sets

For a nonempty and finite set U, which is called the universe, a fuzzy set $\widetilde{X}$ of U has the following form:

$$\widetilde{X} = \left\{ < x, u_{\widetilde{X}}(x) > |x \in U \right\}$$

where $u_{\widetilde{X}} : U \rightarrow [0, 1]$, $u_{\widetilde{X}}(x)$ is called the membership degree to $\widetilde{X}$ of the object $x \in U$. In general, we use $\mathcal{F}(U)$ to denote all fuzzy sets in the universe U. Given two sets $\widetilde{X}_1, \widetilde{X}_2 \in \mathcal{F}(U)$, for any $x \in U$, $\widetilde{X}_1 \subseteq \widetilde{X}_2 \Leftrightarrow u_{\widetilde{X}_1}(x) \leq u_{\widetilde{X}_2}(x)$. If $\widetilde{X}_1 \subseteq \widetilde{X}_2$ and $\widetilde{X}_2 \subseteq \widetilde{X}_1$, then we say $\widetilde{X}_1$ is equal to $\widetilde{X}_2$, denoted by $\widetilde{X}_1 = \widetilde{X}_2$. The universe set and empty set are special fuzzy sets where $\widetilde{U} = \{< x, 1 > |x \in U\}$ and $\widetilde{\emptyset} = \{< x, 0 > |x \in U\}$. Let us denote intersection and union of $\widetilde{X}_1$ and $\widetilde{X}_2$ by $\widetilde{X}_1 \cap \widetilde{X}_2$ and $\widetilde{X}_1 \cup \widetilde{X}_2$, respectively. Moreover, we denote complement of $\widetilde{X}$ by $\sim \widetilde{X}$. Let $\widetilde{X}_1, \widetilde{X}_2 \in \mathcal{F}(U)$, then for $x \in U$

$$\begin{aligned} \sim \widetilde{X} &= \left\{ < x, 1 - u_{\widetilde{X}}(x) > \right\} \\ \widetilde{X}_1 \cap \widetilde{X}_2 &= \left\{ < x, \wedge \left\{ u_{\widetilde{X}_1}(x), u_{\widetilde{X}_2}(x) \right\} > \right\} \\ \widetilde{X}_1 \cup \widetilde{X}_2 &= \left\{ < x, \vee \left\{ u_{\widetilde{X}_1}(x), u_{\widetilde{X}_2}(x) \right\} > \right\}. \end{aligned}$$

B. Basic Notions on FCA

The notion of formal concept provides a convenient basis for the representation of objects in terms of their attributes [7].

FCA introduced by Wille [22] gives comprehensive information and knowledge about a dataset, namely a formal context, which in basic setting is a triplet of set of objects, set of properties, and a relation between the set of objects and the crisp set of the properties. The basic setting, introduced by Wille [22], is well-suited for attributes which are crisp, that is to say each object of the domain of applicability of the attribute either has or does not have the attribute. But for the data where the properties of the objects are fuzzy, the table entries of the data contain truth degrees from a set L (generally L taken as real unit interval [0,1]). To analyze the data in which fuzziness occurs, the procedure of FCA is adopted replacing classical logic by fuzzy logic [8] and the corresponding structures become fuzzy concept lattices [5].

FCA is based on a kind of formal context in datasets. Accordingly, in fuzzy datasets, a fuzzy formal context is a triple $(U, \mathrm{AT}, \widetilde{I})$, where:

1) U is an object set, and $U = \{x_1, x_2, \cdots, x_n\}$;
2) $\mathrm{AT} = \{a_1, a_2, \cdots, a_m\}$ is an attribute set, where $a_j (1 \leq j \leq m)$ is called an attribute;
3) $\widetilde{I} = \{< (x, a), u_{\widetilde{I}}(x, a) > |(x, a) \in U \times \mathrm{AT}\}$, $u_{\widetilde{I}} : U \times \mathrm{AT} \to [0, 1]$.

The complement of $\widetilde{I}$ is denoted by $\sim \widetilde{I} = \{< (x, a), 1 - u_{\widetilde{I}}(x, a) > |(x, a) \in U \times \mathrm{AT}\}$. We denote $\widetilde{I}(x, a) = u_{\widetilde{I}}(x, a)$, then the set of $\widetilde{I}(x, a)$ is denoted by $\mathcal{V} = \{\widetilde{I}(x, a) | x \in U, a \in \mathrm{AT}\}$. Let $\widetilde{I}(x, a), \widetilde{I}(y, a)$, then $\widetilde{I}(x, a) \geq \widetilde{I}(y, a) \Leftrightarrow u_{\widetilde{I}}(x, a) \geq u_{\widetilde{I}}(y, a)$.

In a fuzzy formal context $(U, \mathrm{AT}, \widetilde{I})$, for $X \subseteq U$ and $B \subseteq \mathrm{AT}$, we call $(X, \widetilde{B})$ the fuzzy information granule. With respect to a fuzzy formal context $(U, \mathrm{AT}, \widetilde{I})$, for $X \subseteq U, B \subseteq \mathrm{AT}$ and $\widetilde{A}, \widetilde{B} \in \mathcal{F}(U \times \mathrm{AT})$, where $\forall b, \widetilde{B}(b), \widetilde{A}(b) \in \{\widetilde{I}(x, b) | \forall x \in U\}$. A pair of operators is defined by

$$X^* = \widetilde{\mathrm{AT}} = \left\{< a, u_{\widetilde{X}}(a) > | a \in \mathrm{AT}\right\}$$

where $\widetilde{\mathrm{AT}}(a) = \bigwedge_{x \in X} u_{\widetilde{I}}(x, a)$. We rule $\emptyset^* = \widetilde{A} = \{< a, 0 > | a \in \mathrm{AT}\}$

$$\widetilde{B}^* = \left\{x \in U | \ \widetilde{I}(x, b) \geq \widetilde{B}(b), \forall b \in B\right\}$$

where $\widetilde{I}(x, b) \in \mathcal{V}$. If $b \notin B$, then $\widetilde{B}(b) = 0$. For any $B \subseteq \mathrm{AT}$, denote

$$U^B = \left\{\widetilde{B} | \ \widetilde{B}(b) = \widetilde{I}(x, b), x \in U, b \in B\right\}.$$

Let $(U, \mathrm{AT}, \widetilde{I})$ be a fuzzy formal context, X_1, X_2 and $X \subseteq U$, B_1, B_2 and $B \subseteq \mathrm{AT}$, then above two operators have the following properties.

1) $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*$, $\widetilde{B}_1 \subseteq \widetilde{B}_2 \Rightarrow \widetilde{B}_2^* \subseteq \widetilde{B}_1^*$.
2) $X \subseteq X^{**}$, $\widetilde{B} \subseteq \widetilde{B}^{**}$.
3) $X^* = X^{***}$, $\widetilde{B}^* = \widetilde{B}^{***}$.
4) $X \subseteq \widetilde{B}^* \Leftrightarrow \widetilde{B} \subseteq X^*$.
5) $(X_1 \cup X_2)^* = X_1^* \cap X_2^*$, $(\widetilde{B}_1 \cup \widetilde{B}_2)^* = \widetilde{B}_1^* \cap \widetilde{B}_2^*$.
6) $(X_1 \cap X_2)^* \supseteq X_1^* \cap X_2^*$, $(\widetilde{B}_1 \cap \widetilde{B}_2)^* \supseteq \widetilde{B}_1^* \cup \widetilde{B}_2^*$.

A pair $(X, \widetilde{B})$ is called a fuzzy concept, if $X^* = \widetilde{B}$ and $X = \widetilde{B}^*$ for $X \subseteq U, B \subseteq \mathrm{AT}$. X and $\widetilde{B}$ are called the extension and the intension of $(X, \widetilde{B})$, respectively. It is clear that both (X^{**}, X^*) and $(\widetilde{B}^*, \widetilde{B}^{**})$ are fuzzy concepts.

The fuzzy concept lattice $\widetilde{L}(U, \mathrm{AT}, \widetilde{I})$ is referred to all concepts of a fuzzy formal context $(U, \mathrm{AT}, \widetilde{I})$, and they are ordered by $(X_1, \widetilde{B}_1) \leq (X_2, \widetilde{B}_2) \Leftrightarrow X_1 \subseteq X_2 \Leftrightarrow \widetilde{B}_2 \subseteq \widetilde{B}_1$, where $(X_1, \widetilde{B}_1)$ and $(X_2, \widetilde{B}_2)$ are fuzzy concepts. $(X_1, \widetilde{B}_1)$ is called a sub-concept of $(X_2, \widetilde{B}_2)$, and $(X_2, \widetilde{B}_2)$ is called a super-concept of $(X_1, \widetilde{B}_1)$.

TABLE I
FUZZY DATASET

U	a	b	c	d	e
x_1	0.9	0.7	0.2	0.9	0.8
x_2	0.8	0.8	0.8	0.3	0.2
x_3	0.1	0.2	0.1	0.8	0.2
x_4	0.7	0.8	0.7	0.2	0.2

If $(X_1, \widetilde{B}_1)$ and $(X_2, \widetilde{B}_2)$ are two concepts of a fuzzy formal context $(U, \mathrm{AT}, \widetilde{I})$, then $(X_1 \cap X_2, (\widetilde{B}_1 \cup \widetilde{B}_2)^{**})$ and $((X_1 \cup X_2)^{**}, \widetilde{B}_1 \cap \widetilde{B}_2)$ are also both fuzzy concepts. Hence, if the meet and join are given by

$$\left(X_1, \widetilde{B}_1\right) \wedge \left(X_2, \widetilde{B}_2\right) = \left(X_1 \cap X_2, \left(\widetilde{B}_1 \cup \widetilde{B}_2\right)^{**}\right)$$
$$\left(X_1, \widetilde{B}_1\right) \vee \left(X_2, \widetilde{B}_2\right) = \left((X_1 \cup X_2)^{**}, \widetilde{B}_1 \cap \widetilde{B}_2\right)$$

then the fuzzy concept lattice $\widetilde{L}(U, \mathrm{AT}, \widetilde{I})$ is fuzzy complete lattice.

C. Example

Given a fuzzy dataset in Table I, and the fuzzy formal context $(U, \mathrm{AT}, \widetilde{I})$, where $U = \{x_1, x_2, x_3, x_4\}$ and $\mathrm{AT} = \{a, b, c, d, e\}$. From this fuzzy dataset, one can calculate the all fuzzy concepts of this fuzzy formal context $(U, \mathrm{AT}, \widetilde{I})$ as follows:

$$\left(\emptyset, \left\{< a, 1 >, < b, 1 >, < c, 1 >, < d, 1 >, < e, 1 >\right\}\right)$$
$$\left(\{x_1\}\right.$$
$$\left.\left\{< a, 0.9 >, < b, 0.7 >, < c, 0.2 >, < d, 0.9 >, < e, 0.8 >\right\}\right)$$
$$\left(\{x_2\}\right.$$
$$\left.\left\{< a, 0.8 >, < b, 0.8 >, < c, 0.8 >, < d, 0.3 >, < e, 0.2 >\right\}\right)$$
$$\left(\{x_1, x_2\}\right.$$
$$\left.\left\{< a, 0.8 >, < b, 0.7 >, < c, 0.2 >, < d, 0.3 >, < e, 0.2 >\right\}\right)$$
$$\left(\{x_1, x_3\}\right.$$
$$\left.\left\{< a, 0.1 >, < b, 0.2 >, < c, 0.1 >, < d, 0.8 >, < e, 0.2 >\right\}\right)$$
$$\left(\{x_2, x_4\}\right.$$
$$\left.\left\{< a, 0.7 >, < b, 0.8 >, < c, 0.7 >, < d, 0.2 >, < e, 0.2 >\right\}\right)$$
$$\left(\{x_1, x_2, x_3\}\right.$$
$$\left.\left\{< a, 0.1 >, < b, 0.2 >, < c, 0.1 >, < d, 0.3 >, < e, 0.2 >\right\}\right)$$
$$\left(\{x_1, x_2, x_4\}\right.$$
$$\left.\left\{< a, 0.7 >, < b, 0.7 >, < c, 0.2 >, < d, 0.2 >, < e, 0.2 >\right\}\right)$$
$$\left(U\right.$$
$$\left.\left\{< a, 0.1 >, < b, 0.2 >, < c, 0.1 >, < d, 0.2 >, < e, 0.2 >\right\}\right).$$

In this fuzzy formal context, given an object set $X = \{x_1, x_2, x_4\}$ and a fuzzy attribute set $\widetilde{B} = \{< a, 0.7 >, < b, 0.8 >, < c, 0.7 >, < d, 0.1 >\} \in \mathcal{F}(B)$, where $B = \{a, b, c, d\} \subseteq A$. Then, from the definition we can obtain that

$$\widetilde{B}^* = \{x_2, x_4\}$$
$$X^* =$$
$$\left\{< a, 0.7 >, < b, 0.7 >, < c, 0.2 >, < d, 0.2 >, < e, 0.2 >\right\}.$$

One has proposed the fuzzy concepts in fuzzy datasets above, for any fuzzy information granule $(X, \widetilde{B})$, what we would concern about is the relationship between the objects X and the fuzzy attributes $\widetilde{B}$. Intrinsically, the machine-learning process is the learning process between an object and its attributes. When the object is consistent with its attributes, the laws of the object can be grasped, then we say the machine has learned this information granule, namely the pair $(X, \widetilde{B})$. Thus, some of the necessary or sufficient information about the object can be gradually trained. The object is constantly judged using this information until an understanding of the necessary and sufficient attributes of the object is learned. However, in many cases, machine-learning processes are always never crisp, but fuzzy ones. So, this must need new learning methods, the new two-way learning process is between an object and its fuzzy attributes. When the objects are consistent with their fuzzy attributes, the pair, namely objects and their fuzzy attributes, is called the fuzzy concept, and the machine can grasp laws of these objects, then the necessary and sufficient fuzzy information about objects can be gradually obtained. The objects are constantly judged using this fuzzy information until an understanding of the necessary and sufficient fuzzy attributes of this is gained, namely a fuzzy concept is obtained.

The above all in this section represents the notion about fuzzy sets and basic contents about FCA, next we will introduce two-way learning system and three kinds of information granules in fuzzy datasets.

III. Two-Way Learning System and Information Granules in Fuzzy Datasets

In this section, we propose a two-way learning system in fuzzy datasets and discuss fuzzy information granules in this learning system. At first, we will propose a pair of dual operators between an object and its fuzzy attributes to construct a novel-learning system, namely two-way learning system in fuzzy datasets. Let L be a lattice, where 0_L and 1_L are zero and the unit element, respectively.

Definition 1: Let L_1 and $\widetilde{L}_2$ be complete lattice and fuzzy complete lattice, respectively, for any $a_1, a_2 \in L_1$, $\widetilde{\mathcal{F}} : L_1 \to \widetilde{L}_2$ is a fuzzy operator, for any $\widetilde{b}_1, \widetilde{b}_2 \in \widetilde{L}_2$, $\mathcal{P} : \widetilde{L}_2 \to L_1$ is an operator. We call $\widetilde{\mathcal{F}}$ and $\mathcal{P}$ a pair of dual learning operators if they satisfy the following.

1) $\widetilde{\mathcal{F}}(0_{L_1}) = 1_{\widetilde{L}_2}$, $\widetilde{\mathcal{F}}(1_{L_1}) = 0_{\widetilde{L}_2}$.
2) $\widetilde{\mathcal{F}}(a_1 \vee a_2) = \widetilde{\mathcal{F}}(a_1) \wedge \widetilde{\mathcal{F}}(a_2)$.
3) $\mathcal{P}(0_{\widetilde{L}_2}) = 1_{L_1}$, $\mathcal{P}(1_{\widetilde{L}_2}) = 0_{L_1}$.
4) $\mathcal{P}(\widetilde{b}_1 \vee \widetilde{b}_2) = \mathcal{P}(\widetilde{b}_1) \wedge \mathcal{P}(\widetilde{b}_2)$.

Definition 2: A four-tuple $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ is a two-way learning system in fuzzy datasets if the above two operators $\widetilde{\mathcal{F}}$ and $\mathcal{P}$ further satisfy:

1) $\mathcal{P} \circ \widetilde{\mathcal{F}}(a) \geq a$;
2) $\widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}) \geq \widetilde{b}$;

where $\mathcal{P} \circ \widetilde{\mathcal{F}}(a)$ represents $\mathcal{P}\widetilde{\mathcal{F}}(a)$ and $\widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b})$ represents $\widetilde{\mathcal{F}}\mathcal{P}(\widetilde{b})$.

It should be pointed out that the complete lattice $L_1 = (\cap, \cup, \sim, P(U))$ and the fuzzy complete lattice $L_2 = (\wedge, \vee, \sim, \mathcal{F}(\mathrm{AT}))$. From the above definition, we can find that the fuzzy operator $\widetilde{\mathcal{F}}$ and operator $\mathcal{P}$ characterize an object and its fuzzy attributes for the two-way learning system in fuzzy datasets.

Proposition 1: For any $a_1, a_2 \in L_1$ and $\widetilde{b}_1, \widetilde{b}_2 \in \widetilde{L}_2$, a two-way learning system in fuzzy datasets $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ has the following properties.

1) If $a_1 \leq a_2$, then $\widetilde{\mathcal{F}}(a_2) \leq \widetilde{\mathcal{F}}(a_1)$.
2) If $\widetilde{b}_1 \leq \widetilde{b}_2$, then $\mathcal{P}(\widetilde{b}_2) \leq \mathcal{P}(\widetilde{b}_1)$.
3) $\widetilde{\mathcal{F}}(a_1) \vee \widetilde{\mathcal{F}}(a_2) \leq \widetilde{\mathcal{F}}(a_1 \wedge a_2)$.
4) $\mathcal{P}(\widetilde{b}_1) \vee \mathcal{P}(\widetilde{b}_2) \leq \mathcal{P}(\widetilde{b}_1 \wedge \widetilde{b}_2)$.
5) $\widetilde{b} \leq \widetilde{\mathcal{F}}(a) \Leftrightarrow a \leq \mathcal{P}(\widetilde{b})$, $\widetilde{\mathcal{F}}(a) \leq \widetilde{b} \Leftrightarrow \mathcal{P}(\widetilde{b}) \leq a$.
6) For any $a \in L_1$, $\widetilde{\mathcal{F}} \circ \mathcal{P} \circ \widetilde{\mathcal{F}}(a) = \widetilde{\mathcal{F}}(a)$.
7) For any $\widetilde{b} \in \widetilde{L}_2$, $\mathcal{P} \circ \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}) = \mathcal{P}(\widetilde{b})$.

Proof: 1)–5) can be proved by Definition 1 directly.

Property 6): From 1) and $\mathcal{P} \circ \widetilde{\mathcal{F}}(a) \geq a$, we can obtain $\widetilde{\mathcal{F}}(a) \geq \widetilde{\mathcal{F}} \circ \mathcal{P} \circ \widetilde{\mathcal{F}}(a)$. In contrast, from $\widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}) \geq \widetilde{b}$, if takes $b = \widetilde{\mathcal{F}}(a)$, then we can obtain $\widetilde{\mathcal{F}} \circ \mathcal{P} \circ \widetilde{\mathcal{F}}(a) \geq \widetilde{\mathcal{F}}(a)$. Thus, $\widetilde{\mathcal{F}} \circ \mathcal{P} \circ \widetilde{\mathcal{F}}(a) = \widetilde{\mathcal{F}}(a)$.

Property 7) can be proven in the similar way as 6). ∎

Let $(U, \mathrm{AT}, \widetilde{I})$ be a fuzzy formal context in a fuzzy dataset. If $L_1 = P(U)$ and $\widetilde{L}_2 = P(\widetilde{A})$ are denoted, then the operators $(*, *)$ defined in Section II are a pair of dual learning operators of $(U, \mathrm{AT}, \widetilde{I})$.

From the above discussion, a fuzzy dataset can be viewed as the relationship between the objects and fuzzy attributes in two-way learning process. Some concepts that have been identified have been realized. Concepts that have not yet been learned are not formed. To gain more knowledge about this two-way learning process, we must find the dual learning operators in accordance with Definition 1. In fact, these operators are * in a fuzzy formal context.

In the previous section, we established a pair of dual learning operators between the objects and their fuzzy attributes. When using these two dual operators, the inner relationship of an object and its fuzzy attributes can be trained. When the object is consistent with its fuzzy attributes, we can grasp the nature or the laws of the object. Machines begin to obtain things from the unknown. Thus, sufficient or necessary attributes of the unknown objects can be obtained by using these two operators. Next, we will discuss the relationship between the objects and their fuzzy attributes in our learning process with fuzzy information granules by using these two operators.

To reflect on fuzzy information granule description of this two-way learning system in fuzzy datasets, the pair $(a, \widetilde{b})$ is denoted a fuzzy information granule, where a is an object set and $\widetilde{b}$ is a fuzzy attribute set.

Definition 3: Let $L_1 = P(U)$ be a complete lattice and $\widetilde{L}_2 = P(\widetilde{A})$ be a fuzzy complete lattice. $\widetilde{\mathcal{F}}, \mathcal{P}$ are a pair of dual learning operators, i.e. $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ is a two-way learning system in fuzzy datasets. For any $a \in L_1$ and $\widetilde{b} \in \widetilde{L}_2$, denote

$$\mathcal{G}_1 = \left\{(a, \widetilde{b}) | \widetilde{b} \leq \widetilde{\mathcal{F}}(a), a \leq \mathcal{P}(\widetilde{b})\right\}$$
$$\mathcal{G}_2 = \left\{(a, \widetilde{b}) | \widetilde{\mathcal{F}}(a) \leq \widetilde{b}, \mathcal{P}(\widetilde{b}) \leq a\right\}.$$

If $(a, \widetilde{b}) \in \mathcal{G}_1$, then $(a, \widetilde{b})$ is a necessary fuzzy information granule of this two-way learning system in fuzzy datasets and $\widetilde{b}$ is a necessary fuzzy attribute of object a. Simultaneously, $\mathcal{G}_1$ is a necessary fuzzy information granule set of the two-way learning system in fuzzy datasets.

If $(a, \widetilde{b}) \in \mathcal{G}_2$, then $(a, \widetilde{b})$ is a sufficient fuzzy information granule of this two-way learning system in fuzzy datasets and $\widetilde{b}$ is a sufficient fuzzy attribute of object a. Simultaneously, $\mathcal{G}_2$ is a sufficient fuzzy information granule set of the two-way learning system in fuzzy datasets.

If $(a, \widetilde{b}) \in \mathcal{G}_1 \cap \mathcal{G}_2$, that is to say $\widetilde{b} = \widetilde{\mathcal{F}}(a)$ and $a = \mathcal{P}(\widetilde{b})$, then $(a, \widetilde{b})$ is a necessary and sufficient fuzzy information granule of this two-way learning system in fuzzy datasets and $\widetilde{b}$ is a necessary and sufficient fuzzy attribute of object a.

If $(a, \widetilde{b}) \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $(a, \widetilde{b})$ is a fuzzy information granule of this two-way learning system in fuzzy datasets and $\mathcal{G}_1 \cup \mathcal{G}_2$ is a granule set of the two-way learning system in fuzzy datasets.

If $(a, \widetilde{b}) \notin \mathcal{G}_1 \cup \mathcal{G}_2$, then $(a, \widetilde{b})$ is an inconsistent fuzzy information granule of this two-way learning system in fuzzy datasets.

From the above definition, necessary and sufficient fuzzy information granules are fuzzy concepts of the learning system. In fact, these fuzzy concepts also targets in the machine-learning processes. In fuzzy datasets, for the machine learning, the first step is to learn the necessary or sufficient fuzzy information granules. One can gradually seek for the necessary and sufficient granules, namely the fuzzy concepts, based on the already known fuzzy information granules.

Let $(U, \mathrm{AT}, \widetilde{I})$ be a fuzzy formal context in a fuzzy dataset. If $L_1 = P(U)$ and $\widetilde{L}_2 = P(\widetilde{\mathrm{AT}})$ are denoted, the following hold for any $X \in L_1$ and $\widetilde{B} \in \widetilde{L}_2$.

1) If $X^* \supseteq \widetilde{B}$ and $\widetilde{B}^* \supseteq X$, then $\widetilde{B}$ is a necessary fuzzy attribute of X.
2) If $X^* \subseteq \widetilde{B}$ and $\widetilde{B}^* \subseteq X$, then $\widetilde{B}$ is a sufficient fuzzy attribute of X.

From the above discussion, one may seek to understand sufficient or necessary fuzzy attributes when the fuzzy concept is not precise in two-way learning process.

Proposition 2: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ be a two-way learning system in fuzzy datasets and $\mathcal{G}_1$ be a necessary fuzzy information granule set of this two-way learning system. If $\wedge$ and $\vee$ are defined operators of $\mathcal{G}_1$ and

$$\left(a_1, \widetilde{b}_1\right) \wedge \left(a_2, \widetilde{b}_2\right) = \left(a_1 \wedge a_2, \widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right)\right)$$
$$\left(a_1, \widetilde{b}_1\right) \vee \left(a_2, \widetilde{b}_2\right) = \left(\mathcal{P} \circ \widetilde{\mathcal{F}}\left(a_1 \vee a_2\right), \widetilde{b}_1 \wedge \widetilde{b}_2\right).$$

$(\mathcal{G}_1, \leq)$ is then closed with respect to operators $\wedge$ and $\vee$.

Proof: Assume that $(a_1, \widetilde{b}_1), (a_2, \widetilde{b}_2) \in \mathcal{G}_1$, then

$$\widetilde{b}_1 \leq \widetilde{\mathcal{F}}(a_1), \widetilde{b}_2 \leq \widetilde{\mathcal{F}}(a_2)$$
$$a_1 \leq \mathcal{P}(\widetilde{b}_1), a_2 \leq \mathcal{P}(\widetilde{b}_2)$$

and

$$\begin{aligned} a_1 \wedge a_2 \leq \mathcal{P}\left(\widetilde{b}_1\right) \wedge \mathcal{P}\left(\widetilde{b}_2\right) &= \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right) \\ &= \mathcal{P} \circ \mathcal{F} \circ \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right). \end{aligned}$$

Moreover through Proposition 1, we find that

$$\widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right) = \widetilde{\mathcal{F}}\left(\mathcal{P}\left(\widetilde{b}_1\right) \wedge \mathcal{P}\left(\widetilde{b}_2\right)\right) \leq \widetilde{\mathcal{F}}\left(a_1 \wedge a_2\right).$$

Thus, $(a_1, \widetilde{b}_1) \wedge (a_2, \widetilde{b}_2)$ is a necessary fuzzy information granule, which means $(a_1, \widetilde{b}_1) \wedge (a_2, \widetilde{b}_2) \in \mathcal{G}_1$.

$(a_1, \widetilde{b}_1) \vee (a_2, \widetilde{b}_2) \in \mathcal{G}_1$ can be proven similarly.

The proposition is proven. ■

In fuzzy datasets, Proposition 2 indicates the law of the two operators $\wedge$ and $\vee$ among necessary fuzzy information granules in the two-way learning system.

Proposition 3: Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ be a two-way learning system in fuzzy datasets and $\mathcal{G}_2$ be a sufficient fuzzy information granule set of this two-way learning system. If $\wedge$ and $\vee$ are defined operators of $\mathcal{G}_2$ and

$$\left(a_1, \widetilde{b}_1\right) \wedge \left(a_2, \widetilde{b}_2\right) = \left(a_1 \wedge a_2, \widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right)\right)$$
$$\left(a_1, \widetilde{b}_1\right) \vee \left(a_2, \widetilde{b}_2\right) = \left(\mathcal{P} \circ \widetilde{\mathcal{F}}(a_1 \vee a_2), \widetilde{b}_1 \wedge \widetilde{b}_2\right).$$

$(\mathcal{G}_2, \leq)$ is then closed with respect to operators $\wedge$ and $\vee$.

Proof: Let $(a_1, \widetilde{b}_1), (a_2, \widetilde{b}_2) \in \mathcal{G}_2$, then

$$\widetilde{\mathcal{F}}(a_1) \leq \widetilde{b}_1, \widetilde{\mathcal{F}}(a_2) \leq \widetilde{b}_2$$
$$\mathcal{P}(\widetilde{b}_1) \leq a_1, \mathcal{P}(\widetilde{b}_2) \leq a_2$$

and

$$\begin{aligned} \mathcal{P} \circ \widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right) &= \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right) \\ &= \mathcal{P}\left(\widetilde{b}_1\right) \wedge \mathcal{P}\left(\widetilde{b}_2\right) \leq a_1 \wedge a_2. \end{aligned}$$

Moreover through Proposition 1, we find that

$$\widetilde{\mathcal{F}}\left(a_1 \wedge a_2\right) \leq \widetilde{\mathcal{F}}\left(\mathcal{P}\left(\widetilde{b}_1\right) \wedge \mathcal{P}\left(\widetilde{b}_2\right)\right) = \widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1 \vee \widetilde{b}_2\right).$$

Thus, $(a_1, \widetilde{b}_1) \wedge (a_2, \widetilde{b}_2)$ is a sufficient fuzzy information granule, which means $(a_1, \widetilde{b}_1) \wedge (a_2, \widetilde{b}_2) \in \mathcal{G}_2$.

$(a_1, \widetilde{b}_1) \vee (a_2, \widetilde{b}_2) \in \mathcal{G}_2$ can be proven similarly.

The proposition is proven. ■

In fuzzy datasets, Proposition 3 indicates the law of the two operators $\wedge$ and $\vee$ among sufficient fuzzy information granules in the two-way learning system.

Because $\leq$ is a quasi-order relationship in $(\mathcal{G}_1, \leq)$ and $(\mathcal{G}_2, \leq)$, these relationships are not fuzzy lattices with respect to operators $\wedge$ and $\vee$, termed fuzzy quasi-lattices.

IV. Mechanism of Two-Way Learning System and GrC in Fuzzy Datasets

Essentially, necessary and sufficient fuzzy information granule is a fuzzy concept. As previously described, machine began to obtain fuzzy concepts from fuzzy datasets. In other words, sufficient and necessary fuzzy information granules do not exist at the beginning of the learning system in fuzzy datasets. That is to say, machine can learn the necessary fuzzy information granules, sufficient fuzzy information granules, and necessary and sufficient fuzzy information granules from general fuzzy information granules, respectively.

Case 1: Training of necessary fuzzy information granules from arbitrary fuzzy information granules. Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ be a two-way learning system in fuzzy datasets and $\mathcal{G}_1$ be a necessary fuzzy information granule set of the two-way learning system. If $a \in L_1$, $\widetilde{b} \in \widetilde{L}_2$, then:

1) $(a \wedge \mathcal{P}(\widetilde{b}), \widetilde{b} \vee \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$;
2) $(a \vee \mathcal{P}(\widetilde{b}), \widetilde{b} \wedge \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$;
3) $(\mathcal{P}(\widetilde{b}), \widetilde{b} \wedge \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$;
4) $(a \wedge \mathcal{P}(\widetilde{b}), \mathcal{F}(a)) \in \mathcal{G}_1$;
5) $(\mathcal{P} \circ \widetilde{\mathcal{F}}(a), \widetilde{b} \wedge \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$;
6) $(a \wedge \mathcal{P}(\widetilde{b}), \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b})) \in \mathcal{G}_1$.

Proof:

1) Because $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ is a two-way learning system in fuzzy datasets, from Proposition 1 and Definition 2, we have $\widetilde{\mathcal{F}}(a \wedge \mathcal{P}(\widetilde{b})) \geq \widetilde{\mathcal{F}}(a) \vee \widetilde{\mathcal{F}}(\mathcal{P}(\widetilde{b})) \geq \widetilde{\mathcal{F}}(a) \vee \widetilde{b}$ and $\mathcal{P}(\widetilde{b} \vee \widetilde{\mathcal{F}}(a)) = \mathcal{P} \wedge \mathcal{P}(\widetilde{\mathcal{F}}(a)) \geq a \wedge \mathcal{P}(\widetilde{b})$. Thus, $(a \wedge \mathcal{P}(\widetilde{b}), \widetilde{b} \vee \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$.
2) The way to prove it is similar to 1).
3) Because $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ is a two-way learning system in fuzzy datasets, from Proposition 1 and Definition 2, we have $\widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}) \geq b \geq \widetilde{\mathcal{F}}(a) \wedge \widetilde{b}$ and $\mathcal{P}(\widetilde{B} \wedge \widetilde{\mathcal{F}}(a)) \geq \mathcal{P} \circ \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}) = \mathcal{P}(\widetilde{b})$. Thus, $(\mathcal{P}(\widetilde{b}), \widetilde{b} \wedge \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$.
4) The way to prove it is similar to 3).
5) Because $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ is a two-way learning system in fuzzy datasets, from Proposition 1 and Definition 2, we have $\widetilde{\mathcal{F}} \circ \mathcal{P} \circ \widetilde{\mathcal{F}}(a) = \mathcal{F}(a) \geq \widetilde{\mathcal{F}}(a) \wedge \widetilde{b}$ and $\mathcal{P}(\widetilde{\mathcal{F}}(a) \wedge \widetilde{b}) \geq \mathcal{P} \circ \widetilde{\mathcal{F}}(a) \vee \mathcal{P}(\widetilde{b}) \geq \mathcal{P} \circ \widetilde{\mathcal{F}}(a)$. Thus, $(\mathcal{P} \circ \widetilde{\mathcal{F}}(a), \widetilde{b} \wedge \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_1$.
6) The way to prove it is similar to 5). ■

It is easy to see that necessary fuzzy information granules can be obtained from these six training methods.

Case 2: Training of sufficient fuzzy information granules from arbitrary fuzzy information granules. Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ be a two-way learning system in fuzzy datasets and $\mathcal{G}_2$ be a sufficient fuzzy information granule set of the two-way learning system. If $a \in L_1$, $\widetilde{b} \in \widetilde{L}_2$, then:

1) $(\mathcal{P} \circ \widetilde{\mathcal{F}}(a), \widetilde{b} \vee \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_2$;
2) $(a \vee \mathcal{P}(\widetilde{b}), \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b})) \in \mathcal{G}_2$.

Proof:

1) Because $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ is a two-way learning system in fuzzy datasets, from Proposition 1 and Definition 2, we have $\widetilde{\mathcal{F}} \circ \mathcal{P} \circ \widetilde{\mathcal{F}}(a) = \widetilde{\mathcal{F}}(a) \leq \widetilde{\mathcal{F}}(a) \vee \widetilde{b}$ and $\mathcal{P}(\widetilde{\mathcal{F}}(a) \wedge \widetilde{b}) = \mathcal{P} \circ \widetilde{\mathcal{F}}(a) \wedge \mathcal{P}(\widetilde{b}) \leq \mathcal{P} \circ \widetilde{\mathcal{F}}(a)$. Thus, $(\mathcal{P} \circ \widetilde{\mathcal{F}}(a), \widetilde{b} \vee \widetilde{\mathcal{F}}(a)) \in \mathcal{G}_2$.
2) The way to prove it is similar to 1). ■

It is easy to see that sufficient fuzzy information granules can be obtained from these six training methods.

From the above Cases 1 and 2, the machine can learn the very useful fuzzy information granules from useless fuzzy information granules using two-way learning system in fuzzy datasets. If machine does not receive the necessary and sufficient fuzzy information granules in the learning system, it cannot fully grasp fuzzy information granules given. To fully learn fuzzy information granules, in the following, we will show how to obtain necessary and sufficient fuzzy information granules from necessary, sufficient fuzzy information granules, respectively.

Case 3: Training of necessary and sufficient fuzzy information granules from necessary fuzzy information granules. Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ be a two-way learning system in fuzzy datasets and $\mathcal{G}_1$ be a necessary fuzzy information granule set of this two-way learning system. If $(a_1, \widetilde{b}_1) \in \mathcal{G}_1$, then:

1) $(a_1 \vee \mathcal{P}(\widetilde{b}_1), \widetilde{\mathcal{F}}(a_1 \vee \mathcal{P}(\widetilde{b}_1))) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
2) $(\mathcal{P}(\widetilde{b}_1 \vee \widetilde{\mathcal{F}}(a_1)), \widetilde{b}_1 \vee \widetilde{\mathcal{F}}(a_1)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof:

1) Cause $(a_1, \widetilde{b}_1) \in \mathcal{G}_1$, we can have $a_1 \leq \mathcal{P}(\widetilde{b}_1)$ and $\widetilde{b}_1 \leq \widetilde{\mathcal{F}}(a_1)$. Then, $a_1 \vee \mathcal{P}(\widetilde{b}_1) = \mathcal{P}(\widetilde{b}_1)$, $\widetilde{\mathcal{F}}(a_1 \vee \mathcal{P}(\widetilde{b}_1)) = \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}_1)$. Thus, from the Definition 2 and Proposition 1, we can get the following:

$$\widetilde{\mathcal{F}}\left(a_1 \vee \mathcal{P}\left(\widetilde{b}_1\right)\right) = \widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1\right) = \widetilde{\mathcal{F}}\left(a_1 \vee \mathcal{P}\left(\widetilde{b}_1\right)\right)$$
$$\mathcal{P}\left(\widetilde{\mathcal{F}}\left(a_1 \vee \mathcal{P}\left(\widetilde{b}_1\right)\right)\right) = \mathcal{P} \circ \widetilde{\mathcal{F}} \circ \mathcal{P} = \mathcal{P}\left(\widetilde{b}_1\right) = a_1 \vee \mathcal{P}\left(\widetilde{b}_1\right).$$

Hence, $(a_1 \vee \mathcal{P}(\widetilde{b}_1), \widetilde{\mathcal{F}}(a_1 \vee \mathcal{P}(\widetilde{b}_1)))$ is a necessary and sufficient fuzzy information granule.
2) This item can be obtained similarly. ■

It is easy to see that necessary and sufficient fuzzy information granules can be obtained from necessary fuzzy information granule by the two methods in Case 3.

Case 4: Training of necessary and sufficient fuzzy information granules from sufficient fuzzy information granules. Let $(L_1, \widetilde{L}_2, \widetilde{\mathcal{F}}, \mathcal{P})$ be a two-way learning system in fuzzy datasets and $\mathcal{G}_2$ be a sufficient fuzzy information granule set of this two-way learning system. If $(a_1, \widetilde{b}_1) \in \mathcal{G}_2$, then:

1) $(a_1 \wedge \mathcal{P}(\widetilde{b}_1), \widetilde{\mathcal{F}}(a_1 \wedge \mathcal{P}(\widetilde{b}_1))) \in \mathcal{G}_1 \cap \mathcal{G}_2$;
2) $(\mathcal{P}(b_1 \wedge \widetilde{\mathcal{F}}(a_1)), \widetilde{b}_1 \wedge \widetilde{\mathcal{F}}(a_1)) \in \mathcal{G}_1 \cap \mathcal{G}_2$.

Proof:

1) Cause $(a_1, \widetilde{b}_1) \in \mathcal{G}_2$, we can have $\widetilde{\mathcal{F}}(a_1) \leq \widetilde{b}_1$ and $\mathcal{P}(\widetilde{b}_1) \leq a_1$. Then, $a_1 \wedge \mathcal{P}(\widetilde{b}_1) = \mathcal{P}(\widetilde{b}_1)$, $\widetilde{\mathcal{F}}(a_1 \wedge \mathcal{P}(\widetilde{b}_1)) = \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{b}_1)$. Thus, from the Definition 2 and Proposition 1, we can get the following:

$$\widetilde{\mathcal{F}}\left(a_1 \wedge \mathcal{P}\left(\widetilde{b}_1\right)\right) = \widetilde{\mathcal{F}} \circ \mathcal{P}\left(\widetilde{b}_1\right)$$
$$\mathcal{P}\left(\widetilde{\mathcal{F}}\left(a_1 \wedge \mathcal{P}\left(\widetilde{b}_1\right)\right)\right) = \mathcal{P} \circ \widetilde{\mathcal{F}} \circ \mathcal{P} = \mathcal{P}\left(\widetilde{b}_1\right) = a_1 \vee \mathcal{P}\left(\widetilde{b}_1\right).$$

Hence, $(a_1 \wedge \mathcal{P}(\widetilde{b}_1), \widetilde{\mathcal{F}}(a_1 \wedge \mathcal{P}(\widetilde{b}_1)))$ is a necessary and sufficient fuzzy information granule.
2) This item can be obtained similarly. ■

It is easy to see that necessary and sufficient fuzzy information granules can be obtained from sufficient fuzzy information granule by the two methods in Case 4.

From the above ways in Cases 1–4, we come to the following conclusions.

1) There are six methods to learn necessary fuzzy information granules from a general fuzzy information granule.
2) There are two methods to learn necessary and sufficient fuzzy information granules from a necessary fuzzy information granule.
3) There are two methods to learn sufficient fuzzy information granules from a general fuzzy information granule.
4) There are two methods to learn necessary and sufficient fuzzy information granules from a sufficient fuzzy information granule.

So, we can obtain 16 methods to learn necessary and sufficient information granules from a general fuzzy information granule. The processes of two-way learning approach can be described in Fig. 1.

V. Learning Algorithm and Experiments

According to the above theory proposed, one can train necessary, sufficient, necessary and sufficient fuzzy information granules from an arbitrary fuzzy information granule through

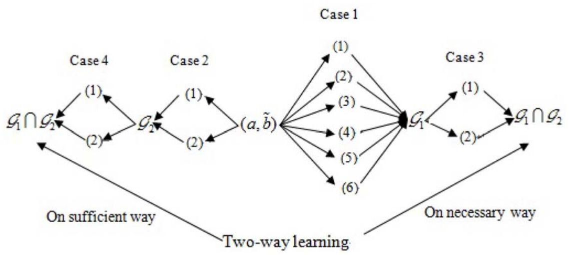

Fig. 1. Processes of two-way learning approach.

the computer programs. In this section, we present the algorithm of the learning methods and a real-life case study, then give experimental evaluation by five datasets from the UCI dataset.

A. Two-Way Learning Algorithm in Fuzzy Datasets

Algorithm: The algorithm of two-way learning in fuzzy datasets is described as follows.

Input: A fuzzy information table and an arbitrary fuzzy information granule.

Output: Necessary fuzzy information granules, sufficient fuzzy information granules, and necessary and sufficient fuzzy information granules.

Step 1: Load the fuzzy information table, do initialized setting and compute number of objects and attributes.

Step 2: Input the arbitrary fuzzy information granule $(a_1, \widetilde{b}_1)$ and select randomly one of the two channels.

Step 3: If the fuzzy information granule is not necessary fuzzy information granule, then skip to the Step 5. Otherwise, turn to the Step 7.

Step 4: If the fuzzy information granule is not sufficient fuzzy information granule, then skip to the Step 6. Otherwise, turn to the Step 8.

Step 5: Learn necessary fuzzy information granules $(a_2^1, \widetilde{b}_2^1)$ from the general fuzzy information granule $(a_1, \widetilde{b}_1)$, $(a_2^2, \widetilde{b}_2^2), \cdots, (a_2^m, \widetilde{b}_2^m)$, $m \leq 6$ by choosing one of six methods in Case 1.

Step 6: Learn sufficient fuzzy information granules $(a_3^1, \widetilde{b}_3^1)$ from the general fuzzy information granule $(a_1, \widetilde{b}_1)$, $(a_3^2, \widetilde{b}_3^2), \cdots, (a_3^n, \widetilde{b}_3^n)$, $n \leq 2$ by choosing one of two methods in Case 2.

Step 7: Learn necessary and sufficient fuzzy information granules $(a_4^2, \widetilde{b}_4^2), \cdots, (a_4^r, \widetilde{b}_4^r)$, $r \leq 12$ from the necessary fuzzy information granules in Step 5 by choosing the methods in Case 3.

Step 8: Learn necessary and sufficient fuzzy information granules $(a_5^2, \widetilde{b}_5^2), \cdots, (a_5^s, \widetilde{b}_5^s)$, $s \leq 4$ from the sufficient fuzzy information granules in Step 6 by choosing the methods in Case 4.

Step 9: Output necessary fuzzy information granules, sufficient fuzzy information granules, and necessary and sufficient fuzzy information granules. And algorithm is finished.

Experimental computing program can be designed and carried out so as to apply the algorithm studied more directly and practical in this paper. Let $N_i(i = 1, 2, 3, 4, 5, 6)$, $S_i(i = 1, 2)$, $C_i(i = 1, 2)$, $D_i(i = 1, 2)$ stand for the methods in Cases 1–4, respectively. The main process of the computing program will be introduced by the flow chart (Fig. 2) and cases are employed to verify the program.

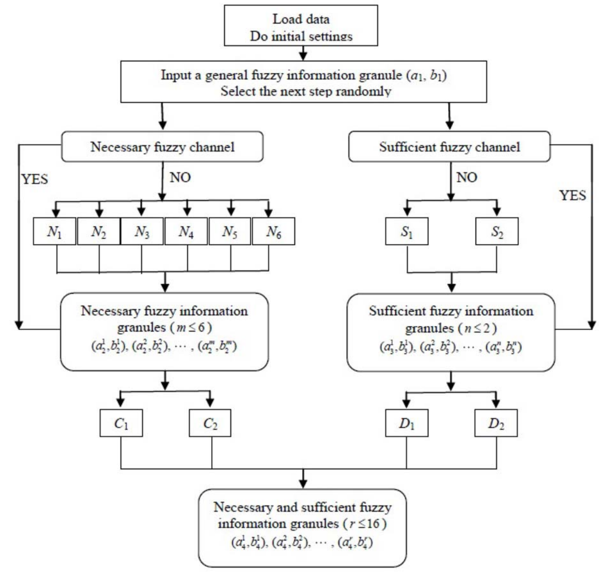

Fig. 2. Program flow graph of learning system based on FCA in fuzzy datasets.

Explanation of Fig. 2: The two-way learning algorithm is in accordance with the flow graph. Input an arbitrary fuzzy information granule (denoted by (a_1, b_1) in the flow graph), we can first choose the necessary fuzzy channel and sufficient fuzzy channel to get the corresponding necessary fuzzy information granules and sufficient fuzzy information granules by the six methods in Case 1 and two methods in Case 3, respectively. Because the different methods may get a same fuzzy information granule, the number of obtaining new necessary and sufficient fuzzy information granules are less than 6 and 2, which indicated in the flow graph $m \leq 6$ and $n \leq 2$. After that, we get in the next step to get the necessary and sufficient fuzzy information granules from results in the previous step, here, methods $C_i(i = 1, 2)$ and $D_i(i = 1, 2)$ which from Cases 3 and 4 can be used to obtain necessary and sufficient fuzzy information granules from necessary fuzzy channel and sufficient fuzzy channel, respectively. The total number of necessary and sufficient fuzzy information granules learning from (a_1, b_1) is 16, while some of them are the same by accident, which indicated in the flow graph $r \leq 16$.

B. Time Complexity Analysis

Let $(U, \mathrm{AT}, \widetilde{I})$ be a fuzzy formal context. The number of objects is denoted by $|U|$. The number of attributes is denoted by $|\mathrm{AT}|$. The cardinal number of the objects of arbitrary fuzzy information granule $(a_1, \widetilde{b}_1)$ is L. We take a variable t_i to stand for the time complexity in an implementation. In the next, we can analyze the time complexity of two-way learning algorithm step-by-step.

The time complexity to do initialized setting and input the arbitrary fuzzy information granule $(a_1, \widetilde{b}_1)$ is 0, then the analysis to Steps 1 and 2 are finished.

For Steps 3 and 4, the time complexity to calculate the $\mathcal{F}(a_1)$ and $\mathcal{P}(\widetilde{b}_1)$ is denoted by $t_1 = (L - 1) \times |\mathrm{AT}|$ and

TABLE II
DATA DESCRIPTION

Dataset	*Samples*	*Features*
Letter recognition	8084	16
Vehicle	846	18
Wholesale customers	440	6
Winequality-Red	1599	12
Winequality-White	4898	12

$t_2 = |U| \times |\mathrm{AT}|$. The time to judge whether $(a_1, \widetilde{b}_1)$ is a necessary fuzzy information granule or a sufficient fuzzy information granule or not the two is $t_3 = (L-1) \times |U| + |\mathrm{AT}|$. Thus, the time complexity to finish Steps 3 and 4 is $t_4 = t_1 + t_2 + t_3 = |U| \times |\mathrm{AT}| + L \times |\mathrm{AT}| + L \times |U| - |U|$. The time complexity of Steps 3 and 4 is completed.

The first four steps are preparations to calculate necessary fuzzy information granule, sufficient fuzzy information granule, and necessary and sufficient fuzzy information granule. The next Step 5 to learn necessary fuzzy information granules from general fuzzy information granule. There are six methods and we will discuss them, respectively. For the methods N_1, N_2, N_3, time complexities are the same, all of these three are $U \times |\mathrm{AT}| + |\mathrm{AT}|$. For the method N_4, the time complexity is $|U| \times |\mathrm{AT}| + (L-1) \times |\mathrm{AT}|$. The time complexity of method N_5 is $(L-1) \times |\mathrm{AT}| + |U| \times |\mathrm{AT}| + |\mathrm{AT}|$. The method N_6 is $2 \times |U| \times |\mathrm{AT}| + (L-1) \times |\mathrm{AT}|$. Then, we have the total time complexity of Step 5 is $t_5 = 7 \times |U| \times |\mathrm{AT}| + 3 \times L \times |\mathrm{AT}| - |\mathrm{AT}|$. The time complexity of obtaining sufficient fuzzy information granules in Step 6 is $t_6 = 3 \times |U| \times |\mathrm{AT}| + 2 \times L \times |\mathrm{AT}| - |\mathrm{AT}|$.

Since, we have calculated the time complexity before Step 7. For every method in Steps 5 and 6, it must be calculated by the two methods in Steps 7 and 8, respectively. Then, we can get the time complexity of Steps 7 and 8 as $t_7 = 12 \times |U| \times |\mathrm{AT}| + 6 \times |U| \times L + 6 \times |\mathrm{AT}|$ and $t_8 = 2 \times |U| \times |\mathrm{AT}| + 2 \times L \times |\mathrm{AT}| + 2 \times |U| \times L$. The time complexity of Step 9 will be not considered, since the output runs fast and direct.

From the above analysis, we can know that the maximum time complexity of the main part in the algorithm (Steps 1–8) is

$$\begin{aligned} t_{\mathrm{main}} &= t_4 + t_5 + t_6 + t_7 + t_8 \\ &= 25 \times |U| \times |\mathrm{AT}| + 8 \times L \times |\mathrm{AT}| + \\ &\quad (9 \times L - 1) \times |U| + 4 \times |\mathrm{AT}|. \end{aligned}$$

As L $(1 \le L \le |U|)$ is the cardinal number of the initial objects, the maximum complexity of the main algorithm is approximately $O(|U|^2 + |U| \times |\mathrm{AT}|)$.

C. Case Study and Experimental Evaluation

In order to interpret the two-way learning approach, a case study about loans to the countries from the United Nations is constructed, and we perform the empirical experiments by the C++ programme about some datasets from the UCI.

Firstly, we give a fuzzy dataset about the basic situations of the developing countries presented in the Appendix. This fuzzy formal context is denoted by $(U, \mathrm{AT}, \widetilde{I})$, where U is combined by the 125 countries and AT is the fuzzy attributes, which includes the following seven elements.

Growth rate of population (GRP) (Data from zh.wikipedia.org/wiki/%E5%90%84%E5%9B%BD%E4%BA%BA%E5%8F%A3%E8%87%AA%E7%84%B6%E5%A2%9E%E9%95%BF%E7%8E%87%E5%88%97%E8%A1%A8).

Urbanization rate (UR) (Data from www.iefnews.com/NewsView.Asp?ID=1296).

Economic growth index (EGI) (Data from bbs.tianya.cn/post-worldlook-710506-1.shtml).

Human development index (HDI) (Data from zh.wikipedia.org/wiki/%E4%BA%BA%E9%A1%9E%E7%99%BC%E5%B1%95%E6%8C%87%E6%95%B8%E5%88%97%E8%A1%A8).

Degree of education (DE) (Data from club.kdnet.net/dispbbs.asp?boardid=1&id=6958472).

Government support (GS) (Artificial construction).

Forest coverage rate (FCR) (Data from www.360doc.com/content/13/0323/17/9159788_273451073.shtml).

To encourage the developing countries to develop their economic, now the United Nations offers grant loans to the countries in need. There are about 125 candidate developing countries in the world. United Nations must consider several conditions to make it much equality, which includes seven fuzzy attributes. The methods proposed in this paper can be used to select the suitable countries. Through several rounds of voting and selection, one may get the initial fuzzy information granule $(X_0, \widetilde{B}_0)$, where X_0 is the countries and $\widetilde{B}_0$ is the fuzzy attributes. While this result $(X_0, \widetilde{B}_0)$ may induce the situations that the countries being selected do not satisfy the given fuzzy attributes and the countries satisfied the given attributes are not selected. Here, now given $X_0 = \{x_1, x_9, x_{15}, x_{31}, x_{38}, x_{46}, x_{55}, x_{82}, x_{88}, x_{99}, x_{100}, x_{117}, x_{125}\}$ and $\widetilde{B}_0 = \{< \mathrm{GRP}, 0.01 >, < \mathrm{EGI}, 0.05 >, < \mathrm{HDI}, 0.55 >, < \mathrm{DE}, 0.70 >, < \mathrm{GS}, 0.75 >\}$. When the fund is abundant, then the United Nations can consider to loan much more countries. The sufficient fuzzy information granule is a good choice. We can compute the sufficient fuzzy information granules of $(X_0, \widetilde{B}_0)$ by the above program in Case 2 as follows.

1) $(\mathcal{P} \circ \widetilde{\mathcal{F}}(X_0), \widetilde{B}_0 \vee \widetilde{\mathcal{F}}(X_0)) = (\{x_1, x_2, x_3, x_4, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{20}, x_{23}, x_{24}, x_{28}, x_{30}, x_{31}, x_{32}, x_{37}, x_{38}, x_{42}, x_{45}, x_{46}, x_{47}, x_{48}, x_{53}, x_{54}, x_{55}, x_{57}, x_{58}, x_{59}, x_{60}, x_{66}, x_{67}, x_{70}, x_{71}, x_{73}, x_{79}, x_{81}, x_{82}, x_{84}, x_{87}, x_{88}, x_{89}, x_{94}, x_{96}, x_{98}, x_{99}, x_{100}, x_{102}, x_{106}, x_{110}, x_{111}, x_{114}, x_{117}, x_{118}, x_{120}, x_{121}\ x_{125}\}, \{< \mathrm{GRP}, 0.01 >, < \mathrm{UR}, 0.21 >, < \mathrm{EGI}, 0.05 >\ < \mathrm{HDI}, 0.55 >, < \mathrm{DE}, 0.70 >, < \mathrm{GS}, 0.75 >, < \mathrm{FCR}, 0.15 >\})$.
2) $(X_0 \vee \mathcal{P}(\widetilde{B}_0), \widetilde{\mathcal{F}} \circ \mathcal{P}(\widetilde{B}_0)) = (\{x_1, x_9, x_{12}, x_4, x_{15}, x_{23}, x_{24}, x_{31}, x_{38}, x_{46}, x_{47}, x_{48}, x_{55}, x_{58}, x_{82}, x_{87}, x_{88}, x_{89}, x_{94}, x_{99}, x_{100}, x_{101}, x_{110}, x_{117}, x_{120}, x_{121}, x_{125}\}, \{< \mathrm{GRP}, 0.01 >, < \mathrm{UR}, 0.15 >, < \mathrm{EGI}, 0.05 >, < \mathrm{HDI}, 0.57 >, < \mathrm{DE}, 0.70 >, < \mathrm{GS}, 0.79 >, < \mathrm{FCR}, 0.15 >\})$.

It should be noted that the learning methods from initial fuzzy information granule may generate the same sufficient fuzzy information granules. Furthermore, if the United Nations hope that the selected countries must satisfy the given fuzzy attributes and all the countries which satisfy the given fuzzy attributes must be selected. Now, the necessary and sufficient fuzzy information granule is a good choice. Thus, we can compute the necessary and sufficient information granules of

TABLE III
NUMBER OF FUZZY CONCEPTS BASED ON MEMBERSHIP FUNCTIONS (LETTER RECOGNITION)

Letter recognition		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-808	2	2	3	3	3	2	3	3	3
	3638-4446	2	2	3	3	3	2	3	2	3
	7276-8083	2	2	3	3	3	2	3	2	3
30%	1-2425	2	2	3	2	2	2	3	2	3
	2829-5255	2	2	3	2	3	2	3	2	3
	5558-8084	2	2	3	2	3	2	2	2	3
60%	1-4850	2	2	3	2	2	2	2	2	3
	1617-6467	2	2	2	2	2	2	2	2	2
	3234-8084	2	2	2	2	2	2	2	2	3
80%	1-6467	2	2	2	2	2	2	2	2	2
	808-7177	2	2	2	2	2	2	2	2	2
	1617-8084	2	2	2	2	2	2	2	2	2

TABLE IV
NUMBER OF FUZZY CONCEPTS BASED ON MEMBERSHIP FUNCTIONS (VEHICLE)

Vehicle		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-85	3	3	4	3	3	3	3	3	3
	383-468	3	3	4	3	3	3	3	3	3
	765-846	2	3	4	2	3	3	3	3	3
30%	1-254	3	3	3	3	3	3	3	3	3
	299-553	3	3	4	3	3	3	3	3	3
	591-846	2	3	3	2	3	3	2	3	3
60%	1-508	2	2	3	2	3	3	3	3	3
	170-676	3	3	3	3	3	3	3	3	3
	337-846	2	3	3	2	3	3	2	3	3
80%	1-677	2	2	2	2	2	2	2	2	2
	84-762	2	2	2	2	2	2	2	2	2
	168-846	2	3	3	2	3	3	2	3	3

TABLE V
NUMBER OF FUZZY CONCEPTS BASED ON MEMBERSHIP FUNCTIONS (WHOLESALE CUSTOMERS)

Wholesale customers		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-44	2	2	3	2	2	3	2	2	3
	198-242	2	2	3	2	2	3	2	2	3
	396-440	2	2	3	2	2	3	2	2	3
30%	1-132	2	2	3	2	2	3	2	2	3
	154-286	2	2	3	2	2	3	2	2	3
	318-440	2	2	3	2	2	3	2	2	3
60%	1-264	2	2	2	2	2	2	2	2	2
	88-352	2	2	2	2	2	2	2	2	2
	176-440	2	2	3	2	2	3	2	2	3
80%	1-352	2	2	2	2	2	2	2	2	2
	44-390	2	2	2	2	2	2	2	2	2
	88-440	2	2	2	2	2	2	2	2	2

TABLE VI
NUMBER OF FUZZY CONCEPTS BASED ON MEMBERSHIP FUNCTIONS (WINEQUALITY-RED)

Winequality-Red		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-160	3	3	5	4	3	3	3	3	3
	675-825	3	4	6	3	4	3	3	2	3
	1440-1599	3	2	6	3	4	3	3	3	2
30%	1-480	3	3	5	3	3	3	3	3	3
	560-1040	3	2	5	3	4	3	3	3	3
	1120-1599	3	2	6	3	4	3	3	2	2
60%	1-960	3	3	5	3	3	3	3	3	3
	320-1280	3	2	5	3	4	3	3	3	3
	640-1599	3	2	6	3	4	3	3	2	2
80%	1-1280	2	2	3	2	2	3	2	2	3
	160-1440	3	2	4	3	3	3	3	3	3
	320-1599	3	2	5	3	3	2	3	3	3

$(X_0, \widetilde{B}_0)$ from the learning methods in Case 4 (it should be noted that different necessary fuzzy information granules may generate the same necessary and sufficient fuzzy information granules by accident).

1) $(\{x_{12}, x_{14}, x_{23}, x_{24}, x_{38}, x_{47}, x_{48}, x_{58}, x_{82}, x_{84}, x_{87}, x_{88}, x_{89}, x_{94}, x_{99}, x_{110}, x_{120}, x_{121}\}, \{< \text{GRP}, 0.01 >, < \text{UR}, 0.15 >, < \text{EGI}, 0.05 >, < \text{HDI}, 0.57 >, < \text{DE}, 0.70 >, < \text{GS}, 0.79 >, < \text{FCR}, 0.15 >\})$.

TABLE VII
NUMBER OF FUZZY CONCEPTS BASED ON MEMBERSHIP FUNCTIONS (WINEQUALITY-WHITE)

Winequality-White		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-490	3	4	7	3	3	3	3	3	4
	2205-2695	3	3	7	3	4	3	3	3	4
	4410-4898	3	3	7	3	3	3	3	3	3
30%	1-1470	3	3	7	3	3	3	3	3	3
	1715-3185	3	3	5	3	3	3	3	3	3
	3327-4898	3	2	5	3	3	2	3	3	3
60%	1-2940	3	3	7	3	3	3	3	3	3
	980-3920	3	3	5	3	3	3	3	3	3
	1960-4898	3	2	5	3	3	2	3	3	3
80%	1-3920	3	3	4	3	3	3	3	3	3
	490-4410	2	2	3	2	2	2	2	2	2
	980-4898	2	2	4	2	2	2	2	2	2

TABLE VIII
RUNNING TIME BASED ON MEMBERSHIP FUNCTIONS (LETTER RECOGNITION)

Letter recognition		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-808	1.1150s	1.1040s	1.0850s	1.1250s	1.0940s	1.2250s	1.1260s	1.1240s	1.1060s
	3638-4446	1.0860s	1.0870s	1.0610s	1.0770s	1.0780s	1.0920s	1.0710s	1.0620s	1.0760s
	7276-8084	1.0460s	1.1180s	1.0930s	1.0610s	1.0610s	1.0770s	1.1280s	1.0910s	1.0780s
30%	1-2425	1.0780s	1.0870s	1.0920s	1.0780s	1.0930s	1.1250s	1.1030s	1.1250s	1.0820s
	2829-5255	1.0920s	1.0880s	1.1250s	1.0610s	1.0780s	1.1400s	1.1020s	1.0930s	1.0790s
	5558-8084	1.0780s	1.0870s	1.0770s	1.0620s	1.0610s	1.0940s	1.1180s	1.0830s	1.0800s
60%	1-4850	1.0730s	1.0870s	1.0930s	1.0610s	1.0780s	1.1080s	1.1030s	1.0890s	1.1120s
	1617-6467	1.0920s	1.0710s	1.0780s	1.0610s	1.0930s	1.0920s	1.1180s	1.0770s	1.0780s
	3234-8084	1.1250s	1.0870s	1.0920s	1.0780s	1.0770s	1.0920s	1.1030s	1.0890s	1.0820s
80%	1-6467	1.0770s	1.1030s	1.0830s	1.1020s	1.0790s	1.0830s	1.0730s	1.0780s	1.0770s
	808-7277	1.0770s	1.1500s	1.0980s	1.0810s	1.0570s	1.0600s	1.1090s	1.1030s	1.1120s
	1617-8084	1.0830s	1.1120s	1.0720s	1.0880s	1.0780s	1.0750s	1.0870s	1.0780s	1.0910s

TABLE IX
RUNNING TIME BASED ON MEMBERSHIP FUNCTIONS (VEHICLE)

Vehicle		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-85	0.1590s	0.1440s	0.1750s	0.1400s	0.1350s	0.2050s	0.1410s	0.2060s	0.1890s
	383-468	0.1090s	0.1090s	0.1140s	0.1230s	0.1050s	0.1180s	0.1430s	0.1390s	0.1200s
	765-846	0.1250s	0.1410s	0.1400s	0.1240s	0.1720s	0.1190s	0.1410s	0.1400s	0.1250s
30%	1-254	0.1730s	0.1350s	0.1700s	0.1420s	0.1440s	0.2050s	0.1590s	0.1600s	0.1740s
	299-553	0.1240s	0.1560s	0.1290s	0.1190s	0.1280s	0.1230s	0.1200s	0.1590s	0.1140s
	591-846	0.1130s	0.1170s	0.1090s	0.1130s	0.1190s	0.1220s	0.1100s	0.1320s	0.1460s
60%	1-508	0.1550s	0.1400s	0.1240s	0.1560s	0.1410s	0.1240s	0.1250s	0.1410s	0.1560s
	170-676	0.1560s	0.1470s	0.1250s	0.1410s	0.1580s	0.1100s	0.1240s	0.1090s	0.1180s
	337-846	0.1250s	0.1210s	0.1080s	0.1140s	0.1230s	0.1590s	0.1250s	0.1170s	0.1270s
80%	1-677	0.1460s	0.1250s	0.1220s	0.1130s	0.1200s	0.1270s	0.1250s	0.1180s	0.1230s
	84-762	0.1260s	0.1190s	0.1430s	0.1370s	0.1090s	0.1560s	0.1570s	0.1390s	0.1100s
	168-846	0.1190s	0.1220s	0.1110s	0.1220s	0.1120s	0.1230s	0.1200s	0.1120s	0.1350s

TABLE X
RUNNING TIME BASED ON MEMBERSHIP FUNCTIONS (WHOLESALE CUSTOMERS)

Wholesale customers		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-44	0.0930s	0.0780s	0.0630s	0.1090s	0.0780s	0.0820s	0.0630s	0.0770s	0.1080s
	198-242	0.0940s	0.0930s	0.0630s	0.0780s	0.0940s	0.0780s	0.1080s	0.0940s	0.0930s
	396-440	0.1020s	0.1180s	0.0870s	0.1030s	0.1190s	0.0880s	0.1330s	0.1340s	0.1330s
30%	1-132	0.0780s	0.0930s	0.0940s	0.0630s	0.0930s	0.1100s	0.0780s	0.0640s	0.0940s
	154-286	0.0660s	0.0770s	0.0780s	0.0790s	0.0940s	0.0980s	0.0780s	0.0790s	0.0930s
	318-440	0.1190s	0.1340s	0.1030s	0.1030s	0.0870s	0.1040s	0.1100s	0.0940s	0.1080s
60%	1-264	0.1080s	0.1090s	0.0930s	0.0780s	0.0940s	0.1250s	0.0980s	0.0930s	0.1070s
	88-352	0.0790s	0.0630s	0.0780s	0.0870s	0.0770s	0.0780s	0.0930s	0.0620s	0.0780s
	176-440	0.0930s	0.1090s	0.0930s	0.1100s	0.1080s	0.0940s	0.1100s	0.0980s	0.0930s
80%	1-352	0.0780s	0.0790s	0.0930s	0.0980s	0.0940s	0.0820s	0.0630s	0.0780s	0.0780s
	44-390	0.0930s	0.1090s	0.0930s	0.0940s	0.0890s	0.0790s	0.0940s	0.0930s	0.1100s
	88-440	0.0780s	0.0930s	0.0980s	0.0940s	0.1080s	0.0980s	0.0930s	0.0940s	0.0780s

From the above necessary and sufficient fuzzy information granule, we can know that the selected countries $\{x_{12}, x_{14}, x_{23}, x_{24}, x_{38}, x_{47}, x_{48}, x_{58}, x_{82}, x_{84}, x_{87}, x_{88}, x_{89}, x_{94}, x_{99}, x_{110}, x_{120}, x_{121}\}$ satisfy the fuzzy attributes $\{<\mathrm{GRP}, 0.01>, <\mathrm{UR}, 0.15>, <\mathrm{EGI}, 0.05>, <\mathrm{HDI}, 0.57>, <\mathrm{DE}, 0.70>, <\mathrm{GS}, 0.79>, <\mathrm{FCR}, 0.15>\}$ and all the countries which satisfy the above fuzzy attributes must be selected.

TABLE XI
RUNNING TIME BASED ON MEMBERSHIP FUNCTIONS (WINEQUALITY-RED)

Winequality-Red		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-160	0.2180s	0.2310s	0.2140s	0.2210s	0.2180s	0.2160s	0.2200s	0.2050s	0.2280s
	675-825	0.2040s	0.2280s	0.2190s	0.2020s	0.2240s	0.2140s	0.2210s	0.2180s	0.2120s
	1440-1599	0.2270s	0.2370s	0.2380s	0.2110s	0.2270s	0.2120s	0.2290s	0.2330s	0.2020s
30%	1-480	0.2200s	0.2400s	0.2180s	0.2120s	0.2210s	0.2370s	0.2190s	0.2220s	0.2370s
	560-1040	0.2150s	0.2120s	0.2120s	0.2090s	0.2110s	0.2110s	0.2010s	0.2280s	0.2220s
	1120-1599	0.2110s	0.2270s	0.2030s	0.2170s	0.2100s	0.2280s	0.2220s	0.2200s	0.2110s
60%	1-960	0.2320s	0.2210s	0.2170s	0.2380s	0.2260s	0.2120s	0.2020s	0.2270s	0.2160s
	320-1280	0.2160s	0.2210s	0.2160s	0.2280s	0.2300s	0.2020s	0.2190s	0.2200s	0.2170s
	640-1599	0.2030s	0.2210s	0.2170s	0.2110s	0.2220s	0.2210s	0.2320s	0.2280s	0.2180s
80%	1-1280	0.2270s	0.2110s	0.2040s	0.2130s	0.2240s	0.2280s	0.2320s	0.2380s	0.2290s
	160-1440	0.2110s	0.2230s	0.2280s	0.2170s	0.2260s	0.2030s	0.2220s	0.2030s	0.2280s
	320-1599	0.2380s	0.2180s	0.2270s	0.2070s	0.2160s	0.2220s	0.2030s	0.2120s	0.2210s

TABLE XII
RUNNING TIME BASED ON MEMBERSHIP FUNCTIONS (WINEQUALITY-WHITE)

Winequality-White		SG	MG	LG	SC	MC	LC	$S\Gamma$	$M\Gamma$	$L\Gamma$
10%	1-490	0.7610s	0.7660s	0.7600s	0.7930s	0.7740s	0.7730s	0.7420s	0.7630s	0.8190s
	2205-2695	0.8060s	0.7630s	0.7760s	0.7770s	0.7900s	0.7740s	0.8050s	0.7750s	0.7760s
	4410-4898	0.7920s	0.7720s	0.7750s	0.7750s	0.8060s	0.8240s	0.8050s	0.7920s	0.7900s
30%	1-1470	0.7910s	0.7610s	0.7760s	0.7740s	0.7900s	0.7740s	0.7900s	0.7900s	0.7900s
	1715-3185	0.8070s	0.7930s	0.7900s	0.7910s	0.7890s	0.7590s	0.8370s	0.7600s	0.7900s
	3327-4898	0.7750s	0.7620s	0.7910s	0.8210s	0.7900s	0.7750s	0.7910s	0.7750s	0.8210s
60%	1-2940	0.7760s	0.7930s	0.7760s	0.7900s	0.7970s	0.8060s	0.7740s	0.8210s	0.7910s
	980-3920	0.7700s	0.7770s	0.7910s	0.7930s	0.7780s	0.8010s	0.7700s	0.7890s	0.7740s
	1960-4898	0.7600s	0.7930s	0.7750s	0.7580s	0.7740s	0.7930s	0.8120s	0.7820s	0.7730s
80%	1-3920	0.8070s	0.8560s	0.7920s	0.7910s	0.7590s	0.7750s	0.7930s	0.7610s	0.8220s
	490-4410	0.7600s	0.7770s	0.7910s	0.7750s	0.8060s	0.7820s	0.7750s	0.7900s	0.7910s
	980-4898	0.7910s	0.7770s	0.7920s	0.7760s	0.8130s	0.7900s	0.7910s	0.8060s	0.7900s

TABLE XIII
HARDWARE AND SOFTWARE CONFIGURATION

Names	*Model*	*Parameters*
CPU	*Intel Core i3-2350M*	*2.3GHz*
Memory	*Samsung DDR3 SDRAM*	*2×2GB 1333MHz*
Hard disk	*West Data*	*500GB*
System	*Windows 7*	*32bit*
Platform	*C++*	*Leasehold*

In the following, we will test the two-way learning approaches on some real-life datasets. Datasets used in the experiments are underlined in Table II, which are downloaded from UCI Repository of machine-learning datasets. To ensure the fuzziness of datasets before testing two-way learning methods, we divide each data by the maximal value of its corresponding attribute. This experimental computing program is running on a personal computer in Table XIII.

In order to test the effect of the algorithm, at the same time, to make comparisons, we randomly take 10%, 30%, 60%, and 80% objects in each dataset as the initial X_0. As for each 10%, 30%, 60%, and 80%, we randomly select three groups of objects in each dataset, respectively. We also take nine different membership functions on the fuzzy attributes. The membership functions are small, middle, large fuzzy Gaussian membership function (SG, MG, LG), small, middle, large fuzzy Cauchy membership function (SC, MC, LC), and small, middle, large fuzzy Γ membership function (SΓ, MΓ, LΓ). Their graphics are shown in Fig. 3. Then the number of resulting fuzzy concepts, namely necessary and sufficient fuzzy information granules of the datasets are shown in Tables III–VII, respectively. Tables VIII–XII are the running time of each dataset.

Given a fuzzy dataset, for an arbitrary fuzzy information granule, we can obtain several necessary and sufficient fuzzy information granules (fuzzy concepts) through two-way learning methods, because different methods may get the same results, the number is no more than theoretical number which is 16.

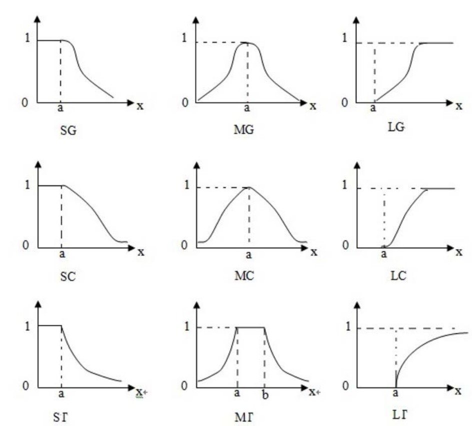

Fig. 3. Nine kinds of membership functions.

Based on the resulting number of fuzzy concepts from Tables III–VII, it is not difficult to draw the following conclusions.

1) In a fuzzy dataset, the number of resulting fuzzy concepts is related to the selected objects and the membership functions.
2) For the same membership function $\widetilde{b}_1$, if the selected initial object sets $a_1^{'} \subseteq a_1^{''}$, then the number of resulting fuzzy concepts from $(a_1^{'}, \widetilde{b}_1)$ is larger than the number of resulting fuzzy concepts from $(a_1^{''}, \widetilde{b}_1)$.

Comparing tables from Tables VIII–XII, we can find that the running time of each fuzzy dataset is irrelevant to the number of selected objects and the kind of selected membership functions. However, it is closely related to the number of whole samples and the number of features.

VI. Conclusion

Machine-learning research has been making a great progress in many directions. Machine-learning techniques are being applied to new kinds of problem, including knowledge discovery in databases, language processing, robot control, and combinatorial optimization, as well as to more traditional problems such as speech recognition, face recognition, medical data analysis, game playing, and so on. The basic element of fuzzy dataset is the fuzzy information granule. By considering fuzzy information granules, this paper investigated carefully the inner relationship among three kinds of fuzzy information granules in fuzzy dataset, namely necessary, sufficient, and necessary and sufficient fuzzy information granules. By the two-way learning system in fuzzy datasets, one can obtain necessary, sufficient, and necessary and sufficient fuzzy information granules from an arbitrary fuzzy information granule in many different methods. As to handling a big data case analysis about the two-way learning system in fuzzy datasets, we used the C++ computer program to fulfill and simplify the computing, the time complexity of the learning algorithm is about $O(|U|^2+|U|\times|\mathrm{AT}|)$. To illustrate our two-way learning system, we constructed a real-life example about loans to the developing countries from the United Nations, and tested the two-way learning algorithm in the UCI datasets. In the future, we will study the information granules in intuitionistic fuzzy datasets and neighborhood fuzzy datasets.

Appendix

Dataset About Developing Countries

	U	*GRP*	*UR*	*EGI*	*HDI*	*DE*	*GS*	*FCR*
x_1	Afgabistan	0.03	0.24	0.11	0.40	0.75	0.67	0.15
x_2	Algeria	0.01	0.45	0.03	0.70	0.84	0.83	0.21
x_3	Angola	0.02	0.57	0.07	0.49	0.85	0.89	0.22
x_4	Argentina	0.01	0.92	0.03	0.80	0.96	0.90	0.31
x_5	Bahamas	0.01	0.73	0.00	0.30	0.92	0.91	0.22
x_6	Bahrain	0.01	0.80	0.02	0.21	0.89	0.87	0.20
x_7	Bengal	0.02	0.29	0.61	0.50	0.87	0.86	0.15
x_8	Barbados	0.01	0.45	0.01	0.80	0.75	0.80	0.25
x_9	Belize	0.02	0.45	0.02	0.70	0.80	0.76	0.16
x_{10}	Benin	0.02	0.46	0.04	0.40	0.76	0.78	0.30
x_{11}	Bhutan	0.01	0.36	0.10	0.50	0.74	0.78	0.18
x_{12}	Bolivia	0.01	0.66	0.05	0.66	0.80	0.87	0.19
x_{13}	Botswana	0.01	0.60	0.04	0.63	0.80	0.88	0.20
x_{14}	Brazil	0.01	0.84	0.06	0.72	0.74	0.90	0.18
x_{15}	Brunei	0.02	0.75	0.03	0.84	0.89	0.92	0.30
x_{16}	Burkina Faso	0.03	0.27	0.07	0.33	0.63	0.80	0.22
x_{17}	Burandi	0.03	0.11	0.04	0.32	0.45	0.68	0.41
x_{18}	Cambodia	0.02	0.20	0.07	0.52	0.49	0.75	0.35
x_{19}	Camcroon	0.02	0.50	0.05	0.48	0.53	0.80	0.34
x_{20}	Cape Verde	0.01	0.60	0.05	0.57	0.67	0.78	0.30
x_{21}	Central African	0.02	0.39	0.04	0.34	0.61	0.79	0.16
x_{22}	Chad	0.00	0.22	0.07	0.33	0.91	0.50	0.32
x_{23}	Chile	0.01	0.88	0.05	0.80	0.88	0.80	0.35
x_{24}	China	0.01	0.52	0.08	0.72	0.84	0.90	0.20
x_{25}	Colombia	0.01	0.74	0.04	0.75	0.82	0.60	0.30
x_{26}	Comoros	0.03	0.28	0.03	0.43	0.44	0.78	0.17
x_{27}	Congo	0.03	0.20	0.05	0.53	0.44	0.88	0.18
x_{28}	Costa Rica	0.01	0.63	0.05	0.74	0.67	0.79	0.24
x_{29}	Cuba	0.00	0.30	0.00	0.78	0.97	0.94	0.20
x_{30}	Djibouti	0.02	0.77	0.05	0.43	0.86	0.85	0.34
x_{31}	Dominica	0.02	0.68	0.04	0.69	0.85	0.80	0.22
x_{32}	Ecuador	0.01	0.66	0.04	0.62	0.84	0.82	0.24
x_{33}	Egypt	0.02	0.44	0.02	0.64	0.76	0.50	0.26
x_{34}	El Salvedor	0.02	0.67	0.00	0.67	0.88	0.86	0.40
x_{35}	Equatorial Gui.	0.03	0.40	0.06	0.54	0.65	0.78	0.16
x_{36}	Ethiopia	0.02	0.17	0.07	0.36	0.54	0.80	0.50
x_{37}	Fiji	0.01	0.51	0.02	0.69	0.76	0.78	0.30
x_{38}	Gabon	0.02	0.85	0.06	0.67	0.89	0.90	0.30
x_{39}	Gambia	0.03	0.55	0.00	0.42	0.78	0.81	0.25
x_{40}	Ghana	0.02	0.50	0.00	0.54	0.75	0.82	0.34
x_{41}	Grenada	0.01	0.39	0.00	0.75	0.69	0.77	0.36
x_{42}	Guatemala	0.02	0.50	0.03	0.57	0.66	0.78	0.19
x_{43}	Guinea	0.03	0.36	0.05	0.34	0.58	0.77	0.18
x_{44}	Gnyana	0.00	0.28	0.04	0.63	0.93	0.90	0.24
x_{45}	Haiti	0.03	0.55	0.05	0.45	0.88	0.88	0.35
x_{46}	Hondaras	0.02	0.50	0.04	0.62	0.80	0.80	0.30
x_{47}	India	0.02	0.32	0.05	0.57	0.79	0.90	0.26
x_{48}	Indonesia	0.01	0.51	0.06	0.61	0.80	0.80	0.28
x_{49}	Iron	0.01	0.68	0.00	0.70	0.84	0.88	0.19
x_{50}	Iraq	0.03	0.67	0.10	0.57	0.65	0.60	0.18
x_{51}	Ivory Coast	0.00	0.16	0.00	0.34	0.60	0.70	0.25
x_{52}	Jamaica	0.01	0.52	0.01	0.72	0.69	0.80	0.37
x_{53}	Jordan	0.02	0.82	0.03	0.70	0.83	0.91	0.20
x_{54}	Kenya	0.03	0.24	0.05	0.51	0.69	0.88	0.20
x_{55}	Kiribati	0.02	0.44	0.03	0.62	0.66	0.78	0.20
x_{56}	North Korea	0.01	0.60	0.01	0.62	0.70	0.40	0.35
x_{57}	South Korea	0.01	0.82	0.03	0.90	0.98	0.90	0.20
x_{58}	Knwait	0.04	0.98	0.06	0.76	0.94	0.88	0.30
x_{59}	Laos	0.02	0.35	0.08	0.52	0.67	0.80	0.40
x_{60}	Lebanon	0.01	0.87	0.04	0.63	0.81	0.80	0.20
x_{61}	Lesotho	0.00	0.28	0.04	0.45	0.78	0.78	0.45
x_{62}	Liberia	0.04	0.49	0.00	0.62	0.80	0.78	0.25
x_{63}	Libya	0.02	0.77	0.03	0.48	0.63	0.80	0.18
x_{64}	Madagascar	0.03	0.33	0.02	0.48	0.50	0.78	0.46
x_{65}	Malawi	0.02	0.16	0.04	0.40	0.54	0.60	0.40
x_{66}	Malaysia	0.02	0.70	0.04	0.76	0.87	0.88	0.30
x_{67}	Maldives	0.03	0.42	0.04	0.66	0.84	0.88	0.35
x_{68}	Mali	0.03	0.36	0.00	0.36	0.70	0.85	0.34
x_{69}	Mauretania	0.03	0.20	0.05	0.53	0.65	0.82	0.40
x_{70}	Mauri	0.01	0.25	0.03	0.73	0.82	0.84	0.40
x_{71}	Mexico	0.01	0.78	0.02	0.77	0.85	0.88	0.25
x_{72}	Mongolia	0.02	0.66	0.13	0.65	0.91	0.50	0.46
x_{73}	Morocco	0.01	0.56	0.03	0.58	0.76	0.80	0.30
x_{74}	Mozambique	0.02	0.31	0.08	0.32	0.54	0.76	0.30
x_{75}	Myanmar	0.01	0.33	0.06	0.48	0.57	0.78	0.32
x_{76}	Namibia	0.01	0.39	0.00	0.63	0.71	0.79	0.30
x_{77}	Naoro	0.02	0.35	0.00	0.40	0.67	0.78	0.25
x_{78}	Nepal	0.02	0.17	0.05	0.46	0.70	0.78	0.40
x_{79}	Nicaragua	0.02	0.57	0.04	0.59	0.74	0.80	0.25
x_{80}	Niger	0.03	0.18	0.15	0.46	0.56	0.70	0.30
x_{81}	Nigeria	0.02	0.50	0.07	0.46	0.78	0.80	0.30
x_{82}	Oman	0.03	0.73	0.05	0.70	0.80	0.82	0.23
x_{83}	Pakistan	0.02	0.37	0.04	0.50	0.65	0.80	0.25
x_{84}	Panama	0.02	0.76	0.09	0.77	0.90	0.90	0.32
x_{85}	Papua New Gui.	0.02	0.13	0.08	0.54	0.68	0.78	0.25
x_{86}	Paraguay	0.02	0.60	0.00	0.67	0.81	0.82	0.23
x_{87}	Peru	0.01	0.76	0.06	0.73	0.89	0.85	0.21
x_{88}	Philippines	0.02	0.49	0.05	0.64	0.86	0.88	0.22
x_{89}	Qatar	0.02	0.98	0.06	0.83	0.91	0.95	0.15
x_{90}	Reunion	0.00	0.25	0.00	0.43	0.54	0.79	0.35
x_{91}	Rwanda	0.03	0.40	0.08	0.44	0.60	0.80	0.23
x_{92}	Samoa	0.00	0.20	0.03	0.69	0.64	0.84	0.30
x_{93}	Sao Tome e Pri.	0.03	0.60	0.05	0.51	0.63	0.80	0.23

x_{94}	Sandi Arabia	0.02	0.82	0.06	0.77	0.87	0.90	0.30
x_{95}	Senegal	0.03	0.43	0.04	0.52	0.58	0.80	0.22
x_{96}	Seyebelles	0.01	0.53	0.03	0.77	0.84	0.84	0.24
x_{97}	Sierra Leone	0.00	0.40	0.21	0.63	0.79	0.80	0.30
x_{98}	Singapore	0.01	1.00	0.02	0.87	0.98	0.96	0.50
x_{99}	Solomon Islands	0.03	0.21	0.07	0.60	0.81	0.79	0.34
x_{100}	Somalia	0.03	0.38	0.03	0.60	0.74	0.88	0.28
x_{101}	Sri Lanka	0.01	0.15	0.07	0.63	0.75	0.86	0.35
x_{102}	St Kitts	0.01	0.45	0.03	0.73	0.86	0.85	0.30
x_{103}	St Lucia	0.01	0.17	0.01	0.72	0.80	0.80	0.40
x_{104}	St Vineent Gre.	0.00	0.50	0.01	0.72	0.82	0.90	0.32
x_{105}	Sadan	0.02	0.33	0.00	0.41	0.60	0.79	0.25
x_{106}	Surinam	0.01	0.68	0.04	0.68	0.66	0.80	0.30
x_{107}	Swaziland	0.00	0.21	0.00	0.34	0.54	0.76	0.36
x_{108}	Syria	0.02	0.31	0.00	0.44	0.56	0.80	0.22
x_{109}	Tanzania	0.00	0.27	0.07	0.59	0.64	0.80	0.24
x_{110}	Thailand	0.01	0.34	0.06	0.68	0.70	0.89	0.26
x_{111}	Togo	0.02	0.39	0.05	0.44	0.68	0.80	0.26
x_{112}	Tonga	0.02	0.24	0.01	0.70	0.92	0.90	0.26
x_{113}	Trinidad Tob.	0.00	0.14	0.01	0.76	0.86	0.80	0.42
x_{114}	Tunisia	0.01	0.66	0.03	0.70	0.88	0.86	0.26
x_{115}	Tuwalu	0.02	0.51	0.00	0.49	0.73	0.88	0.22
x_{116}	Uganda	0.04	0.16	0.04	0.45	0.55	0.86	0.30
x_{117}	Emirates	0.04	0.83	0.04	0.87	0.97	0.90	0.30
x_{118}	Uruguay	0.01	0.92	0.04	0.78	0.96	0.90	0.20
x_{119}	Vanuatu	0.00	0.25	0.00	0.44	0.54	0.88	0.30
x_{120}	Venezuela	0.02	0.93	0.06	0.63	0.82	0.86	0.18
x_{121}	Vietnam	0.01	0.29	0.05	0.59	0.74	0.80	0.31
x_{122}	Yemen	0.04	0.30	0.00	0.46	0.68	0.78	0.30
x_{123}	Zaire	0.00	0.20	0.00	0.50	0.60	0.76	0.34
x_{124}	Zambia	0.02	0.40	0.07	0.43	0.60	0.80	0.21
x_{125}	Zimbabwe	0.01	0.39	0.05	0.37	0.70	0.90	0.20

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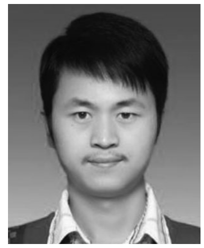

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