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Generalized multigranulation double-quantitative decision-theoretic rough set

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A R T I C L E I N F O

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A B S T R A C T

The principle of the minority subordinate to the majority is the most feasible and credible when people make decisions in real world. So generalized multigranulation rough set theory is a desirable fusion method, in which upper and lower approximations are approximated by granular structures satisfying a certain level of information. However, the relationship between a equivalence class and a concept under each granular structure is very strict. Therefore, more attention are paid to fault tolerance capabilities of double-quantitative rough set theory and the feasibility of majority principle. By considering relative and absolute quantitative information between the class and concept, we propose two kinds of generalized multigranulation double-quantitative decision-theoretic rough sets(GMDq-DTRS). Firstly, we define upper and lower approximations of generalized multigranulation double-quantitative rough sets by introducing upper and lower support characteristic functions. Then, important properties of two kinds of GMDq-DTRS models are explored and corresponding decision rules are given. Moreover, internal relations between the two models under certain constraints and GMDq-DTRS and other models are explored. The definition of the approximation accuracy in GMDq-DTRS is proposed to show the advantage of GMDq-DTRS. Finally, an illustrative case is shown to elaborate the theories advantage of GMDq-DTRS which are valuable to deal with practical problems. Generalized multigranulation double-quantitative decision-theoretic rough set theory will be more feasible when making decisions in real life.

1. Introduction

Rough set theory, proposed by Pawlak in his seminal paper [21], is a new mathematical tool for processing uncertain information. Correlational studies spread across many fields [31,48], such as artificial intelligence, machine learning, neural computing, data mining, cloud computing, information security, knowledge discovery, internet of things, biological information processing and so on.

Compared with classical set theory, Pawlak's rough set theory does not require any transcendental knowledge about data, such as membership functions of fuzzy sets, or probability distribution [7,8,39]. The basic idea of rough sets is to describe a concept by the upper and lower approximate definable sets. The lower approximation consists of elements whose equivalence class is completely contained in the concept and the upper approximation is made up of elements whose equivalence class is partially contained in the concept. Without considering intersection degree, so rough sets have no fault tolerance capability. A large number of generalized models have been put forward, such as the grade rough set model(GRS) [44], the rough set model based on tolerance relation [9,37], the dominance-based rough set model [2], the fuzzy rough set model and the rough fuzzy set model [1] and so on. Particularly, many probabilistic rough set models are presented. Wong et al.[32] put forward the definition of probabilistic rough sets by the introduction of probability approximation spaces into rough sets. Pawlak et al [22] proposed a model of probabilistic approaches versus the deterministic approach. Yao et al.[46] presented the decision-theoretic rough set (DTRS) based on conditional probability and two parameters, which provides reasonable semantic interpretation for decision-making process and gives an effective approach for selecting the threshold parameters. Ziarko [50] constructed the variable precision rough set model when the sum of two parameters is equal to 1. Ślezak studied the Bayesian rough set model [28]. Herbert and Yao [4] explored the game-theoretic rough set model by combining game theory with decision making. Yao et al. [49] constructed a model of web-based medical decision support systems based on DTRS model. Liu et al. [10] proposed a multiple-category classification approach with decision-theoretic rough sets, which can effectively reduce misclassification rate. Yu et al. [45] studied a automatic method of clustering analysis with the decision-theoretic rough set theory. Jia [5,6] raised an optimization problem and attribute reduction about

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DTRS model under considering the minimization of the decision cost. Yao et al. [43] constructed a model of web-based medical decision support systems based on DTRS model. Liu et al. [11] proposed a method of policy analysis with three-way decisions. Zhao et al. [53] made an intensive study of email information filtering system by using three-way decisions.

In general, the DTRS model mainly describes approximate spaces in terms of relative quantitative information. The GRS model [15,33,44] mainly describes approximate spaces from absolute quantitative information by introducing absolute rough membership. They are two fundamental expansion models which have strong fault tolerance capabilities due to quantitative descriptions, so none can be neglected. Hence, Zhang et al. [52] made a comparative study of variable precision rough set model and graded rough set model. Greco et al. [3] presented a generalized variable precision rough set model using the absolute and relative rough membership. Combining relative and absolute quantitative information, Li and Xu [18] proposed a framework of double-quantitative decision-theoretic rough sets (Dq-DTRS) based on the Bayesian decision procedure and GRS model.

From the perspective of granular computing, either classical rough sets or double-quantitative rough sets are based on single indiscernibility relations. In many circumstances, however, a target concept needs to be described through multiple binary relations on the basis of a user's requirements or goals of problem solving. Therefore, Qian et al. [23–25] introduced multigranulation rough set theory(MGRS). Multigranulation theoretical framework has been greatly enriched, and a lot of generalized models about multigranulation have also been put forward and deeply studied. Wu and Leung [30] proposed a formal approach to granular computing with multi-scale data decision information systems. Raghavan and Tripathy [26] explored topological properties of multigranulation rough sets for the first time. Xu et al. [33–37] considered variable, fuzzy and ordered multigranulation rough set models, respectively. Liu and Miao [14] presented a multigranulation rough set method in covering contexts. Liang et al. [17] established an efficient feature selection algorithm with a multi-granulation view. She et al. [27] deeply studied explored topological structures and properties of multigranulation rough sets. Considering the principle of the minority subordinate to the majority, Xu [38] proposed the generalized multigranulation rough set model(GMGRS). In the multigranulation rough set theory, each of various binary relation determines a corresponding information granulation, which largely impacts the commonality between each of the granulations and the fusion among all granulations. Qian et al. [30] therefore introduced the idea of multigranulation into DTRS, and further proposed three kinds of the multigranulation DTRS model. And Li and Xu [19,20] studied the multigranulation DTRS in an ordered information system.

In fact, there are so many factors need to be considered in the process of making decisions, and every aspect taken into account is unpractical in terms of time, energy, money and material resources. So the whole decision process is divided into model partition. Each part makes decision according to required granulations and the comprehension evaluation is finally made based on the the principle of the minority subordinate to the majority. For example, singing contest judges come from different industries, which have their own aesthetic standards. A record company may consider from an economic point of view. Music producers pay more attention to the ability of expressing the soul of the music. Then the winner is supported by majority people after the vote. Decisions come from different granular structures, and each decision may have a deviation in terms of actual situation throughout the process. Therefore, double-quantitative decision-theoretic rough sets with strong fault tolerance capabilities are consistent with real world situations, and more attention should be paid to the theory. Meanwhile, it is necessary to introduce the idea of generalized multigranulation into decision-theoretic rough sets. Then we further emphasize comparative advantages of Dq-DTRS and GMGRS, which can be illustrated from the following aspects:

- Compared with classical decision-theoretic rough sets, Dq-DTRS [18] exhibit strong double fault tolerance capabilities in terms of both relative and absolute fault tolerance, and have further advantage of completeness.
- A generalized variable precision rough set model using the absolute and relative rough membership [3] has been used extensively in the study of measures, reasoning, applications of uncertainty and approximate spaces.
- Considering the principle of the minority subordinate to the majority, GMGRS [38] theory is a kind of information fusion strategies through single granulation rough sets.
- For some special information systems, such as multi-source information systems, distributive information systems and groups of intelligent agents, the classical decision-theoretic rough sets can not be used to data mining from these information systems, but GMGRS can.

So the motivation of this paper is to explore double-quantitative decision-theoretic rough sets theory in multiple granular structures. Then we develop a new multigranulation decision model, called generalized multigranulation double-quantitative decision-theoretic rough sets (GMDq-DTRS). In accordance with the type of the double-quantitative decision-theoretic rough sets, two kinds of generalized multigranulation double-quantitative decision-theoretic rough set models are constructed.

The rest of this paper is organized as follows. Section 2 provides a review of basic concepts of Pawlak's rough sets, decision-theoretic rough sets, double-quantitative decision-theoretic rough sets and generalized multigranulation rough sets. In Section 3, we define the lower and upper approximations of generalized multigranulation double-quantitative decision-theoretic rough sets, and discuss the basic relation among two kinds of GMDq-DTRS models under certain constraints. Meanwhile, the comparison between GMDq-DTRS and other models is made. The approximation accuracy in GMDq-DTRS is proposed to show the advantage of GMDq-DTRS. In Section 4, an illustrative case was presented to interpret the theory and advantage of GMDq-DTRS. Finally, Section 5 gets conclusions.

2. Preliminary

In this section, we provide a review of some basic concepts such as rough sets, decision-theoretic rough sets, double-quantitative decision-theoretic rough sets, generalized multigranulation rough sets.

2.1. Pawlak's rough sets

Suppose U be a non-empty finite universe and R be an equivalence relation of $U \times U$. The equivalence relation R induces a partition of U, denoted by $U/R = \{[x]_R | x \in U\}$, where $[x]_R$ represents the equivalence class of x with regard to R. Then (U, R) is the Pawlak approximation space. For an arbitrary subset X of U, the lower and upper approximations of X are defined as follows [21]:

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\} = \cup\{[x]_R | [x]_R \cap X \neq \emptyset\},$$
$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\} = \cup\{[x]_R | [x]_R \subseteq X\}.$$

And $pos(X) = \underline{R}(X), neg(X) = \sim \overline{R}(X), bnd(X) = \overline{R}(X) - \underline{R}(X)$ are called the positive region, negative region, and boundary region of X, respectively. Objects definitely and not definitely contained in the set X form positive region $pos(X)$ and negative region $neg(X)$.

Objects that may be contained in the set X constitute boundary region $bnd(X)$.

Uncertainty measures which can provide new viewpoints for analyzing data is a key topic in rough set theory. The approximation accuracy proposed by Pawlak provides the percentage of possible correct decisions when classifying objects by employing the attribute set R. Let $DS = \{U, AT \cup DT, V, f\}$ be a decision system, where U is a nonempty finite universe; AT is the set of condition attributes and DT is the set of decision attributes; V is the union of attribute value domain, i.e., $V = \cup_{a \in AT \cup DT} V_a$; and f: $U \times \{AT \cup DT\} \to V$ is an information function, i.e., $\forall a \in AT \cup DT$, $x \in U$, that $f(x, a) \in V_a$, where $f(x, a)$ is the value of the object x about the attribute a. Let $U/DT = \{Y_1, Y_2, \cdots, Y_m\}$ be a classification of universe U, and R be an attribute set. Then the approximation accuracy of U/DT by R is defined as

$$\alpha_R(U/DT) = \frac{\sum_{Y_i \in U/DT} |\underline{R}(Y_i)|}{\sum_{Y_i \in U/D} |\overline{R}(Y_i)|}.$$

Decision-theoretic rough sets proposed by Yao give a way about how to make decisions under minimum Bayesian expectation risk. Based on the idea of three-way decisions, decision-theoretic rough sets use a state set Ω and an action set A to describe the decision-making process [39–42]. $\Omega = \{X, X^C\}$ indicating that an object is in a decision class X and not in X. The set of actions with respect to a state is given by $A = \{a_P, a_B, a_N\}$, where a_P, a_B and a_N represent three actions about deciding $x \in pos(X)$, deciding $x \in bnd(X)$, and deciding $x \in neg(X)$, respectively. Let λ_{PP}, λ_{BP} and λ_{NP} denote the losses caused by taking actions a_P, a_B and a_N, respectively, when an object belongs to X; and λ_{PN}, λ_{BN} and λ_{NN} denote the losses incurred for taking the same actions when the object does not belong to X.

Given the loss function, the expected loss associated with taking the particular actions for the objects in $[x]_R$ can be expressed as:

$$R(a_P|[x]_R) = \lambda_{PP} P(X|[x]_R) + \lambda_{PN} P(X^C|[x]_R);$$
$$R(a_B|[x]_R) = \lambda_{BP} P(X|[x]_R) + \lambda_{BN} P(X^C|[x]_R);$$
$$R(a_N|[x]_R) = \lambda_{NP} P(X|[x]_R) + \lambda_{NN} P(X^C|[x]_R).$$

where $P(X|[x]_R) = |X \cap [x]_R|/|[x]_R|$ represents condition probability of x with regard to X and $P(X^C|[x]_R) = 1 - P(X|[x]_R)$, $|\bullet|$ denotes the cardinality of a set.

By Bayesian decision procedure, minimum-risk decision rules are displayed as:

(P) If $R(a_P|[x]_R) \le R(a_B|[x]_R)$ and $R(a_P|[x]_R) \le R(a_N|[x]_R)$, decide $x \in pos(X)$;

(B) If $R(a_B|[x]_R) \le R(a_P|[x]_R)$ and $R(a_B|[x]_R) \le R(a_N|[x]_R)$, decide $x \in bnd(X)$;

(N) If $R(a_N|[x]_R) \le R(a_P|[x]_R)$ and $R(a_N|[x]_R) \le R(a_B|[x]_R)$, decide $x \in neg(X)$.

According to actual situations, it is a reasonable hypothesis that the cost of $pos(X)$ is smallest and the cost of $pos(X)$ and $bnd(X)$ are strictly smaller than the cost of $neg(X)$ when $x \in X$, the reverse of the order of loss is used for $x \in X^C$, namely, $\lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$. Then we can rewrite above rules as follows:

(P) If $P(X|[x]_R) \ge \alpha$ and $P(X|[x]_R) \ge \gamma$, decide $x \in pos(X)$;

(B) If $P(X|[x]_R) \le \alpha$ and $P(X|[x]_R) \ge \beta$, decide $x \in bnd(X)$;

(N) If $P(X|[x]_R) \ge \beta$ and $P(X|[x]_R) \le \gamma$, decide $x \in neg(X)$.

Where parameters α, β and γ are defined as:

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}; \beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})};$$
$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}.$$

If a loss function further satisfies the condition: $(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) \ge (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$, then we can get $0 \le \beta < \gamma < \alpha \le 1$. DTRS has the following decision rules:

(P) If $P(X|[x]_R) \ge \alpha$, decide $x \in pos(X)$;

(B) If $\beta < P(X|[x]_R) < \alpha$, decide $x \in bnd(X)$;

(N) If $P(X|[x]_R) \le \beta$, decide $x \in neg(X)$.

Meanwhile, we can get the probabilistic approximations, namely the upper and lower approximations of the DTRS model:

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\};$$
$$\underline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \ge \alpha\}.$$

If $\underline{R}_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is rough. If $\alpha = 1, \beta = 0$, then $\overline{R}_{(\alpha,\beta)}(X) = \overline{R}(X)$, $\underline{R}_{(\alpha,\beta)}(X) = \underline{R}(X)$. Therefore, the DTRS model is a generalization of Pawlak's model.

Here, $pos_{(\alpha,\beta)}(X) = \underline{R}_{(\alpha,\beta)}(X)$, $neg_{(\alpha,\beta)}(X) = \sim \overline{R}_{(\alpha,\beta)}(X)$, $bnd_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_{(\alpha,\beta)}(X)$ are the positive region, negative region and boundary region of X, respectively.

2.2. Generalized multigranulation rough sets

Generalized multigranulation rough sets are different from the classical model, because the former is constructed on the basic of a family of indiscernibility relations instead of single indiscernibility relation. Considering the principle of the minority subordinate to the majority, generalized multigranulation rough sets use a level of information $\varphi \in (0.5, 1]$ to select objects [38].

Let $I = (U, AT, V, F)$ be an information system, where U is a nonempty finite universe; AT is a set of condition attributes; V is the union of attribute value domain, i.e., $V = \cup_{a \in A} V_a$; F: $U \times A \to V$ is an information function,i.e., $\forall a \in A$, $x \in U$, that $F(x, a) \in V_a$, where $F(x, a)$ is the value of the object x about the attribute a. Unless otherwise specified, all information systems in this paper are analogous to that defined above.

Suppose an arbitrary subset A_i of a condition attribute set AT, where $i = 1, 2, \cdots, s (s \le 2^{AT})$, $\varphi \in (0.5, 1]$. For an arbitrary subset X of U, the lower and upper approximations of $X$with respect to $\sum_{i=1}^{S} A_i$ can be defined as

$$\overline{GM}_{\sum_{i=1}^{S} A_i}(X) = \left\{ x \in U : \left(\sum_{i=1}^{S} (1 - S^{A_i}_{\sim X}(x)) \right) / s > 1 - \varphi \right\},$$

$$\underline{GM}_{\sum_{i=1}^{S} A_i}(X) = \left\{ x \in U : \left(\sum_{i=1}^{S} S^{A_i}_X(x) \right) / s \ge \varphi \right\},$$

respectively, where $S^{A_i}_X(x)$ is support characteristic function of $x \in U$ with respect to concept X under A_i; $if [x]_{A_i} \subseteq X, then\ S^{A_i}_X(x) = 1, else\ S^{A_i}_X(x) = 0$. X is called a definable set with respect to $\sum_{i=1}^{S} A_i$ if and only if $\overline{GM}_{\sum_{i=1}^{S} A_i}(X) = \underline{GM}_{\sum_{i=1}^{S} A_i}(X)$; otherwise X is called a rough set with respect to $\sum_{i=1}^{S} A_i$. φ is called a level of information with respect to $\sum_{i=1}^{S} A_i$. Positive region $pos(X)$, negative region $neg(X)$, and boundary region $bnd(X)$ are defined as follows:

$$pos(X) = \underline{GM}_{\sum_{i=1}^{S} A_i}(X); neg(X) = \sim \overline{GM}_{\sum_{i=1}^{S} A_i}(X);$$
$$bnd(X) = \overline{GM}_{\sum_{i=1}^{S} A_i}(X) - \underline{GM}_{\sum_{i=1}^{S} A_i}(X).$$

2.3. Double-quantitative decision-theoretic rough sets

Considering absolute quantitative information in the Bayesian decision procedure of the DTRS model, two fundamental Dq-DTRS models (DqI-DTRS and DqII-DTRS) are proposed [18]. In the following, it should be point out that $0 \le k \le |U|$, where $|U|$ is the cardinality of U.

The first kind of double-quantitative decision-theoretic rough set (DqI-DTRS) is denoted by $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k)$, where $\overline{R}_{(\alpha,\beta)}$ and $\underline{R}_k$ are the approximation operators [17]. For an arbitrary subset X of U

can be characterized by a pair of upper and lower approximations which are

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\};$$
$$\underline{R}_k(X) = \{x \in U| \ |[x]_R| - |[x]_R \cap X| \le k\},$$

and the positive region, negative region, upper and lower boundary region of $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k)$ are defined as follows:

$$pos'(X) = \overline{R}_{(\alpha,\beta)}(X) \cap \underline{R}_k(X); neg'(X) = \sim (\overline{R}_{(\alpha,\beta)}(X) \cup \underline{R}_k(X));$$
$$Ubn'(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_k(X); Lbn'(X) = \underline{R}_k(X) - \overline{R}_{(\alpha,\beta)}(X),$$

naturally, we have the followng decision rules:

(P') If $P(X|[x]_R) > \beta$, $|[x]_R| - |[x]_R \cap X| \le k$, decide $x \in pos'(X)$;
(N') If $P(X|[x]_R) \le \beta$, $|[x]_R| - |[x]_R \cap X| > k$, decide $x \in neg'(X)$;
(UB') If $P(X|[x]_R) > \beta$, $|[x]_R| - |[x]_R \cap X| > k$, decide $x \in Ubn'(X)$;
(LB') If $P(X|[x]_R) \le \beta$, $|[x]_R| - |[x]_R \cap X| \le k$, decide $x \in Lbn'(X)$.

The second kind of double-quantitative decision-rough set (DqII-DTRS) denoted by $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)})$ is defined by using approximation operators $\overline{R}_k$ and $\underline{R}_{(\alpha,\beta)}$, where the core mapping are presented by the following approximations:

$$\overline{R}_k(X) = \{x \in U | |[x]_R \cap X| > k\}; \underline{R}_{(\alpha,\beta)}(X) = \{x \in U| \ P(X|[x]_R) \ge \alpha\}.$$

Accordingly, the positive region, negative region, upper and lower boundary region of $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)})$ are stated as follows:

$$pos''(X) = \overline{R}_k(X) \cap \underline{R}_{(\alpha,\beta)}(X); neg''(X) = \sim (\overline{R}_k(X) \cup \underline{R}_{(\alpha,\beta)}(X));$$
$$Ubn''(X) = \overline{R}_k(X) - \underline{R}_{(\alpha,\beta)}(X); Lbn''(X) = \underline{R}_{(\alpha,\beta)}(X) - \overline{R}_k(X).$$

Naturally, we have the decision rules:

(P'') If $P(X|[x]_R) \ge \alpha$, $|[x]_R \cap X| > k$, decide $x \in pos''(X)$;
(N'') If $P(X|[x]_R) < \alpha$, $|[x]_R \cap X| \le k$, decide $x \in neg''(X)$;
(UB'') If $P(X|[x]_R) < \alpha$, $|[x]_R \cap X| > k$, decide $x \in Ubn''(X)$;
(LB'') If $P(X|[x]_R) \ge \alpha$, $|[x]_R \cap X| \le k$, decide $x \in Lbn''(X)$.

3. Generalized multigranulation double-quantitative decision-theoretic rough sets

DqI-DTRS and DqII-DTRS introduce a pair of relative and absolute quantitative measures into the classical model. They have idiographic quantitative semantics and strong double fault tolerance capabilities, and can adapt to complex environments. In many real applications such as multi-source data analysis, knowledge discovery from data with high dimensions and distributive information systems, the multigranulation version of Dq-DTRS will be very desirable when decision-theoretic rough sets are applied to these cases. In this section, we will establish a generalized multigranulation double-quantitative decision-theoretic rough set framework. In the following, it should be point out that k (k is a non-negative integer) represents the grade of overlap between an equivalence class and a set to be approximated, α_i and β_i represent parameters of the DTRS model.

3.1. The first kind of generalized multigranulation double-quantitative decision-theoretic rough set

According to the literature [38], we know that the classical generalized multigranulation lower approximate consists of all objects, whose number of granulations satisfied $x \in \underline{A}_i(X)$ not greater than $s\varphi$, and the upper approximation consists of all objects, whose number of granulations satisfied $x \in \overline{A}_i(X)$ greater than $s(1-\varphi)$. Combining the above idea, the lower and upper approximations of generalized multigranulation double-quantitative rough sets shown as following.

Definition 3.1. Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \le 2^{AT})$, $\varphi \in (0.5, 1]$. In the first kind of generalized multigranulation double-quantitative rough set(GMDqIRS), the lower and upper approximations of an arbitrary subset X with respect to $\sum_{i=1}^{S} A_i$ can be defined as

$$\overline{GM}^I_{\sum_{i=1}^S A_i}(X) = \{x \in U| \ (\sum_{i=1}^{S} USI_X^{A_i}(x))/s > 1-\varphi\},$$

$$\underline{GM}^I_{\sum_{i=1}^S A_i}(X) = \{x \in U| \ (\sum_{i=1}^{S} LSI_X^{A_i}(x))/s \ge \varphi\},$$

respectively, where $USI_X^{A_i}(x)$ is the first kind of upper support characteristic function of $x \in U$ with respect to concept X under A_i,

$$USI_X^{A_i}(x) = \begin{cases} 1, & \text{if } P(X|[x]_{A_i}) > \beta_i; \\ 0, & \text{other.} \end{cases} \tag{1}$$

And $LSI_X^{A_i}(x)$ is the first kind of lower support characteristic function of $x \in U$ with respect to concept X under A_i;

$$LSI_X^{A_i}(x) = \begin{cases} 1, & \text{if } |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k; \\ 0, & \text{other.} \end{cases} \tag{2}$$

φ is called a level of information with respect to $\sum_{i=1}^{S} A_i$. X is called a definable set with respect to $\sum_{i=1}^{S} A_i$, if and only if $\overline{GM}^I_{\sum_{i=1}^S A_i}(X) = \underline{GM}^I_{\sum_{i=1}^S A_i}(X)$; otherwise, X is called a rough set with respect to $\sum_{i=1}^{S} A_i$.

By the lower approximation $\underline{GM}^I_{\sum_{i=1}^S A_i}(X)$ and upper approximation $\overline{GM}^I_{\sum_{i=1}^S A_i}(X)$, the positive region, negative region, upper and lower boundary region of X are expressed as:

$$pos^I(X) = \overline{GM}^I_{\sum_{i=1}^S A_i}(X) \cap \underline{GM}^I_{\sum_{i=1}^S A_i}(X);$$
$$neg^I(X) = \sim (\overline{GM}^I_{\sum_{i=1}^S A_i}(X) \cup \underline{GM}^I_{\sum_{i=1}^S A_i}(X));$$
$$Ubn^I(X) = \overline{GM}^I_{\sum_{i=1}^S A_i}(X) - \underline{GM}^I_{\sum_{i=1}^S A_i}(X);$$
$$Lbn^I(X) = \underline{GM}^I_{\sum_{i=1}^S A_i}(X) - \overline{GM}^I_{\sum_{i=1}^S A_i}(X).$$

Combining the extreme types of optimism and pessimism, we can get the first kind of optimistic and pessimistic multigranulation lower and upper approximations of x with respect to $\sum_{i=1}^{S} A_i$, which can be expressed as follows:

$$\overline{OM}^I_{\sum_{i=1}^S A_i}(X) = \{x \in U| \ \wedge_{i=1}^{S} (P(X|[x]_{A_i}) > \beta_i)\};$$
$$\underline{OM}^I_{\sum_{i=1}^S A_i}(X) = \{x \in U| \ \vee_{i=1}^{s} (|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k)\}$$

$$\overline{PM}^I_{\sum_{i=1}^S A_i}(X) = \{x \in U| \ \vee_{i=1}^{S} (P(X|[x]_{A_i}) > \beta_i)\};$$
$$\underline{PM}^I_{\sum_{i=1}^S A_i}(X) = \{x \in U| \ \wedge_{i=1}^{s} (|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k)\}$$

and expressions of other regions are analogous to that used above.

Considering the relationship during optimistic, pessimistic and generalized multigranulation ,the following conclusions are true.

Proposition 3.1. *Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \le 2^{AT})$. For an arbitrary subset X of U, the following conclusions hold:*

$$\underline{PM}^I_{\sum_{i=1}^S A_i}(X) \subseteq \underline{GM}^I_{\sum_{i=1}^S A_i}(X) \subseteq \underline{OM}^I_{\sum_{i=1}^S A_i}(X), \qquad \overline{OM}^I_{\sum_{i=1}^S A_i}(X) \subseteq \overline{GM}^I_{\sum_{i=1}^S A_i}(X) \subseteq \overline{PM}^I_{\sum_{i=1}^S A_i}(X).$$

Proof. It can be easily verified by definitions of generalized, optimistic and pessimistic multigranulation double-quantitative rough sets.

Considering the relationship between the multigranulation and single granular structure, we have the following conclusions. □

Proposition 3.2. *Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \leq 2^{AT})$. For an arbitrary subset X of U, the following conclusions hold:*

$\overline{OM}^I_{\sum_{i=1}^S A_i}(X) \subseteq \overline{A}_{i(\alpha_i,\beta_i)}(X), \underline{OM}^I_{\sum_{i=1}^S A_i}(X) \supseteq \underline{A}_{i_k}(X),$

$\overline{PM}^I_{\sum_{i=1}^S A_i}(X) \supseteq \overline{A}_{i(\alpha_i,\beta_i)}(X), \underline{PM}^I_{\sum_{i=1}^S A_i}(X) \subseteq \underline{A}_{i_k}(X).$

Proof. The result holds trivially according to definitions of optimistic and pessimistic multigranulation double-quantitative rough sets.

Considering the relationship between the multigranulation and all the single granular structure, the following conclusions hold. □

Proposition 3.3. *Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \leq 2^{AT})$. For an arbitrary subset X of U, we have*

$\overline{OM}^I_{\sum_{i=1}^S A_i}(X) = \bigcap_{i=1}^s \overline{A}_{i(\alpha_i,\beta_i)}(X),\ \underline{OM}^I_{\sum_{i=1}^S A_i}(X) = \bigcup_{i=1}^s \underline{A}_{i_k}(X),$

$\overline{PM}^I_{\sum_{i=1}^S A_i}(X) = \bigcup_{i=1}^s \overline{A}_{i(\alpha_i,\beta_i)}(X),\ \underline{PM}^I_{\sum_{i=1}^S A_i}(X) = \bigcap_{i=1}^s \underline{A}_{i_k}(X).$

Proof. The assertion follows immediately from definitions of optimistic and pessimistic multigranulation double-quantitative rough sets.

Based on the idea of three-way decisions and definition of the first kind of the generalized multigranulation double-quantitative rough set, we have the decision rules as follows: □

Rules 3.1. Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \leq 2^{AT})$. For an arbitrary subset X of U, decision rules can be expressed as follows:

(P^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi)$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| \geq s\varphi$, decide $x \in pos^I(X)$;

(N^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| \leq s(1-\varphi)$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| < s\varphi$, decide $x \in neg^I(X)$;

(UB^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi)$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| < s\varphi$, decide $x \in Ubn^I(X)$;

(LB^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| \leq s(1-\varphi)$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| \geq s\varphi$, decide $x \in Lbn^I(X)$.

According to rules 3.1, if the number of granulations satisfying $P(X|[x]_{A_i}) > \beta_i$ is greater than $s(1-\varphi)$, and the number of granulations satisfying$|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$ is not smaller than $s\varphi$, decide $x \in pos^I(X)$; if the number of granulations satisfying $P(X|[x]_{A_i}) > \beta_i$ is not greater than $s(1-\varphi)$, and the number of granulations satisfying $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$ is smaller than $s\varphi$, decide $x \in neg^I(X)$; if the number of granulations satisfying $P(X|[x]_{A_i}) > \beta_i$ is greater than $s(1-\varphi)$, and the number of granulations satisfying $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$ is smaller than $s\varphi$, decide $x \in Ubn^I(X)$; if the number of granulations satisfying $P(X|[x]_{A_i}) > \beta_i$ is not greater than $s(1-\varphi)$, and the number of granulations satisfying $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$ is not smaller than $s\varphi$, decide $x \in Lbn^I(X)$.

The optimistic multigranulation rough set only need one granular structure to satisfy with corresponding relationship between equivalence class and the approximated target. Combining the idea of optimism, we can obtain decision rules, which are

(P^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| = s$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| \geq 1$, decide $x \in pos^I(X)$;

(N^I) If $|A_i : P(X|[x]_{A_i}) \leq \beta_i| \geq 1$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| > k| = s$, decide $x \in neg^I(X)$;

(UB^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| = s$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| > k| = s$, decide $x \in Ubn^I(X)$;

(LB^I) If $|A_i : P(X|[x]_{A_i}) \leq \beta_i| \geq 1$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| \geq 1$, decide $x \in Lbn^I(X)$.

Particularly, if all granulations satisfy $P(X|[x]_{A_i}) > \beta_i$, and at least one granulation satisfies $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$, decide $x \in pos^I(X)$; if at least one granulation satisfies $P(X|[x]_{A_i}) \leq \beta_i$, and all granulations satisfy $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$, decide $x \in neg^I(X)$; if all granulations satisfy $P(X|[x]_{A_i}) > \beta_i$ and $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$, decide $x \in Ubn^I(X)$; if at least one granulation satisfies $P(X|[x]_{A_i}) \leq \beta_i$ and $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$, decide $x \in Lbn^I(X)$.

Based on SCED (seeking common ground while eliminating differences) strategy, the pessimistic multigranulation rough set require all granular structures to satisfy with the corresponding relationship between equivalence class and the approximated target. Combining the idea of pessimism, we can gain the following decision rules:

(P^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| \geq 1$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| = s$, decide $x \in pos^I(X)$;

(N^I) If $|A_i : P(X|[x]_{A_i}) \leq \beta_i| = s$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| > k| \geq 1$, decide $x \in neg^I(X)$;

(UB^I) If $|A_i : P(X|[x]_{A_i}) > \beta_i| \geq 1$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| > k| \geq 1$, decide $x \in Ubn^I(X)$;

(LB^I) If $|A_i : P(X|[x]_{A_i}) \leq \beta_i| = s$, $|A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k| = s$, decide $x \in Lbn^I(X)$.

Correspondingly, if at least one granulation satisfies $P(X|[x]_{A_i}) > \beta_i$, and all granulations satisfy $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$, decide $x \in pos^I(X)$; if all granulations satisfy $P(X|[x]_{A_i}) \leq \beta_i$, and at least one granulation satisfies $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$, decide $x \in neg^I(X)$; if at least one granulation satisfies $P(X|[x]_{A_i}) > \beta_i$ and $|[x]_{A_i}| - |[x]_{A_i} \cap X| > k$, decide $x \in Ubn^I(X)$; if all granulations satisfy $P(X|[x]_{A_i}) \leq \beta_i$ and $|[x]_{A_i}| - |[x]_{A_i} \cap X| \leq k$, decide $x \in Lbn^I(X)$.

In order to measure the classification ability, the definition of the approximation accuracy under multiple granular structures is proposed.

Definition 3.2. Let $DS = (U, AT \cup DT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \leq 2^{AT})$. And $U/DT = \{Y_1, Y_2, \cdots, Y_m\}$ be a classification of universe U. In the first kind of generalized multigranulation double-quantitative rough set(GMDqI-DTRS), the approximation accuracy of U/DT with respect to $\sum_{i=1}^S A_i$ is defined as

$$\alpha_{\sum_{i=1}^S A_i}(U/DT) = \frac{\sum_{Y_i \in U/DT} |\underline{GM}^I_{\sum_{i=1}^S A_i}(Y_i)|}{\sum_{Y_i \in U/D} |\overline{GM}^I_{\sum_{i=1}^S A_i}(Y_i)|},$$

where $\overline{GM}^I_{\sum_{i=1}^S A_i}(Y_i) = \{x \in U | \ (\sum_{i=1}^S USI^{A_i}_{Y_i}(x))/s > 1-\varphi\}$ and $\underline{GM}^I_{\sum_{i=1}^S A_i}(Y_i) = \{x \in U | \ (\sum_{i=1}^S LSI^{A_i}_{Y_i}(x))/s \geq \varphi\}$.

3.2. The second kind of generalized multigranulation double-quantitative decision-theoretic rough set

First of all, we present the definition of the second kind of lower and upper approximations.

Definition 3.3. Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \leq 2^{AT})$. In the second kind of generalized multigranulation double-quantitative rough set (GMDqIIRS), the lower and upper approximations of an arbitrary subset X with respect to $\sum_{i=1}^S A_i$ can be defined as

$$\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U: \ (\sum_{i=1}^{S} USII_X^{A_i}(x))/s > 1-\varphi\};$$

$$\underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U: \ (\sum_{i=1}^{S} LSII_X^{A_i}(x))/s \geq \varphi\}$$

respectively, where $USII_X^{A_i}(x)$ is the second kind of upper support characteristic function of $x \in U$ with respect to concept X under A_i,

$$USII_X^{A_i}(x) = \begin{cases} 1, & \text{if } |[x]_{A_i} \cap X| > k; \\ 0, & \text{other.} \end{cases} \tag{3}$$

And $LSII_X^{A_i}(x)$ is the second kind of lower support characteristic function of $x \in U$ with respect to concept X under A_i,

$$LSII_X^{A_i}(x) = \begin{cases} 1, & \text{if } P(X|[x]_{A_i}) \geq \alpha_i; \\ 0, & \text{other.} \end{cases} \tag{4}$$

φ is called a level of information with respect to $\sum_{i=1}^{S} A_i$. X is called a definable set with respect to $\sum_{i=1}^{S} A_i$, if and only if $\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X)$; otherwise X is rough.

By the lower approximation $\underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X)$ and upper approximation $\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X)$, the positive region, negative region, upper and lower boundary region of X are as following:

$$pos^{II}(X) = \overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) \cap \underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X);$$
$$neg^{II}(X) = \sim (\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) \cup \underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X));$$
$$Ubn^{II}(X) = \overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) - \underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X);$$
$$Lbn^{II}(X) = \underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X) - \overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(X).$$

Then the second kind of optimistic and pessimistic multigranulation lower and upper approximations of X with respect to $\sum_{i=1}^{S} A_i$ can be stated as follows:

$$\overline{OM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U| \ \wedge_{i=1}^{S} (|[x]_{A_i} \cap X| > k)\};$$
$$\underline{OM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U| \ \vee_{i=1}^{s} (P(X|[x]_{A_i}) \geq \alpha_i)\}$$

$$\overline{PM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U| \ \vee_{i=1}^{S} (|[x]_{A_i} \cap X| > k)\};$$
$$\underline{PM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U| \ \wedge_{i=1}^{s} (P(X|[x]_{A_i}) \geq \alpha_i)\}$$

and other regions can be obtained by the same way like above.

From the definitions of generalized, optimistic and pessimistic multigranulation double-quantitative rough set, there are propositions can be induced as follows:

Proposition 3.4. *Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \leq 2^{AT})$. For an arbitrary subset X of U, the following conclusions hold trivially.*

$\underline{PM}^{II}_{\sum_{i=1}^{S} A_i} \subseteq \underline{GM}^{II}_{\sum_{i=1}^{S} A_i} \subseteq \underline{OM}^{II}_{\sum_{i=1}^{S} A_i}$, $\overline{OM}^{II}_{\sum_{i=1}^{S} A_i} \subseteq \overline{GM}^{II}_{\sum_{i=1}^{S} A_i} \subseteq \overline{PM}^{II}_{\sum_{i=1}^{S} A_i}$.

When considering the relationship between multiple granular structures and single granular structure, the following conclusions hold trivially, namely,

$\overline{OM}^{II}_{\sum_{i=1}^{S} A_i}(X) \subseteq \overline{A}_{ik}(X), \underline{OM}^{II}_{\sum_{i=1}^{S} A_i}(X) \supseteq \underline{A}_{i_k}(X)$,

$\overline{PM}^{II}_{\sum_{i=1}^{S} A_i}(X) \supseteq \overline{A}_{ik}(X), \underline{PM}^{II}_{\sum_{i=1}^{S} A_i}(X) \subseteq \underline{A_i}_{(\alpha_i, \beta_i)}(X)$.

When thinking about the relationship between multiple granular structures and the set of single granular structure, the following conclusions hold obviously:

$\overline{OM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \bigcap_{i=1}^{s} \overline{A}_{ik}(X)$, $\underline{OM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \bigcup_{i=1}^{s} \underline{A_i}_{(\alpha_i, \beta_i)}(X)$,

$\overline{PM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \bigcup_{i=1}^{s} \overline{A}_{ik}(X)$, $\underline{PM}^{II}_{\sum_{i=1}^{S} A_i}(X) = \bigcap_{i=1}^{s} \underline{A_i}_{(\alpha_i, \beta_i)}(X)$

Based on the idea of three-way decisions and definition of the second kind of the generalized multigranulation double-quantitative rough set, we have the following decision rules:

Rules 3.2. Let $I = (U, AT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s(s \leq 2^{AT})$. For an arbitrary subset X of U, decision rules can be expressed as follows:

(P^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| > s(1-\varphi)$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| \geq s\varphi$, decide $x \in pos^{II}(X)$;

(N^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| \leq s(1-\varphi)$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| < s\varphi$, decide $x \in neg^{II}(X)$;

(UB^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| > s(1-\varphi)$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| < s\varphi$, decide $x \in Ubn^{II}(X)$;

(LB^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| \leq s(1-\varphi)$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| \geq s\varphi$, decide $x \in Lbn^{II}(X)$.

According to rules 3.2, if the number of granulations satisfying $|[x]_{A_i} \cap X| > k$ is greater than $s(1-\varphi)$, and the number of granulations satisfying $P(X|[x]_{A_i}) \geq \alpha_i$ is not smaller than $s\varphi$, decide $x \in pos^{II}(X)$; if the number of granulations satisfying $|[x]_{A_i} \cap X| > k$ is not greater than $s(1-\varphi)$, and the number of granulations satisfying $P(X|[x]_R) \geq \alpha_i$ smaller than $s\varphi$, decide $x \in neg^{II}(X)$; if the number of granulations satisfying $|[x]_{A_i} \cap X| > k$ is greater than $s(1-\varphi)$, and the number of granulations satisfying $P(X|[x]_{A_i}) \geq \alpha_i$ is smaller than $s\varphi$, decide $x \in Ubn^{II}(X)$; if the number of granulations satisfying $|[x]_{A_i} \cap X| > k$ is not greater than $s(1-\varphi)$, and the number of granulations satisfying $P(X|[x]_{A_i}) \geq \alpha_i$ is not smaller than $s\varphi$, decide $x \in Lbn^{II}(X)$.

When thinking over the idea of optimism, we can obtain decision rules as follows:

(P^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| = s$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| \geq 1$, decide $x \in pos^{II}(X)$;

(N^{II}) If $|A_i : |[x]_{A_i} \cap X| \leq k| \geq 1$, $|A_i : P(X|[x]_{A_i}) < \alpha_i| = s$, decide $x \in neg^{II}(X)$;

(UB^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| = s$, $|A_i : P(X|[x]_R) < \alpha_i| = s$, decide $x \in Ubn^{II}(X)$;

(LB^{II}) If $|A_i : |[x]_{A_i} \cap X| \leq k| \geq 1$, $|A_i$: $P(X|[x]_R) \geq \alpha_i| \geq 1$, decide $x \in Lbn^{II}(X)$.

Accordingly, if all granulations satisfy $|[x]_{A_i} \cap X| > k$, and at least one granulation satisfies $P(X|[x]_{A_i}) \geq \alpha_i$, decide $x \in pos^{II}(X)$; if at least one granulation satisfies $|[x]_{A_i} \cap X| \leq k$, and all granulations satisfy $P(X|[x]_{A_i}) < \alpha_i$, decide $x \in neg^{II}(X)$; if all granulations satisfy $|[x]_{A_i} \cap X| > k$ and $P(X|[x]_{A_i}) < \alpha_i$, decide $x \in Ubn^{II}(X)$; if at least one granulation satisfies $|[x]_{A_i} \cap X| \leq k$ and $P(X|[x]_{A_i}) \geq \alpha_i$, decide $x \in Lbn^{II}(X)$.

When considering the idea of pessimism, we can obtain decision rules as follows:

(P^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| \geq 1$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| = s$, decide $x \in pos^{II}(X)$;

(N^{II}) If $|A_i : |[x]_{A_i} \cap X| \leq k| = s$, $|A_i : P(X|[x]_{A_i}) < \alpha_i| \geq 1$, decide $x \in neg^{II}(X)$;

(UB^{II}) If $|A_i : |[x]_{A_i} \cap X| > k| \geq 1$, $|A_i : P(X|[x]_{A_i}) < \alpha_i| \geq 1$, decide $x \in Ubn^{II}(X)$;

(LB^{II}) If $|A_i : |[x]_\nu \cap X| \leq k| = s$, $|A_i : P(X|[x]_{A_i}) \geq \alpha_i| = s$, decide $x \in Lbn^{II}(X)$.

Correspondingly, if at least one granulation satisfies $|[x]_{A_i} \cap X| > k$, and all granulations satisfy $P(X|[x]_{A_i}) \geq \alpha_i$, decide $x \in pos^{II}(X)$; if all granulations satisfy $|[x]_{A_i} \cap X| \leq k$, and at least one granulation satisfies $P(X|[x]_R) < \alpha_i$, decide $x \in neg^{II}(X)$; if at least one granulation satisfies $|[x]_{A_i} \cap X| > k$ and $P(X|[x]_{A_i}) < \alpha_i$, decide $x \in Ubn^{II}(X)$;

if all granulations satisfy $|[x]_{A_i} \cap X| \le k$ and $P(X|[x]_{A_i}) \ge \alpha_i$, decide $x \in Lbn^{II}(X)$.

The Dq-DTRS model degenerates to the Pawlak model when $\alpha = 1, \beta = 0, k = 0$. So we can known two kinds of the generalized multigranulation double-quantitative decision-theoretic rough set are all equivalent to the generalized multigranulation rough set when $\alpha_i = 1, \beta_i = 0, k = 0$. Certainly, the two kinds of the generalized multigranulation double-quantitative decision-theoretic rough set are also equivalent. Especially,optimistic and pessimistic multigranulation double-quantitative decision-theoretic rough sets degenerate to the optimistic multigranulation rough set and pessimistic multigranulation rough set.

At the same time, the uncertainty measure of the second kind of generalized multigranulation double-quantitative rough set(GMDqII-DTRS) is also proposed.

Definition 3.4. Let $DS = (U, AT \cup DT, V, F)$ be an information system, $A_i \subseteq AT, i = 1, 2, \cdots, s (s \le 2^{AT})$. And $U/DT = \{Y_1, Y_2, \cdots, Y_m\}$ be a classification of universe U. In the second kind of generalized multigranulation double-quantitative rough set(GMDqII-DTRS), the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ is defined as

$$\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{\sum_{Y_i \in U/DT} |\underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(Y_i)|}{\sum_{Y_i \in U/D} |\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(Y_i)|},$$

where $\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}(Y_i) = \{x \in U : \ (\sum_{i=1}^{S} USII^{A_i}_{Y_i}(x))/s > 1 - \varphi\}$ and $\underline{GM}^{II}_{\sum_{i=1}^{S} A_i}(Y_i) = \{x \in U : \ (\sum_{i=1}^{S} LSII^{A_i}_{Y_i}(x))/s \ge \varphi\}$.

3.3. Comparison

In the subsection, we deeply explored the relationship between GMDqI-DTRS and GMDqII-DTRS, the internal connection between the generalized multigranulation double-quantitative decision-theoretic rough set(GMDq-DTRS) and the generalized multigranulation rough set(GMRS), the inherent relations between GMDq-DTRS and double-quantitative decision-theoretic rough set (Dq-DTRS), and the relationship between GMDq-DTRS and variable precision rough set(VRS).

(1) The relationship between GMDqI-DTRS and GMDqII-DTRS

According to decision-theoretic rough set, if the foss function satisfies $\lambda^i_{PP} \le \lambda^i_{BP} < \lambda^i_{NP}$, $\lambda^i_{NN} \le \lambda^i_{BN} < \lambda^i_{PN}$ and $(\lambda^i_{BP} - \lambda^i_{PP})(\lambda^i_{BN} - \lambda^i_{NN}) \le (\lambda^i_{NP} - \lambda^i_{BP})(\lambda^i_{PN} - \lambda^i_{BN})$, we have $\alpha_i > \beta_i, i = 1, 2, \cdots, s$. At the same time, for the same k, we discuss the relationship between the value of $\alpha_i + \beta_i$ and 1.

When $\alpha_i + \beta_i = 1, \quad i = 1, 2, \cdots, s,$ there are following conclusions

(1) $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| \ge s\varphi$
$\Leftrightarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| < s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$.
(2) $|A_i : P(X|[x]_{A_i}) > \beta_i| \le s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| < s\varphi$
$\Leftrightarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| \ge s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| > s(1-\varphi)$.
(3) $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| < s\varphi$
$\Leftrightarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| < s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| > s(1-\varphi)$.
(4) $|A_i : P(X|[x]_{A_i}) > \beta_i| \le s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| \ge s\varphi$
$\Leftrightarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| \ge s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$.

The reasoning process is stated as follows. According to $\alpha_i + \beta_i = 1$, it is true for $i = 1, 2, \cdots, s$ that $\alpha_i = 1 - \beta_i$, and $P(X|[x]_{A_i}) > \beta_i$, $|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k$ is equivalent to $P(\sim X|[x]_{A_i}) < 1 - \beta_i$, $\||[x]_{A_i} \cap (\sim X)| \le k$. So it is true that $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| \ge s\varphi$ is equivalent to $|A_i : P(\sim X|[x]_{A_i}) < 1 - \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i} \cap (\sim X)| \le k| \ge s\varphi$. By substituting $\alpha_i = 1 - \beta_i$ into the latter, we obtain $|A_i : P(\sim X|[x]_{A_i}) < \alpha_i| > s(1-\varphi), |A_i : |[x]_{A_i} \cap (\sim X)| \le k| \ge s\varphi$. At the same time, $|A_i : P(\sim X|[x]_{A_i}) < \alpha_i| > s(1-\varphi)$ is equivalent to $|A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| < s\varphi$, and $|A_i : |[x]_{A_i} \cap (\sim X)| \le k| \ge s\varphi$ is equivalent to $|A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$. Then the proof of the first conclusion is now completed. Other conclusions can be proved by the same method as employed in the first conclusion.

At above case, the loss function must satisfy $(\lambda^i_{BP} - \lambda^i_{PP})(\lambda^i_{NP} - \lambda^i_{BP}) = (\lambda^i_{PN} - \lambda^i_{BN})(\lambda^i_{BN} - \lambda^i_{NN})$. From above conclusions, we can know that the accepted region of X in GMDqI-DTRS is equivalent to the rejective region of $\sim X$ in GMDqII-DTRS, the rejective region of X in GMDqI-DTRS is equivalent to the accepted region of $\sim X$ in GMDqII-DTRS, the upper and lower delayed regions of X and $\sim X$ are identical for both GMDqI-DTRS and GMDqII-DTRS.

When $\alpha_i + \beta_i < 1, \ i = 1, 2, \cdots, s,$ there are following conclusions

(1) $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| \ge s\varphi$
$\Leftarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| < s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$.
(2) $|A_i : P(X|[x]_{A_i}) > \beta_i| \le s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| < s\varphi$
$\Rightarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| \ge s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| > s(1-\varphi)$.
(3) $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| < s\varphi$
$\Leftarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| < s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| > s(1-\varphi)$.
(4) $|A_i : P(X|[x]_{A_i}) > \beta_i| \le s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| \ge s\varphi$
$\Rightarrow |A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| \ge s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$.

The analysis can be stated as follows. In accordance with $\alpha_i + \beta_i < 1$, it is true for $i = 1, 2, \cdots, s$ that $\alpha_i < 1 - \beta_i$, and $P(X|[x]_{A_i}) > \beta_i$, $|[x]_{A_i}| - |[x]_{A_i} \cap X| \le k$ is equivalent to $P(\sim X|[x]_{A_i}) < 1 - \beta_i$, $\||[x]_{A_i} \cap (\sim X)| \le k$. So we can get that $|A_i : P(X|[x]_{A_i}) > \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i}| - |[x]_{A_i} \cap X| \le k| \ge s\varphi$ is equivalent to $|A_i : P(\sim X|[x]_{A_i}) < 1 - \beta_i| > s(1-\varphi), |A_i : |[x]_{A_i} \cap (\sim X)| \le k| \ge s\varphi$. At the same time, $|A_i : P(\sim X|[x]_{A_i}) < 1 - \beta_i| > s(1-\varphi)$ is equivalent to $|A_i : P(\sim X|[x]_{A_i}) \ge 1 - \beta_i| < s\varphi$, and $|A_i : |[x]_{A_i} \cap (\sim X)| \le k| \ge s\varphi$ is equivalent to $|A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$. On the basis of $\alpha_i < 1 - \beta_i$, it is true that $|A_i : P((\sim X)|[x]_{A_i}) \ge \alpha_i| < s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$ implies $|A_i : P(\sim X|[x]_{A_i}) \ge 1 - \beta_i| < s\varphi, |A_i : |[x]_{A_i} \cap (\sim X)| > k| \le s(1-\varphi)$. Then the proof of the first conclusion is now completed. Other conclusions can be proved by the same method as employed in the first conclusion.

At above case, the foss function must satisfy $(\lambda^i_{BP} - \lambda^i_{PP})(\lambda^i_{NP} - \lambda^i_{BP}) > (\lambda^i_{PN} - \lambda^i_{BN})(\lambda^i_{BN} - \lambda^i_{NN})$. From above conclusions, we can obtain that the rejective region and lower delayed region of X in GMDqI-DTRS are contained in the accepted region and lower delayed region of $\sim X$ in GMDqII-DTRS, respectively; the rejective region and upper delayed region of $\sim X$ in GMDqII-DTRS are contained in the accepted region and upper delayed region of X in GMDqI-DTRS, respectively Table 1.

Table 1
The relationship between GMDqI-DTRS and GMDqII-DTRS.

Cases	relationships			
$\alpha_i+\beta_i=1$	$pos^I(X)=neg^{II}(\sim X)$	$neg^I(X)=pos^{II}(\sim X)$	$Ubn^I(X)=Ubn^{II}(\sim X)$	$Lbn^I(X)=Lbn^{II}(\sim X)$
$\alpha_i+\beta_i<1$	$pos^I(X)\supseteq neg^{II}(\sim X)$	$neg^I(X)\subseteq pos^{II}(\sim X)$	$Ubn^I(X)\supseteq Ubn^{II}(\sim X)$	$Lbn^I(X)\subseteq Lbn^{II}(\sim X)$
$\alpha_i+\beta_i>1$	$pos^I(X)\subseteq neg^{II}(\sim X)$	$neg^I(X)\supseteq pos^{II}(\sim X)$	$Ubn^I(X)\subseteq Ubn^{II}(\sim X)$	$Lbn^I(X)\supseteq Lbn^{II}(\sim X)$

When $\alpha_i+\beta_i>1,\quad i=1,2,\cdots,s,$ there are following conclusions

(1) $|A_i:P(X|[x]_{A_i})>\beta_i|>s(1-\varphi),|A_i:|[x]_{A_i}|-|[x]_{A_i}\cap X|\le k|\ge s\varphi$
$\Rightarrow|A_i:P((\sim X)|[x]_{A_i})\ge\alpha_i|<s\varphi,|A_i:|[x]_{A_i}\cap(\sim X)|>k|\le s(1-\varphi)$.
(2) $|A_i:P(X|[x]_{A_i})>\beta_i|\le s(1-\varphi),|A_i:|[x]_{A_i}|-|[x]_{A_i}\cap X|\le k|<s\varphi$
$\Leftarrow|A_i:P((\sim X)|[x]_{A_i})\ge\alpha_i|\ge s\varphi,|A_i:|[x]_{A_i}\cap(\sim X)|>k|>s(1-\varphi)$.
(3) $|A_i:P(X|[x]_{A_i})>\beta_i|>s(1-\varphi),|A_i:|[x]_{A_i}|-|[x]_{A_i}\cap X|\le k|<s\varphi$
$\Rightarrow|A_i:P((\sim X)|[x]_{A_i})\ge\alpha_i|<s\varphi,|A_i:|[x]_{A_i}\cap(\sim X)|>k|>s(1-\varphi)$.
(4) $|A_i:P(X|[x]_{A_i})>\beta_i|\le s(1-\varphi),|A_i:|[x]_{A_i}|-|[x]_{A_i}\cap X|\le k|\ge s\varphi$
$\Leftarrow|A_i:P((\sim X)|[x]_{A_i})\ge\alpha_i|\ge s\varphi,|A_i:|[x]_{A_i}\cap(\sim X)|>k|\le s(1-\varphi)$.

The reasoning method analogous to that used above. At above case, the foss function must satisfy $(\lambda^i_{BP}-\lambda^i_{PP})(\lambda^i_{NP}-\lambda^i_{BP})<(\lambda^i_{PN}-\lambda^i_{BN})(\lambda^i_{BN}-\lambda^i_{NN})$. Above all, we can understand that the accepted region and upper delayed region of X in GMDqI-DTRS are contained in the rejective region and upper delayed region of $\sim X$ in GMDqII-DTRS, respectively; the accepted region and lower delayed region of $\sim X$ in GMDqII-DTRS are contained in the rejective region and lower delayed region of X in GMDqI-DTRS, respectively.

Intuitively, internal relations of the two models are shown in different cases as follows:

(2) The internal connection between GMDq-DTRS and GMRS

The internal connection between GMDq-DTRS and GMRS can be clearly obtained by the following description.

Let $I=(U,AT,V,F)$ be an information system, $A_i\subseteq AT,i=1,2,\cdots,s(s\le 2^{AT})$. For an arbitrary subset X of U, when $\beta_i=0,k=0$, the first kind of upper support characteristic function $USI^{A_i}_X(x)$ degenerates to characteristic function $1-S^{A_i}_{\sim X}(x)$ and lower support characteristic function $LSI^{A_i}_X(x)$ degenerates to support characteristic function $S^{A_i}_X(x)$. Therefore, it is true that $\overline{GM}^I_{\sum_{i=1}^S A_i}(X)=\overline{GM}_{\sum_{i=1}^S A_i}(X)$, $\underline{GM}^I_{\sum_{i=1}^S A_i}(X)=\underline{GM}_{\sum_{i=1}^S A_i}(X)$ when $\beta_i=0,k=0$. So GMDqI-DTRS is equivalent to GMRS. That is to say, GMDqI-DTRS is a generalized model of GMRS.

Furthermore, when $\beta>0$, $k>0$, conclusions $\overline{R}_{(\alpha,\beta)}(X)\subseteq\overline{R}(X)$, $\underline{R}_{\ k}(X)\supseteq\underline{R}\ (X)$ hold in the first kind of double-quantitative rough set. Therefore, for the same φ, it is true that $\overline{GM}^I_{\sum_{i=1}^S A_i}(X)\subseteq\overline{GM}_{\sum_{i=1}^S A_i}(X)$, $\underline{GM}^I_{\sum_{i=1}^S A_i}(X)\supseteq\underline{GM}_{\sum_{i=1}^S A_i}(X)$ when $\beta>0$, $k>0$. So the positive region of GMDqI-DTRS is bigger than the positive region of GMRS, the negative region of GMDqI-DTRS is bigger than the negative region of GMRS. Accordingly, GMDqI-DTRS has a certain probability of error. In other words, GMDqI-DTRS inherits the advantage of Dq-DTRS with a certain probability of error. Moreover, for an information system $DS=(U,AT\cup DT,V,F)$, the approximation accuracy of U/DT with respect to $\sum_{i=1}^S A_i$ in GMDqI-DTRS is higher than the approximation accuracy of GMRS according to the definition 3.2. Similar results can be obtained in the GMDqII-DTRS.

In the information system $I=(U,AT,V,F)$, for an arbitrary subset X of U, when $\alpha_i=1,k=0$, the second kind of upper support characteristic function $USII^{A_i}_X(x)$ degenerates to characteristic function $1-S^{A_i}_{\sim X}(x)$ and lower support characteristic function $LSII^{A_i}_X(x)$ degenerates to support characteristic function $S^{A_i}_X(x)$. Therefore, it is true that $\overline{GM}^{II}_{\sum_{i=1}^S A_i}(X)=\overline{GM}_{\sum_{i=1}^S A_i}(X)$, $\underline{GM}^{II}_{\sum_{i=1}^S A_i}(X)=\underline{GM}_{\sum_{i=1}^S A_i}(X)$ when $\alpha_i=1,k=0$. So GMDqII-DTRS is equivalent to GMRS. That is to say, GMDqII-DTRS is a generalized model of GMRS.

When $\alpha_i<1$, $k>0$, conclusions $\overline{R}_k(X)\subseteq\overline{R}(X)$ and $\underline{R}_{\ (\alpha,\ \beta)}(X)\supseteq\underline{R}\ (X)$ hold in the second kind of double-quantitative rough set. Therefore, for the same φ, it is true that $\overline{GM}^{II}_{\sum_{i=1}^S A_i}(X)\subseteq\overline{GM}_{\sum_{i=1}^S A_i}(X)$, $\underline{GM}^{II}_{\sum_{i=1}^S A_i}(X)\supseteq\underline{GM}_{\sum_{i=1}^S A_i}(X)$ when $\alpha_i<1$, $k>0$. So the positive region of GMDqII-DTRS is bigger than the positive region of GMRS, the negative region of GMDqII-DTRS is bigger than the negative region of GMRS. Therefore, GMDqII-DTRS has a certain probability of error and the approximation accuracy of U/DT with respect to $\sum_{i=1}^S A_i$ in GMDqII-DTRS is higher than the approximation accuracy of GMRS according to the Definition 3.4.

It is true that two kinds of the generalized multigranulation double-quantitative decision-theoretic rough set are all equivalent to the generalized multigranulation rough set when $\alpha_i=1,\beta_i=0,k=0$. Certainly, when $\alpha_i=1,\beta_i=0,k=0$, the two kinds of the generalized multigranulation double-quantitative decision-theoretic rough set are also equivalent. Especially, optimistic and pessimistic multigranulation double-quantitative decision-theoretic rough sets degenerate to the optimistic multigranulation rough set and pessimistic multigranulation rough set, respectively. When $\alpha_i<1$, $\beta_i>0$, $k>0$, both GMDqI-DTRS and GMDqII-DTRS have a certain probability of error. Moreover, the approximation accuracy of GMDq-DTRS is higher than the approximation accuracy of GMRS. That is to say , the classification ability of GMDq-DTRS is better than the classification ability of GMRS from the perspective of the approximation accuracy. Therefore, GMDq-DTRS may be more practical in daily life.

(3) The inherent relation between GMDq-DTRS and Dq-DTRS

For the level of information φ, if $x\in\overline{GM}^I_{\sum_{i=1}^S A_i}(X)$ and $x\in\underline{GM}^I_{\sum_{i=1}^S A_i}(X)$, then there is at least one granular structure A_i which makes $x\in\overline{R}_{(\alpha_i,\beta_i)}(X)$ and $x\in\underline{R}_{\ k}(X)$. Therefore, an object belongs to the lower and upper approximations of GMDqI-DTRS implies this object at least belongs to the lower and upper approximations of DqI-DTRS under the granular structure. Accordingly, if $x\in\overline{GM}^{II}_{\sum_{i=1}^S A_i}(X)$ and $x\in\underline{GM}^{II}_{\sum_{i=1}^S A_i}(X)$, then there is at least one granular structure A_j which makes $x\in\overline{R}_{(\alpha_j,\beta_j)}(X)$ and $x\in\underline{R}_{\ k}(X)$. In other words, an object belongs to the lower and upper approximations of GMDqII-DTRS implies that this object at least belongs to the lower and upper approximations of DqII-DTRS under the granular structure. However, an object belongs to the lower and upper approximations of Dq-DTRS under one granular structure does not imply that this object belongs to the lower and upper approximations of GMDq-DTRS. Viewed from the overall model, GMDq-DTRS provides a more detailed characterization of approximate space.

From the perspective of fusion, GMDq-DTRS is the information fusion of many Dq-DTRS models.

(4) The relationship between GMDq-DTRS and variable precision rough set(VRS)

In the lower and upper approximations of rough set model, four kinds of models can be obtained by considering relative quantitative and absolute quantitative information, namely

① The upper approximation is quantified by the relative quantitative information and the lower approximation is quantified by the relative quantitative information. The formula is expressed as $\overline{R}(X) = \{x \in U | P(X|[x]_R) > \beta\}, \underline{R}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\}$.
② The upper approximation is quantified by the absolute quantitative information and the lower approximation is quantified by the absolute quantitative information. The formula is expressed as $\overline{R}(X) = \{x \in U | |[x]_R \cap X| > k\}, \underline{R}(X) = \{x \in U | |[x]_R| - |[x]_R \cap X| \leq k\}$.
③ The upper approximation is quantified by the relative quantitative information and the lower approximation is quantified by the absolute quantitative information. The formula is expressed as $\overline{R}(X) = \{x \in U | P(X|[x]_R) > \beta\}, \underline{R}(X) = \{x \in U | |[x]_R| - |[x]_R \cap X| \leq k\}$.
④ The upper approximation is quantified by the absolute quantitative information and the lower approximation is quantified by the relative quantitative information. The formula is expressed as $\overline{R}(X) = \{x \in U | |[x]_R \cap X| > k\}, \underline{R}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\}$.

It is obvious that the first is the decision-theoretic rough set model(DTRS), the second is grade rough set model(GRS), the third is the first kind of double-quantitative decision-theoretic rough set model(DqI-DTRS), and the fourth is the second kind of double-quantitative decision-theoretic rough set model(DqII-DTRS). It is well known that the variable precision rough set(VRS) is a special model of DTRS in which $\alpha + \beta = 1$. It is evident that the variable precision rough set is the model of relative quantitative information and Dq-DTRS is the model of relative quantitative and absolute quantitative information by the combined consideration of relative and absolute quantification in the lower and upper approximations. GMDq-DTRS is also a double-quantitative model by the combined consideration of relative and absolute quantification in the lower and upper approximations under multiple granular structures. Therefore, from the perspective of quantization index, GMDq-DTRS provides a more comprehensive characterization for the approximate space than the variable precision rough set. From granular structures, GMDq-DTRS provides a more detailed characterization of approximate space than the variable precision rough set.

(5) The comparison between GMDq-DTRS and other models

With the development of information technology, more and more data is released every day, and the amount of data is more and more large. One of the most urgent things is how to make full use of data to make decisions. Based on the principle of the minority subordinate to the majority and the combination of relative and absolute quantification, generalized multigranulation double-quantitative decision-theoretic rough set(GMDq-DTRS) theory may provide a comprehensive decision method for mass data. It is well known that GMDq-DTRS is a generalization of generalized multigranulation rough set(GMRS). Recently, there are a lot of research about multigranulation rough set. Therefore, the detailed comparison between GMDq-DTRS and some models is made.

Feng et al. [29] explored variable precision multigranulation fuzzy rough sets by using the maximal and minimal membership degrees of an object with respect to a fuzzy set based on multi-fuzzy tolerance relations and the decision theory of Type-1 variable precision multigranulation fuzzy rough set was discussed. Their focus are variable precision multigranulation fuzzy rough sets and decision-theoretic rough set. The emphasis of this paper are generalized multigranulation and double-quantitative decision-theoretic rough set.

Zhang et al. [51] established four kinds of constructive methods of rough approximation operators from the view point of the union and intersection operations of rough approximation pairs. From the paper, we know that many rough sets(include optimistic and pessimistic multigranulation rough sets) are essentially direct applications of these constructive methods. Their focus are constructive methods of rough approximation operators and multigranulation rough sets. The emphasis of this paper is generalized multigranulation rough approximations which is a general multigranulation rough approximations, and the study is based on double-quantitative decision-theoretic rough set theory.

Li et al. [16] investigated the relationship between optimistic and pessimistic multigranulation rough sets and concept lattices via rule acquisition by the comparison and combination of rough set theory, granular computing and formal concept analysis. Their focus is the relationship of decision rules of optimistic and pessimistic multigranulation rough sets and the rules of concept lattices. Our paper also study relationships, but they are the relationship of GMDqI-DTRS and GMDqII-DTRS, the relationship of GMDq-DTRS and GMRS, the relationship of GMDq-DTRS and other models.

Lin et al. [13] proposed the fuzzy multigranulation decision-theoretic rough set and a comparative study between the fuzzy model and Qian's multigranulation decision-theoretic rough set model was made. Their focus is decision-theoretic rough set. The emphasis of this paper is the combination of decision-theoretic rough set and grade rough set, namely the double-quantitative decision-theoretic rough set. From the perspective of multi-source, both Lin's paper and ours provide methods for multi-source data analysis.

Lin et al. [12] proposed a feature selection method by fusing all individual feature rank lists which were obtained based on the significance of features in different granular structures. In terms of classification performance, the proposed method can effectively select a discriminative feature subset and perform as well as or better than other popular feature selection algorithms. Their focus is the feature selection by fusion the significance of features under all granular structures. Considering the decisions fusion based on the principle of the minority subordinate to the majority, the emphasis of this paper is exploration of the double-quantitative decision-theoretic rough set model with strong fault tolerance ability under granular structures.

Yang et al. [47] proposed naive and fast algorithms for updating the multigranulation rough approximations with the increasing of the granular structures. The most important thing is the fast algorithm based on the monotonic property of the multigranulation rough approximations can effectively reduce the computational time when facing high dimensional data sets, traditional reduction and attribute clustering based reduction. It is mainly focus on fast updating the optimistic and pessimistic multigranulation rough approximations. The emphasis of this paper is the construction of a new model of generalized multigranulation rough approximations based on the combined consideration of relative and absolute quantification in the lower and upper approximations.

4. Case study and application

Compared with classical rough set theory, generalized multigranulation double-quantitative decision-theoretic rough sets have a certain fault tolerance capability. Compared with the generalized

Table 2
Cars data.

Cars	*Design*	*Model*	*Color*	*Beauty*	Cars	*Design*	*Model*	*Color*	*Beauty*
x_1	1	0	0	0	x_{11}	1	1	0	0
x_2	0	0	1	1	x_{12}	0	1	2	1
x_3	1	2	2	0	x_{13}	2	0	1	0
x_4	1	1	2	1	x_{14}	0	0	2	0
x_5	1	0	0	1	x_{15}	1	0	1	1
x_6	2	2	2	1	x_{16}	2	1	2	1
x_7	2	1	1	0	x_{17}	1	1	1	0
x_8	0	2	0	0	x_{18}	0	1	1	0
x_9	2	2	1	1	x_{19}	2	0	0	1
x_{10}	0	2	1	1	x_{20}	2	2	0	0

Table 3
Statistical results of car classes under the granular structureA_1.

(i, j)	$[x]_{A_1}$	$\|[x]_{A_1}\|$	$[x]_{A_1} \cap X$	$\|[x]_{A_1} \cap X\|$	$P(X\|[x]_{A_1})$	$\|[x]_{A_1}\| - \|[x]_{A_1} \cap X\|$
(0, 0)	$x_{2,\ 14}$	2	x_2	1	1/2	1
(0, 1)	$x_{12,\ 18}$	2	x_{12}	1	1/2	1
(0, 2)	$x_{8,\ 10}$	2	x_{10}	1	1/2	1
(1, 0)	$x_{1,\ 5,\ 15}$	3	$x_{5,\ 15}$	2	2/3	1
(1, 1)	$x_{4,\ 11,\ 17}$	3	x_4	1	1/3	2
(1, 2)	x_3	1	$\emptyset$	0	0	1
(2, 0)	$x_{13,\ 19}$	2	x_{19}	1	1/2	1
(2, 1)	$x_{7,\ 16}$	2	x_{16}	1	1/2	1
(2, 2)	$x_{6,\ 9,\ 20}$	3	$x_{6,\ 9}$	2	2/3	1

multigranulation rough set, generalized multigranulation double-quantitative decision-theoretic rough sets consider relative and absolute quantitative information between the class and concept, so the classification ability is better than GMRS from the perspective of the approximation accuracy. Meanwhile, compared with the double-quantitative decision-theoretic rough set, generalized multigranulation double-quantitative decision-theoretic rough sets provide a feasible decision method that is the minority subordinate to the majority. In order to show the advantage of using relative and absolute quantitative simultaneously under multiple granular structures, a specific case is introduced in this paper.

In this section, a knowledge representation system of cars is introduced to illustrate the theory and advantage of the new model. Detailed description is shown in the following. Let $S = (U, AT, D, F)$ be a decision table, where U is composed of 20 cars, and $AT = \{Design, Model, Color\}$ ia a conditional attribute set and $D = \{Beauty\}$ is a decision attribute set. Let $A_i \subseteq AT, i = 1, 2, 3.$ denote equivalence relations about condition attributes, where $A_1 = \{Design, Model\}$, $A_2 = \{Design, Color\}$, $A_3 = \{Model, Color\}$. Based on the measured car data in Table 2, Table 3–5 show the statistical results of car classes under different granular structures, where $(i,\ j)$ $(i,\ j \in [0,\ 2])$ denotes the rank of condition attributes and $X = \{x_2, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{15}, x_{16}, x_{19}\}$ denotes a decision class in which cars are beautiful. The rough set regions will be calculated in the case that $k = 1, \varphi = 2/3$.

4.1. Description of the GMDq-DTRS theory

Firstly, results of car classes under A_1, A_2, A_3 granular structures can be get from Table 3–5.

Then we can get the generalized multigranulation double-quantitative lower and upper approximations of X with respect to $\sum_{i=1}^{3} A_i$ under different constraint conditions. In the Bayesian decision procedure [37], experts will give values of the loss function, namely, $\lambda_{iP} = \lambda(a_i|X)$, $\lambda_{iN} = \lambda(a_i|X^C)$, and $i = P, B, N$.

Case 1: Consider loss functions of Table 6, there are $\alpha_1 = 0.6, \beta_1 = 0.4, \alpha_2 = 0.7, \beta_2 = 0.3, \alpha_3 = 0.8, \beta_3 = 0.2$. It is true that $\alpha_i + \beta_i = 1$ for $i = 1, 2, 3$. According to Table 2–4, we can obtain the upper and lower approximations of DqI-DTRS model.

Under the granular structure A_1, according to the definition of DqI-DTRS, we can get

$$\overline{A_1}_{(0.6,0.4)}(X) = U - \{x_3, x_4, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\},$$
$$\underline{A_1}_1(X) = U - \{x_4, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

Under the granular structure A_2, according to the definition of DqI-DTRS, we can get

$$\overline{A_2}_{(0.7,0.3)}(X) = U - \{x_8\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\},$$
$$\underline{A_2}_1(X) = U - \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}$$
$$= \{x_2, x_3, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.$$

Under the granular structure A_3, according to the definition of DqI-DTRS, we can get

$$\overline{A_3}_{(0.8,0.2)}(X) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\},$$
$$\underline{A_3}_1(X) = U - \{x_7, x_8, x_{17}, x_{18}, x_{20}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{19}\}.$$

Then on the basis of the definition of $\overline{GM}^{I}_{\sum_{i=1}^{S} A_i}$ and $\underline{GM}^{I}_{\sum_{i=1}^{S} A_i}$, there are

$$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X) = U - \{x_8, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}$$
$$\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X) = U - \{x_7, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}$$

By the lower approximation $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X)$ and upper approximation $\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X)$, the positive region, negative region, upper

Table 4
Statistical results of car classes under the granular structure A_2.

(i, j)	$[x]_{A_2}$	$\|[x]_{A_2}\|$	$[x]_{A_2} \cap X$	$\|[x]_{A_2} \cap X\|$	$P(X\|[x]_{A_2})$	$\|[x]_{A_2}\| - \|[x]_{A_2} \cap X\|$
(0, 0)	x_8	1	$\emptyset$	0	0	1
(0, 1)	$x_{2,\,10,\,18}$	3	$x_{2,\,10}$	2	2/3	1
(0, 2)	$x_{12,\,14}$	2	x_{12}	1	1/2	1
(1, 0)	$x_{1,\,5,\,11}$	3	x_5	1	1/3	2
(1, 1)	$x_{15,\,17}$	2	x_{15}	1	1/2	1
(1, 2)	$x_{3,\,4}$	2	x_4	1	1/2	1
(2, 0)	$x_{19,\,20}$	2	x_{19}	1	1/2	1
(2, 1)	$x_{7,\,9,\,13}$	3	x_9	1	1/3	2
(2, 2)	$x_{6,\,16}$	2	$x_{6,\,16}$	2	1	0

Table 5
Statistical results of car classes under the granular structure A_3.

(i, j)	$[x]_{A_3}$	$\|[x]_{A_3}\|$	$[x]_{A_3} \cap X$	$\|[x]_{A_3} \cap X\|$	$P(X\|[x]_{A_3})$	$\|[x]_{A_3}\| - \|[x]_{A_3} \cap X\|$
(0, 0)	$x_{1,\,5,\,19}$	3	$x_{5,\,19}$	2	2/3	1
(0, 1)	$x_{2,\,13,\,15}$	3	$x_{2,\,15}$	2	2/3	1
(0, 2)	x_{14}	1	$\emptyset$	0	0	1
(1, 0)	x_{11}	1	$\emptyset$	0	0	1
(1, 1)	$x_{7,\,17,\,18}$	3	$\emptyset$	0	0	3
(1, 2)	$x_{4,\,12,\,16}$	3	$x_{4,\,12,\,16}$	3	1	0
(2, 0)	$x_{8,\,20}$	2	$\emptyset$	0	0	2
(2, 1)	$x_{9,\,10}$	2	$x_{9,\,10}$	2	1	0
(2, 2)	$x_{3,\,6}$	2	x_6	1	1/2	1

Table 6
Loss functions of A_1, A_2, A_3 granular structures.

	A_1		A_2		A_3	
a_P	0	22	0	13	0	36
a_B	12	4	3	6	8	4
a_N	18	0	17	0	24	0

and lower boundary region of X are as follows:

$$pos^I(X) = U - \{x_7, x_8, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\};$$
$$neg^I(X) = \{x_{11}, x_{17}\}; Ubn^I(X) = \{x_7\}; Lbn^I(X) = \{x_8\}.$$

Similarly, according to results from Table 7–9 and definitions of DqII-DTRS, we can obtain the upper and lower approximations of $\sim X = \{x_1, x_3, x_7, x_8, x_{11}, x_{13}, x_{14}, x_{17}, x_{18}, x_{20}\}$ about DqII-DTRS.

Under the granular structure A_1, according to the definition of DqII-DTRS, we can get

$$\overline{A_1}_1(\sim X) = \{x_4, x_{11}, x_{17}\}, \underline{A_1}_{(0.6,0.4)}(\sim X) = \{x_3, x_4, x_{11}, x_{17}\}.$$

Under the granular structure A_2, according to the definition of DqII-DTRS, we can get

$$\overline{A_2}_1(\sim X) = \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}, \underline{A_2}_{(0.7,0.3)}(\sim X) = \{x_8\}.$$

Under the granular structure A_3, according to the definition of DqII-DTRS, we can get

$$\overline{A_3}_1(\sim X) = \{x_7, x_8, x_{17}, x_{18}, x_{20}\},$$
$$\underline{A_3}_{(0.8,0.2)}(\sim X) = \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}.$$

Then according to the definition of $\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}$ and $\underline{GM}^{II}_{\sum_{i=1}^{S} A_i}$, there are

$$\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X) = \{x_7, x_{11}, x_{17}\}, \underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X) = \{x_8, x_{11}, x_{17}\}.$$

By the lower approximation $\underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X)$ and upper approximation $\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X)$, the positive region, negative region, upper boundary region and lower boundary region of $\sim X$ are as following:

$$pos^{II}(\sim X) = \{x_7, x_{11}\}; Ubn^{II}(\sim X) = \{x_7\}; Lbn^{II}(\sim X) = \{x_8\};$$
$$neg^{II}(\sim X) = U - \{x_7, x_8, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

Compared with above results of the positive, negative and boundary regions of GMDqIRS and GMDqIIRS, we can obtain that the accepted region, rejective region, delayed region of X in GMDqI-DTRS are equivalent to the rejective region, accepted region, delayed region of $\sim X$ in GMDqII-DTRS, respectively.

Case 2: Consider loss functions of Table 10, then it is true that $\alpha_i + \beta_i < 1$ for $i = 1, 2, 3$. We can obtain the upper and lower approximations of DqI-DTRS model.

From the loss founction of the granular structure A_1, there are $\alpha_i = 0.6$, and $\beta_i = 0.3$. By the definition of DqI-DTRS and the result of Table 3, we can get

$$\overline{A_1}_{(0.6,0.3)}(X) = U - \{x_3\}$$
$$= \{x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12},$$
$$x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\},$$
$$\underline{A_1}_1(X) = U - \{x_4, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{12},$$
$$x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

From the loss founction of the granular structure A_2, there are $\alpha_i = 0.7$, and $\beta_i = 0.2$. By the definition of DqI-DTRS and the result of Table 4, we can obtain

$$\overline{A_2}_{(0.7,0.2)}(X) = U - \{x_8\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12},$$
$$x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\},$$
$$\underline{A_2}_1(X) = U - \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}$$
$$= \{x_2, x_3, x_4, x_6, x_8, x_{10}, x_{12}, x_{14},$$
$$x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.$$

From the loss founction of the granular structure A_3, there are $\alpha_i = 0.8$, and $\beta_i = 0.1$. By the definition of DqI-DTRS and the result of Table 5, we can have

Table 7
Statistical results of car classes under the granular structure A_1.

(i, j)	$[x]_{A_1}$	$\|[x]_{A_1}\|$	$[x]_{A_1} \cap (\sim X)$	$\|[x]_{A_1} \cap (\sim X)\|$	$P((\sim X)\|[x]_{A_1})$	$\|[x]_{A_1}\| - \|[x]_{A_1} \cap (\sim X)\|$
(0, 0)	$x_{2,\ 14}$	2	x_{14}	1	1/2	1
(0, 1)	$x_{12,\ 18}$	2	x_{18}	1	1/2	1
(0, 2)	$x_{8,\ 10}$	2	x_8	1	1/2	1
(1, 0)	$x_{1,\ 5,\ 15}$	3	x_1	1	1/3	2
(1, 1)	$x_{4,\ 11,\ 17}$	3	$x_{11,\ 17}$	2	2/3	1
(1, 2)	x_3	1	x_3	1	1	0
(2, 0)	$x_{13,\ 19}$	2	x_{13}	1	1/2	1
(2, 1)	$x_{7,\ 16}$	2	x_7	1	1/2	1
(2, 2)	$x_{6,\ 9,\ 20}$	3	x_{20}	1	1/3	2

Table 8
Statistical results of car classes under the granular structure A_2.

(i, j)	$[x]_{A_2}$	$\|[x]_{A_2}\|$	$[x]_{A_2} \cap (\sim X)$	$\|[x]_{A_2} \cap (\sim X)\|$	$P((\sim X)\|[x]_{A_2})$	$\|[x]_{A_2}\| - \|[x]_{A_2} \cap (\sim X)\|$
(0, 0)	x_8	1	x_8	1	1	0
(0, 1)	$x_{2,\ 10,\ 18}$	3	x_{18}	1	1/3	2
(0, 2)	$x_{12,\ 14}$	2	x_{14}	1	1/2	1
(1, 0)	$x_{1,\ 5,\ 11}$	3	$x_{1,\ 11}$	2	2/3	1
(1, 1)	$x_{15,\ 17}$	2	x_{17}	1	1/2	1
(1, 2)	$x_{3,\ 4}$	2	x_3	1	1/2	1
(2, 0)	$x_{19,\ 20}$	2	x_{20}	1	1/2	1
(2, 1)	$x_{7,\ 9,\ 13}$	3	$x_{7,\ 13}$	2	2/3	1
(2, 2)	$x_{6,\ 16}$	2	$\emptyset$	0	0	2

Table 9
Statistical results of car classes under the granular structureA_3.

(i, j)	$[x]_{A_3}$	$\|[x]_{A_3}\|$	$[x]_{A_3} \cap (\sim X)$	$\|[x]_{A_3} \cap (\sim X)\|$	$P((\sim X)\|[x]_{A_3})$	$\|[x]_{A_3}\| - \|[x]_{A_3} \cap (\sim X)\|$
(0, 0)	$x_{1,\ 5,\ 19}$	3	x_1	1	1/3	2
(0, 1)	$x_{2,\ 13,\ 15}$	3	x_{13}	1	1/3	2
(0, 2)	x_{14}	1	x_{14}	1	1	0
(1, 0)	x_{11}	1	x_{11}	1	1	0
(1, 1)	$x_{7,\ 17,\ 18}$	3	$x_{7,\ 17,\ 18}$	3	1	0
(1, 2)	$x_{4,\ 12,\ 16}$	3	$\emptyset$	0	0	3
(2, 0)	$x_{8,\ 20}$	2	$x_{8,\ 20}$	2	1	0
(2, 1)	$x_{9,\ 10}$	2	$\emptyset$	0	0	2
(2, 2)	$x_{3,\ 6}$	2	x_3	1	1/2	1

Table 10
Loss functions of A_1, A_2, A_3 granular structures.

	A_1		A_2		A_3	
a_P	0	9	0	19	0	18
a_B	2	6	6	5	4	2
a_N	16	0	26	0	22	0

$$\overline{A_3}_{(0.8,0.1)}(X) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\},$$
$$\underline{A_3}_1(X) = U - \{x_7, x_8, x_{17}, x_{18}, x_{20}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{19}\}.$$

According to the definition of $\overline{GM}^I_{\sum_{i=1}^S A_i} and\, \underline{GM}^I_{\sum_{i=1}^S A_i}$, there are

$$\overline{GM}^I_{\sum_{i=1}^3 A_i}(X) = U - \{x_8\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12},$$
$$x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}$$
$$\underline{GM}^I_{\sum_{i=1}^3 A_i}(X) = U - \{x_7, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

By the lower approximation $\underline{GM}^I_{\sum_{i=1}^3 A_i}(X)$ and upper approximation $\overline{GM}^I_{\sum_{i=1}^3 A_i}(X)$, the positive region, negative region, upper boundary region and lower boundary region of X are as following:

$$pos^I(X) = U - \{x_7, x_8, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\};$$
$$neg^I(X) = \emptyset; Ubn^I(X)$$
$$= \{x_7, x_{11}, x_{17}\}; Lbn^I(X) = \{x_8\}.$$

Similarly, we can obtain the upper and lower approximations of $\sim X = \{x_1, x_3, x_7, x_8, x_{11}, x_{13}, x_{14}, x_{17}, x_{18}, x_{20}\}$ about DqII-DTRS.

When $\alpha_1 = 0.6$, $\beta_1 = 0.3$, there are $\overline{A_1}_1(\sim X) = \{x_4, x_{11}, x_{17}\}$, $\underline{A_1}_{(0.6,0.3)}(\sim X) = \{x_3, x_4, x_{11}, x_{17}\}$.

When $\alpha_2 = 0.7$, $\beta_2 = 0.2$, there are $\overline{A_2}_1(\sim X) = \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}, \underline{A_2}_{(0.7,0.2)}(\sim X) = \{x_8\}$.

When $\alpha_3 = 0.8$, $\beta_3 = 0.1$, there are $\overline{A_3}_1(\sim X) = \{x_7, x_8, x_{17}, x_{18}, x_{20}\}, \underline{A_3}_{(0.8,0.1)}(\sim X) = \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$.

According to the definition of $\overline{GM}^{II}_{\sum_{i=1}^S A_i} and \underline{GM}^{II}_{\sum_{i=1}^S A_i}$, there are

$$\overline{GM}^{II}_{\sum_{i=1}^3 A_i}(\sim X)$$
$$= \{x_7, x_{11}, x_{17}\}, \underline{GM}^{II}_{\sum_{i=1}^3 A_i}(\sim X) = \{x_8, x_{11}, x_{17}\}.$$

By the lower approximation $\underline{GM}^{II}_{\sum_{i=1}^3 A_i}(\sim X)$ and upper approximation $\overline{GM}^{II}_{\sum_{i=1}^3 A_i}(\sim X)$, the positive region, negative region, upper boundary region and lower boundary region of $\sim X$ are as following:

Table 11
Loss functions of A_1, A_2, A_3 granular structures.

	A_1		A_2		A_3	
a_P	0	13	0	13	0	19
a_B	6	4	3	6	4	3
a_N	10	0	12	0	11	0

$$pos^{II}(\sim X) = \{x_7, x_{11}\}; Ubn^{II}(\sim X) = \{x_7\}; Lbn^{II}(\sim X) = \{x_8\}$$
$$neg^{II}(\sim X) = U - \{x_7, x_8, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

Comparison of results of the positive, negative and boundary regions between GMDqIRS and GMDqIIRS , we can obtain that the rejective region and lower delayed region of X in GMDqI-DTRS are contained in the accepted region and lower delayed region of $\sim X$ in GMDqII-DTRS, respectively; the rejective region and upper delayed region of $\sim X$ in GMDqII-DTRS are included in the accepted region and upper delayed region of X in GMDqI-DTRS, respectively.

Case 3: Considering loss functions of Table 11, then conclusion $\alpha_i + \beta_i > 1,\ i = 1, 2, 3$ hold. We can obtain the upper and lower approximations of X about DqI-DTRS.

When $\alpha_i = 0.6,\ \beta_i = 0.5,$ there are

$$\overline{A_1}_{(0.6,0.5)}(X) = \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\},$$
$$\underline{A_1}_1(X) = U - \{x_4, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

When $\alpha_i = 0.7,\ \beta_i = 0.4,$ there are

$$\overline{A_2}_{(0.7,0.4)}(X) = U - \{x_1, x_5, x_7, x_8, x_9, x_{11}, x_{13}\}$$
$$= \{x_2, x_3, x_4, x_6, x_{10}, x_{12}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\},$$
$$\underline{A_2}_1(X) = U - \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}$$
$$= \{x_2, x_3, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}\}.$$

When $\alpha_i = 0.8,\ \beta_i = 0.3,$ there are

$$\overline{A_3}_{(0.8,0.3)}(X) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\},$$
$$\underline{A_3}_1(X) = U - \{x_7, x_8, x_{17}, x_{18}, x_{20}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{19}\}.$$

According to the definition of $\overline{GM}^{I}_{\sum_{i=1}^{S} A_i}$ and $\underline{GM}^{I}_{\sum_{i=1}^{S} A_i}$, there are

$$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{15}, x_{16}, x_{19}, x_{20}\}$$
$$\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X) = U - \{x_7, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}$$

By the lower approximation $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X)$ and upper approximation $\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X)$, the positive region, negative region, upper boundary region and lower boundary region of X are as following:

$$pos^{I}(X) = U - \{x_7, x_8, x_{11}, x_{13}, x_{14}, x_{17}, x_{18}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{15}, x_{16}, x_{19}, x_{20}\};$$
$$neg^{I}(X) = \{x_7, x_{11}, x_{17}\}; Ubn^{I}(X) = \emptyset; Lbn^{I}(X) = \{x_8, x_{13}, x_{14}, x_{18}\}.$$

Similarly, we can obtain the upper and lower approximations of $\sim X = \{x_1, x_3, x_7, x_8, x_{11}, x_{13}, x_{14}, x_{17}, x_{18}, x_{20}\}$ about DqII-DTRS.

When $\alpha_i = 0.6,\ \beta_i = 0.5,$ there are $\overline{A_1}_1(\sim X) = \{x_4, x_{11}, x_{17}\}, \underline{A_1}_{(0.6,0.5)}(\sim X) = \{x_3, x_4, x_{11}, x_{17}\}.$

When $\alpha_i = 0.7,\ \beta_i = 0.4,$ there are $\overline{A_2}_1(\sim X) = \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}, \underline{A_2}_{(0.7,0.4)}(\sim X) = \{x_8\}.$

When $\alpha_i = 0.8,\ \beta_i = 0.3,$ there are $\overline{A_3}_1(\sim X) = \{x_7, x_8, x_{17}, x_{18}, x_{20}\}, \underline{A_3}_{(0.8,0.3)}(\sim X) = \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}.$

According to the definition of $\overline{GM}^{II}_{\sum_{i=1}^{S} A_i}$ and $\underline{GM}^{II}_{\sum_{i=1}^{S} A_i}$, there are

$$\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X) = \{x_7, x_{11}, x_{17}\}, \underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X) = \{x_8, x_{11}, x_{17}\}.$$

By the lower approximation $\underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X)$ and upper approximation $\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(\sim X)$, the positive region, negative region, upper and lower boundary region of $\sim X$ are as following:

$$pos^{II}(\sim X) = \{x_7, x_{11}\}; Ubn^{II}(\sim X) = \{x_7\}; Lbn^{II}(\sim X) = \{x_8\};$$
$$neg^{II}(\sim X) = U - \{x_7, x_8, x_{11}, x_{17}\}$$
$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}, x_{20}\}.$$

Observing above results of the positive, negative and boundary regions in GMDqIRS and GMDqIIRS , we can known that the accepted region and upper delayed region of X in GMDqI-DTRS are contained in the rejective region and upper delayed region of $\sim X$ in GMDqII-DTRS, respectively; the accepted region and lower delayed region of $\sim X$ in GMDqII-DTRS are included in rejective region and lower delayed region of X in GMDqI-DTRS, respectively.

When $\alpha_i < 1$, $\beta_i > 0$, $k \geq 1$, by comparing related results of generalized multigranulation double-quantitative rough sets and generalized multigranulation rough sets, we find generalized multigranulation double-quantitative decision-theoretic rough set theory has a strong fault tolerance ability, and can provide a more detailed description of the approximate space.

4.2. Description of the relationship between GMDq-DTRS and GMRS

According to the Pawlak rough set theory, the lower and upper approximations of X under different granular structures can be obtained. Detailed results are as follows:

$\overline{A_1}(X) = U - \{x_3\},\ \underline{A_1}(X) = \emptyset;$

$\overline{A_2}(X) = U - \{x_8\},\ \underline{A_2}(X) = \{x_6, x_{16}\};$

$\overline{A_3}(X) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\},\quad \underline{A_3}(X) = \{x_4, x_9, x_{10}, x_{12}, x_{16}\}.$

When the information level $\varphi = 2/3,$ the generalized multigranulation lower and upper approximations of X with respect to $\sum_{i=1}^{3} A_i$ can be obtained, namely

$\overline{GM}_{\sum_{i=1}^{3} A_i}(X) = U - \{x_8\}, \underline{GM}_{\sum_{i=1}^{3} A_i}(X) = \{x_{16}\}.$

On the other hand, when $\alpha_i = 1, \beta_i = 0, k = 0,$ the lower and upper approximations of X under different granular structures can be obtained in the DqI-DTRS model. Detailed results are as follows:

$\overline{A_1}_{(1,0)}(X) = U - \{x_3\} = \overline{A_1}(X),\ \underline{A_1}_0(X) = \emptyset = \underline{A_1}(X);$

$\overline{A_2}_{(1,0)}(X) = U - \{x_8\} = \overline{A_2}(X),\ \underline{A_2}_0(X) = \{x_6, x_{16}\} = \underline{A_2}(X);$

$\overline{A_3}_{(1,0)}(X) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\} = \overline{A_3}(X),\quad \underline{A_3}_0(X) = \{x_4, x_9, x_{10}, x_{12}, x_{16}\} = \underline{A_3}(X).$

When the information level $\varphi = 2/3,$ the lower and upper approximations of an arbitrary subset X with respect to $\sum_{i=1}^{3} A_i$ can be obtained in the GMDqI-DTRS, namely

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X) = U - \{x_8\} = \overline{GM}_{\sum_{i=1}^{3} A_i}(X), \underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X) = \{x_{16}\} = \underline{GM}_{\sum_{i=1}^{3} A_i}(X).$

And the lower and upper approximations of X under different granular structures can be obtained in the DqII-DTRS model. Detailed results are as follows:

$\overline{A_1}_0(X) = U - \{x_3\} = \overline{A_1}(X),\ \underline{A_1}_{(1,0)}(X) = \emptyset = \underline{A_1}(X);$

$\overline{A_2}_0(X) = U - \{x_8\} = \overline{A_2}(X),\ \underline{A_2}_{(1,0)}(X) = \{x_6, x_{16}\} = \underline{A_2}(X);$

$\overline{A_3}_0(X) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\} = \overline{A_3}(X),\quad \underline{A_3}_{(1,0)}(X) = \{x_4, x_9, x_{10}, x_{12}, x_{16}\} = \underline{A_3}(X).$

When the information level $\varphi = 2/3$, the lower and upper approximations of an arbitrary subset X with respect to $\sum_{i=1}^{3} A_i$ can be obtained in the GMDqII-DTRS, namely

$\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X) = U - \{x_8\} = \overline{GM}_{\sum_{i=1}^{3} A_i}(X)$, $\underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X) = \{x_{16}\} = \underline{GM}_{\sum_{i=1}^{3} A_i}(X)$.

By comparing the results of generalized multigranulation double-quantitative rough sets and generalized multigranulation rough sets, we can find the two kinds of the generalized multigranulation double-quantitative decision-theoretic rough set are also equivalent when $\alpha_i = 1, \beta_i = 0, k = 0$. The two kinds of the generalized multigranulation double-quantitative decision-theoretic rough set are all equivalent to the generalized multigranulation rough set.

4.3. Description of the advantage of GMDq-DTRS

The calculation method of the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the generalized multigranulation rough set(GMRS) is as follows:

$$\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{\sum_{Y_i \in U/DT} |\underline{GM}_{\sum_{i=1}^{S} A_i}(Y_i)|}{\sum_{Y_i \in U/D} |\overline{GM}_{\sum_{i=1}^{S} A_i}(Y_i)|},$$

where $\overline{GM}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U : \quad (\sum_{i=1}^{S}(1 - S^{A_i}_{\sim X}(x)))/s > 1 - \varphi\}$, $\underline{GM}_{\sum_{i=1}^{S} A_i}(X) = \{x \in U : \ (\sum_{i=1}^{S} S^{A_i}_X(x))/s \geq \varphi\}$.

The lower and upper approximations of decision classes X_1 and X_2 under granular structures A_1, A_2 and A_3 can be obtained as follows:

$\overline{A_1}(X_1) = U - \{x_3\}$, $\underline{A_1}(X_1) = \emptyset$; $\overline{A_1}(X_2) = U$, $\underline{A_1}(X_2) = \{x_3\}$;

$\overline{A_2}(X_1) = U - \{x_8\}$, $\underline{A_2}(X_1) = \{x_6, x_{16}\}$; $\overline{A_2}(X_2) = U - \{x_6, x_{16}\}$, $\underline{A_2}(X_2) = \{x_8\}$;

$\overline{A_3}(X_1) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$, $\underline{A_3}(X_1) = \{x_4, x_9, x_{10}, x_{12}, x_{16}\}$;

$\overline{A_3}(X_2) = U - \{x_4, x_9, x_{10}, x_{12}, x_{16}\}$, $\underline{A_3}(X_2) = \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$.

In the GMRS, the lower and upper approximations of X_1 and $X_2$with respect to $\sum_{i=1}^{S} A_i$ can be obtained as follows:

$\overline{GM}_{\sum_{i=1}^{S} A_i}(X_1) = U - \{x_8\}$, $\underline{GM}_{\sum_{i=1}^{S} A_i}(X_1) = \{x_{16}\}$;

$\overline{GM}_{\sum_{i=1}^{S} A_i}(X_2) = U - \{x_{16}\}$, $\underline{GM}_{\sum_{i=1}^{S} A_i}(X_2) = \{x_8\}$.

Therefore, the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the generalized multigranulation rough set(GMRS) can be calculated as $\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{|\underline{GM}_{\sum_{i=1}^{3} A_i}(X_1)| + |\underline{GM}_{\sum_{i=3}^{S} A_i}(X_2)|}{|\overline{GM}_{\sum_{i=1}^{3} A_i}(X_1)| + |\overline{GM}_{\sum_{i=1}^{3} A_i}(X_2)|} = \frac{1}{19}$.

In the first kind of double-quantitative decision-theoretic rough set(DqI-DTRS), the lower and upper approximations of decision classes X_1 and X_2 under granular structures A_1, A_2 and A_3 can be obtained as follows:

when $\alpha_i + \beta_i = 1$,

$\overline{A_1}_{(0.6,0.4)}(X_1) = U - \{x_3, x_4, x_{11}, x_{17}\}$, $\underline{A_1}_1(X_1) = U - \{x_4, x_{11}, x_{17}\}$;

$\overline{A_1}_{(0.6,0.4)}(X_2) = U - \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}$, $\underline{A_1}_1(X_2) = U - \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}$.

$\overline{A_2}_{(0.7,0.3)}(X_1) = U - \{x_8\}$, $\underline{A_2}_1(X_1) = U - \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}$;

$\overline{A_2}_{(0.7,0.3)}(X_2) = U - \{= x_6, x_{16}\}$, $\underline{A_2}_1(X_2) = U - \{x_2, x_6, x_{10}, x_{16}, x_{18}\}$.

$\overline{A_3}_{(0.8,0.2)}(X_1) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$, $\underline{A_3}_1(X_1) = U - \{x_7, x_8, x_{17}, x_{18}, x_{20}\}$

$\overline{A_3}_{(0.8,0.2)}(X_2) = U - \{x_4, x_9, x_{10}, x_{12}, x_{16}\}$, $\underline{A_3}_1(X_2) = U - \{x_1, x_2, x_4, x_5, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\}$.

In the GMDqI-DTRS, when $\varphi = 2/3$, the lower and upper approximations of X_1 and $X_2$with respect to $\sum_{i=1}^{S} A_i$ can be obtained as follows:

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1) = U - \{x_8, x_{11}, x_{17}\}$, $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1) = U - \{x_7, x_{11}, x_{17}\}$;

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2) = U - \{x_6, x_9, x_{16}\}$, $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2) = U - \{x_1, x_2, x_5, x_6, x_9, x_{10}, x_{15}, x_{16}\}$

Therefore, the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the GMDqI-DTRS can be calculated as $\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{|\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1)| + |\underline{GM}^{I}_{\sum_{i=3}^{S} A_i}(X_2)|}{|\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1)| + |\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2)|} = \frac{29}{34}$.

When $\alpha_i + \beta_i < 1$,

$\overline{A_1}_{(0.6,0.3)}(X_1) = U - \{x_3\}$, $\underline{A_1}_1(X_1) = U - \{x_4, x_{11}, x_{17}\}$;

$\overline{A_1}_{(0.6,0.3)}(X_2) = U$, $\underline{A_1}_1(X_2) = U - \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}$.

$\overline{A_2}_{(0.7,0.2)}(X_1) = U - \{x_8\}$, $\underline{A_2}_1(X_1) = U - \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}$;

$\overline{A_2}_{(0.7,0.2)}(X_2) = U - \{x_6, x_{16}\}$, $\underline{A_2}_1(X_2) = U - \{x_2, x_6, x_{10}, x_{16}, x_{18}\}$.

$\overline{A_3}_{(0.8,0.1)}(X_1) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$, $\underline{A_3}_1(X_1) = U - \{x_7, x_8, x_{17}, x_{18}, x_{20}\}$

$\overline{A_3}_{(0.8,0.1)}(X_2) = U - \{x_4, x_9, x_{10}, x_{12}, x_{16}\}$, $\underline{A_3}_1(X_2) = U - \{x_1, x_2, x_4, x_5, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\}$.

In the GMDqI-DTRS, when $\varphi = 2/3$, the lower and upper approximations of X_1 and $X_2$with respect to $\sum_{i=1}^{S} A_i$ can be obtained as follows:

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1) = U - \{x_8\}$, $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1) = U - \{x_7, x_{11}, x_{17}\}$;

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2) = U - \{x_{16}\}$, $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2) = U - \{x_1, x_2, x_5, x_6, x_9, x_{10}, x_{15}, x_{16}\}$.

Therefore, the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the GMDqI-DTRS can be calculated as $\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{|\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1)| + |\underline{GM}^{I}_{\sum_{i=3}^{S} A_i}(X_2)|}{|\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1)| + |\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2)|} = \frac{29}{38}$.

When $\alpha_i + \beta_i > 1$,

$\overline{A_1}_{(0.6,0.5)}(X_1) = \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}$, $\underline{A_1}_1(X_1) = U - \{x_4, x_{11}, x_{17}\}$;

$\overline{A_1}_{(0.6,0.5)}(X_2) = \{x_3, x_4, x_{11}, x_{17}\}$, $\underline{A_1}_1(X_2) = U - \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}$.

$\overline{A_2}_{(0.7,0.4)}(X_1) = U - \{x_1, x_5, x_7, x_8, x_9, x_{11}, x_{13}\}$, $\underline{A_2}_1(X_1) = U - \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}$;

$\overline{A_2}_{(0.7,0.4)}(X_2) = U - \{x_2, x_6, x_{10}, x_{16}, x_{18}\}$, $\underline{A_2}_1(X_2) = U - \{x_2, x_6, x_{10}, x_{16}, x_{18}\}$.

$\overline{A_3}_{(0.8,0.3)}(X_1) = U - \{x_7, x_8, x_{11}, x_{14}, x_{17}, x_{18}, x_{20}\}$, $\underline{A_3}_1(X_1) = U - \{x_7, x_8, x_{17}, x_{18}, x_{20}\}$

$\overline{A_3}_{(0.8,0.3)}(X_2) = U - \{x_4, x_9, x_{10}, x_{12}, x_{16}\}$, $\underline{A_3}_1(X_2) = U - \{x_1, x_2, x_4, x_5, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\}$.

In the GMDqI-DTRS, when $\varphi = 2/3$, the lower and upper approximations of X_1 and $X_2$with respect to $\sum_{i=1}^{S} A_i$ can be obtained as follows:

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1) = U - \{x_7, x_8, x_{11}, x_{13}, x_{14}, x_{17}, x_{18}\}$, $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1) = U - \{x_7, x_{11}, x_{17}\}$;

$\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2) = U - \{x_2, x_6, x_9, x_{10}, x_{12}, x_{16}, x_{18}\}$, $\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2) = U - \{x_1, x_2, x_5, x_6, x_9, x_{10}, x_{15}, x_{16}\}$.

Therefore, the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the GMDqI-DTRS can be calculated as $\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{|\underline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1)| + |\underline{GM}^{I}_{\sum_{i=3}^{S} A_i}(X_2)|}{|\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_1)| + |\overline{GM}^{I}_{\sum_{i=1}^{3} A_i}(X_2)|} = \frac{29}{26}$.

By comparing the results of the approximation accuracy of GMDqI-DTRS and GMRS, it is evident that the approximation accuracy of GMDqI-DTRS is higher than the approximation accuracy of GMRS no matter what kind of constraints.

In the second kind of double-quantitative decision-theoretic rough set(DqII-DTRS), the lower and upper approximations of decision classes X_1 and X_2 under granular structures A_1, A_2 and A_3 can be obtained as follows:

when $\alpha_i + \beta_i = 1$,

$\overline{A_1}_1(X_1) = \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}, \underline{A_1}_{(0.6,0.4)}(X_1) = \{x_1, x_5, x_6, x_9, x_{15}, x_{20}\}$;

$\overline{A_1}_1(X_2) = \{x_4, x_{11}, x_{17}\}, \underline{A_1}_{(0.6,0.4)}(X_2) = \{x_3, x_4, x_{11}, x_{17}\}$.

$\overline{A_2}_1(X_1) = \{x_2, x_6, x_{10}, x_{16}, x_{18}\}, \underline{A_2}_{(0.7,0.3)}(X_1) = \{x_6, x_{16}\}$;

$\overline{A_2}_1(X_2) = \{x_1, x_5, x_7, x_9, x_{11}, x_{13}\}, \underline{A_2}_{(0.6,0.4)}(X_2) = \{x_8\}$.

$\overline{A_3}_1(X_1) = \{x_1, x_2, x_4, x_5, x_9, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}, x_{19}\}, \underline{A_3}_{(0.8,0.2)}(X_1) = \{x_4, x_9, x_{10}, x_{12}, x_{16}\}$;

$\overline{A_3}_1(X_2) = \{x_7, x_8, x_{17}, x_{18}, x_{20}\}, \underline{A_3}_{(0.6,0.4)}(X_2) = \{x_4, x_7, x_8, x_{11}, x_{17}, x_{18}, x_{20}\}$.

In the GMDqI-DTRS, when $\varphi = 2/3$, the lower and upper approximations of X_1 and $X_2$with respect to $\sum_{i=1}^{S} A_i$ can be obtained as follows:

$\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_1) = \{x_1, x_2, x_5, x_6, x_9, x_{15}, x_{16}, x_{20}\}, \underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_1) = \{x_6, x_9, x_{16}\}$;

$\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_2) = \{x_7, x_{11}, x_{17}\}, \underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_2) = \{x_8, x_{11}, x_{17}\}$.

Therefore, the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the GMDqI-DTRS can be calculated as $\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{|\underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_1)| + |\underline{GM}^{II}_{\sum_{i=3}^{S} A_i}(X_2)|}{|\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_1)| + |\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_2)|} = \frac{6}{11}$.

When $\alpha_i + \beta_i < 1$ and $\alpha_i + \beta_i > 1$, the approximation accuracy of U/DT with respect to $\sum_{i=1}^{S} A_i$ in the GMDqI-DTRS is also $\alpha_{\sum_{i=1}^{S} A_i}(U/DT) = \frac{|\underline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_1)| + |\underline{GM}^{II}_{\sum_{i=3}^{S} A_i}(X_2)|}{|\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_1)| + |\overline{GM}^{II}_{\sum_{i=1}^{3} A_i}(X_2)|} = \frac{6}{11}$.

By comparing the results of the approximation accuracy of GMDqII-DTRS and GMRS, it is evident that the approximation accuracy of GMDqII-DTRS is higher than the approximation accuracy of GMRS no matter what kind of constraints.

In conclusion, approximate classification capability of generalized multigranulation double-quantitative rough sets is better than approximate classification capability of the generalized multigranulation rough set.

5. Conclusions

By weakening constraint conditions, double-quantitative rough sets are more consistent with the reality of the approximation space and provide enough information for making decisions. And the principle of the minority subordinate to the majority is the most feasible and credible when people make decisions in real world. The research on combining generalized multigranulation with double-quantitative decision-theoretic is significant. In this paper, we propose the definition of lower and upper approximations of generalized multigranulation double-quantitative rough sets. Through the pair, we get basic concepts of two kinds of generalized multigranulation double-quantitative rough sets and obtain corresponding decision rules based on the idea of three-way decisions. Then the relationship between these two kinds of rough sets is discussed under different constraint conditions. Moreover, the relationship between GMDq-DTRS and other models is compared in detail. Finally, the theory and advantage of the new model are interpreted by an illustrative case study.

Generalized multigranulation double-quantitative decision-theoretic rough sets provide theoretical foundation for making decisions and extend generalized multigranulation rough sets. This paper just provides a framework of generalized multigranulation double-quantitative decision-theoretic rough sets. Like uncertainty measures and properties of models with respect to concepts and parameters need to be explored. Generalized multigranulation double-quantitative decision-theoretic rough sets provide a new method for information fusion. Especially, applications of the models proposed in the paper to real life should be studied in the future.

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