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Multigranulation Decision-theoretic Rough Set in Ordered Information System

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Abstract. The decision-theoretic rough set model based on Bayesian decision theory is a main development tendency in the research of rough sets. To extend the theory of decision-theoretic rough set, the article devotes this study to presenting multigranulation decision-theoretic rough set model in ordered information systems. This new multigranulation decision-theoretic rough set approach is characterized by introducing the basic set assignment function in an ordered information system. It is addressed about how to construct probabilistic rough set and multigranulation decision-theoretic rough set models in an ordered information system. Moreover, three kinds of multigranulation decision-theoretic rough set model are analyzed carefully in an ordered information system. In order to explain probabilistic rough set model and multigranulation decision-theoretic rough set models in an ordered information system, an illustrative example is considered, which is helpful for applying these theories to deal with practical issues.

Keywords: Decision-theoretic rough set, multigranulation, ordered information system, partition function, probability measure

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1. Introduction

Rough set theory proposed by Pawlak [17], is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. This relatively new soft computing methodology has received great attention in recent years, and its effectiveness has been confirmed successful applications in many science and engineering fields, such as pattern recognition, data mining, image processing, medical diagnosis and so on. Rough set theory is built on the basis of the classification mechanism, it is classified as the equivalence relation in a specific universe, and the equivalence relation constitutes a partition of the universe. A concept, or more precisely the extension of a concept, is represented by a subset of a universe of objects and is approximated by a pair of definable concepts of a logic language. The main idea of rough set theory is the use of a known knowledge in knowledge base to approximate the inaccurate and uncertain knowledge.

Due to the existence of uncertainty and complexity of particular problems, several extensions of the rough set model have been proposed in terms of various requirements, such as the variable precision rough set model [46, 43], rough set model based on tolerance relation [11], the Bayesian rough set model [25], the decision-theoretic rough set model [37], the fuzzy rough set model and the rough fuzzy set model [3] and many others investigations [19, 28]. In many circumstances, relations in ordered information systems are not equivalence relations, but partial relations. Such as the dominance relation. It is vital to propose an extension called the dominance-based rough set approach [26] to take account into the ordering properties of criteria. The innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. Studies have been made on properties and algorithmic implementations of dominance-based rough set approach. In recent years, researchers have enriched the ordered information system theories and obtained many achievements. For instance, Shao et al. further explored an extension of the dominance relation in an inconsistent ordered information system [21]. Xu et al. constructed a method of attribute reduction based on evidence theory in an ordered information system [31], and others [2, 35].

Pawlak and its generalized rough sets are constructed based on one set of information granule, which are induced from a partition or a covering. In 1985, Hobbs proposed the concept of granularity [8], and Zadeh first explored the concept of granular computing [45] between 1996 and 1997. They all think that information granules refer to pieces, classes, and groups into which complex information are divided in accordance with the characteristics and processes of the understanding and decision-making. Currently, granular computing is an emerging computing paradigm of information processing. It concerns the processing of complex information entities called information granules [27]. Information granules, as encountered in natural language, are implicit in their nature. To make them fully operational so that they become effectively used in the analysis and design of intelligent systems, we need to make information granules explicit. This is possible through a prudent formalization available within the realm of granular computing. Pal et al. presented the relationship among granular computing, rough entropy and object extraction [16]. Skowron et al. introduced basic notions related to granular computing on the information granule syntax and semantics as well as the inclusion and closeness (similarity) relations of granules [24], the foundations of rough-neural computing [23]. Yao first promoted the relationship between information granulation and rough set approximation theory [38]. Peters et al. proposed an approach to measures of information granules based on rough set theory [18]. In order to make rough set theory have a wider range of applications, Qian et al. extended Pawlak's single-granulation rough set to a multigranulation rough set model [20]. And later, many researchers have extended the multigranulation rough sets. Xu et al. developed a fuzzy multigranulation rough set model [29], a generalized multigranulation rough set approach [30] and a multigranulation rough set model in ordered information systems [28]. Yang et al. proposed the hierarchical structure properties of the multigranulation rough sets [34], multigranulation rough set in incomplete information system [33], and presented a test cost sensitive multigranulation rough set model [32]. Lin et al. presented a neighborhood-based multigranulation rough set [14]. She et al. explored the topological structures and the properties of multigranulation rough sets [22]. Li et al. developed a further study of multigranulation T-fuzzy rough sets, relationships between multigranulation and classical T-fuzzy rough sets were studied carefully [13].

Recently, decision-theoretic rough set has been paid close attentions. The acceptance of decision-theoretic rough sets is merely due to the fact that they are defined by using probabilistic information, which is more general and flexible. Yao presented a new decision making method based on the decision-theoretic rough set, which is called three-way decision theory [39, 40]. Decision-theoretic rough sets can derive various rough set models through setting the thresholds. Professor Yao gave a decision theoretic framework for approximating concepts in 1992 [36] and later applied this model to attribute reduction [41]. Azam and Yao proposed a threshold configuration mechanism for reducing the overall uncertainty of probabilistic regions in probabilistic rough sets [1]. Jia et al. proposed an optimization representation of decision-theoretic rough set model and raised an optimization problem by considering the minimization of the decision cost [9, 10]. Liu et al. combined the logistic regression and the decision-theoretic rough set into a new classification approach, which can effectively reduce the misclassification rate [15]. Yu et al. applied decision-theoretic rough set model for automatically determining the number of clusters with much smaller time cost [42]. Qian et al. discussed the decision-theoretic rough set theory based on Bayesian decision procedure into the multigranulation perspective [19]. Zhou et al. investigated a comparative study of two kinds of probabilistic rough set model, namely the decision-theoretic rough set model and the confirmation-theoretic rough set model [44]. These studies represents a snapshot of recent achievements and developments on the decision-theoretic rough set theory.

Probabilistic rough sets in the classical information system are based on the equivalence relation. In many real situations, one may often face the problems in which the ordering of properties of the considered attributes play a crucial role. Relevantly, Greco et al. discussed a Bayesian decision theory for dominance-based rough set model in 2007 [7], Kusunoki et al. studied an empirical risk associated with the classification function [12], their approach aimed to take account into costs of misclassification in fixing parameters of the dominance-based rough set approach, while didn't transact the essence of the problems about how to construct a probability measure space in an ordered information system. Once one uses the probabilistic rough set theory into ordered information systems, it may face with the problems that the relations are dominance relations, namely can't induce probability measure spaces. In this paper, our objective is to explore how to apply the probabilistic rough set theory into ordered information systems and develop the multigranulation decision-theoretic rough set theory in an ordered information system through combining multigranulation idea with the Bayesian decision theories. The rest of this paper is organized as follows. Some preliminary concepts about the probabilistic rough set, the Bayesian decision procedure and the multigranulation rough set in an ordered information system are briefly reviewed in Section 2. In Section 3, we developed the probabilistic rough set in an ordered information system. In Section 4, we proposed the multigranulation decision-theoretic rough set in an ordered information system, and discussed three kinds of this multigranulation decision-theoretic rough set model in an ordered information system. Then in Section 5, an illustrative example was presented in an ordered information system. Finally, Section 6 gets the conclusions.

2. Preliminaries

In this section, we review some basic concepts about rough sets in an ordered information system, probabilistic approaches to rough set theory, the decision-theoretic rough set based on Bayesian decision theory and the multigranulation rough set in an ordered information system. Throughout this paper, we assume that the universe U is a non-empty and finite set.

The notion of information system provides a convenient basis for the representation of objects in terms of their attributes. An information system is a triple $I = (U, AT, F)$, where

- U is a non-empty and finite set of objects, and $U = \{x_1, x_2, \cdots, x_n\}$;
- AT is a non-empty and finite set of attributes, and $AT = \{a_1, a_2, \cdots, a_m\}$;
- $F = \{f_l | U \to V_l, l \leq m\}$, f_l is the value of a_l on $x \in U$, V_l is the domain of a_l, $a_l \in AT$.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion. An information system is called an ordered information system if all condition attributes are criteria [4, 5, 6]. As the decreasing preference can be converted to increasing preference, in this paper we only consider the increasing preference without any loss of generality.

Assume that the domain of a criterion $a_l \in AT$ is complete pre-ordered by an outranking relation $\geq_{a_l}$, then $x \geq_{a_l} y$ means that x is at least as good as y with respect to criterion a_l. And we can say that x dominates y. We define $x \geq_{a_l} y$ by $f_l(x) \geq f_l(y)$ according to increasing preference, where $a_l \in AT$ and $x, y \in U$. For a subset of attributes $A \subseteq AT$, $x \geq y$ means that $x \geq_{a_l} y$ for any $a_l \in A$, and that is to say x dominates y with respect to all attributes in A. In general, we denote an ordered information system by $I^{\geq} = (U, AT, F)$.

Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$ and $R_A^{\geq}$ is a dominance relation in $I^{\geq}$, denote $R_A^{\geq} = \{(x, y) \in U \times U | f_l(x) \geq f_l(y), \forall a_l \in A\}$, and $U/R_A^{\geq} = \{[x]_{R_A^{\geq}} | x \in U\}$ is the set of dominance classes induced by a dominance relation $R_A^{\geq}$, where $[x]_{R_A^{\geq}}$ is called dominance class containing x, and $[x]_{R_A^{\geq}} = \{y \in U | (y, x) \in R_A^{\geq}\}$.

For any $X \subseteq U$, $\underline{R_A^{\geq}}(X) = \{x \in U | [x]_{R_A^{\geq}} \subseteq X\}$ and $\overline{R_A^{\geq}}(X) = \{x \in U | [x]_{R_A^{\geq}} \bigcap X \neq \emptyset\}$ are the lower and upper approximations of X in the ordered information system $I^{\geq} = (U, AT, F)$. When $\underline{R_A^{\geq}}(X) \neq \overline{R_A^{\geq}}(X)$, one may call X is a rough set.

Especially, decision ordered information system is a special case of an ordered information system in which, among the attributes, we distinguish decision attributes. The other attributes are called condition attributes.

As one has introduced in the former section, probabilistic approaches to rough sets have many forms, such as the Bayesian rough set model, the variable precision rough set model, the decision-theoretic rough set model and other related studies including multigranulation decision-theoretic rough set. In classical information systems, Pawlak's rough set is based on certainty knowledge base, namely its approximation space is completely certain. It means that Pawlak rough set ignores the uncertainty of the available information system. If one still handles the data analysis with Pawlak rough set in this knowledge base, it may not reflect the essence. To overcome this issue, we need to discuss the probabilistic rough set.

Let U be a non-empty and finite set of objects, one can define P as probability measure if the set-valued function P maps from 2^U to $[0,1]$. P satisfies the two conditions: $P(U)=1$; if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$. Then P is a probability measure of $\sigma - algebra$ which is combined by the family subset of U.

In an information system, given U as a non-empty and finite set of objects, where R is an equivalence relation in U. Denote $[x]_R$ is the equivalence class with respect to x. And P is a probability measure of $\sigma - algebra$ which is combined by the family subset of U. The triple $A_P = (U, R, P)$ is called probability approximation space. And $P = (X|Y)$ is the conditional probability of whether concept X happens or not depends on Y.

Definition 2.1 [36] Let $0 \leq \beta < \alpha \leq 1$, for any $X \subseteq U$, the lower and upper approximations based on thresholds α, β with respect to $A_P = (U, R, P)$ are defined as follows

$$\underline{pr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\},$$

$$\overline{pr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\}.$$

If $\underline{pr}_R^{(\alpha,\beta)}(X) = \overline{pr}_R^{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a rough set.
Accordingly, the probabilistic positive, negative and boundary region are

$$\begin{aligned} pos(X) &= \underline{pr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\}; \\ neg(X) &= U - \overline{pr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \leq \beta\}; \\ bn(X) &= \overline{pr}_R^{(\alpha,\beta)}(X) - \underline{pr}_R^{(\alpha,\beta)}(X) = \{x \in U | \beta < P(X|[x]_R) < \alpha\}. \end{aligned}$$

The parameters α, β in the probabilistic rough set theory above can be determined by special methods according to some additional conditions. From Bayesian decision theory as one will introduce next, the parameters α, β can be obtained.

In the Bayesian decision produce, a finite set of states can be written as $\Omega = \{\omega_1, \omega_2, \cdots, \omega_s\}$, and a finite set of m possible actions can be denoted by $A = \{a_1, a_2, \cdots, a_r\}$. Let $P(\omega_j|x)$ be the conditional probability of an object x being in state ω_j given that the object is described by x. Let $\lambda(a_i|\omega_j)$ denote the loss, or cost for taking action a_i when the state is ω_j, the expected loss function associated with taking action a_i is given by

$$R(a_i|x) = \sum_{j=1}^{s} \lambda(a_i|\omega_j) P(\omega_j|x)$$

In Pawlak's rough set theory, lower and upper approximation operators partition the universe U into three disjoint sets. Using the conditional probability $P(X|[x]_R)$, the Bayesian decision procedure can decide how to assign x into these disjoint three regions.

With respect to the membership of an object in X, we have a set of two states and a set of three actions for each state. The set of states is given by $\Omega = \{X, X^C\}$ indicating that an element is in X and not in X, respectively. The set of actions with respect to a state is given by $A = \{a_P, a_B, a_N\}$, where P, B and N represent the three actions in deciding $x \in pos(X)$, deciding $x \in bn(X)$, and deciding $x \in neg(X)$, respectively. The loss function regarding the risk or cost of actions in different states is given by the 3×2 matrix:

	X (P)	X^C (N)
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

In the matrix, λ_{PP}, λ_{BP} and λ_{NP} denote the losses incurred for taking actions a_P, a_B and a_N, respectively, when an object belongs to X, and λ_{PN}, λ_{BN} and λ_{NN} denote the losses incurred for taking the same actions when the object does not belong to X.

The expected loss $R(a_i|[x]_R)$ associated with taking the individual actions can be expressed as [37, 39]

$$R(a_P|[x]_R) = \lambda_{PP}P(X|[x]_R) + \lambda_{PN}P(X^C|[x]_R);$$
$$R(a_B|[x]_R) = \lambda_{BP}P(X|[x]_R) + \lambda_{BN}P(X^C|[x]_R);$$
$$R(a_N|[x]_R) = \lambda_{NP}P(X|[x]_R) + \lambda_{NN}P(X^C|[x]_R).$$

When $\lambda_{PP} \leq \lambda_{NP} < \lambda_{BP}$ and $\lambda_{BN} \leq \lambda_{NN} < \lambda_{PN}$, the Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) If $P(X|[x]_R) \geq \gamma$ and $P(X|[x]_R) \geq \alpha$, decide $pos(X)$;

(N) If $P(X|[x]_R) \leq \beta$ and $P(X|[x]_R) \leq \gamma$, decide $neg(X)$;

(B) If $\beta \leq P(X|[x]_R) \leq \alpha$, decide $bn(X)$.

Where the parameters α, β and γ are defined as:

$$\alpha = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{NP} - \lambda_{PN}) + (\lambda_{PP} - \lambda_{NN})};$$
$$\beta = \frac{\lambda_{NN} - \lambda_{BN}}{(\lambda_{BP} - \lambda_{NP}) + (\lambda_{NN} - \lambda_{BN})};$$
$$\gamma = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{NP} - \lambda_{PP}) + (\lambda_{PN} - \lambda_{BN})}.$$

If a loss function further satisfies the condition: $(\lambda_{PN} - \lambda_{NN})(\lambda_{BP} - \lambda_{NP}) \geq (\lambda_{NP} - \lambda_{PP})(\lambda_{NN} - \lambda_{BN})$, then we can get $\alpha \geq \gamma \geq \beta$.

When $\alpha > \beta$, we have $\alpha > \gamma > \beta$. The decision-theoretic rough set has the decision rules:

(P) If $P(X|[x]_R) \geq \alpha$, decide $pos(X)$;

(N) If $P(X|[x]_R) \leq \beta$, decide $neg(X)$;

(B) If $\beta < P(X|[x]_R) < \alpha$, decide $bn(X)$.

Using these three decision rules, we get the probabilistic approximations:

$$\underline{apr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\},$$
$$\overline{apr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\}.$$

If $\underline{apr}_R^{(\alpha,\beta)}(X) = \overline{apr}_R^{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a rough set.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. Then the decision-theoretic rough set has the decision rules:

(P) If $P(X|[x]_R) > \alpha$, decide $pos(X)$;

(N) If $P(X|[x]_R) < \alpha$, decide $neg(X)$;

(B) If $P(X|[x]_R) = \alpha$, decide $bn(X)$.

We get the probabilistic approximations:

$$\underline{apr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \alpha\},$$
$$\overline{apr}_R^{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geq \alpha\}.$$

In real application of the probabilistic rough set models, we obtain the thresholds α, β based on an intuitive understanding the levels of tolerance for errors. Just like we confirm the value of parameters α and β included in the Section 3, Section 4 and the illustration in Section 5. And the calculation methods of the conditional probability can also meet for demands in application. Based on the well-established Bayesian decision procedure, the decision-theoretic rough set model is a kind of probabilistic rough set model. Decision-theoretic rough set provides systematic methods for deriving the required thresholds on probabilistic rough set.

The multigranulation rough set was first proposed by Qian et al. [20], and generalized into the ordered information system by Xu et al. [28]. From the view of granular computing, the approximation of a set is described by using a single relation (granulation) on the universe. And the multigranulation means at least two or two more relations, in the ordered information system, the relations are dominance relations, which is what we will study next.

In the multigranulation decision-theoretic rough sets, we assume that the values of $\lambda_k(a_i|\omega_j), k \leq m$ are all equal to each other. It also means that the thresholds α, β and γ in each granular structure are also same to each other. The determined procedure of the parameters α, β and γ is consistent with that of classical decision-theoretic rough sets.

Definition 2.2 [28] Let $I^{\geq} = (U, A, F)$ be an ordered information system, set $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$ as the dominance relations., $\forall X \subseteq U$, $[x]_{R_i^{\geq}}$ is called dominance class contains x with respect to $R_i^{\geq}$, the pessimistic multigranulation lower and upper approximations of X are denoted by

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}^{PES}(X) = \{x \in U | \bigwedge_{i=1}^{m}([x]_{R_i^{\geq}} \subseteq X)\},$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}^{PES}(X) = \{x \in U | \bigvee_{i=1}^{m}([x]_{R_i^{\geq}} \cap X \neq \emptyset)\}.$$

Similarly, we have the optimistic multigranulation lower and upper approximations of X in ordered information systems as following

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}^{OPT}(X) = \{x \in U | \bigvee_{i=1}^{m}([x]_{R_i^{\geq}} \subseteq X)\},$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}^{OPT}(X) = \{x \in U | \bigwedge_{i=1}^{m}([x]_{R_i^{\geq}} \cap X \neq \emptyset)\}.$$

Multigranulation is a new and interesting topic in the theory of rough set. It provides a new perspective for decision making analysis based on the rough set theory. The multigranulation rough set in an ordered information system is different from Pawlak's rough set approach since the former is constructed on the basis of a family of dominance relations instead of single equivalence relation. In the following, we will discuss the probabilistic rough set in an ordered information system.

3. Probabilistic rough sets in an ordered information system

Probabilistic rough set models allow a tolerance inaccuracy in lower and upper approximations. While in an ordered information system, the relations are never equivalence relations but dominance relations, which will not produce the probability measure space. We can handle the dominance classes with an operator, and transform the non-probability measure into a probability measure space.

In the ordered information system $I^{\geq} = (U, AT, F)$, $A \subseteq AT$ and $R_A^{\geq}$ is a dominance relation in $I^{\geq}$. $[x]_{R_A^{\geq}}$ is the dominance class containing x. And $P(X|Y)$ is the conditional probability of whether concept X happens or not depends on Y. We get the following definition.

Definition 3.1 [31] Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$ and $R_A^{\geq}$ is a dominance relation in $I^{\geq}$.The basic set assignment function j is from 2^U to 2^U, is defined as

$$j(X) = \{x \in U | [x]_{R_A^{\geq}} = X\}, X \in 2^U.$$

Obviously, $x \in j(X) \Leftrightarrow [x]_{R_A^{\geq}} = X$.

The basic set assignment function $j([x]_{R_A^{\geq}})$ contains these two properties:

- $\bigcup_{X \subseteq U} j(X) = U$;
- For $X \neq Y$, $j(X) \bigcap j(Y) = \emptyset$.

It is easy to notice that the function $j([x]_{R_A^{\geq}})$ is a partition function of the universe U, one can also call the partition function as set-valued mapping approximation operator. Accordingly, in the ordered information system, this operator transforms the triple $A_P = (U, R_A^{\geq}, P)$ into probability measure approximation space.

Definition 3.2 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $A \subseteq AT$ and $R_A^{\geq}$ is a dominance relation in $I^{\geq}$. Set $0 \leq \beta < \alpha \leq 1$, for any $X \subseteq U$, the lower and upper approximations based on parameters α, β with respect to $A_P = (U, R_A^{\geq}, P)$ are defined as follows

$$\underline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|j([x]_{R_A^{\geq}})) \geq \alpha\},$$

$$\overline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|j([x]_{R_A^{\geq}})) > \beta\}.$$

If $\underline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \overline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a rough set.

Accordingly, the probabilistic positive, negative and boundary region are

$$pos(X) = \underline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|j([x]_{R_A^{\geq}})) \geq \alpha\};$$

$$neg(X) = U - \overline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | P(X|j([x]_{R_A^{\geq}})) \leq \beta\};$$

$$bn(X) = \overline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) - \underline{jpr}_{R_A^{\geq}}^{(\alpha,\beta)}(X) = \{x \in U | \beta < P(X|j([x]_{R_A^{\geq}})) < \alpha\}.$$

In the following, an example is employed to present the probabilistic rough sets in an ordered information system.

Example 3.1 Table 1 is an ordered information system, $U = \{x_1, x_2, \cdots, x_7\}$ is a universe which consists of 7 objects, a_1, a_2, a_3, a_4 are the conditional attributes of the system. One uses A, B, C, D to denote the values of these attributes. Moreover, A $\geq$ B $\geq$ C $\geq$ D.

Table 1. An ordered information system

U	a_1	a_2	a_3	a_4
x_1	B	C	C	D
x_2	C	B	B	A
x_3	B	B	C	B
x_4	A	D	A	C
x_5	C	B	B	A
x_6	B	A	D	B
x_7	B	C	C	D

Here we consider all of these four conditions: a_1, a_2, a_3, a_4, accordingly, $R^{\geq}$ is the dominance relation induced by these four attributes. Then one can obtain that the dominance classes are as following

$[x_1]_{R^{\geq}} = \{x_1, x_3, x_7\} = X_1$, $[x_2]_{R^{\geq}} = \{x_2, x_5\} = X_2$, $[x_3]_{R^{\geq}} = \{x_3\} = X_3$, $[x_4]_{R^{\geq}} = \{x_4\} = X_4$, $[x_5]_{R^{\geq}} = \{x_2, x_5\} = X_2$,

$[x_6]_{R^{\geq}} = \{x_6\} = X_5$, $[x_7]_{R^{\geq}} = \{x_1, x_3, x_7\} = X_1$.

It is obvious that these seven classes form a covering of the universe, but not a partition. Accordingly, one may use the partition function j. Then we can get $j(X_1) = \{x_1, x_7\}$, $j(X_2) = \{x_2, x_5\}$, $j(X_3) = \{x_3\}$, $j(X_4) = \{x_4\}$, $j(X_5) = \{x_6\}$.

It is easy to notice that $j(X_1)$, $j(X_2)$, $j(X_3)$, $j(X_4)$ and $j(X_5)$ form a partition of the universe U.

Given $X = \{x_2, x_3, x_5\}$ is a subset of universe U. Assume that $\alpha = 2/3, \beta = 1/4$. Conditional probability is $P(X|Y)$, where

$$P(X|Y) = \frac{|X \bigcap Y|}{|Y|}.$$

Then the conditional probabilities with respect to $R^{\geq}$ are shown as following:

$P(X|j([x_1]_{R^{\geq}})) = 1/3$, $P(X|j([x_7]_{R^{\geq}})) = 1/3$,

$P(X|j([x_2]_{R^{\geq}})) = 1$, $P(X|j([x_5]_{R^{\geq}})) = 1$,

$P(X|j([x_3]_{R^{\geq}})) = 1$,

$P(X|j([x_4]_{R^{\geq}})) = 0$,

$P(X|j([x_6]_{R^{\geq}})) = 0$.

The lower and upper approximations based on parameters α, β with respect to $A_P = (U, R^{\geq}, P)$ are computed as

$$\underline{jpr}_{R^{\geq}}^{(\frac{2}{3},\frac{1}{4})}(X) = \{x \in U | P(X|j([x]_{R^{\geq}})) \geq 2/3\} = \{x_2, x_3, x_5\},$$
$$\overline{jpr}_{R^{\geq}}^{(\frac{2}{3},\frac{1}{4})}(X) = \{x \in U | P(X|j([x]_{R^{\geq}})) > 1/4\} = \{x_1, x_2, x_3, x_5, x_7\}.$$

And then the probabilistic positive, negative and boundary region are

$$pos(X) = \underline{jpr}_{R^{\geq}}^{(\frac{2}{3},\frac{1}{4})}(X) = \{x_2, x_3, x_5\};$$
$$neg(X) = U - \overline{jpr}_{R^{\geq}}^{(\frac{2}{3},\frac{1}{4})}(X) = \{x_4, x_6\};$$
$$bn(X) = \overline{jpr}_{R^{\geq}}^{(\frac{2}{3},\frac{1}{4})}(X) - \underline{jpr}_{R^{\geq}}^{(\frac{2}{3},\frac{1}{4})}(X) = \{x_1, x_7\}.$$

In an ordered information system, through the basic set assignment function j, one can easily achieve the probability approximation space. Furthermore, in the ordered information system, one can propose multigranulation decision-theoretic rough set theory, which will be introduced in the next section.

4. Multigranulation decision-theoretic rough set in an ordered information system

Multigranulation decision-theoretic rough set in an ordered information system is different from rough sets in an ordered information system because the former is constructed based on a family of probability measure. Dominance relation induces a covering rather than a partition of the universe in an ordered information system, and dominance classes may not product a probability measure, where the equivalence classes really do. As one has studied in Section 3, we use set-valued mapping approximation operators to transform non-probability measure into probability measure spaces.

In the ordered information system, set $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$ as the m granular structures. For any $X \subseteq U$, the lower and upper approximations in a multigranulation rough set approach to ordered information system can be represented as two fusion functions, respectively.

$$\underline{\sum_{i=1}^{m} R_i^{\geq}} = f_l(R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}),$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}} = f_u(R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}).$$

Where f_l is called a lower fusion function, and f_u is called an upper fusion function [19]. These two fusions are used to compute the lower and upper approximations of a multigranulation rough set through these m granular dominance relations.

And in practical applications of multigranulation rough sets, the fusion function has many forms according to various semantics and requirements. Through the probabilistic way, let $\lambda_k(a_i|\omega_j)$ denote the loss, or cost, which means taking action a_i when the state is ω_j by the $k-th$ dominance relation $R_k^{\geq}$.

Let $P(\omega_j|x_k)$ be the conditional probability of an object x being in state ω_j, given that the object is described by x_k under $k-th$ dominance relation $R_k^{\geq}$, obviously, the $k-th$ $(k = 1, 2, \cdots, m)$ dominance relation $R_k^{\geq}$ may not induces the probability measure space. One also need to transform the non-probability measure into probability measure by using basic set assignment function. And then the expected loss associated with taking action a_i is given by

$$R(a_i|x_1, x_2, \cdots, x_m) = \sum_{i=1}^{m}\sum_{j=1}^{s} \lambda_k(a_i|\omega_j)P(\omega_j|x_k).$$

The expected loss $R(a_i|x_1, x_2, \cdots, x_m)$ is a conditional risk. $\tau(x_1, x_2, \cdots, x_m)$ specifies which action to take, and its value is one of the actions $a_1, a_2, \cdots, a_r$. The overall risk **R** is the expected loss with the decision rule $\tau(x_1, x_2, \cdots, x_m)$, the overall risk is defined by

$$\mathbf{R} = \sum_{x_1, x_2, \cdots, x_m} R(\tau(x_1, x_2, \cdots, x_m)|x_1, x_2, \cdots, x_m)P(x_1, x_2, \cdots, x_m).$$

Where $P(x_1, x_2, \cdots, x_m)$ is a joint probability, which is calculated through fusing $(P(x_1), P(x_2), \cdots,$ $P(x_m))$ induced by m granular structures and induced by the same universe.

In the ordered information system, given multiple dominance relations $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$, the multigranulation decision-theoretic rough sets aim to select a series of actions for which the overall risk is as small as possible, in which the actions include deciding positive region, deciding negative region and deciding boundary region.

For each single element $R_i^{\geq}$ among $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$, one can convent the dominance class by $R_k^{\geq}$ into a partition by the basic set assignment function, namely the partition function, where the new definable approximation in correspondence with the dominance relation approximation.

Definition 4.1 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$ are all dominance relations, $[x]_{R_i^{\geq}}$ is a dominance class induced by $R_i^{\geq}$. The partition function $j : 2^U \rightarrow 2^U$ is defined as following: for any $R_i^{\geq}$,

$$j(X) = \{x \in U | [x]_{R_i^{\geq}} = X\}, X \in 2^U.$$

It is obvious that $x \in j(X) \Leftrightarrow [x]_{R_i^{\geq}} = X$. Here we can use $j([x]_{R_i^{\geq}})$ instead of $j(X)$, respectively. Accordingly, each $j([x]_{R_i^{\geq}})$ forms a partition of U, one can satisfy the condition of probability measure.

In the following, we will discuss three kinds of multigranulation decision-theoretic rough set model in an ordered information system. It should be pointed out that the parameters α, β are derived from the Bayesian decision procedure and Bayesian decision principle.

Definition 4.2 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$ are all dominance relations, $[x]_{R_i^{\geq}}$ is a dominance class induced by $R_i^{\geq}$. For any $X \subseteq U$, thresholds $0 \leq \beta < \alpha \leq 1$, the mean multigranulation lower and upper approximations are denoted by

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{M}(X) = \{x \in U | [\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m \geq \alpha\},$$
$$\overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{M}(X) = \{x \in U | [\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m > \beta\}.$$

Where $P(X|j([x]_{R_i^{\geq}}))$ is the conditional probability of the equivalence class $j([x]_{R_i^{\geq}})$, with respect to X.

We can also define the mean multigranulation positive, negative and boundary region of X in the ordered information system.

$$pos(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{M}(X) = \{x \in U | [\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m \geq \alpha\};$$

$$neg(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{M}(X) = \{x \in U | [\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m \leq \beta\};$$

$$bn(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{M}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{M}(X) = \{x \in U | \beta < [\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m < \alpha\}.$$

Similar to the classical decision-theoretic rough sets, we can obtain the decision rules

(P_M) If $[\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m \geq \alpha$, decide $pos(X)$;

(N_M) If $[\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m \leq \beta$, decide $neg(X)$;

(B_M) If $\beta < [\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m < \alpha$, decide $bn(X)$.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. Then the mean multigranulation decision-theoretic rough set in the ordered information system has the following decision rules:

(P_M) If $[\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m > \alpha$, decide $pos(X)$;

(N_M) If $[\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m < \alpha$, decide $neg(X)$;

(B_M) If $[\sum_{i=1}^{m} P(X|j([x]_{R_i^{\geq}}))]/m = \alpha$, decide $bn(X)$.

Definition 4.3 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$ are all dominance relations, $[x]_{R_i^{\geq}}$ is a dominance class induced by $R_i^{\geq}$. For any $X \subseteq U$, thresholds $0 \leq \beta < \alpha \leq 1$, the pessimistic multigranulation lower and upper approximations are denoted by

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{PES}(X) = \{x \in U | \bigwedge_{i=1}^{m} (P(X|j([x]_{R_i^{\geq}})) \geq \alpha)\},$$

$$\overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{PES}(X) = \{x \in U | \bigvee_{i=1}^{m} (P(X|j([x]_{R_i^{\geq}})) > \beta)\}.$$

Where $P(X|j([x]_{R_i^{\geq}})$ is the conditional probabilistic of the equivalence class $j([x]_{R_i^{\geq}})$ with respect to X.

We can define the pessimistic multigranulation positive, negative and boundary region of X in the ordered information system.

$$pos(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{PES}(X) = \{x \in U | \bigwedge_{i=1}^{m}(P(X|j([x]_{R_i^{\geq}})) \geq \alpha)\};$$

$$neg(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{PES}(X) = \{x \in U | \bigvee_{i=1}^{m}(P(X|j([x]_{R_i^{\geq}})) \leq \beta)\};$$

$$bn(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{PES}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{PES}(X).$$

Similar to the classical decision-theoretic rough sets, we can obtain the decision rules:

(P_{PES}) If $\forall i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) \geq \alpha$, decide $pos(X)$;

(N_{PES}) If $\exists i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) \leq \beta$, decide $neg(X)$;

(B_{PES}) Otherwise, decide $bn(X)$.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. Then the pessimistic multigranulation decision-theoretic rough set in the ordered information system has the following decision rules:

(P_{PES}) If $\forall i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) > \alpha$, decide $pos(X)$;

(N_{PES}) If $\exists i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) < \alpha$, decide $neg(X)$;

(B_{PES}) Otherwise, decide $bn(X)$.

Definition 4.4 Let $I^{\geq} = (U, AT, F)$ be an ordered information system, $R_1^{\geq}, R_2^{\geq}, \cdots, R_m^{\geq}$ are all dominance relations, $[x]_{R_i^{\geq}}$ is a dominance class induced by $R_i^{\geq}$. For any $X \subseteq U$, thresholds $0 \leq \beta < \alpha \leq 1$, the optimistic multigranulation lower and upper approximations are denoted by

$$\underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{OPT}(X) = \{x \in U | \bigvee_{i=1}^{m}(P(X|j([x]_{R_i^{\geq}})) \geq \alpha)\},$$

$$\overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{OPT}(X) = U - \{x \in U | \bigwedge_{i=1}^{m}(P(X|j([x]_{R_i^{\geq}})) \leq \beta)\}.$$

Where $Pr(X|j([x]_{R_i^{\geq}})$ is the conditional probability of the equivalence class $j([x]_{R_i^{\geq}})$ with respect to X.

We can define the optimistic multigranulation positive, negative and boundary region of X in the ordered information system.

$$pos(X) = \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{OPT}(X) = \{x \in U | \bigvee_{i=1}^{m} (P(X|j([x]_{R_i^{\geq}})) \geq \alpha)\};$$

$$neg(X) = U - \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{OPT}(X) = \{x \in U | \bigwedge_{i=1}^{m} (P(X|j([x]_{R_i^{\geq}})) \leq \beta)\};$$

$$bn(X) = \overline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{OPT}(X) - \underline{\sum_{i=1}^{m} R_i^{\geq}}_{(\alpha,\beta)}^{OPT}(X).$$

Similar to the classical decision-theoretic rough sets, we can obtain the decision rules:

(P_{OPT}) If $\exists i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) \geq \alpha$, decide $pos(X)$;

(N_{OPT}) If $\forall i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) \leq \beta$, decide $neg(X)$;

(B_{OPT}) Otherwise, decide $bn(X)$.

When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. Then the optimistic multigranulation decision-theoretic rough set in the ordered information system has the following decision rules:

(P_{OPT}) If $\exists i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) > \alpha$, decide $pos(X)$;

(N_{OPT}) If $\forall i \in \{1, 2, \cdots, m\}$, such that $P(X|j([x]_{R_i^{\geq}})) < \alpha$, decide $neg(X)$;

(B_{OPT}) Otherwise, decide $bn(X)$.

5. Illustrative example

The concept of three-way decision plays an important role in many real world decision-making problems. One usually makes a decision based on available information and evidence. When the evidence is insufficient or weak, it might be impossible to make either a positive or a negative decision. One therefore chooses an alternative decision that is neither yes nor no. The idea of multigranulation and the theory of decision-theoretic rough set can be combined to data mining and decision-making in real-life applications. According to multigranulation decision-theoretic rough set model in an ordered information system, we can make more comprehensive decisions in the actual decision-making processes.

Table 2 is an ordered information system about the bird flu (H1N1), where $U = \{x_1, x_2, \cdots, x_{10}\}$ is a universe which consists of 10 patients with the clinical features degree; *Hyperpyrexia, Cough, Rhinorrhoea, Myodynia, Diarrhea, Nausea* are the conditional attributes of the system. One uses 1, 2, 3, 4 to describe the degree of these six features, where the numbers 1, 2, 3, 4 are respectively for None, Slight, Middling, Serious. Clearly, $4 \geq 3 \geq 2 \geq 1$.

We consider the situation that when $m = 2$, the dominance relations contain $R_1^{\geq}$ and $R_2^{\geq}$, where $R_1^{\geq}$ is a dominance relation formed by three attributes: *Hyperpyrexia, Cough, Rhinorrhoea* and $R_2^{\geq}$ is a dominance relation formed by three attributes: *Myodynia, Diarrhea, Nausea*. $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$ is the set of patients who really get the bird flu (H1N1).

Table 2. An ordered information system about the bird flu (H1N1)

U	*Hyperpyrexia*	*Cough*	*Rhinorrhoea*	*Myodynia*	*Diarrhea*	*Nausea*
x_1	1	2	1	2	2	3
x_2	3	4	2	3	3	2
x_3	2	2	3	3	1	4
x_4	4	1	4	2	4	3
x_5	3	4	2	2	4	3
x_6	1	2	1	3	1	4
x_7	1	2	1	3	3	2
x_8	4	1	4	2	4	3
x_9	2	2	3	2	2	3
x_{10}	4	1	4	3	3	2

Next we will make sure the patients who need to get treatments, who need not to get treatments and who need a deep observation. Then computing the above three kinds of multigranulation decision-theoretic rough sets in the ordered information system is a natural. From Bayesian decision procedure, one can give the values $\lambda_{iP} = \lambda(a_i|X)$, $\lambda_{iN} = \lambda(a_i|X^C)$, and $i = P, B, N$, accordingly, one either assumes that $\alpha = 3/4, \beta = 1/2$. Similar to Example 3.1, we set the conditional probability

$$P(X|Y) = \frac{|X \bigcap Y|}{|Y|}.$$

We can get the dominance classes based on $R_1^{\geq}$ as follows
$X_{11} = [x_1]_{R_1^{\geq}} = [x_6]_{R_1^{\geq}} = [x_7]_{R_1^{\geq}} = \{x_1, x_2, x_3, x_5, x_6, x_7, x_9\}$,
$X_{12} = [x_2]_{R_1^{\geq}} = [x_5]_{R_1^{\geq}} = \{x_2, x_5\}$,
$X_{13} = [x_3]_{R_1^{\geq}} = [x_9]_{R_1^{\geq}} = \{x_2, x_3, x_5, x_9\}$,
$X_{14} = [x_4]_{R_1^{\geq}} = [x_8]_{R_1^{\geq}} = [x_{10}]_{R_1^{\geq}} = \{x_4, x_8, x_{10}\}$.
And for $R_2^{\geq}$, the dominance classes are
$X_{21} = [x_1]_{R_2^{\geq}} = [x_9]_{R_2^{\geq}} = \{x_1, x_4, x_5, x_8, x_9\}$,
$X_{22} = [x_2]_{R_2^{\geq}} = [x_7]_{R_2^{\geq}} = [x_{10}]_{R_2^{\geq}} = \{x_2, x_7, x_{10}\}$,
$X_{23} = [x_3]_{R_2^{\geq}} = [x_6]_{R_2^{\geq}} = \{x_3, x_6\}$,
$X_{24} = [x_4]_{R_2^{\geq}} = [x_5]_{R_2^{\geq}} = [x_8]_{R_2^{\geq}} = \{x_4, x_5, x_8\}$.

It is obvious that these classes $[x_i]_{R_1^{\geq}} (i = 1, 2, \cdots, 10)$ with respect to $R_1^{\geq}$ form a covering of the universe, but not a partition, so as to $[x_i]_{R_2^{\geq}} (i = 1, 2, \cdots, 10)$. Accordingly, one may use the partition function j to construct the partitions of universe U. Then we can get the equivalence classes in terms of $R_1^{\geq}$ as
$j(X_{11}) = \{x_1, x_6, x_7\}, j(X_{12}) = \{x_2, x_5\}, j(X_{13}) = \{x_3, x_9\}, j(X_{14}) = \{x_4, x_8, x_{10}\}$.
And the equivalence classes in terms of $R_2^{\geq}$ as
$j(X_{21}) = \{x_1, x_9\}, j(X_{22}) = \{x_2, x_7, x_{10}\}, j(X_{23}) = \{x_3, x_6\}, j(X_{24}) = \{x_4, x_5, x_8\}$.

Then the conditional probabilities with respect to $R_1^{\geq}$ are shown as following:
$P(X|j(X_{11})) = P(X|j([x_1]_{R_1^{\geq}})) = P(X|j([x_6]_{R_1^{\geq}})) = P(X|j([x_7]_{R_1^{\geq}})) = 1/3$,
$P(X|j(X_{12})) = P(X|j([x_2]_{R_1^{\geq}})) = P(X|j([x_5]_{R_1^{\geq}})) = 1$,
$P(X|j(X_{13})) = P(X|j([x_3]_{R_1^{\geq}})) = P(X|j([x_9]_{R_1^{\geq}})) = 1/2$,
$P(X|j(X_{14})) = P(X|j([x_4]_{R_1^{\geq}})) = P(X|j([x_8]_{R_1^{\geq}})) = P(X|j([x_{10}]_{R_1^{\geq}})) = 2/3$.

Accordingly, the lower and upper approximations based on parameters α, β with respect to $A_P = (U, R_1^{\geq}, P)$ are computed as

$$\underline{jpr}_{R_1^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x \in U | P(X|j([x]_{R_1^{\geq}})) \geq \frac{3}{4}\} = \{x_2, x_5\},$$
$$\overline{jpr}_{R_1^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x \in U | P(X|j([x]_{R_1^{\geq}})) > \frac{1}{2}\} = \{x_2, x_4, x_5, x_8, x_{10}\}.$$

Then the probabilistic positive, negative and boundary region are

$$pos(X) = \underline{jpr}_{R_1^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_5\};$$
$$neg(X) = U - \overline{jpr}_{R_1^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x_1, x_3, x_6, x_7, x_9\};$$
$$bn(X) = \overline{jpr}_{R_1^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) - \underline{jpr}_{R_1^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x_4, x_8, x_{10}\}.$$

Similarly, we can get the following conditional probabilistic in term of $R_2^{\geq}$ are
$P(X|j(X_{21})) = P(X|j([x_1]_{R_2^{\geq}})) = P(X|j([x_9]_{R_2^{\geq}})) = 1/2$,
$P(X|j(X_{22})) = P(X|j([x_2]_{R_2^{\geq}})) = P(X|j([x_7]_{R_2^{\geq}})) = P(X|j([x_{10}]_{R_2^{\geq}})) = 2/3$,
$P(X|j(X_{23})) = P(X|j([x_3]_{R_2^{\geq}})) = P(X|j([x_6]_{R_2^{\geq}})) = 0$,
$P(X|j(X_{24})) = P(X|j([x_4]_{R_2^{\geq}})) = P(X|j([x_5]_{R_2^{\geq}})) = P(X|j([x_8]_{R_2^{\geq}})) = 1$.

The lower and upper approximations based on parameters α, β with respect to $A_P = (U, R_2^{\geq}, P)$ are computed as

$$\underline{jpr}_{R_2^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x \in U | P(X|j([x]_{R_2^{\geq}})) \geq \frac{3}{4}\} = \{x_4, x_5, x_8\},$$
$$\overline{jpr}_{R_2^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x \in U | P(X|j([x]_{R_2^{\geq}})) > \frac{1}{2}\} = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}.$$

Then the probabilistic positive, negative and boundary region are

$$pos(X) = \underline{jpr}_{R_2^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x_4, x_5, x_8\};$$
$$neg(X) = U - \overline{jpr}_{R_2^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x_1, x_3, x_6, x_9\};$$
$$bn(X) = \overline{jpr}_{R_2^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) - \underline{jpr}_{R_2^{\geq}}^{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_7, x_{10}\}.$$

According to the three multigranulation decision-theoretic rough sets in ordered information system, we can ensure a patient belongs to which region within the positive, negative and boundary region, which determines whether this patient need the treatments or not.

(1) By the mean multigranulation decision-theoretic rough set theory in ordered information system, the lower and upper approximation are computed as

$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{M}_{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_4, x_5\},$$

$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{M}_{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

We can obtain three decision regions:
$(P_M)\, pos(X) = \{x_2, x_4, x_5\}$;
$(N_M)\, neg(X) = \{x_1, x_6\}$;
$(B_M)\, bn(X) = \{x_3, x_7, x_8, x_9, x_{10}\}$.

(2) By the pessimistic multigranulation decision-theoretic rough set theory in ordered information system, the lower and upper approximations are computed as

$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{PES}_{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2\},$$

$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{PES}_{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

We can obtain three decision regions:
$(P_{PES})\, pos(X) = \{x_2\}$;
$(N_{PES})\, neg(X) = \{x_1, x_6\}$;
$(B_{PES})\, bn(X) = \{x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$.

(3) By the optimistic multigranulation decision-theoretic rough set theory in ordered information system, the lower and upper approximations are computed as

$$\underline{\sum_{i=1}^{2} R_i^{\geq}}^{OPT}_{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_4, x_5, x_7, x_8, x_{10}\},$$

$$\overline{\sum_{i=1}^{2} R_i^{\geq}}^{OPT}_{(\frac{3}{4},\frac{1}{2})}(X) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}.$$

We can obtain three decision regions:
$(P_{OPT})\, pos(X) = \{x_2, x_4, x_5, x_7, x_8, x_{10}\}$;
$(N_{OPT})\, neg(X) = \{x_1, x_6\}$;
$(B_{OPT})\, bn(X) = \{x_3, x_9\}$.

In Section 4, we introduced three kinds of fusion function and produced three pairs of lower and upper approximation operators. Table 3 is the upper and lower approximations of X in three types of multigranulation decision-theoretic rough set models.

Table 3. Multigranulation decision-theoretic approximations in an ordered information system

Type	Lower approximation	Upper approximation
Mean multigranulation	x_2, x_4, x_5	$x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}$
Pessimistic multigranulation	x_2	$x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}$
Optimistic multigranulation	$x_2, x_4, x_5, x_7, x_8, x_{10}$	$x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}$

These three models have different range of application in real life. For the example about bird flu (H1N1), we can make a decision whether a patient needs the treatments by the above three kinds of multigranulation decision-theoretic rough set model in an ordered information system. The positive, boundary and negative regions are shown in Table 4.

Table 4. Regions of multigranulation decision-theoretic rough sets in an ordered information system

Type	Positive region	Boundary region	Negative region
Mean multigranulation	x_2, x_4, x_5	$x_3, x_7, x_8, x_9, x_{10}$	x_1, x_6
Pessimistic multigranulation	x_2	$x_3, x_4, x_5, x_7, x_8, x_9, x_{10}$	x_1, x_6
Optimistic multigranulation	$x_2, x_4, x_5, x_7, x_8, x_{10}$	x_3, x_9	x_1, x_6

We have the following evaluations and analyses:

(1) According to the decision regions of mean multigranulation decision-theoretic rough set model in an ordered information system, the patients x_2, x_4 and x_5 belong to positive region means that these three patients really need the deep treatments; x_1 and x_6 belong to negative region means that they need not to get treatments; x_3, x_7, x_8, x_9 and x_{10} belong to the boundary region means that they need further observations to make the decisions.

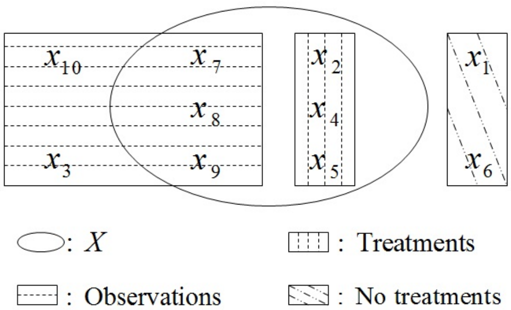

Figure 1. Results of mean multigranulation decision-theoretic rough set in an ordered information system

Comparing with the given set $X = \{x_2, x_4, x_5, x_7, x_8, x_9\}$, even though patients x_7, x_8 and x_9 infect with bird flu (H1N1), they may need further observations to make sure whether they need the treatments; persons x_3 and x_{10} are who don't infect with bird flu (H1N1) also need further observations to determine whether they need the treatments (see Figure 1).

(2) According to the decision regions of pessimistic multigranulation decision-theoretic rough set model in an ordered information system, the patient x_2 belongs to positive region means that these

two patients really need the deep treatments; patients x_1 and x_6 belong to negative region means that they need not to get treatments; patients $x_3, x_4, x_5, x_7, x_8, x_9$ and x_{10} belong to the boundary region means that they need further observations to make the decisions.

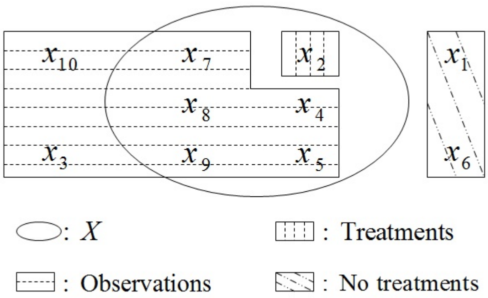

Figure 2. Results of pessimistic multigranulation decision-theoretic rough set in an ordered information system

Comparing with the given set X, even though x_4, x_5, x_7, x_8 and x_9 infect with bird flu (H1N1), they need further observations to make sure whether they need treatments; persons x_3 and x_{10} is who doesn't infect with bird flu (H1N1) also need further observations (see Figure 2).

(3) According to the decision regions of optimistic multigranulation decision-theoretic rough set model in an ordered information system, the patients x_2, x_4, x_5, x_7, x_8 and x_{10} belong to positive region means that these six patients really need the deep treatments; patients x_1 and x_6 belong to negative region means that they need not to get treatments; patients x_3 and x_9 belong to the boundary region means that they need further observations to make the decision.

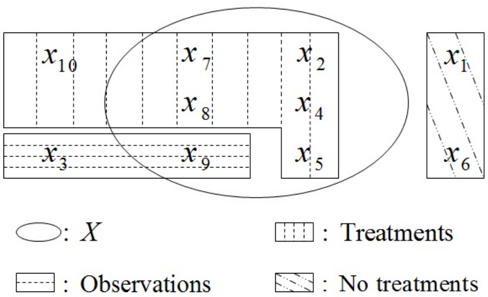

Figure 3. Results of optimistic multigranulation decision-theoretic rough set in an ordered information system

Comparing with the given set X, even though x_9 infects with bird flu (H1N1), he (or she) needs further observations to make sure whether they need treatments; while person x_{10} is who doesn't infect with bird flu (H1N1) may need to get the treatments; x_3 is who doesn't infect with bird flu (H1N1) need the further observations (see Figure 3).

In real applications of the probabilistic rough set models, one may directly supply the parameters α and β based on an intuitive understanding the levels of tolerance for errors [46]. This means that one indeed uses an intermediate result of the decision-theoretic rough set model without an in-depth understanding of its theoretical foundations [37]. It should be pointed out that the uses of special parameters α and β may largely due to an unawareness of the well-established Bayesian decision procedure [40]. As a final remark, one may find it much easier to give loss functions that can be related to more intuitive terms such as costs, benefits, and risks, than to give abstract threshold values [40]. This is particular true in situations where the costs can be translated into monetary values. This study develops a framework of multigranulation decision-theoretic rough set in an ordered information system, in which there are many interesting issues to be explored.

6. Conclusions

The Bayesian decision procedure is a useful tool to generalize the probabilistic rough sets in classical information systems. By considering the probabilistic rough sets in ordered information systems, the basic set assignment function is introduced into our work. In ordered information systems, the relations are always dominance relation, but not equivalence relation, it results in a non-probability measure space. Using the basic set assignment function, we can transact the covering of universe U induced by a dominance relation into a partition of the universe U. Relevantly, we have applied the partition function into the multigranulation probabilistic rough sets, which contains the multigranulation decision-theoretic rough set. This paper mainly discusses the probability measure, Bayesian decision principle and the multigranulation decision-theoretic rough sets in ordered information systems. As to handling the multigranulation decision-theoretic rough sets in an ordered information system, we use three forms of multigranulation decision-theoretic approximations spaces which are mean, pessimistic as well as optimistic lower and upper approximation spaces. Among this article, we construct a real life example about the bird flu (H1NI) to present the theories about the multigranulation decision-theoretic rough sets in an ordered information system to be much easy to accept.

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