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Double-quantitative decision-theoretic rough set

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ABSTRACT

The probabilistic rough set (PRS) and the graded rough set (GRS) are two quantification models that measure relative and absolute quantitative information between the equivalence class and a basic concept, respectively. As a special PRS model, the decision-theoretic rough set (DTRS) mainly utilizes the conditional probability to express relative quantification. However, it ignores absolute quantitative information of the overlap between equivalence class and the basic set, and it cannot reflect the distinctive degrees of information and extremely narrow their applications in real life. In order to overcome these defects, this paper proposes a framework of double-quantitative decision-theoretic rough set (Dq-DTRS) based on Bayesian decision procedure and GRS. Two kinds of Dq-DTRS model are constructed, which essentially indicate the relative and absolute quantification. After further studies to discuss decision rules and the inner relationship between these two models, we introduce an illustrative case study about the medical diagnosis to interpret and express the theories, which is valuable for applying these theories to deal with practical issues.

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1. Introduction

Rough set theory proposed by Pawlak [17], is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. This relatively new soft computing methodology has received great attention in recent years, and its effectiveness has been confirmed successful applications in many science and engineering fields, such as pattern recognition, data mining, image processing, and medical diagnosis. Rough set theory is built on the basis of the classification mechanism, it is classified as the equivalence relation in a specific universe, and the equivalence relation constitutes a partition of the universe. A concept, or more precisely the extension of a concept, is represented by a subset of a universe of objects and is approximated by a pair of definable concepts of a logic language. Rough set models give rise to a construct that highlights some items endowed with uncertainty [18]. The main idea of rough set is the use of a known knowledge in knowledge base to approximate the inaccurate and uncertain knowledge.

Pawlak rough set has a severe limitation. The relationship between equivalence classes and the basic set are strict that there are no fault tolerance mechanisms. Quantitative information about the degree of overlap of the equivalence classes

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and the basic set is not taken into consideration. Therefore, neither wider relationships nor quantitative information can be utilized. In fact, there are some degrees of inclusion relations between sets, and the extent of overlap of sets is important information to consider in applications. The classical rough set model must be improved and expansions of the model that include quantification are of particular value. Improved models are called quantitative rough set model, and among them are the PRS model [30] and GRS model [29].

Recently, the PRS has been paid much attentions. The acceptance of PRS is merely due to the fact that they are defined by using probabilistic information and are more general and flexible. The introduction of probability enables the models to treat the universe of objects as samples from a much larger universe [19]. Probability is an important tool for describing uncertainty. The PRS model [25] has many merits, such as the measurability of the probabilistic information, the generality and flexibility of the model and its insensitivity to noise. The PRS model has been investigated extensively, and many concrete realizations of the model are available, including DTRS [24], game-theoretic rough sets [4], variable precision rough sets [39], 0.5-PRS [25], parameterized rough set [3], Bayesian rough sets [38] and fuzzy PRS [10,36]. The main differences among those models are their different, but equivalent, formulations of probabilistic approximations and interpretations of the required parameters. PRS models use conditional probability to quantify the degree of set inclusion. Notions can be interpreted in terms of probabilities or a posteriori probabilities [25]. Thresholds on the probability are used to define rough set approximations. The threshold values, known as parameters, are applied to a rough membership function or a rough inclusion to obtain probabilistic or parameterized approximations. However, when we apply the PRS models to the real-life issues, the conditional probability is calculated based on the rough membership functions. It is a kind of relative error from the form, in other words, the conditional probability reflects the relative quantitative information [12,20,33]. All of the existing PRS model using the rough membership functions are related to the relative quantitative information.

Because the problem of noisy data is substantially mitigated, the DTRS model is highly useful in data acquisition and analysis, it is an expansion of the Pawlak rough set model. The DTRS model has become increasingly popular in a variety of theoretical and practical fields, producing many thorough results. DTRS implies the ideal of three-way decisions [5,27]. Yao presented a new decision making method based on the DTRS, which is called the three-way decision theory, namely, the decision rules obtained from positive region, negative region and boundary region [15,25,27]. In fact, PRS is developed based on the Bayesian decision principle and Bayesian decision procedure, in which its parameters can be learned from a decision table. The three-way decision rules have much more superiority than both two-way decision and Pawlak's classical decision rules [26]. The DTRS can derive various rough set models through setting the thresholds. Since the DTRS was proposed by Yao in 1990 [30], it has attracted much more attentions. Professor Yao gave a decision theoretic framework for approximating concepts in 1992 [23] and later applied this model to attribute reduction [31]. Azam and Yao proposed a threshold configuration mechanism for reducing the overall uncertainty of probabilistic regions in the PRS [1]. Jia et al. proposed an optimization representation of DTRS model and raised an optimization problem by considering the minimization of the decision cost [6,7]. Liu et al. combined the logistic regression and the DTRS into a new classification approach, which can effectively reduce the misclassification rate [14]. Ma et al. explored the PRS model by considering two universes and accordingly discussed the rough entropy [15]. Yu et al. applied DTRS model for automatically determining the number of clusters with much smaller time cost [32]. Qian et al. combined the thought of multigranulation into DTRS, then proposed three kinds of multigranulation DTRS model [19]. Later on that, Li et al. developed a probabilistic rough set model by considering dominance relations other than equivalence relations [9], and then further studied multigranulation DTRS in an ordered information system [8]. Professor Zhou introduced a kind of DTRS model for an information table with more than two decision classes, which is the multi-class DTRS [37]. These studies represents a snapshot of recent achievements and developments on the DTRS theory. In particular, Greco et al. presented a generalized variable precision rough set model using the absolute and relative rough membership [3]. The inclusion degree, as a generalization of the rough membership, has been used extensively in the study of measures, reasoning, applications of uncertainty and approximate spaces.

GRS model [13,22,29] has many features in common with DTRS model and functions as a typical expansion model by including quantification. The DTRS based on Bayesian decision procedure and the GRS are two fundamental expansion models that achieve strong fault tolerance capabilities by utilizing quantitative descriptions. Since Yao and Lin explored the relationships between rough sets and modal logics, they proposed the GRS model based on graded modal logics [29]. GRS model primarily considers the absolute quantitative information regarding the basic concept and knowledge granules and is a generalization of the Pawlak rough set model. The regions of the GRS model are extensions of grade approximations. Because the inclusion relation of the grade approximations does not hold any longer, positive and negative regions, upper and lower boundary regions are naturally proposed. Obviously, regions of the GRS model also extend the corresponding notions of the classical rough set model. They classify the universe more precisely and have their own logical meanings related to the grade quantitative index, in another way, GRS models consider absolute quantitative information between equivalence classes and the basic concept [34,35].

DTRS model and GRS model can respectively reflect relative and absolute quantitative information about the degree of overlap between equivalence classes and a basic set. The relative and absolute quantitative information are two distinctive objective sides that describe approximate space, and each has its own virtues and application environments, so that none can be neglected. Relative quantitative information and absolute quantitative information are two kinds of quantification mythology in certain applications. Usually, most researchers prefer using the relative quantitative information [7,11,15,16,24,25,38,39]. However, the absolute quantitative information is more important than or as important as the relative quantitative information in some specific fields or special cases, many corresponding examples can be found in practice. We introduce three examples to highlight the motivation that lead to considering the relative quantification and absolute quantification, and explain the importance of these two types of quantitative information in different scenarios [35].

(1) You want an iPhone 5s, and nobody gives you as a present. So you want to borrow money from friend *A* and friend *B*. The price of iPhone 5s (32 G) is about 650 dollars. *A* has 500 dollars and can lend you 400 dollars, while *B* has 1000 dollars and can lend you 600 dollars. If you can borrow from only one of them, who would you choose? Of course friend *B* is the preferable choice, although the relative proportion of your friend's wealth is only 60%, which is lower than that 80% of *A*. In this example, we specifically focus more on the absolute quantitative information and thus give low priority to the relative quantitative information, whose comparability is actually very weak.
(2) Tsinghua University is the best one in China, enrolling only the most excellent students from middle schools, and its enrollment rate is very low. There are two middle schools *A* and *B*. And in a year, only 1 student from *A* (with 1500 students) are enrolled by Tsinghua University, while 2 students from *B* (with 3500 students) are admitted. Here is the question: which middle school is better in terms of enrollment? Due to the strict admission requirement, the enrollment rate is so low that the relative quantitative information (1/1500 and 2/3500) has little or not very meaningful significance. Therefore, one may simply take the absolute quantitative information factor into consideration in assessments. $2 > 1$ although $2/3500 < 1/1500$, so one may conclude that *B* is better than *A*. In practice, people may pay more attention to the number of enrolled students from a middle school into Tsinghua, rather than the total number of students at the middle school itself.
(3) Company *A* and Company *B* have 40 and 20 application projects, respectively, but only 30 establishment projects are required. How does one choose between them? If only the relative quantitative information is considered, one can conclude that A and B will obtain 20 and 10 establishment projects, respectively. Is this fair in reality? If the 2 companies have almost the same research levels, then this may be reasonable. However, if the research level of A is much higher than that of *B*, then A should obtain more than 20 establishment projects, and B should obtain less than 10. Obviously, the number of establishment projects is the pivotal index here.

In approximate space, elements in different equivalence classes may have large differences, so different equivalence classes can reflect distinctive degrees of information. In fact, there are usually large information gaps between equivalence classes. This situation should be emphasized and utilized. However, the DTRS model neglects such gaps, which can be seen in the analysis of Example (3), possibly making it less accurate. On the contrary, the GRS model is concerned with the absolute quantitative information of the overlap between equivalence classes and the basic set, which can reflect the distinctive degrees of information. Therefore, it is in this situation that the GRS model is a key model in reflecting the quantitative information of the approximation space. It is also an important supplementary model to the DTRS model in certain applications, particularly those in which there are large information gaps between equivalence classes. Three regions in Pawlak rough set model are of a qualitative nature, both acceptance and rejection decisions are made without any error. In practical applications, we rarely have such extreme cases. One typically makes a decision of acceptance or rejection by allowing certain level of errors [16,21]. This calls for quantitative generalizations of the Pawlak rough set model. Relevantly, Yao et al. proposed a framework of quantitative rough sets based on subsethood measures [28]. The framework enables us to classify and unify existing generalized rough set models (e.g., decision-theoretic rough sets, probabilistic rough sets, and variable precision rough sets). By considering an absolute quantitative information of the equivalence and the basic set in DTRS model, we introduce a framework of Dq-DTRS. In this study, our objective is to explore the quantification in the DTRS models and introduce a pair of quantitative information which are relative and absolute measures based on DTRS and GRS. We introduce the absolute quantitative information into DTRS models to improve the results obtained by DTRS models more specific.

The rest of this paper is organized as follows. In Section 2, we proposed our motivation that lead to considering the relative quantification and absolute quantification. Moreover, we presented the related works with our paper. We defined two kinds of Dq-DTRS model, and discussed the basic relation among these two models and Pawlak rough set model in Section 3. In Section 4, an illustrative example was presented and analyzed. Finally, Section 5 gets the conclusions.

2. Related work

For a non-empty set U, we call it the universe of discourse. The class of all subsets of U is denoted by $P(U)$. For $X \in P(U)$, the equivalence relation R in a Pawlak approximation space (U, R) partitions the universe U into disjoint subsets. Such a partition of the universe is a quotient set of U and is denoted by $U/R = \{[x]_R | x \in U\}$, where $[x]_R = \{y \in U | (x, y) \in R\}$ is the equivalence class containing x with respect to R. In the view of granular computing, equivalence classes are the basic building blocks for the representation and approximation of any subset of the universe of discourse. Each equivalence class may be viewed as a granule consisting of indistinguishable elements. The basic concept $X \in P(U)$, one can characterize X by a pair of upper and lower approximations which are

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\} = \cup\{[x]_R | [x]_R \cap X \neq \emptyset\},$$
$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\} = \cup\{[x]_R | [x]_R \subseteq X\}.$$

Here, $pos(X) = \underline{R}(X), neg(X) = \sim \overline{R}(X), bn(X) = \overline{R}(X) - \underline{R}(X)$ are called the positive region, negative region, and boundary region of X, respectively.

The GRS is different from the DTRS in the description of quantification. Suppose k is a non-negative integer and is called "grade".

$$\overline{R}_k(X) = \{x \in U|\ |[x]_R \cap X| > k\} = \cup\{[x]_R|\ |[x]_R \cap X| > k\},$$
$$\underline{R}_k(X) = \{x \in U|\ |[x]_R| - |[x]_R \cap X| \leqslant k\} = \cup\{[x]_R|\ |[x]_R| - |[x]_R \cap X| \leqslant k\}$$

are called grade k upper and lower approximations of X, respectively. If $\overline{R}_k(X) = \underline{R}_k(X)$, then X is called a definable set by grade k; otherwise, X is called a rough set by grade k. $\overline{R}_k$ and $\underline{R}_k$ are called grade k upper and lower approximation operators, respectively. If $k = 0$, then $\overline{R}_k(X) = \overline{R}(X), \underline{R}_k(X) = \underline{R}(X)$. Therefore, the classical rough set model is a special case of GRS model. In other words, the GRS model extends the classical model. Accordingly, we can get the following regions.

$$pos(X) = \overline{R}_k(X) \cap \underline{R}_k(X);$$
$$neg(X) = \sim (\overline{R}_k(X) \cup \underline{R}_k(X));$$
$$Ubn(X) = \overline{R}_k(X) - \underline{R}_k(X);$$
$$Lbn(X) = \underline{R}_k(X) - \overline{R}_k(X);$$
$$bn(X) = Ubn_k(X) \cup Lbn_k(X),$$

where $pos(X), neg(X), Ubn(X), Lbn(X)$ and $bn(X)$ are called grade k positive region, negative region, upper boundary region, lower boundary region, and boundary region of X.

Bayesian decision procedure deals mainly with making decisions with minimum risk or cost under probabilistic uncertainty. The following processes can be found in the textbook by Duda and Hart [2]. In the Bayesian decision produce, a finite set of states can be written as $\Omega = \{\omega_1, \omega_2, \ldots, \omega_s\}$, and a finite set of m possible actions can be denoted by $A = \{a_1, a_2, \ldots, a_r\}$. Let $P(\omega_j|x)$ be the conditional probability of an object x being in state ω_j given that the object is described by x. Let $\lambda(a_i|\omega_j)$ denote the loss, or cost for taking action a_i when the state is ω_j, the expected loss function associated with taking action a_i is given by

$$R(a_i|x) = \sum_{j=1}^{s} \lambda(a_i|\omega_j) P(\omega_j|x).$$

With respect to the membership of an object in X, we have a set of two states and a set of three actions for each state. The set of states is given by $\Omega = \{X, X^C\}$ indicating that an element is in X and not in X, respectively. The set of actions with respect to a state is given by $A = \{a_P, a_B, a_N\}$, where P, B and N represent the three actions in deciding $x \in pos(X)$, deciding $x \in bn(X)$, and deciding $x \in neg(X)$, respectively. The loss function regarding the risk or cost of actions in different states is given in Table 1.

In the matrix, $\lambda_{PP}, \lambda_{NP}$ and λ_{BP} denote the losses incurred for taking actions a_P, a_N and a_B, respectively, when an object belongs to X, and $\lambda_{PN}, \lambda_{NN}$ and λ_{BN} denote the losses incurred for taking the same actions when the object does not belong to X.

The expected loss $R(a_i|[x]_R)$ associated with taking the individual actions can be expressed as [24,25]

$$R(a_P|[x]_R) = \lambda_{PP} P(X|[x]_R) + \lambda_{PN} P(X^C|[x]_R);$$
$$R(a_N|[x]_R) = \lambda_{NP} P(X|[x]_R) + \lambda_{NN} P(X^C|[x]_R);$$
$$R(a_B|[x]_R) = \lambda_{BP} P(X|[x]_R) + \lambda_{BN} P(X^C|[x]_R).$$

When $\lambda_{PP} \leqslant \lambda_{NP} < \lambda_{BP}$ and $\lambda_{BN} \leqslant \lambda_{NN} < \lambda_{PN}$, the Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) If $P(X|[x]_R) \geqslant \gamma$ and $P(X|[x]_R) \geqslant \alpha$, decide $pos(X)$;
(N) If $P(X|[x]_R) \leqslant \beta$ and $P(X|[x]_R) \leqslant \gamma$, decide $neg(X)$;
(B) If $\beta \leqslant P(X|[x]_R) \leqslant \alpha$, decide $bn(X)$.

Table 1
The loss function.

	X (P)	X^C (N)
a_P	λ_{PP}	λ_{PN}
a_N	λ_{NP}	λ_{NN}
a_B	λ_{BP}	λ_{BN}

Where the parameters α, β and γ are defined as:

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})};$$
$$\beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})};$$
$$\gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}.$$

If a loss function further satisfies the condition: $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) \geqslant (\lambda_{BN} - \lambda_{NN})(\lambda_{BP} - \lambda_{PP})$, then we can get $\alpha \geqslant \gamma \geqslant \beta$. When $\alpha > \beta$, we have $\alpha > \gamma > \beta$. The DTRS has the decision rules:

(P) If $P(X|[x]_R) \geqslant \alpha$, decide $pos(X)$;
(N) If $P(X|[x]_R) \leqslant \beta$, decide $neg(X)$;
(B) If $\beta < P(X|[x]_R) < \alpha$, decide $bn(X)$.

Using these three decision rules, we get the probabilistic approximations, namely the upper and lower approximations of the DTRS model:

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\},$$
$$\underline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geqslant \alpha\}.$$

If $\underline{R}_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X)$, then X is a definable set, otherwise X is a rough set.
When $\alpha = \beta$, we have $\alpha = \gamma = \beta$. Then the DTRS has the decision rules:

(P) If $P(X|[x]_R) > \alpha$, decide $pos(X)$;
(N) If $P(X|[x]_R) < \alpha$, decide $neg(X)$;
(B) If $P(X|[x]_R) = \alpha$, decide $bn(X)$.

We get the probabilistic approximations:

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) \geqslant \alpha\} = \cup\{[x]_R | P(X|[x]_R) \geqslant \alpha\},$$
$$\underline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \alpha\} = \cup\{[x]_R | P(X|[x]_R) > \alpha\}.$$

Here, $pos_{(\alpha,\beta)}(X) = \underline{R}_{(\alpha,\beta)}(X), neg_{(\alpha,\beta)}(X) = \sim \overline{R}_{(\alpha,\beta)}(X), bn_{(\alpha,\beta)}(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_{(\alpha,\beta)}(X)$ are the positive region, negative region and boundary region, respectively.

DTRS model based on Bayesian decision principle was initially proposed by Yao. The conditional probability in this model is determined by the rough membership functions $P(X|[x]_R) = |[x]_R \cap X| / |[x]_R|$, which implies the relative quantitative information. The decision-theoretic approximations are made with $0 \leqslant \beta < \alpha \leqslant 1$. The parameters α and β were obtained from the losses of the Bayesian decision procedure, which are related to the relative quantitative information. The loss function can be considered as the standard threshold values, they are not abstract notions, but have a intuitive interpretation. One can easily interpret and measure loss or cost in a real application.

From the above discussions, DTRS models utilize conditional probability related to $P(X|[x]_R)$, there is no information regarding the absolute quantitative information. The relative information has the merits of being macroscopic, statistical, and convenient for applications. However, without the absolute information, the relative information is incomplete for quantification. The quantitative completeness determines the degrees of accuracy and certainty and is a primary and fundamental factor in many cases. The complexity of the model is another important factor to consider. In practice, the computational complexity of the composite system (evaluating both the relative and absolute information) has the same feasibility level as that of the individual systems to within a factor of approximately two. The combined quantification is therefore necessary and valuable. However, most DTRS models have the balanced or linear feature, and the relative measurement is unified for all cases [19,24–27]. In other words, in performing the relative extraction, the relative single quantification does not consider the absolute information environment and also roughens and neglects the objective environment (related to $[x]_R$). For example, for two knowledge granules $[x]_R^1 \neq [x]_R^2$, if $P(X|[x]_R^1) = P(X|[x]_R^2)$, then $[x]_R^1$ and $[x]_R^2$ are indiscernible or equal in DTRS, which the conditional probability is determined by rough membership function. However, $[x]_R^1 \neq [x]_R^2$ or $[x]_R^1 \cap X \neq [x]_R^2 \cap X$, and $[x]_R^1$ and $[x]_R^2$ can be discerned by introducing the absolute quantitative information $[x]_R$ or $[x]_R \cap X$. The double quantification formed by adding the absolute quantitative information can improve the descriptive abilities of DTRS models and expand their range of applicability.

The conditional probability is calculated by the rough membership function, namely the calculation of probability only considers the relative quantitative information, and the GRSs are related to the absolute quantitative information. Using these two indexes, single quantification models have been extended to capture more general relationships and complex situation. However, the two indexes are not equivalent, and the relationship between them is often close, complementary and dialectical. Therefore, given the completeness and complementary of the relative information and the absolute information, their double quantification has substantial value. This double quantification method can provide a thorough description of the approximate space, yield new models with strong double fault tolerance capabilities to adapt to increasing complex environments, accelerate the development of both DTRS and GRS models, and promote knowledge discovery based on double-quantitative information.

The comprehensive description of relative and absolute quantitative information and the composite study of the DTRS model and GRS model should provide more and better results. As far as the current situation is concerned, the introduction of the GRS model into the DTRS model, may improve the results obtained by the DTRS model in many application fields.

3. Dq-DTRS model

It is necessary to implement the double quantification using the relative and absolute information in DTRS and GRS, respectively. When considering the absolute quantitative information into the Bayesian decision procedure in the DTRS model, one can get two kinds of Dq-DTRS model. When we consider relative quantitative information and absolute quantitative information in the upper and lower approximations, there are four forms, which can be shown as follows:

(1) Upper approximation quantifies relative quantitative information, lower approximation quantifies relative quantitative information.
(2) Upper approximation quantifies absolute quantitative information, lower approximation quantifies absolute quantitative information.
(3) Upper approximation quantifies relative quantitative information, lower approximation quantifies absolute quantitative information.
(4) Upper approximation quantifies absolute quantitative information, lower approximation quantifies relative quantitative information.

It is easy to see that (1) and (2) forms the DTRS model and GRS model, respectively, which quantifies single relative or absolute quantitative information. As to (3) and (4), the upper and lower approximations quantify the relative and absolute together, then two fundamental Dq-DTRS models (DqI-DTRS and DqII-DTRS) can be constructed. It should be point out that $0 \leqslant k \leqslant |U|$, where $|U|$ is the cardinal number of U.

Definition 1. If we denote

$$\overline{R}_{(\alpha,\beta)}(X) = \{x \in U | P(X|[x]_R) > \beta\},$$
$$\underline{R}_k(X) = \{x \in U| \ |[x]_R| - |[x]_R \cap X| \leqslant k\},$$

then $\overline{R}_{(\alpha,\beta)}$ and $\underline{R}_k$ are the approximation operators. From these operators, a rough set model can be determined, called the first kind of double-quantitative decision-theoretic rough set (DqI-DTRS), and denoted by $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k)$. The positive region, negative region, upper boundary region and lower boundary region of $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k)$ are as following:

$$pos'(X) = \overline{R}_{(\alpha,\beta)}(X) \cap \underline{R}_k(X);$$
$$neg'(X) = \sim (\overline{R}_{(\alpha,\beta)}(X) \cup \underline{R}_k(X));$$
$$Ubn'(X) = \overline{R}_{(\alpha,\beta)}(X) - \underline{R}_k(X);$$
$$Lbn'(X) = \underline{R}_k(X) - \overline{R}_{(\alpha,\beta)}(X).$$

Definition 2. The model $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)})$ called the second kind of double-quantitative decision-rough set (DqII-DTRS), is defined using the dual approximation operators $\overline{R}_k$ and $\underline{R}_{(\alpha,\beta)}$, where the core mapping are given by the following approximations:

$$\overline{R}_k(X) = \{x \in U | |[x]_R \cap X| > k\};$$
$$\underline{R}_{(\alpha,\beta)}(X) = \{x \in U| \ P(X|[x]_R) \geqslant \alpha\}.$$

Accordingly, the positive region, negative region, upper boundary region and lower boundary region of $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)})$ are as following:

$$pos''(X) = \overline{R}_k(X) \cap \underline{R}_{(\alpha,\beta)}(X);$$
$$neg''(X) = \sim (\overline{R}_k(X) \cup \underline{R}_{(\alpha,\beta)}(X));$$
$$Ubn''(X) = \overline{R}_k(X) - \underline{R}_{(\alpha,\beta)}(X);$$
$$Lbn''(X) = \underline{R}_{(\alpha,\beta)}(X) - \overline{R}_k(X).$$

The relative information similarly complements the absolute description and can be used to improve the GRS model. The sharp contrast between the relative and grade environments is typical of double quantification applications. For example, if the relative quantification varies over a small range while the grade changes significantly, then the double quantification can play an effective role.

For DqI-DTRS, we have the decision rules:

(P') If $P(X|[x]_R) > \beta, |[x]_R| - |[x]_R \cap X| \leqslant k$, decide $pos'(X)$;
(N') If $P(X|[x]_R) \leqslant \beta, |[x]_R| - |[x]_R \cap X| > k$, decide $neg'(X)$;
(UB') If $P(X|[x]_R) > \beta, |[x]_R| - |[x]_R \cap X| > k$, decide $Ubn'(X)$;
(LB') If $P(X|[x]_R) \leqslant \beta, |[x]_R| - |[x]_R \cap X| \leqslant k$, decide $Lbn'(X)$.

For DqII-DTRS, we have the decision rules:

(P'') If $P(X|[x]_R) \geqslant \alpha, |[x]_R \cap X| > k$, decide $pos''(X)$;
(N'') If $P(X|[x]_R) < \alpha, |[x]_R \cap X| \leqslant k$, decide $neg''(X)$;
(UB'') If $P(X|[x]_R) < \alpha, |[x]_R \cap X| > k$, decide $Ubn''(X)$;
(LB'') If $P(X|[x]_R) \geqslant \alpha, |[x]_R \cap X| \leqslant k$, decide $Lbn''(X)$.

Based on the well-established Bayesian decision procedure, the DTRS model is derived from probability. That is to say, the DTRS model is a kind of PRS model. The DTRS provides systematic methods for deriving the required thresholds on PRS. In real application of the PRS models, we can obtain the thresholds α, β based on an intuitive understanding the levels of tolerance for errors.

Both the DTRS and GRS models are expansions of the Pawlak rough set model. The nature of the expansions is first analyzed, and this analysis leads to a formulation of the basic model expansions. The DTRS model degenerates to the Pawlak model when $\alpha = 1$ and $\beta = 0$. Therefore, the DTRS model extends the Pawlak model by lessening the upper approximation and enlarging the lower approximation, and the boundary region in the decision-theoretic model becomes a subset of the boundary region in the Pawlak model. An expansion with a subset boundary is preferable because the size of the boundary region determines the roughness degree to a great extent. Similarly, when the Pawlak rough set model is extended to the GRS model, the upper approximation becomes smaller, while the lower approximation becomes larger. The relevant formula is $k \geqslant 0 \Longleftrightarrow \overline{R}_k(X) \subseteq \overline{R}_0(X),\ \underline{R}_k(X) \supseteq \underline{R}_0(X)$.

Theorem 1. *Model expansion*

(1) *In DqI-DTRS, if* $\alpha = 1, \beta = 0$ *and* $k = 0$, *then* $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k) = (U, \overline{R}, \underline{R})$;
moreover, $\overline{R}_{(\alpha,\beta)}(X) \subseteq \overline{R}(X),\ \underline{R}_k(X) \supseteq \underline{R}(X)$.
In other words, $(U, \overline{R}_{(\alpha,\beta)}, \underline{R}_k)$ *is a directional expansion of the Pawlak model.*
(2) *In DqII-DTRS, if* $\alpha = 1, \beta = 0$ *and* $k = 0$, *then* $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)}) = (U, \overline{R}, \underline{R})$;
moreover, $\overline{R}_k(X) \subseteq \overline{R}(X),\ \underline{R}_{(\alpha,\beta)}(X) \supseteq \underline{R}(X)$.
In other words, $(U, \overline{R}_k, \underline{R}_{(\alpha,\beta)})$ *is also a directional expansion of the Pawlak model.*

According to Theorem 1, the two models with their thresholds exhibit favorable directional expansion properties, which originate from the basic expansions of both the DTRS model and GRS model. In contrast, the Pawlak model is merely a special case of the two models, with $\alpha = 1, \beta = 0$ and $k = 0$. The generalizations represented by the new models therefore have favorable theoretical properties.

DqI-DTRS and DqII-DTRS perform double quantification of the relative and absolute information. They have concrete quantitative semantics and therefore thoroughly describe the approximate space. They also exhibit strong double fault tolerance capabilities (in terms of both relative and absolute fault tolerance) and can therefore adapt to complex environments. Moreover, they correspond to positive expansions of the Pawlak model and therefore exhibit a more encompassing theoretical structure, and they have further advantage of completeness. The new models are therefore promising for both theoretical studies and practical applications involving double quantification. In addition, the two models are similar, parallel and symmetric. By imposing various conditions on the loss function, and the grade k, we can easily derive other Dq-DTRS approximations.

In the following, we provide a brief summary of rules for a single concept and discusses the inner relationship between the two double-quantitative decision-theoretic tough set models. A positive rule is used for accepting, a negative rule for rejecting and a boundary rule for abstaining. We accept an object as being an instance of a concept X based on a positive, reject an object as being an instance of X based on a negative rule, and abstain based on a boundary rule. That is the Three-way decision rules.

According to the four decision regions, one can make decisions based on the following positive, upper boundary, lower boundary and negative rules.

(1) For DqI-DTRS model, we have the following decisions:
$Des([x]_R) \rightarrow Des_{P'}(X)$, for $x \in pos'(X)$,
$Des([x]_R) \rightarrow Des_{N'}(X)$, for $x \in neg'(X)$,
$Des([x]_R) \rightarrow Des_{UB'}(X)$, for $x \in Ubn'(X)$,
$Des([x]_R) \rightarrow Des_{LB'}(X)$, for $x \in Lbn'(X)$.
(2) For DqII-DTRS model, we have the following decisions:
$Des([x]_R) \rightarrow Des_{P''}(X)$, for $x \in pos''(X)$,
$Des([x]_R) \rightarrow Des_{N''}(X)$, for $x \in neg''(X)$,
$Des([x]_R) \rightarrow Des_{UB''}(X)$, for $x \in Ubn''(X)$,
$Des([x]_R) \rightarrow Des_{LB''}(X)$, for $x \in Lbn''(X)$.

Unlike rules in the classical rough set theory, all the types of rules may be uncertain. The conditional probability and the grade represent the levels and grade of tolerance in making incorrect decisions. When the conditional probability is too low for acceptance (below threshold α) and at the same time too high for rejection (above threshold β), we choose a boundary rule for an abstain decision, an indecision or a delayed decision.

Case 1. $\alpha + \beta = 1$. If a loss function satisfies $\lambda_{PP} \leqslant \lambda_{NP} < \lambda_{BP}, \lambda_{BN} \leqslant \lambda_{NN} < \lambda_{PN}$ and $(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) > (\lambda_{BN} - \lambda_{NN})(\lambda_{BP} - \lambda_{PP})$, we have $\alpha > \beta$. Thus, $\beta < 0.5$ and $\alpha > 0.5$ hold.

From the fact that $\alpha = 1 - \beta$ and $\beta = 1 - \alpha$. At the same time, for the k, it follows that

(1) $P(X|[x]_R) > \beta,\ |[x]_R| - |[x]_R \cap X| \leqslant k \Longleftrightarrow P(\sim X|[x]_R) < 1 - \beta,\ |[x]_R \cap (\sim X)| \leqslant k$
$\Longleftrightarrow P(\sim X|[x]_R) < \alpha,\ |[x]_R \cap (\sim X)| \leqslant k.$
(2) $P(X|[x]_R) \leqslant \beta,\ |[x]_R| - |[x]_R \cap X| > k \Longleftrightarrow P(\sim X|[x]_R) \geqslant 1 - \beta,\ |[x]_R \cap (\sim X)| > k$
$\Longleftrightarrow P(\sim X|[x]_R) \geqslant \alpha,\ |[x]_R \cap (\sim X)| > k.$
(3) $P(X|[x]_R) > \beta,\ |[x]_R| - |[x]_R \cap X| > k \Longleftrightarrow P(\sim X|[x]_R) < 1 - \beta,\ |[x]_R \cap (\sim X)| > k$
$\Longleftrightarrow P(\sim X|[x]_R) < \alpha,\ |[x]_R \cap (\sim X)| > k.$
(4) $P(X|[x]_R) \leqslant \beta,\ |[x]_R| - |[x]_R \cap X| \leqslant k \Longleftrightarrow P(\sim X|[x]_R) \geqslant 1 - \beta,\ |[x]_R \cap (\sim X)| \leqslant k$
$\Longleftrightarrow P(\sim X|[x]_R) \geqslant \alpha,\ |[x]_R \cap (\sim X)| \leqslant k.$

By the definition of the positive, negative, upper boundary and lower boundary regions of DqI-DTRS and DqII-DTRS, we have:

$pos'(X) = neg''(\sim X),$
$neg'(X) = pos''(\sim X),$
$Ubn'(X) = Ubn''(\sim X),$
$Lbn'(X) = Lbn''(\sim X).$

For the condition with $\alpha + \beta = 1$, the following double implications hold between the decisions about acceptance and rejection in the two kinds of Dq-DTRS model:

P': accept an object for X with $P(X|[x]_R) > \beta$ and $|[x]_R| - |[x]_R \cap X| \leqslant k$.
$\Longleftrightarrow$ N'': reject an object for $\sim X$ with $P(\sim X|[x]_R) < \alpha$ and $|[x]_R \cap (\sim X)| \leqslant k$.
N': reject an object for X with $P(X|[x]_R) \leqslant \beta$ and $|[x]_R| - |[x]_R \cap X| > k$.
$\Longleftrightarrow$ P'': accept an object for $\sim X$ with $P(\sim X|[x]_R) \geqslant \alpha$ and $|[x]_R \cap (\sim X)| > k$.

That is to say, for any object the acceptance of X in DqI-DTRS is equivalent to the rejection of $\sim X$ in DqII-DTRS and the rejection of X in DqI-DTRS is equivalent to the acceptance of $\sim X$ in DqII-DTRS, and the vice versa. Furthermore, the abstained decisions are the same for X and $\sim X$ in both DqI-DTRS and DqII-DTRS.

For $\alpha + \beta = 1$, a loss function must satisfy $\lambda_{PP} \leqslant \lambda_{BP} < \lambda_{NP}, \lambda_{NN} \leqslant \lambda_{BN} < \lambda_{PN}$ and the condition:

$$\frac{(\lambda_{PN} - \lambda_{BN})}{(\lambda_{BP} - \lambda_{PP})} = \frac{(\lambda_{NP} - \lambda_{BP})}{(\lambda_{BN} - \lambda_{NN})}.$$

The classical rough set model is defined by $\alpha = 1,\ \beta = 0$ and $k = 0$, which satisfies the condition $\alpha + \beta = 1$. Thus it is a special case of the Case 1.

It should be pointed out that the two models proposed by Zhang et al. [34] is a special circumstance of Case 1 in the section. The reason: when $\alpha + \beta = 1$, the DTRS can be degenerated to variable precision rough set (VPRS), which is investigated in Ref. [34].

Case 2. $\alpha + \beta < 1$. The condition $\alpha > \beta,\ \beta < 0.5$ holds for Case 2. A loss function must satisfy the condition:

$$\frac{(\lambda_{PN} - \lambda_{BN})}{(\lambda_{BP} - \lambda_{PP})} < \frac{(\lambda_{NP} - \lambda_{BP})}{(\lambda_{BN} - \lambda_{NN})}.$$

For Case 2, we no longer have the equivalence of accepting X in DqI-DTRS model and rejecting $\sim X$ in DqII-DTRS model, nor the equivalence of rejecting X in DqI-DTRS model and accepting $\sim X$ in DqII-DTRS model, only the following double implications hold between the decisions about acceptance and rejection in the two kind of Dq-DTRS models:

N'': reject an object for $\sim X$ with $P(X|[x]_R) < \alpha$ and $|[x]_R \cap (\sim X)| \leqslant k$.
$\Rightarrow P'$: accept an object for X with $P(X|[x]_R) > \beta$ and $|[x]_R| - |[x]_R \cap X| \leqslant k$.
N': reject an object for X with $P(X|[x]_R) \leqslant \beta$ and $|[x]_R| - |[x]_R \cap X| > k$.
$\Rightarrow P''$: accept an object for $\sim X$ with $P(X|[x]_R) \geqslant \alpha$ and $|[x]_R \cap (\sim X)| > k$.

That is to say, from the rejection of one class in a kind of Dq-DTRS model, we can conclude the acceptance of this class in the second kind of Dq-DTRS model.

Case 3. $\alpha + \beta > 1$. The condition $\alpha > \beta,\ \alpha > 0.5$ holds for Case 3. A loss function must satisfy the condition:

$$\frac{(\lambda_{PN} - \lambda_{BN})}{(\lambda_{BP} - \lambda_{PP})} > \frac{(\lambda_{NP} - \lambda_{BP})}{(\lambda_{BN} - \lambda_{NN})}.$$

For Case 3, the following mirror implications of Case 2 hold in Case 3:

P': accept an object for X with $P(X|[x]_R) > \beta$ and $|[x]_R| - |[x]_R \cap X| \leqslant k$
$\Rightarrow N''$: reject an object for $\sim X$ with $P(X|[x]_R) < \alpha$ and $|[x]_R \cap (\sim X)| \leqslant k$.
P'': accept an object for $\sim X$ with $P(X|[x]_R) \geqslant \alpha$ and $|[x]_R \cap (\sim X)| > k$
$\Rightarrow N'$: reject an object for X with $P(X|[x]_R) \leqslant \beta$ and $|[x]_R| - |[x]_R \cap X| > k$.

That is to say, from the acceptance of one class in a kind of Dq-DTRS model, we can conclude the rejection of this class in the second kind of Dq-DTRS model. While the reverse of the implications does not hold.

4. Comparison and analysis

In this section, the medical example [35] is introduced to illustrate the utilization of the new models by comparing with DTRS and GRS. Let $S = (U, AT, D, F)$ be a decision table, where U is composed of 36 patients, and the condition and decision attributes are *fever*, *headache* and *cold*, respectively. Let R denote the equivalence relation on the condition attributes. Based on the measured medical data in Tables 2 and 3 provides the statistical results of the patient classes, where (i,j) $(i,j \in [0,2])$ denote the rank of condition attributes and X denotes the cold patient set. The rough set regions will be calculated in the case that $k = 1$.

Based on the condition attributes *Fever* and *Headache*, the universe is classified into nine classes. From Table 2, the cold patient set $X = \{x_3, x_5, x_6, x_9, x_{10}, x_{11}, x_{14}, x_{15}, x_{18}, x_{20}, x_{21}, x_{24}, x_{26}, x_{28}, x_{29}, x_{33}, x_{34}\}$. In the following, we will discuss the limitations of DTRS and GRS as well as advantage of the two models proposed in the paper.

Table 2
Initial medical data.

Patient	*Fever*	*Headache*	*Cold*	Patient	*Fever*	*Headache*	*Cold*
x_1	0	0	0	x_{19}	0	0	0
x_2	1	1	0	x_{20}	1	2	1
x_3	0	2	1	x_{21}	2	0	1
x_4	2	1	0	x_{22}	0	0	0
x_5	1	0	1	x_{23}	2	1	0
x_6	2	2	1	x_{24}	1	2	1
x_7	0	0	0	x_{25}	0	2	0
x_8	1	2	0	x_{26}	2	2	1
x_9	2	2	1	x_{27}	1	1	0
x_{10}	1	1	1	x_{28}	2	0	1
x_{11}	1	2	1	x_{29}	2	1	1
x_{12}	2	0	0	x_{30}	0	0	0
x_{13}	0	0	0	x_{31}	1	2	0
x_{14}	2	1	1	x_{32}	0	1	0
x_{15}	0	1	1	x_{33}	2	1	1
x_{16}	1	1	0	x_{34}	1	1	1
x_{17}	0	2	0	x_{35}	0	0	0
x_{18}	2	1	1	x_{36}	2	0	0

Table 3
Statistical results of the patient classes.

(i,j)	$[x]_R$	$\|[x]_R\|$	$[x]_R \cap X$	$\|[x]_R \cap X\|$	$P(X\|[x]_R)$	$\|[x]_R\| - \|[x]_R \cap X\|$
$(0,0)$	$x_{1,7,13,19,22,30,35}$	7	$\emptyset$	0	0	7
$(0,1)$	$x_{15,32}$	2	x_{15}	1	1/2	1
$(0,2)$	$x_{3,17,25}$	3	x_3	1	1/3	2
$(1,0)$	x_5	1	x_5	1	1	0
$(1,1)$	$x_{2,10,16,27,34}$	5	$x_{10,34}$	2	2/5	3
$(1,2)$	$x_{8,11,20,24,31}$	5	$x_{11,20,24}$	3	3/5	2
$(2,0)$	$x_{12,21,28,36}$	4	$x_{21,28}$	2	2/4	2
$(2,1)$	$x_{4,14,18,23,29,33}$	6	$x_{14,18,23,29,33}$	4	2/3	2
$(2,2)$	$x_{6,9,26}$	3	$x_{6,9,26}$	3	1	0

Table 4
Regions of GRS.

GRS	Positive region	Negative region	Upper boundary region	Lower boundary region
$k = 1$	$[x_6]_R$	$[x_1]_R, [x_3]_R$	$[x_2]_R, [x_4]_R, [x_8]_R, [x_{12}]_R$	$[x_5]_R, [x_{15}]_R$

Limitations of DTRS and GRS:
We will calculate the GRS and DTRS models.
As $k = 1$, we can get the grade k upper and lower approximations of X:

$$\overline{R}_k(X) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$\underline{R}_k(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R.$$

From the regions of GRS (see Table 4), it is easy to see that $[x_6]_R$ in the positive region means these patients have colds, $[x_1]_R, [x_3]_R$ in the negative region means these patients do not have colds, $[x_2]_R, [x_4]_R, [x_8]_R, [x_{12}]_R$ in the upper boundary region means the uncertainties are closer to do not have colds, and $[x_5]_R, [x_{15}]_R$ in the lower boundary region means the uncertainties are closer to having colds. The regions of DTRS and Dq-DTRS can be analyzed similarly.

In the Bayesian decision procedure, from the losses, one can give the values $\lambda_{i1} = \lambda(a_i|X), \lambda_{i2} = \lambda(a_i|X^C)$, and $i = 1, 2, 3$.
Consider the following loss function:

$$\lambda_{PP} = 0, \quad \lambda_{PN} = 27,$$
$$\lambda_{BP} = 12, \quad \lambda_{BN} = 4,$$
$$\lambda_{NP} = 18, \quad \lambda_{NN} = 0.$$

Then we can get $\alpha = 0.6,\ \beta = 0.4 \Rightarrow \alpha + \beta = 1$. We can obtain the decision-theoretic upper and lower approximations.

$$\overline{R}_{(0.6,0.4)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R;$$
$$\underline{R}_{(0.6,0.4)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R.$$

Accordingly, one can get the positive region, negative region and boundary region as following:

$$pos_{(0.6,0.4)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R;$$
$$neg_{(0.6,0.4)}(X) = [x_1]_R \cup [x_2]_R \cup [x_3]_R;$$
$$bn_{(0.6,0.4)}(X) = [x_{12}]_R \cup [x_{15}]_R.$$

Consider the following loss function:

$$\lambda_{PP} = 0, \quad \lambda_{PN} = 28,$$
$$\lambda_{BP} = 10, \quad \lambda_{BN} = 2,$$
$$\lambda_{NP} = 12, \quad \lambda_{NN} = 0.$$

Then we can get $\alpha = 0.7,\ \beta = 0.5 \Rightarrow \alpha + \beta > 1$. We can obtain the decision-theoretic upper and lower approximations.

$$\overline{R}_{(0.7,0.5)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R;$$
$$\underline{R}_{(0.7,0.5)}(X) = [x_5]_R \cup [x_6]_R.$$

Accordingly, one can get the positive region, negative region and boundary region as following:

$$pos_{(0.7,0.5)}(X) = [x_5]_R \cup [x_6]_R;$$
$$neg_{(0.7,0.5)}(X) = [x_1]_R \cup [x_2]_R \cup [x_3]_R \cup [x_{12}]_R \cup [x_{15}]_R;$$
$$bn_{(0.7,0.5)}(X) = [x_4]_R \cup [x_8]_R.$$

Consider the following loss function:

$$\lambda_{PP} = 0, \quad \lambda_{PN} = 14,$$
$$\lambda_{BP} = 7, \quad \lambda_{BN} = 3,$$
$$\lambda_{NP} = 14, \quad \lambda_{NN} = 0.$$

Then we can get $\alpha = 0.5,\ \beta = 0.3 \Rightarrow \alpha + \beta < 1$. We can obtain the decision-theoretic upper and lower approximations.

$$\overline{R}_{(0.5,0.3)}(X) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R;$$
$$\underline{R}_{(0.5,0.3)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R.$$

Accordingly, one can get the positive region, negative region and boundary region as following:

$$pos_{(0.5,0.3)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R;$$
$$neg_{(0.5,0.3)}(X) = [x_1]_R;$$
$$bn_{(0.5,0.3)}(X) = [x_2]_R \cup [x_3]_R \cup [x_{12}]_R \cup [x_{15}]_R.$$

It is easy to see that for $[x_5]_R$ and $[x_6]_R$ (see Table 3), $P(X|[x_5]_R) = P(X|[x_6]_R) = 1$. $[x_5]_R$ and $[x_6]_R$ become indiscernible and equal in the DTRS model (see Table 5). However, $|[x_5]_R| = 1 \neq 3 = |[x_6]_R|$ and $|[x_5]_R \cap X| = 1 \neq 3 = |[x_6]_R \cap X|$. And for $[x_{12}]_R$ and $[x_{15}]_R, P(X|[x_{12}]_R) = P(X|[x_{15}]_R) = 1/2, [x_{12}]_R$ and $[x_{15}]_R$ are also indiscernible and equal in DTRS model. For the threshold values $\alpha = 0.6,\ \beta = 0.4$, both $[x_5]_R$ and $[x_6]_R$ belong to the positive region, at the same time, both $[x_{12}]_R$ and $[x_{15}]_R$ belong to boundary region. For the threshold values $\alpha = 0.7,\ \beta = 0.5$, both $[x_5]_R$ and $[x_6]_R$ belong to the positive region, at the same time, both $[x_{12}]_R$ and $[x_{15}]_R$ belong to negative region. For the threshold values $\alpha = 0.5,\ \beta = 0.3$, both $[x_5]_R$ and $[x_6]_R$ belong to the positive region, at the same time, both $[x_{12}]_R$ and $[x_{15}]_R$ belong to boundary region.

For the grade $k = 1$ (see Tables 3 and 4), $|[x_5]_R \cap X| = |[x_{15}]_R \cap X| = 1$. $[x_5]_R$ and $[x_{15}]_R$ become indiscernible and equal in the GRS model, $[x_5]_R$ and $[x_{15}]_R$ belong to lower boundary region. However, $P(X|[x_5]_R) = 1 \neq P(X|[x_{15}]_R) = 1/2$. And for $[x_4]_R$ and $[x_8]_R, |[x_4]_R| - |[x_4]_R \cap X| = |[x_8]_R| - |[x_8]_R \cap X| = 2, [x_4]_R$ and $[x_8]_R$ are also indiscernible and equal in the GRS model. Both $[x_4]_R$ and $[x_8]_R$ belong to upper boundary region. However, $P(X|[x_4]_R) = 2/3 \neq P(X|[x_8]_R) = 3/5$.

Therefore, neither DTRS nor GRS can discern a complete and valuable description in some circumstances.

Advantage of proposed Dq-DTRS models:

In the following, we will calculate the proposed models in Case 1, Case 2 and Case 3 (see Table 6).

Case 1: the upper and lower approximations of DqI-DTRS model are

$$\overline{R}_{(0.6,0.4)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R,$$
$$\underline{R}_k(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of DqI-DTRS:

$$pos'(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R;$$
$$neg'(X) = [x_1]_R \cup [x_2]_R \cup [x_3]_R;$$
$$Ubn'(X) = [x_4]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$Lbn'(X) = \emptyset.$$

The upper and lower approximations of DqII-DTRS model are

$$\overline{R}_k(X) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R,$$
$$\underline{R}_{(0.6,0.4)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of DqII-DTRS:

$$pos''(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$$
$$neg''(X) = [x_1]_R \cup [x_3]_R \cup [x_{15}]_R;$$
$$Ubn''(X) = [x_2]_R \cup [x_{12}]_R;$$
$$Lbn''(X) = [x_5]_R.$$

Table 5
Regions of DTRS.

DTRS	Positive region	Negative region	Boundary region
$(\alpha = 0.6, \beta = 0.4)$	$[x_4]_R, [x_5]_R, [x_6]_R, [x_8]_R$	$[x_1]_R, [x_2]_R, [x_3]_R$	$[x_{12}]_R, [x_{15}]_R$
$(\alpha = 0.7, \beta = 0.5)$	$[x_5]_R, [x_6]_R$	$[x_1]_R, [x_2]_R, [x_3]_R, [x_{12}]_R, [x_{15}]_R$	$[x_4]_R, [x_8]_R$
$(\alpha = 0.5, \beta = 0.3)$	$[x_4]_R, [x_5]_R, [x_6]_R, [x_8]_R$	$[x_1]_R$	$[x_2]_R, [x_3]_R, [x_{12}]_R, [x_{15}]_R$

Table 6
Regions of DqI-DTRS and DqII-DTRS.

Dq-DTRS		Positive region	Negative region	Upper boundary region	Lower boundary region
Case 1	I	$[x_5]_R, [x_6]_R, [x_{15}]_R$	$[x_1]_R, [x_2]_R, [x_3]_R$	$[x_4]_R, [x_8]_R, [x_{12}]_R$	$\emptyset$
	II	$[x_4]_R, [x_6]_R, [x_8]_R$	$[x_1]_R, [x_3]_R, [x_{15}]_R$	$[x_2]_R, [x_{12}]_R$	$[x_5]_R$
Case 2	I	$[x_5]_R, [x_6]_R$	$[x_1]_R, [x_2]_R, [x_3]_R, [x_{12}]_R$	$[x_4]_R, [x_8]_R$	$[x_{15}]_R$
	II	$[x_6]_R$	$[x_1]_R, [x_3]_R, [x_{15}]_R$	$[x_2]_R, [x_4]_R, [x_8]_R, [x_{12}]_R$	$[x_5]_R$
Case 3	I	$[x_5]_R, [x_6]_R, [x_{15}]_R$	$[x_1]_R$	$[x_2]_R, [x_3]_R, [x_4]_R, [x_8]_R, [x_{12}]_R$	$\emptyset$
	II	$[x_4]_R, [x_6]_R, [x_8]_R$	$[x_1]_R, [x_3]_R, [x_{15}]_R$	$[x_2]_R, [x_{12}]_R$	$[x_5]_R$

For $\alpha = 0.6,\ \beta = 0.4$, the two Dq-DTRS models have their own quantitative semantics for the relative and absolute degree quantification. In DqI-DTRS, $pos'(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R$ denotes the relative degree of the patients belong to cold patient set exceeds 0.6 and the external grade with respect to the cold patient set does not exceed 1. In DqII-DTRS, $pos''(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$ denoted the relative degree of the patients belong to cold patient set at least 0.4 and the internal grade with respect to the cold patient set exceed 1. The same analysis for the negative region, upper boundary region and lower boundary region in both two Dq-DTRS models with the thresholds $\alpha = 0.6,\ \beta = 0.4$, and the grade $k = 1$.

Case 2: the upper and lower approximations of DqI-DTRS model are

$$\overline{R}_{(0.7,0.5)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R,$$
$$\underline{R}_k(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of DqI-DTRS:

$$pos'(X) = [x_5]_R \cup [x_6]_R;$$
$$neg'(X) = [x_1]_R \cup [x_2]_R \cup [x_3]_R \cup [x_{12}]_R;$$
$$Ubn'(X) = [x_4]_R \cup [x_8]_R;$$
$$Lbn'(X) = [x_{15}]_R.$$

The upper and lower approximations of DqII-DTRS model are

$$\overline{R}_k(X) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R,$$
$$\underline{R}_{(0.7,0.5)}(X) = [x_5]_R \cup [x_6]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of DqII-DTRS:

$$pos''(X) = [x_6]_R;$$
$$neg''(X) = [x_1]_R \cup [x_3]_R \cup [x_{15}]_R;$$
$$Ubn''(X) = [x_2]_R \cup [x_4]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$Lbn''(X) = [x_5]_R.$$

For $\alpha = 0.7,\ \beta = 0.5$. In DqI-DTRS, $pos'(X) = [x_6]_R$ denotes the relative degree of the patients belong to cold patient set exceeds 0.7 and the external grade with respect to the cold patient set does not exceed 1. In DqII-DTRS, $pos''(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$ denoted the relative degree of the patients belong to cold patient set at least 0.5 and the internal grade with respect to the cold patient set exceed 1. The same analysis for the negative region, upper boundary region and lower boundary region in both two Dq-DTRS models with the thresholds $\alpha = 0.7,\ \beta = 0.5$, and the grade $k = 1$.

Case 3: the upper and lower approximations of DqI-DTRS model are

$$\overline{R}_{(0.5,0.3)}(X) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R \cup [x_{15}]_R,$$
$$\underline{R}_k(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of DqI-DTRS:

$$pos'(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R;$$
$$neg'(X) = [x_1]_R;$$
$$Ubn'(X) = [x_2]_R \cup [x_3]_R \cup [x_4]_R \cup [x_8]_R \cup [x_{12}]_R;$$
$$Lbn'(X) = \emptyset.$$

The upper and lower approximations of DqII-DTRS model are

$$\overline{R}_k(X) = [x_2]_R \cup [x_4]_R \cup [x_6]_R \cup [x_8]_R \cup [x_{12}]_R,$$
$$\underline{R}_{(0.5,0.3)}(X) = [x_4]_R \cup [x_5]_R \cup [x_6]_R \cup [x_8]_R.$$

We can also get the positive region, negative region, upper boundary region and lower boundary region of DqII-DTRS:

$pos''(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R;$
$neg''(X) = [x_1]_R \cup [x_3]_R \cup [x_{15}]_R;$
$Ubn''(X) = [x_2]_R \cup [x_{12}]_R;$
$Lbn''(X) = [x_5]_R.$

For $\alpha = 0.5,\ \beta = 0.3$, the two Dq-DTRS models have their own quantitative semantics for the relative and absolute degree quantification. In DqI-DTRS, $pos'(X) = [x_5]_R \cup [x_6]_R \cup [x_{15}]_R$ denotes the relative degree of the patients belong to cold patient set exceeds 0.5 and the external grade with respect to the cold patient set does not exceed 1. In DqII-DTRS, $pos''(X) = [x_4]_R \cup [x_6]_R \cup [x_8]_R$ denoted the relative degree of the patients belong to cold patient set at least 0.3 and the internal grade with respect to the cold patient set exceed 1. The same analysis for the negative region, upper boundary region and lower boundary region in both two Dq-DTRS models with the thresholds $\alpha = 0.5,\ \beta = 0.3$, and the grade $k = 1$.

$[x_5]_R$ and $[x_6]_R$ are indiscernible and equal in the DTRS model, the same to $[x_{12}]_R$ and $[x_{15}]_R$.

For the threshold values $\alpha = 0.6, \beta = 0.4$, and grade $k = 1$ in case 1, it easy to see that $[x_5]_R$ and $[x_6]_R$ belong to the lower boundary region and positive region in DqII-DTRS, respectively. We can find that $[x_{12}]_R$ and $[x_{15}]_R$ belong to the upper boundary region and positive region in DqI-DTRS, respectively, and belong to upper boundary region and negative region in DqII-DTRS, respectively.

For the threshold values $\alpha = 0.7, \beta = 0.5$, and grade $k = 1$ in case 2, $[x_5]_R$ and $[x_6]_R$ belong to lower boundary region and positive region in DqII-DTRS, respectively.

For the threshold values $\alpha = 0.5, \beta = 0.3$, and grade $k = 1$ in case 3, $[x_5]_R$ and $[x_6]_R$ belong to lower boundary region and positive region in DqII-DTRS, respectively. The same analysis to $[x_{12}]_R$ and $[x_{15}]_R$.

Therefore, the double-quantitative of the relative and absolute information provides a valuable description in transacting real life applications.

5. Conclusions

The relative and absolute quantitative information of the approximate space are two fundamental quantitative indexes, which represent two distinct objective descriptors. For two knowledge granules $[x]_R^1 \neq [x]_R^2$, if $P(X|[x]_R^1) = P(X|[x]_R^2)$, then $[x]_R^1$ and $[x]_R^2$ are equal in DTRS. However, $[x]_R^1 \neq [x]_R^2$ or $[x]_R^1 \cap X \neq [x]_R^2 \cap X$, and $[x]_R^1$ and $[x]_R^2$ can be discerned by introducing the absolute quantitative information $[x]_R$ or $[x]_R \cap X$. And also for $[x]_R^1 \neq [x]_R^2$, if $[x]_R^1 \cap X = [x]_R^2 \cap X$, at the same time $[x]_R^1 \cap \sim X = [x]_R^2 \cap \sim X$, then $[x]_R^1$ and $[x]_R^2$ are indiscernible in GRS. While their conditional probabilities may satisfy $P(X|[x]_R^1) \neq P(X|[x]_R^2)$, in this case, $[x]_R^1$ and $[x]_R^2$ can be discerned by relative quantitative information $P(X|[x]_R)$. The double quantification formed by adding the absolute quantitative information can improve the descriptive abilities of DTRS models and expand their range of applicability. The proposed models, DqI-DTRS and DqII-DTRS, perform a basic double quantification of the relative information and absolute information. The new models are directional expansions of Pawlak rough set model and satisfy the quantitative completeness properties, exhibit strong double fault tolerance capabilities.

This paper mainly investigates double quantification, namely the relative and absolute information combining DTRS theory and GRS theory together. Moreover, after proposing the decision rules containing both relative quantification and absolute quantification in two kinds of Dq-DTRS model, we study their inner relationship between these two models. Among this article, we introduce the example on medical diagnose to illustrate and express the Dq-DTRSs. This paper gives a framework of Dq-DTRS model, in the future work, several aspects of these two models are worth investigating, including the uncertainty measures and the deep properties of the models with respect to the concept and parameters.

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