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# Concept learning via granular computing: A cognitive viewpoint

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ABSTRACT

Concepts are the most fundamental units of cognition in philosophy and how to learn concepts from various aspects in the real world is the main concern within the domain of conceptual knowledge presentation and processing. In order to improve efficiency and flexibility of concept learning, in this paper we discuss concept learning via granular computing from the point of view of cognitive computing. More precisely, cognitive mechanism of forming concepts is analyzed based on the principles from philosophy and cognitive psychology, including how to model concept-forming cognitive operators, define cognitive concepts and establish cognitive concept structure. Granular computing is then combined with the cognitive concept structure to improve efficiency of concept learning. Furthermore, we put forward a cognitive computing system which is the initial environment to learn composite concepts and can integrate past experiences into itself for enhancing flexibility of concept learning. Also, we investigate cognitive processes whose aims are to deal with the problem of learning one exact or two approximate cognitive concepts from a given object set, attribute set or pair of object and attribute sets.



## 1. Introduction

*Cognitive computing* is the development of computer systems modeled on the human brain [38]. It embodies major natural intelligence behaviors of the brain including perception, attention, thinking, etc. As an emerging paradigm of intelligent computing methodologies, cognitive computing has the characteristic of integrating past experiences into itself [22]. Nowadays, this theory has become an interdisciplinary research and application field and absorbed methods from psychology, information theory, mathematics and so on [37,39].

*Concepts* are the most fundamental units of cognition in philosophy and they carry certain meanings in almost all cognitive processes such as inference, learning and reasoning [37,50]. In this sense, a concept is in fact a cognitive unit to identify and/or model a real-world concrete entity and a perceived-world abstract subject. As is well known, how to efficiently learn concepts from various aspects in the real world is the main concern within the domain of conceptual knowledge presentation and processing. Up to now, for meeting the requirements of data analysis and knowledge manipulation, all kinds of concepts which carry certain meanings have been proposed such as *abstract concepts* [37], *Wille's concepts* [42], *object-oriented concepts* [48,49] and *property-oriented concepts* [9].

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In philosophy, a concept can generally be identified by its extension part (often called *extent*) and intension part (often called *intent*) which can be determined with each other [9,37,42,48,49]. The extent of a concept is the set of all objects or instances that the concept denotes, and the intent of a concept is the set of attributes or properties that a concept connotes [37,42]. In order to reflect the relationship of specialization and generalization among concepts, a structure of concepts can further be built by defining a partial order on the concepts under consideration. By this means, the obtained concept structure often forms a *complete lattice* no matter what kinds of meanings we give to the concepts, and hence some scholars from the community of *formal concept analysis* [42] called it certain *concept lattice* instead. For example, different concept structures have been proposed by specifying certain meanings of the concepts in the real world such as Wille's concept lattice [42], object-oriented concept lattice [48,49], property-oriented concept lattice [9], *AFS-concept lattice* [40], *power concept lattice* [11] and others [6,15,16,20,21]. Note that these certain concept structures or lattices establish rigorous mathematical models and provide formal semantics for data analysis in practice. In other words, meanings of real-world concrete entities can be represented and semantics of abstract subjects can be embodied by these certain concept structures. All in all, learning concepts (sometimes including their corresponding structure) has been investigated from various aspects. However, the current paper focuses on this issue from a novel aspect (i.e., a cognitive viewpoint).

Note that, generally speaking, learning a certain concept structure from a given dataset is computationally expensive when its size is large. The reason is that the number of concepts in the structure will increase exponentially in the worst case. Considering that *granular computing* gives rise to processing that is less time demanding than the one required when dealing with detailed numeric processing [2–4]. *Information granule*, the basic notion in the theory of granular computing which can broadly be viewed as a collection of information granules and the area of intelligent computing revolving around them [26], was introduced into Wille's concept lattice as an attempt to decrease computation time [43]. In a general sense, by information granule, one regards a collection of elements drawn together by their closeness (resemblance, proximity, functionality, etc.) articulated in terms of some useful spatial, temporal, or functional relationships [52,53]. In fact, information granules are intuitively appealing constructs, which play a pivotal role in human cognitive and decision-making activities [24]. It is also worth stressing that information granules permeate almost all human endeavors [2–5,24,25,27,53,54]. For example, information granules have been studied in *rough set theory* [17,23,29,30,35,44,46,47] extensively which is considered as one of the approaches of granular computing, and applied in *formal concept analysis* [10], *evidence analysis* [33], etc. Recently, studies on combination of granular computing with formal concept analysis have been made by several researchers [7,12,19,31,41,43,45,56]. And what is particularly worth mentioning is that information granules in formal concept analysis mean granular concepts [43] which are the basic concepts used to deduce others. As a matter of fact, in order to improve efficiency sharply, learning concept structure indeed needs the idea of granular computing no matter how we specify the certain meanings of the concepts in the real world, no exception to cognitive concepts to be discussed in the current paper.

Note that the aforementioned concepts were learned using constructive methods, which means that the concepts were formed by defining certain *concept-forming operators* [9,11,16,40,42,48,49]. In the meanwhile, axiomatic methods are also needed in terms of learning methodologies in which concepts are learned by establishing *axiomatic systems* (i.e., sets of axioms). To the best of our knowledge, axiomatic systems of concept learning were often called *concept systems* instead. In recent years, there have been several concept systems proposed for certain concept learning such as *cognitive system* [45], *concept granular computing system* [31], *generalized concept system* [19] and *generalized dual concept system* [18]. At the same time, it should be pointed out that these concept systems can also be used to learn the certain concepts which were obtained by using the constructive methods in [9,11,16,40,42,48,49]. Compared with constructive methods, axiomatic methods try to look beyond appearance for the essence of concept learning. However, the existing concept systems cannot integrate past experiences into itself. In other words, they are not able to deal with e.g., dynamic data and thereby are lack of flexibility for data analysis in practice. Besides, no explanation was provided to the background of the axioms of the existing concept systems, which means that they are too abstract to be understood. But, as far as we know, it is possible to solve these problems to some extent by means of concept learning based on cognitive computing because this kind of computing approach has the characteristic of integrating past experiences into itself and the background of simulating intelligence behaviors of the brain including perception, attention and learning.

To sum up, concept learning deserves to be studied based on granular computing from the perspective of cognitive computing, which may be beneficial to understanding and describing human cognitive processes in a conceptual knowledge way. Our current study mainly focuses on this issue. More precisely, the problems to be discussed are analysis of cognitive mechanism of forming concepts, integration of granular computing into cognitive concept structure, establishment of cognitive computing system, and implementation of cognitive processes. Note that the proposed cognitive computing system not only can integrate past experiences into itself by recursive thinking, but also is easy to be understood because cognitive mechanism of forming concepts is analyzed in advance based on the principles from philosophy and cognitive psychology.

The rest of this paper is organized as follows. In Section 2, cognitive mechanism of forming concepts is analyzed based on the principles from philosophy and cognitive psychology, including how to define concept-forming cognitive operators, construct cognitive concepts and induce their hierarchical structure. In Section 3, granular computing is integrated into the induced cognitive concept structure. In Section 4, we put forward a cognitive computing system in which the notions of a cognitive computing state, an object-oriented cognitive computing state and an attribute-oriented cognitive computing state are proposed. In Section 5, we investigate the cognitive processes whose aims are to deal with the problem of learning one exact or two approximate cognitive concepts from a given object set, attribute set or pair of object and attribute sets. In Section 6, we discuss the main differences and relations between the proposed concept learning approach and the existing

**Table 1**
A SARS dataset.

| Patient | Fever | Cough | Headache | Difficulty breathing |
|---|---|---|---|---|
| 1 | Yes | Yes | No | Yes |
| 2 | No | Yes | No | Yes |
| 3 | No | No | Yes | No |
| 4 | Yes | No | Yes | No |

ones, and give explanations on some obtained results in our study. The paper is then concluded with a brief summary and an outlook for further research.

## 2. Cognitive mechanism of forming concepts

In this section, we analyze cognitive mechanism of forming concepts based on the principles from philosophy and cognitive psychology.

Let $U$ be an object set and $A$ be an attribute set. We denote the power sets of $U$ and $A$ by $2^U$ and $2^A$, respectively. Hereinafter, suppose $\mathcal{L}: 2^U \to 2^A$ and $\mathcal{H}: 2^A \to 2^U$ are two set-valued mappings which are rewritten as $\mathcal{L}$ and $\mathcal{H}$ for short when there is no confusion. If the mappings $\mathcal{L}$ and $\mathcal{H}$ are used to derive concepts from a given object-attribute relation in the sense of cognition, then what do they need to obey? In what follows, we address this problem based on the principles from philosophy and cognitive psychology.

From the point of view of philosophy, a concept has two constituent parts: extent $X$ and intent $B$, where $X$ is a set of objects and $B$ is a set of attributes. In general, the more objects a concept denotes, the less attributes it connotes, and vice versa. By this principle, given object sets $X_1, X_2$ and attribute sets $B_1, B_2$, we have

$$X_1 \subseteq X_2 \Rightarrow \mathcal{L}(X_2) \subseteq \mathcal{L}(X_1), \tag{1}$$

$$B_1 \subseteq B_2 \Rightarrow \mathcal{H}(B_2) \subseteq \mathcal{H}(B_1), \tag{2}$$

where $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ denote the corresponding intents of $X_1$ and $X_2$, respectively; $\mathcal{H}(B_1)$ and $\mathcal{H}(B_2)$ denote the corresponding extents of $B_1$ and $B_2$, respectively.

From the perspective of cognitive psychology, the principle for perception can be used to restrain the mapping $\mathcal{L}$, while that for attention can be used to restrain the mapping $\mathcal{H}$. The details are described below.

According to Gestalt psychology [13,14], the perception of the whole is more than the integration of those of its parts. By this principle, we obtain

$$\mathcal{L}(X_1 \cup X_2) \supseteq \mathcal{L}(X_1) \cap \mathcal{L}(X_2). \tag{3}$$

Here, the intersection of attribute sets on the right side represents "the integration of perceptions of parts".

**Remark 1.** By considering that the value on the left of Eq. (3) will equal the one on the right when $X_1 = X_2$, the proper inclusion "$\supset$" which should have been used between the left and right sides so as to strictly obey the principle for perception, has to be weakened to "$\supseteq$".

Moreover, in terms of the principle for attention in cognitive psychology, the selection model of Deutsch and Deutsch [8] says that all information (attended and unattended) should be analyzed for meaning in order to select some inputs for full awareness. Whether or not the information is selected is dependent on how relevant it is at the time. By this principle, we get

$$\mathcal{H}(B) = \{x \in U | B \subseteq \mathcal{L}(\{x\})\}. \tag{4}$$

That is, the information at least relevant to all attributes in $B$ will be selected.

It is easy to verify that Eq. (2) can be implied by Eq. (4). Thus, combining the principles from philosophy with those from cognitive psychology for perception and attention, we claim that Eqs. (1), (3) and (4) should be satisfied by the set-valued mappings $\mathcal{L}$ and $\mathcal{H}$ when they are used to learn concepts from a given object-attribute relation.

**Definition 1.** Let $\mathcal{L}$ and $\mathcal{H}$ be two set-valued mappings. If for any $X_1, X_2 \subseteq U$ and $B \subseteq A$, the following properties hold:

(i) $X_1 \subseteq X_2 \Rightarrow \mathcal{L}(X_2) \subseteq \mathcal{L}(X_1)$,
(ii) $\mathcal{L}(X_1 \cup X_2) \supseteq \mathcal{L}(X_1) \cap \mathcal{L}(X_2)$,
(iii) $\mathcal{H}(B) = \{x \in U | B \subseteq \mathcal{L}(\{x\})\}$,

then $\mathcal{L}$ and $\mathcal{H}$ are called concept-forming cognitive operators (or simply cognitive operators).

**Example 1.** Table 1 depicts a dataset of four patients who suffer from *severe acute respiratory syndrome* (SARS) [1].

Let $U$ be the set of four patients and $A$ be the set of four symptoms. For convenience, we denote the four patients by 1, 2, 3 and 4, respectively, and the four symptoms (*Fever*, *Cough*, *Headache*, *Difficulty breathing*) by $a$, $b$, $c$ and $d$, respectively. That is, $U = \{1,2,3,4\}$ and $A = \{a,b,c,d\}$. Then, by intuitive perception and attention, we can obtain the following set-valued mappings:

$$\begin{array}{lllll} \mathcal{L}: & \emptyset \mapsto \{a,b,c,d\}, & \{1\} \mapsto \{a,b,d\}, & \{2\} \mapsto \{b,d\}, & \{3\} \mapsto \{c\}, \\ & \{4\} \mapsto \{a,c\}, & \{1,2\} \mapsto \{b,d\}, & \{1,3\} \mapsto \emptyset, & \{1,4\} \mapsto \{a\}, \\ & \{2,3\} \mapsto \emptyset, & \{2,4\} \mapsto \emptyset, & \{3,4\} \mapsto \{c\}, & \{1,2,3\} \mapsto \emptyset, \\ & \{1,2,4\} \mapsto \emptyset, & \{1,3,4\} \mapsto \emptyset, & \{2,3,4\} \mapsto \emptyset, & \{1,2,3,4\} \mapsto \emptyset \end{array}$$

and

$$\begin{array}{lllll} \mathcal{H}: & \emptyset \mapsto \{1,2,3,4\}, & \{a\} \mapsto \{1,4\}, & \{b\} \mapsto \{1,2\}, & \{c\} \mapsto \{3,4\}, \\ & \{d\} \mapsto \{1,2\}, & \{a,b\} \mapsto \{1\}, & \{a,c\} \mapsto \{4\}, & \{a,d\} \mapsto \{1\}, \\ & \{b,c\} \mapsto \emptyset, & \{b,d\} \mapsto \{1,2\}, & \{c,d\} \mapsto \emptyset, & \{a,b,c\} \mapsto \emptyset, \\ & \{a,b,d\} \mapsto \{1\}, & \{a,c,d\} \mapsto \emptyset, & \{b,c,d\} \mapsto \emptyset, & \{a,b,c,d\} \mapsto \emptyset, \end{array}$$

where $\mathcal{L}(X) = B$ means that $B$ is the set of symptoms possessed by all patients in $X$, and $\mathcal{H}(B) = X$ means that $X$ is the set of patients at least suffering from all symptoms in $B$. Then, by Definition 1, it is easy to verify that $\mathcal{L}$ and $\mathcal{H}$ are cognitive operators.

For brevity, hereinafter we write $\mathcal{L}(\{x\})$ $(x \in U)$ as $\mathcal{L}(x)$ and $\mathcal{H}(\{a\})$ $(a \in A)$ as $\mathcal{H}(a)$.

**Proposition 1.** *Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators. Then for any $X \subseteq U$ and $B \subseteq A$, we have*

$$\mathcal{L}(X) = \bigcap_{x \in X} \mathcal{L}(x), \tag{5}$$

$$\mathcal{H}(B) = \bigcap_{a \in B} \mathcal{H}(a). \tag{6}$$

**Proof.** It is immediate from Definition 1. □

The formula in Eq. (5) can be explained as "the perception of the whole is equal to the integration of those of its parts", which is not surprising for the cognitive operator $\mathcal{L}$. The reason is that on one hand the principle "the perception of the whole is more than the integration of those of its parts" is weakened to "the perception of the whole is more than or equal to the integration of those of its parts" in Eq. (3), and on the other hand the principle for a concept in philosophy has also been embodied into the cognitive operator $\mathcal{L}$. Similarly, the formula in Eq. (6) can be explained as "the information at least relevant to all attributes under consideration is equal to the integration of those at least relevant to its parts".

**Proposition 2.** *Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators. Then for any $X \subseteq U$ and $B \subseteq A$, we have*

$$X \subseteq \mathcal{HL}(X), \tag{7}$$

$$B \subseteq \mathcal{LH}(B), \tag{8}$$

*where $\mathcal{HL}(\bullet)$ and $\mathcal{LH}(\bullet)$ represent the compositions $\mathcal{H}(\mathcal{L}(\bullet))$ and $\mathcal{L}(\mathcal{H}(\bullet))$, respectively.*

**Proof.** It is immediate from Definition 1. □

The formula in Eq. (7) can be explained as "other objects (if any) analogous to $X$ can be recognized by $\mathcal{HL}$-cognition of $X$", and the formula in Eq. (8) can be explained as "other attributes (if any) analogous to $B$ can be recognized by $\mathcal{LH}$-cognition of $B$".

Here, we are extremely interested in the pair $(X,B)$ satisfying $X = \mathcal{H}(B)$ and $B = \mathcal{L}(X)$ since in this case both $X$ and $B$ reach the balance with respect to $\mathcal{HL}$ and $\mathcal{LH}$-cognitions, respectively. In other words, $X = \mathcal{HL}(X)$ and $B = \mathcal{LH}(B)$ can be satisfied simultaneously. In fact, such pairs $(X,B)$ satisfying $X = \mathcal{H}(B)$ and $B = \mathcal{L}(X)$ are a kind of useful conceptual knowledge in the sense of cognition.

**Definition 2.** Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators. For $X \subseteq U$ and $B \subseteq A$, if $\mathcal{L}(X) = B$ and $\mathcal{H}(B) = X$, we say that the pair $(X,B)$ is a concept under the cognitive operators $\mathcal{L}$ and $\mathcal{H}$ (or simply a cognitive concept). In this case, $X$ and $B$ are referred to as the extent and intent of the cognitive concept $(X,B)$, respectively.

**Example 2.** Let $\mathcal{L}$ and $\mathcal{H}$ be the cognitive operators shown in Example 1. Then by Definition 2, it is easy to verify that $(\{1,2,3,4\},\emptyset)$, $(\{1,2\},\{b,d\})$, $(\{1,4\},\{a\})$, $(\{3,4\},\{c\})$, $(\{1\},\{a,b,d\})$, $(\{4\},\{a,c\})$ and $(\emptyset,\{a,b,c,d\})$ are cognitive concepts which can be viewed to some extent as the learning results after perception and attention.

Moreover, in the real world, it is necessary to make the correlation analysis between cognitive concepts. This motivates us to establish generalization and specialization relationships between the cognitive concepts. More precisely, for two cognitive concepts $(X_1,B_1)$ and $(X_2,B_2)$ under $\mathcal{L}$ and $\mathcal{H}$, if $X_1 \subseteq X_2$, then $(X_1,B_1)$ is called a subconcept of $(X_2,B_2)$, or equivalently, $(X_2,B_2)$ is called a superconcept of $(X_1,B_1)$, which is denoted by $(X_1,B_1) \preceq (X_2,B_2)$. The set of all cognitive concepts together with the partial order relation $\preceq$ forms a complete lattice. We call it a cognitive concept structure or a cognitive concept lattice which is denoted by $\mathfrak{B}(U,A,\mathcal{L},\mathcal{H})$. The infimum $(\bigwedge)$ and supremum $(\bigvee)$ of a set of cognitive concepts $\{(X_t,B_t\ **)|\ t\in T\}$ ($T$ is an index set) are respectively defined as:

$$\begin{aligned}\bigwedge_{t\in T}(X_t,B_t) &= \left(\bigcap_{t\in T}X_t, \mathcal{LH}\left(\bigcup_{t\in T}B_t\right)\right),\\ \bigvee_{t\in T}(X_t,B_t) &= \left(\mathcal{HL}\left(\bigcup_{t\in T}X_t\right), \bigcap_{t\in T}B_t\right).\end{aligned} \tag{9}$$

## 3. Integrating granular computing into cognitive concept lattice

According to the discussion in Section 2, given cognitive operators $\mathcal{L}$ and $\mathcal{H}$, we can find in theory all cognitive concepts. However, in practice, it may be hard to implement such an action since the exhaustion of all elements of $\mathcal{L}$ and $\mathcal{H}$ is difficult when the cardinalities of $U$ and $A$ are large, let alone finding all cognitive concepts. For instance, in Example 1, the total number of $X_i \mapsto \mathcal{L}(X_i)$ $(X_i \subseteq U)$ in $\mathcal{L}$ and $B_j \mapsto \mathcal{H}(B_j)$ $(B_j \subseteq A)$ in $\mathcal{H}$ is $2^{|U|}+2^{|A|}$, which is exponential with respect to the cardinalities $|U|$ and $|A|$. However, fortunately, according to Eqs. (5) and (6), every $\mathcal{L}(X_i)$ can be represented as the intersection of $\mathcal{L}(x)$ $(x\in X_i)$ and every $\mathcal{H}(B_j)$ can be represented as the intersection of $\mathcal{H}(a)$ $(a\in B_j)$, which means that it is sufficient to list $\{x\}\mapsto\mathcal{L}(x)$ $(x\in U)$ in the mapping $\mathcal{L}$ and $\{a\}\mapsto\mathcal{H}(a)$ $(a\in A)$ in the mapping $\mathcal{H}$. In other words, $\{x\}\mapsto\mathcal{L}(x)$ $(x\in U)$ are basic but sufficient enough for inducing $\mathcal{L}$, and $\{a\}\mapsto\mathcal{H}(a)$ $(a\in A)$ are basic but sufficient enough for inducing $\mathcal{H}$. Therefore, $\{x\}\mapsto\mathcal{L}(x)$ $(x\in U)$ and $\{a\}\mapsto\mathcal{H}(a)$ $(a\in A)$ can respectively be viewed as the information granules of $\mathcal{L}$ and $\mathcal{H}$ in terms of knowledge representation. Moreover, by considering that information granules are the basic notion in the theory of granular computing, it is natural for us to integrate granular computing into cognitive concept lattice for decreasing the computation time. Besides, such an integration is also in accordance with characteristics of human thinking in which complex information is often divided into pieces, classes and groups [28].

In what follows, we put forward the notion of information granules of cognitive operators and that of a granular concept.

**Definition 3.** Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators. Then $\mathcal{L}^G=\{\{x\}\mapsto\mathcal{L}(x)|\ x\in U\}$ and $\mathcal{H}^G=\{\{a\}\mapsto\mathcal{H}(a)|\ a\in A\}$ are called information granules of $\mathcal{L}$ and $\mathcal{H}$, respectively.

According to Eqs. (5) and (6), the information granules $\mathcal{L}^G$ and $\mathcal{H}^G$ can respectively be used to form any $X\mapsto\mathcal{L}(X)$ and $B\mapsto\mathcal{H}(B)$ as follows:

$$\begin{aligned}\mathcal{L}(X) &= \bigcap_{x\in X}\mathcal{L}^G(x),\\ \mathcal{H}(B) &= \bigcap_{a\in B}\mathcal{H}^G(a).\end{aligned}$$

It should be pointed out that $X\mapsto\mathcal{L}(X)$ and $B\mapsto\mathcal{H}(B)$ may not be information granules when $X$ and $B$ are not singleton sets. Take Example 1 for instance, $\mathcal{L}(X)=\emptyset$ when $X=\{2,4\}$, and $\mathcal{H}(B)=\emptyset$ when $B=\{b,c\}$. Thus, $X\mapsto\mathcal{L}(X)\notin\mathcal{L}^G$ and $B\mapsto\mathcal{H}(B)\notin\mathcal{H}^G$.

Considering that the exhaustion of all elements of $\mathcal{L}$ and $\mathcal{H}$ is quite hard in practice when the cardinalities of $U$ and $A$ are large, we store and remember the information of $\mathcal{L}$ and $\mathcal{H}$ in their granular forms, i.e., $\mathcal{L}^G$ and $\mathcal{H}^G$.

Moreover, in preparation for presenting the notion of a granular concept, we need the following proposition.

**Proposition 3.** *Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators. Then for any $X\subseteq U$ and $B\subseteq A$, both $(\mathcal{HL}(X),\mathcal{L}(X))$ and $(\mathcal{H}(B),\mathcal{LH}(B))$ are cognitive concepts.*

**Proof.** It is immediate from Definitions 1 and 2 and Proposition 2. □

**Definition 4.** Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators. Then for any $x\in U$ and $a\in A$, we say that $(\mathcal{HL}(x),\mathcal{L}(x))$ and $(\mathcal{H}(a),\mathcal{LH}(a))$ are granular concepts.

It is easy to observe that granular concepts are basic but sufficient enough to induce others, and we can confirm this by the following proposition.

**Proposition 4.** *Let $\mathcal{L}$ and $\mathcal{H}$ be cognitive operators and $\underline{\mathfrak{B}}(U, A, \mathcal{L}, \mathcal{H})$ be the cognitive concept lattice. Then for any $(X, B) \in \underline{\mathfrak{B}}(U, A, \mathcal{L}, \mathcal{H})$, we have*

$$(X, B) = \bigvee_{x \in X} (\mathcal{H}\mathcal{L}(x), \mathcal{L}(x)) = \bigwedge_{a \in B} (\mathcal{H}(a), \mathcal{L}\mathcal{H}(a)). \tag{10}$$

**Proof.** It follows directly from Eqs. (5), (6) and (9). □

Considering that the exhaustion of all cognitive concepts under $\mathcal{L}$ and $\mathcal{H}$ is quite hard in practice, we store and remember the conceptual knowledge $\underline{\mathfrak{B}}(U, A, \mathcal{L}, \mathcal{H})$ in the form of its granular concepts $(\mathcal{H}\mathcal{L}(x), \mathcal{L}(x))$ $(x \in U)$ and $(\mathcal{H}(a), \mathcal{L}\mathcal{H}(a))$ $(a \in A)$ which are able to induce $\underline{\mathfrak{B}}(U, A, \mathcal{L}, \mathcal{H})$.

Note that in fact either $(\mathcal{H}\mathcal{L}(x), \mathcal{L}(x))$ $(x \in U)$ or $(\mathcal{H}(a), \mathcal{L}\mathcal{H}(a))$ $(a \in A)$ is enough to induce all cognitive concepts based on Eq. (10). However, we still need both of them since in practice it may start with an object set, an attribute set or a pair of object and attribute sets for concept learning or set approximation (see Section 5 for details).

To facilitate subsequent discussion, we denote

$$G_{\mathcal{L}\mathcal{H}} = \{(\mathcal{H}\mathcal{L}(x), \mathcal{L}(x)) |\ x \in U\} \cup \{(\mathcal{H}(a), \mathcal{L}\mathcal{H}(a)) |\ a \in A\}. \tag{11}$$

That is, $G_{\mathcal{L}\mathcal{H}}$ is the set of all granular concepts under the cognitive operators $\mathcal{L}$ and $\mathcal{H}$.

**Example 3.** Let $\mathcal{L}$ and $\mathcal{H}$ be the cognitive operators shown in Example 1. By Definition 3, we have that $\mathcal{L}^G = \{\{1\} \mapsto \{a, b, d\}, \{2\} \mapsto \{b, d\}, \{3\} \mapsto \{c\}, \{4\} \mapsto \{a, c\}\}$ and $\mathcal{H}^G = \{\{a\} \mapsto \{1, 4\}, \{b\} \mapsto \{1, 2\}, \{c\} \mapsto \{3, 4\}, \{d\} \mapsto \{1, 2\}\}$ are the information granules of $\mathcal{L}$ and $\mathcal{H}$, respectively. With these information granules, we can obtain

$$\begin{aligned}
&\mathcal{L}(1) = \mathcal{L}^G(1) = \{a, b, d\}, && \mathcal{H}\mathcal{L}(1) = \mathcal{H}(\{a, b, d\}) = \mathcal{H}^G(a) \cap \mathcal{H}^G(b) \cap \mathcal{H}^G(d) = \{1\},\\
&\mathcal{L}(2) = \mathcal{L}^G(2) = \{b, d\}, && \mathcal{H}\mathcal{L}(2) = \mathcal{H}(\{b, d\}) = \mathcal{H}^G(b) \cap \mathcal{H}^G(d) = \{1, 2\},\\
&\mathcal{L}(3) = \mathcal{L}^G(3) = \{c\}, && \mathcal{H}\mathcal{L}(3) = \mathcal{H}(c) = \mathcal{H}^G(c) = \{3, 4\},\\
&\mathcal{L}(4) = \mathcal{L}^G(4) = \{a, c\}, && \mathcal{H}\mathcal{L}(4) = \mathcal{H}(\{a, c\}) = \mathcal{H}^G(a) \cap \mathcal{H}^G(c) = \{4\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}(a) = \mathcal{H}^G(a) = \{1, 4\}, && \mathcal{L}\mathcal{H}(a) = \mathcal{L}(\{1, 4\}) = \mathcal{L}^G(1) \cap \mathcal{L}^G(4) = \{a\},\\
&\mathcal{H}(b) = \mathcal{H}^G(b) = \{1, 2\}, && \mathcal{L}\mathcal{H}(b) = \mathcal{L}(\{1, 2\}) = \mathcal{L}^G(1) \cap \mathcal{L}^G(2) = \{b, d\},\\
&\mathcal{H}(c) = \mathcal{H}^G(c) = \{3, 4\}, && \mathcal{L}\mathcal{H}(c) = \mathcal{L}(\{3, 4\}) = \mathcal{L}^G(3) \cap \mathcal{L}^G(4) = \{c\},\\
&\mathcal{H}(d) = \mathcal{H}^G(d) = \{1, 2\}, && \mathcal{L}\mathcal{H}(d) = \mathcal{L}(\{1, 2\}) = \mathcal{L}^G(1) \cap \mathcal{L}^G(2) = \{b, d\}.
\end{aligned}$$

Then, by Definition 4, we conclude that $(\{1\}, \{a, b, d\})$, $(\{1, 2\}, \{b, d\})$, $(\{3, 4\}, \{c\})$, $(\{4\}, \{a, c\})$ and $(\{1, 4\}, \{a\})$ are granular concepts. That is, $G_{\mathcal{L}\mathcal{H}} = \{(\{1\}, \{a, b, d\}),\ (\{1, 2\}, \{b, d\}),\ (\{3, 4\}, \{c\}),\ (\{4\}, \{a, c\}),\ (\{1, 4\}, \{a\})\}$.

## 4. Cognitive computing system

In the previous section, we have discussed how to derive granular concepts from cognitive operators $\mathcal{L}$ and $\mathcal{H}$. In the real world, the information on the object set $U$ and attribute set $A$ will be updated as time goes by, which means that the obtained granular concepts need to be updated accordingly. For instance, in Example 1, four symptoms (*Fever*, *Cough*, *Headache*, *Difficulty breathing*) related to SARS have been found from the current four patients. As time goes by, there will appear more patients from whom additional symptoms (e.g., *Diarrhea*, *Muscle aches*, *Nausea and vomiting*) will be observed (see Example 4 for details). So, it is necessary to update the granular concepts obtained in Example 3.

In what follows, we put forward a cognitive computing system which can be viewed as the initial environment to learn composite concepts by updating the current granular concepts with the newly input information.

For convenience, hereinafter $n$ object sets $U_1, U_2, \ldots, U_n$ with $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n$ are denoted by $\{U_t\}^{\uparrow}$, and similarly $n$ attribute sets $A_1, A_2, \ldots, A_n$ with $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ are denoted by $\{A_t\}^{\uparrow}$.

**Definition 5.** Let $U_{i-1}$, $U_i$ be object sets of $\{U_t\}^{\uparrow}$ and $A_{i-1}$, $A_i$ be attribute sets of $\{A_t\}^{\uparrow}$. Denote $\Delta U_{i-1} = U_i - U_{i-1}$ and $\Delta A_{i-1} = A_i - A_{i-1}$. Suppose

$$\begin{aligned}
&(i) && \mathcal{L}_{i-1} : 2^{U_{i-1}} \to 2^{A_{i-1}}, && \mathcal{H}_{i-1} : 2^{A_{i-1}} \to 2^{U_{i-1}},\\
&(ii) && \mathcal{L}_{\Delta U_{i-1}} : 2^{\Delta U_{i-1}} \to 2^{A_{i-1}}, && \mathcal{H}_{\Delta U_{i-1}} : 2^{A_{i-1}} \to 2^{\Delta U_{i-1}},\\
&(iii) && \mathcal{L}_{\Delta A_{i-1}} : 2^{U_i} \to 2^{\Delta A_{i-1}}, && \mathcal{H}_{\Delta A_{i-1}} : 2^{\Delta A_{i-1}} \to 2^{U_i},\\
&(iv) && \mathcal{L}_i : 2^{U_i} \to 2^{A_i}, && \mathcal{H}_i : 2^{A_i} \to 2^{U_i}
\end{aligned}$$

are four pairs of cognitive operators whose information granules: $(i^*)$ $\mathcal{L}_{i-1}^G$, $\mathcal{H}_{i-1}^G$, $(ii^*)$ $\mathcal{L}_{\Delta U_{i-1}}^G, \mathcal{H}_{\Delta U_{i-1}}^G$, $iii^*)$ $\mathcal{L}_{\Delta A_{i-1}}^G, \mathcal{H}_{\Delta A_{i-1}}^G$ and $(iv^*)$ $\mathcal{L}_i^G$, $\mathcal{H}_i^G$ satisfy the following properties:

$$\mathcal{L}_i^G(x) = \begin{cases} \mathcal{L}_{i-1}^G(x) \cup \mathcal{L}_{\Delta A_{i-1}}^G(x), & \text{if } x \in U_{i-1}, \\ \mathcal{L}_{\Delta U_{i-1}}^G(x) \cup \mathcal{L}_{\Delta A_{i-1}}^G(x), & \text{otherwise,} \end{cases} \tag{12}$$

$$\mathcal{H}_i^G(a) = \begin{cases} \mathcal{H}_{i-1}^G(a) \cup \mathcal{H}_{\Delta U_{i-1}}^G(a), & \text{if } a \in A_{i-1}, \\ \mathcal{H}_{\Delta A_{i-1}}^G(a), & \text{otherwise,} \end{cases} \tag{13}$$

where $\mathcal{L}_{\Delta U_{i-1}}^G(x)$ and $\mathcal{H}_{\Delta U_{i-1}}^G(a)$ are set to be empty when $\Delta U_{i-1} = \emptyset$, and $\mathcal{L}_{\Delta A_{i-1}}^G(x)$ and $\mathcal{H}_{\Delta A_{i-1}}^G(a)$ are set to be empty when $\Delta A_{i-1} = \emptyset$. Then we say that $\mathcal{L}_i$ and $\mathcal{H}_i$ are extended cognitive operators of $\mathcal{L}_{i-1}$ and $\mathcal{H}_{i-1}$ with the newly input information $\mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}}$.

As for the background of extended cognitive operators, $\mathcal{L}_{i-1}, \mathcal{H}_{i-1}$ can be considered as last state of knowledge expressing, and $\mathcal{L}_i, \mathcal{H}_i$ can be considered as the current state of knowledge expressing which is the result of updating the last state of knowledge expressing with the newly input information $\mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}}$. Under this background, from the last state to the current state, granular concepts need to be updated once.

**Definition 6.** Let $U_{i-1}, U_i$ be object sets of $\{U_t\}^{\uparrow}$, $A_{i-1}, A_i$ be attribute sets of $\{A_t\}^{\uparrow}$, $\Delta U_{i-1} = U_i - U_{i-1}$, $\Delta A_{i-1} = A_i - A_{i-1}$ and (i) $\mathcal{L}_{i-1}, \mathcal{H}_{i-1}$, (ii) $\mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}$, (iii) $\mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}}$, (iv) $\mathcal{L}_i, \mathcal{H}_i$ be four pairs of cognitive operators. If $\mathcal{L}_i$ and $\mathcal{H}_i$ are the extended cognitive operators of $\mathcal{L}_{i-1}$ and $\mathcal{H}_{i-1}$ with the newly input information $\mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}}$, then we call $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ a cognitive computing state, where $G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}$ is the set of all granular concepts under the cognitive operators $\mathcal{L}_{i-1}$ and $\mathcal{H}_{i-1}$. Moreover, a collection of cognitive computing states, denoted by $\mathcal{S} = \bigcup_{i=2}^{n} \{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$, is called a cognitive computing system.

As for the background of cognitive computing system, every cognitive computing state can be viewed as the result of updating the information under consideration once, and a collection of cognitive computing states can be viewed as the result of updating a series of information successively.

Then it is important to compute the final granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n} \{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$ based on the initial granular concepts $G_{\mathcal{L}_1\mathcal{H}_1}$ and a series of newly input information "$\mathcal{L}_{\Delta U_1}, \mathcal{H}_{\Delta U_1}, \mathcal{L}_{\Delta A_1}, \mathcal{H}_{\Delta A_1}$", "$\mathcal{L}_{\Delta U_2}, \mathcal{H}_{\Delta U_2}, \mathcal{L}_{\Delta A_2}, \mathcal{H}_{\Delta A_2}$", ..., "$\mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}}$". Note that this problem is called transformation between information granules in the theory of granular computing. Moreover, considering that recursive approach can be applied here, it is sufficient to solve the subproblem of determining $G_{\mathcal{L}_i\mathcal{H}_i}$ with $G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$. In other words, we only need to discuss cognitive computing state $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$. To achieve this task, we continue to propose the notions of an object-oriented cognitive computing state and an attribute-oriented cognitive computing state.

**Definition 7.** Let $U_{i-1}, U_i$ be object sets of $\{U_t\}^{\uparrow}$, $A_{i-1}$ be an attribute set, $\Delta U_{i-1} = U_i - U_{i-1}$ and (i) $\mathcal{L}_{i-1} : 2^{U_{i-1}} \to 2^{A_{i-1}}$, $\mathcal{H}_{i-1} : 2^{A_{i-1}} \to 2^{U_{i-1}}$, (ii) $\mathcal{L}_{\Delta U_{i-1}} : 2^{\Delta U_{i-1}} \to 2^{A_{i-1}}$, $\mathcal{H}_{\Delta U_{i-1}} : 2^{A_{i-1}} \to 2^{\Delta U_{i-1}}$, (iii) $\mathcal{L}_0 : 2^{U_i} \to 2^{A_{i-1}}$, $\mathcal{H}_0 : 2^{A_{i-1}} \to 2^{U_i}$ be three pairs of cognitive operators. If $\mathcal{L}_0$ and $\mathcal{H}_0$ are the extended cognitive operators of $\mathcal{L}_{i-1}$ and $\mathcal{H}_{i-1}$ with the newly input information $\mathcal{L}_{\Delta U_{i-1}}$ and $\mathcal{H}_{\Delta U_{i-1}}$ (i.e., only object information being updated), then $\mathcal{OS}_{\mathcal{L}_0\mathcal{H}_0} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}})$ is called an object-oriented cognitive computing state.

**Proposition 5.** *Let $\mathcal{OS}_{\mathcal{L}_0\mathcal{H}_0} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}})$ be an object-oriented cognitive computing state. Then the following statements hold:*

(i) *For any $x \in U_i$, if $x \in U_{i-1}$, then*

$$(\mathcal{H}_0\mathcal{L}_0(x), \mathcal{L}_0(x)) = (\mathcal{H}_{i-1}\mathcal{L}_{i-1}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{i-1}(x), \mathcal{L}_{i-1}(x)); \tag{14}$$

*otherwise,*

$$(\mathcal{H}_0\mathcal{L}_0(x), \mathcal{L}_0(x)) = (\mathcal{H}_{i-1}\mathcal{L}_{\Delta U_{i-1}}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{\Delta U_{i-1}}(x), \mathcal{L}_{\Delta U_{i-1}}(x)). \tag{15}$$

(ii) *For any $a \in A_{i-1}$, we have*

$$(\mathcal{H}_0(a), \mathcal{L}_0\mathcal{H}_0(a)) = (\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a), \mathcal{L}_{i-1}\mathcal{H}_{i-1}(a) \cap \mathcal{L}_{\Delta U_{i-1}}\mathcal{H}_{\Delta U_{i-1}}(a)). \tag{16}$$

**Proof.** (i) If $x \in U_{i-1}$, by Definition 5, we have $\mathcal{L}_0(x) = \mathcal{L}_{i-1}(x)$ due to $\mathcal{L}_{\Delta A_{i-1}}(x) = \emptyset$. Based on Eqs. (6) and (13), we conclude

$$\begin{aligned} \mathcal{H}_0(\{a_1, a_2\}) &= \mathcal{H}_0^G(a_1) \cap \mathcal{H}_0^G(a_2) = \left(\mathcal{H}_{i-1}^G(a_1) \cup \mathcal{H}_{\Delta U_{i-1}}^G(a_1)\right) \cap \left(\mathcal{H}_{i-1}^G(a_2) \cup \mathcal{H}_{\Delta U_{i-1}}^G(a_2)\right) \\ &= \left(\mathcal{H}_{i-1}^G(a_1) \cap \mathcal{H}_{i-1}^G(a_2)\right) \cup \left(\mathcal{H}_{\Delta U_{i-1}}^G(a_1) \cap \mathcal{H}_{i-1}^G(a_2)\right) \cup \left(\mathcal{H}_{i-1}^G(a_1) \cap \mathcal{H}_{\Delta U_{i-1}}^G(a_2)\right) \cup \left(\mathcal{H}_{\Delta U_{i-1}}^G(a_1) \cap \mathcal{H}_{\Delta U_{i-1}}^G(a_2)\right) \\ &= \left(\mathcal{H}_{i-1}^G(a_1) \cap \mathcal{H}_{i-1}^G(a_2)\right) \cup \left(\mathcal{H}_{\Delta U_{i-1}}^G(a_1) \cap \mathcal{H}_{\Delta U_{i-1}}^G(a_2)\right) = \mathcal{H}_{i-1}(\{a_1, a_2\}) \cup \mathcal{H}_{\Delta U_{i-1}}(\{a_1, a_2\}). \end{aligned}$$

Moreover, using recursive approach, we obtain $\mathcal{H}_0\mathcal{L}_0(x) = \mathcal{H}_0\mathcal{L}_{i-1}(x) = \mathcal{H}_{i-1}\mathcal{L}_{i-1}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{i-1}(x)$. To sum up, it follows $(\mathcal{H}_0\mathcal{L}_0(x), \mathcal{L}_0(x)) = (\mathcal{H}_{i-1}\mathcal{L}_{i-1}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{i-1}(x), \mathcal{L}_{i-1}(x))$. In a similar manner, if $x \in \Delta U_{i-1}$, we can prove $(\mathcal{H}_0\mathcal{L}_0(x), \mathcal{L}_0(x)) = (\mathcal{H}_{i-1}\mathcal{L}_{\Delta U_{i-1}}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{\Delta U_{i-1}}(x), \mathcal{L}_{\Delta U_{i-1}}(x))$.

(ii) By Definition 5, for any $a \in A_{i-1}$, we have $\mathcal{H}_0(a) = \mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a)$. Furthermore, based on Eqs. (5) and (12), we conclude

$$\mathcal{L}_0(\mathcal{H}_0(a)) = \mathcal{L}_0(\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a)) = \mathcal{L}_0(\mathcal{H}_{i-1}(a)) \cap \mathcal{L}_0(\mathcal{H}_{\Delta U_{i-1}}(a)) = \left(\bigcap_{x\in\mathcal{H}_{i-1}(a)} \mathcal{L}_0^G(x)\right) \bigcap \left(\bigcap_{x\in\mathcal{H}_{\Delta U_{i-1}}(a)} \mathcal{L}_0^G(x)\right)$$

$$= \left(\bigcap_{x\in\mathcal{H}_{i-1}(a)} \mathcal{L}_{i-1}^G(x)\right) \bigcap \left(\bigcap_{x\in\mathcal{H}_{\Delta U_{i-1}}(a)} \mathcal{L}_{\Delta U_{i-1}}^G(x)\right) = \mathcal{L}_{i-1}\mathcal{H}_{i-1}(a) \cap \mathcal{L}_{\Delta U_{i-1}}\mathcal{H}_{\Delta U_{i-1}}(a).$$

As a result, we obtain $(\mathcal{H}_0(a), \mathcal{L}_0\mathcal{H}_0(a)) = (\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a), \mathcal{L}_{i-1}\mathcal{H}_{i-1}(a) \cap \mathcal{L}_{\Delta U_{i-1}}\mathcal{H}_{\Delta U_{i-1}}(a))$. □

Moreover, we put forward the notion of an attribute-oriented cognitive computing state.

**Definition 8.** Let $U_i$ be an object set, $A_{i-1}$, $A_i$ be attribute sets of $\{A_t\}^{\uparrow}$, $\Delta A_{i-1} = A_i - A_{i-1}$ and (i) $\mathcal{L}_0 : 2^{U_i} \to 2^{A_{i-1}}$, $\mathcal{H}_0 : 2^{A_{i-1}} \to 2^{U_i}$, (ii) $\mathcal{L}_{\Delta A_{i-1}} : 2^{U_i} \to 2^{\Delta A_{i-1}}$, $\mathcal{H}_{\Delta A_{i-1}} : 2^{\Delta A_{i-1}} \to 2^{U_i}$, (iii) $\mathcal{L}_i : 2^{U_i} \to 2^{A_i}$, $\mathcal{H}_i : 2^{A_i} \to 2^{U_i}$ be three pairs of cognitive operators. If $\mathcal{L}_i$ and $\mathcal{H}_i$ are the extended cognitive operators of $\mathcal{L}_0$ and $\mathcal{H}_0$ with the newly input information $\mathcal{L}_{\Delta A_{i-1}}$ and $\mathcal{H}_{\Delta A_{i-1}}$ (i.e., only attribute information being updated), then $\mathcal{AS}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_0\mathcal{H}_0}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ is called an attribute-oriented cognitive computing state.

**Proposition 6.** *Let $\mathcal{AS}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_0\mathcal{H}_0}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ be an attribute-oriented cognitive computing state. Then the following statements hold:*

(i) *For any $a \in A_i$, if $a \in A_{i-1}$, then*

$$(\mathcal{H}_i(a), \mathcal{L}_i\mathcal{H}_i(a)) = (\mathcal{H}_0(a), \mathcal{L}_0\mathcal{H}_0(a) \cup \mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_0(a)); \tag{17}$$

*otherwise,*

$$(\mathcal{H}_i(a), \mathcal{L}_i\mathcal{H}_i(a)) = (\mathcal{H}_{\Delta A_{i-1}}(a), \mathcal{L}_0\mathcal{H}_{\Delta A_{i-1}}(a) \cup \mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_{\Delta A_{i-1}}(a)). \tag{18}$$

(ii) *For any $x \in U_i$, we have*

$$(\mathcal{H}_i\mathcal{L}_i(x), \mathcal{L}_i(x)) = (\mathcal{H}_0\mathcal{L}_0(x) \cap \mathcal{H}_{\Delta A_{i-1}}\mathcal{L}_{\Delta A_{i-1}}(x), \mathcal{L}_0(x) \cup \mathcal{L}_{\Delta A_{i-1}}(x)). \tag{19}$$

**Proof.** We can prove it in a manner similar to Proposition 5. □

Combining Definitions 7 and 8 with Propositions 5 and 6, we know that a cognitive computing state $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ can be decomposed into the object-oriented cognitive computing state $\mathcal{OS}_{\mathcal{L}_0\mathcal{H}_0} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}})$ and the attribute-oriented cognitive computing state $\mathcal{AS}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_0\mathcal{H}_0}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$. Such a decomposition is beneficial to the computation of granular concepts $G_{\mathcal{L}_i\mathcal{H}_i}$. More precisely, (1) we firstly decompose $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}$ into $\mathcal{OS}_{\mathcal{L}_0\mathcal{H}_0}$ and $\mathcal{AS}_{\mathcal{L}_i\mathcal{H}_i}$; (2) we further use Proposition 5 to calculate the granular concepts $G_{\mathcal{L}_0\mathcal{H}_0}$; (3) we finally employ Proposition 6 to compute $G_{\mathcal{L}_i\mathcal{H}_i}$. The detailed results are shown in the following proposition.

**Proposition 7.** *Let $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ be a cognitive computing state. Then the following statements hold:*

(1) *For any $x \in U_i$, if $x \in U_{i-1}$, then*

$$(\mathcal{H}_i\mathcal{L}_i(x), \mathcal{L}_i(x)) = \big((\mathcal{H}_{i-1}\mathcal{L}_{i-1}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{i-1}(x)) \cap \mathcal{H}_{\Delta A_{i-1}}\mathcal{L}_{\Delta A_{i-1}}(x), \mathcal{L}_{i-1}(x) \cup \mathcal{L}_{\Delta A_{i-1}}(x)\big); \tag{20}$$

*otherwise,*

$$(\mathcal{H}_i\mathcal{L}_i(x), \mathcal{L}_i(x)) = \big((\mathcal{H}_{i-1}\mathcal{L}_{\Delta U_{i-1}}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{\Delta U_{i-1}}(x)) \cap \mathcal{H}_{\Delta A_{i-1}}\mathcal{L}_{\Delta A_{i-1}}(x), \mathcal{L}_{\Delta U_{i-1}}(x) \cup \mathcal{L}_{\Delta A_{i-1}}(x)\big). \tag{21}$$

(2) *For any $a \in A_i$, if $a \in A_{i-1}$, then*

$$(\mathcal{H}_i(a), \mathcal{L}_i\mathcal{H}_i(a)) = \big(\mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a), (\mathcal{L}_{i-1}\mathcal{H}_{i-1}(a) \cap \mathcal{L}_{\Delta U_{i-1}}\mathcal{H}_{\Delta U_{i-1}}(a)) \cup (\mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_{i-1}(a) \cap \mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_{\Delta U_{i-1}}(a))\big); \tag{22}$$

*otherwise,*

$$(\mathcal{H}_i(a), \mathcal{L}_i\mathcal{H}_i(a)) = \big(\mathcal{H}_{\Delta A_{i-1}}(a), (\mathcal{L}_{i-1}(\mathcal{H}_{\Delta A_{i-1}}(a) \cap U_{i-1}) \cap \mathcal{L}_{\Delta U_{i-1}}(\mathcal{H}_{\Delta A_{i-1}}(a) \cap \Delta U_{i-1})) \cup \mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_{\Delta A_{i-1}}(a)\big). \tag{23}$$

Proposition 7 gives a simple transformation way from the information granules $G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}$ to $G_{\mathcal{L}_i\mathcal{H}_i}$ with the newly input information $\mathcal{L}_{\Delta U_{i-1}}$, $\mathcal{H}_{\Delta U_{i-1}}$ and $\mathcal{L}_{\Delta A_{i-1}}$, $\mathcal{H}_{\Delta A_{i-1}}$.

Based on the above discussion, we are now ready to propose a procedure (called Algorithm 1 to compute the granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$, where each $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ represents a cognitive computing state.

**Algorithm 1.** Computing the granular concepts of a cognitive computing system

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**Require:** $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$, where $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i} = (G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ is a cognitive computing state.
**Ensure** The granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of $\mathcal{S}$.
1: Initialize $G_{\mathcal{L}_1\mathcal{H}_1} = \{(\mathcal{H}_1\mathcal{L}_1(x), \mathcal{L}_1(x)) \mid x \in U_1\} \cup \{(\mathcal{H}_1(a), \mathcal{L}_1\mathcal{H}_1(a)) \mid a \in A_1\}$ and $i = 2$;
2: **While** $i \leqslant n$
3: Denote the object-oriented cognitive computing state $(G_{\mathcal{L}_{i-1}\mathcal{H}_{i-1}}, \mathcal{L}_{\Delta U_{i-1}}, \mathcal{H}_{\Delta U_{i-1}})$ by $\mathcal{OS}_{\mathcal{L}_0\mathcal{H}_0}$;
4: **For** each $x \in U_i$
5: **If** $x \in U_{i-1}$
6: set $\mathcal{L}_0(x) = \mathcal{L}_{i-1}(x)$, $\mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{i-1}(x) = \bigcap_{a \in \mathcal{L}_{i-1}(x)} \mathcal{H}_{\Delta U_{i-1}}(a)$ and $\mathcal{H}_0\mathcal{L}_0(x) = \mathcal{H}_{i-1}\mathcal{L}_{i-1}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{i-1}(x)$;
7: **Else**
8: do $\mathcal{L}_0(x) = \mathcal{L}_{\Delta U_{i-1}}(x), \mathcal{H}_{i-1}\mathcal{L}_{\Delta U_{i-1}}(x) = \bigcap_{a \in \mathcal{L}_{\Delta U_{i-1}}(x)} \mathcal{H}_{i-1}(a)$ and $\mathcal{H}_0\mathcal{L}_0(x) = \mathcal{H}_{i-1}\mathcal{L}_{\Delta U_{i-1}}(x) \cup \mathcal{H}_{\Delta U_{i-1}}\mathcal{L}_{\Delta U_{i-1}}(x)$;
9: **End If**
10: **End For**
11: **For** each $a \in A_{i-1}$
12: compute $\mathcal{H}_0(a) = \mathcal{H}_{i-1}(a) \cup \mathcal{H}_{\Delta U_{i-1}}(a)$ and $\mathcal{L}_0\mathcal{H}_0(a) = \mathcal{L}_{i-1}\mathcal{H}_{i-1}(a) \cap \mathcal{L}_{\Delta U_{i-1}}\mathcal{H}_{\Delta U_{i-1}}(a)$;
13: **End For**
14: Denote the attribute-oriented cognitive computing state $(G_{\mathcal{L}_0\mathcal{H}_0}, \mathcal{L}_{\Delta A_{i-1}}, \mathcal{H}_{\Delta A_{i-1}})$ by $\mathcal{AS}_{\mathcal{L}_i\mathcal{H}_i}$;
15: **For** each $a \in A_i$
16: **If** $a \in A_{i-1}$
17: set $\mathcal{H}_i(a) = \mathcal{H}_0(a)$, $\mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_0(a) = \bigcap_{x \in \mathcal{H}_0(a)} \mathcal{L}_{\Delta A_{i-1}}(x)$ and $\mathcal{L}_i\mathcal{H}_i(a) = \mathcal{L}_0\mathcal{H}_0(a) \cup \mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_0(a)$;
18: **Else**
19: do $\mathcal{H}_i(a) = \mathcal{H}_{\Delta A_{i-1}}(a), \mathcal{L}_0\mathcal{H}_{\Delta A_{i-1}}(a) = \bigcap_{x \in \mathcal{H}_{\Delta A_{i-1}}(a)} \mathcal{L}_0(x)$ and $\mathcal{L}_i\mathcal{H}_i(a) = \mathcal{L}_0\mathcal{H}_{\Delta A_{i-1}}(a) \cup \mathcal{L}_{\Delta A_{i-1}}\mathcal{H}_{\Delta A_{i-1}}(a)$;
20: **End If**
21: **End For**
22: **For** each $x \in U_i$
23: let $\mathcal{L}_i(x) = \mathcal{L}_0(x) \cup \mathcal{L}_{\Delta A_{i-1}}(x)$ and $\mathcal{H}_i\mathcal{L}_i(x) = \mathcal{H}_0\mathcal{L}_0(x) \cap \mathcal{H}_{\Delta A_{i-1}}\mathcal{L}_{\Delta A_{i-1}}(x)$;
24: **End For**
25: Set $G_{\mathcal{L}_i\mathcal{H}_i} = \{(\mathcal{H}_i\mathcal{L}_i(x), \mathcal{L}_i(x)) \mid x \in U_i\} \cup \{(\mathcal{H}_i(a), \mathcal{L}_i\mathcal{H}_i(a)) \mid a \in A_i\}$;
26: $i \leftarrow i + 1$;
27: **End While**
28: **Return** $G_{\mathcal{L}_i\mathcal{H}_i}$.

---

According to Propositions 5 and 6, the computations in Steps 3–13 are to find the granular concepts of the object-oriented cognitive computing state $\mathcal{OS}_{\mathcal{L}_0\mathcal{H}_0}$, and those in Steps 14–24 are to find the granular concepts of the attribute-oriented cognitive computing state $\mathcal{AS}_{\mathcal{L}_i\mathcal{H}_i}$. By Proposition 7, we know that $G_{\mathcal{L}_i\mathcal{H}_i}$ obtained in Step 25 is the granular concepts of the cognitive computing state $\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}$. Consequently, $G_{\mathcal{L}_n\mathcal{H}_n}$ output by Algorithm 1 is the granular concepts of the input cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$.

In fact, from the point of view of granular computing, Algorithm 1 can be viewed as a transformation way from the information granules $G_{\mathcal{L}_1\mathcal{H}_1}$ to $G_{\mathcal{L}_n\mathcal{H}_n}$ by means of recursive approach.

Now, we analyze the time complexity of Algorithm 1. Suppose $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$ is the input cognitive computing system. Then, based on the discussion in Section 2, running Step 1 takes $O((|U_1| + |A_1|)|\ U_1||A_1|)$. Furthermore, we know that the time complexity of Steps 3–13 is $O((|U_i| + |A_i|)|\ U_i||A_i|)$ and so is that of Steps 14–24. Thus, running Steps 2–27 takes

$O(n(|U_n| + |A_n|)|\ U_n||A_n|)$, where $n$ is the number of cognitive computing states. To sum up, the time complexity of Algorithm 1 is $O(n(|U_n| + |A_n|)|\ U_n||A_n|)$ which is polynomial.

**Example 4.** In Example 1, four symptoms (*Fever*, *Cough*, *Headache*, *Difficulty breathing*) related to SARS have been found from the four patients. As time goes by, there will appear more patients (e.g., patients 5, 6, 7, 8 and 9) from whom additional symptoms (e.g., *Diarrhea*, *Muscle aches*, *Nausea and vomiting*) will be observed. We suppose the information updating on the patients and symptoms is shown in Table 2. Note that in the table, the values of patients 1, 2, 3 and 4 under the new symptoms *Diarrhea*, *Muscle aches*, and *Nausea and vomiting* being "No" are a very special assumption which cannot be applied to all situations since four original patients with the four original symptoms may have gone and could not be contacted when additional symptoms can be tested using new technologies, i.e., sometimes no chance of obtaining the values of the new symptoms from them.

Now we update the granular concepts obtained in Example 3. We denote the set of original patients in Table 1 by $U_1$ (i.e., $U_1 = \{1,2,3,4\}$), that of original symptoms by $A_1$ (i.e., $A_1 = \{a,b,c,d\}$), and the cognitive operators between $2^{U_1}$ and $2^{A_1}$ by $\mathcal{L}_1$ and $\mathcal{H}_1$ (see Example 1 for details). Then the information granules of $\mathcal{L}_1$ and $\mathcal{H}_1$ and their granular concepts $G_{\mathcal{L}_1\mathcal{H}_1}$ can be found in Example 3.

Besides, we denote the new patients by 5, 6, 7, 8, 9, and the new symptoms by $e,f$ and $g$, respectively. Let $U_2 = \{1,2,3,4,5,6,7,8,9\}$, $A_2 = \{a,b,c,d,e,f,g\}$, $\Delta U_1 = U_2 - U_1 = \{5,6,7,8,9\}$, $\Delta A_1 = A_2 - A_1 = \{e,f,g\}$. Similar to Example 1, by intuitive perception and attention, we can obtain the cognitive operators $\mathcal{L}_{\Delta U_1} : 2^{\Delta U_1} \to 2^{A_1}$, $\mathcal{H}_{\Delta U_1} : 2^{A_1} \to 2^{\Delta U_1}$ and $\mathcal{L}_{\Delta A_1} : 2^{U_2} \to 2^{\Delta A_1}$, $\mathcal{H}_{\Delta A_1} : 2^{\Delta A_1} \to 2^{U_2}$. The information granules of $\mathcal{L}_{\Delta U_1}$ and $\mathcal{H}_{\Delta U_1}$ are

$$\mathcal{L}^G_{\Delta U_1} = \{\{5\} \mapsto \{a,c\},\ \{6\} \mapsto \{b\},\ \{7\} \mapsto \{a,d\},\ \{8\} \mapsto \{b\}, \{9\} \mapsto \{c\}\},$$
$$\mathcal{H}^G_{\Delta U_1} = \{\{a\} \mapsto \{5,7\},\ \{b\} \mapsto \{6,8\},\ \{c\} \mapsto \{5,9\},\ \{d\} \mapsto \{7\}\},$$

and those of $\mathcal{L}_{\Delta A_1}$ and $\mathcal{H}_{\Delta A_1}$ are

$$\mathcal{L}^G_{\Delta A_1} = \{\{1\} \mapsto \emptyset,\ \{2\} \mapsto \emptyset,\ \{3\} \mapsto \emptyset,\ \{4\} \mapsto \emptyset,\ \{5\} \mapsto \{e\},\ \{6\} \mapsto \{f\},\ \{7\} \mapsto \{g\},\ \{8\} \mapsto \{f,g\},\ \{9\} \mapsto \{e\}\},$$

$$\mathcal{H}^G_{\Delta A_1} = \{\{e\} \mapsto \{5,9\},\ \{f\} \mapsto \{6,8\},\ \{g\} \mapsto \{7,8\}\}.$$

We denote by $\mathcal{L}_2$ and $\mathcal{H}_2$ the extended cognitive operators of $\mathcal{L}_1$ and $\mathcal{H}_1$ with the newly input information $\mathcal{L}_{\Delta U_1}$, $\mathcal{H}_{\Delta U_1}$ and $\mathcal{L}_{\Delta A_1}$, $\mathcal{H}_{\Delta A_1}$ (see Eqs. (12) and (13) for construction of $\mathcal{L}_2$ and $\mathcal{H}_2$). Then $\mathcal{S}_{\mathcal{L}_2\mathcal{H}_2} = (G_{\mathcal{L}_1\mathcal{H}_1}, \mathcal{L}_{\Delta U_1}, \mathcal{H}_{\Delta U_1}, \mathcal{L}_{\Delta A_1}, \mathcal{H}_{\Delta A_1})$ is a cognitive computing state which also forms a cognitive computing system $\mathcal{S}$ with $n = 2$.

Using Algorithm 1, we obtain the granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$ of the cognitive computing system $\mathcal{S}$ as follows:

$$(\{1\},\{a,b,d\}),\ (\{1,2\},\{b,d\}),\ (\{3,4,5,9\},\{c\}),\ (\{4,5\},\{a,c\}),\ (\{5\},\{a,c,e\}),\ (\{6,8\},\{b,f\}),\ (\{7\},\{a,d,g\}),$$
$$(\{8\},\{b,f,g\}),\ (\{5,9\},\{c,e\}),\ (\{1,4,5,7\},\{a\}),\ (\{1,2,6,8\},\{b\}),\ (\{1,2,7\},\{d\}),\ (\{7,8\},\{g\}).$$

## 5. Cognitive processes

In the real world, it is important to learn cognitive concept(s) based on the current ones when additional information is given to object set, attribute set or both of them. Solving this problem is often called cognitive process [38,45]. For instance, continued with Example 4, we suppose patients 3, 4 and 5 are children suffering from SARS. Then which symptoms characterize these children exactly in terms of suffering from SARS? Unfortunately, the current granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$ listed at the end of Example 4 cannot answer this question since no granular concept has the extent $X_1 = \{3,4,5\}$. In order to deal with this issue, we need to learn additional cognitive concept(s) according to the given object set $X_1 = \{3,4,5\}$ based on the obtained granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$. Similarly, it is also necessary to learn additional cognitive concept(s) when new information is given to an attribute set or a pair of object and attribute sets.

**Table 2**
A SARS dataset with information updating on patients and symptoms.

| Patient | Fever | Cough | Headache | Difficulty breathing | Diarrhea | Muscle aches | Nausea and vomiting |
|---|---|---|---|---|---|---|---|
| 1 | Yes | Yes | No | Yes | No | No | No |
| 2 | No | Yes | No | Yes | No | No | No |
| 3 | No | No | Yes | No | No | No | No |
| 4 | Yes | No | Yes | No | No | No | No |
| 5 | Yes | No | Yes | No | Yes | No | No |
| 6 | No | Yes | No | No | No | Yes | No |
| 7 | Yes | No | No | Yes | No | No | Yes |
| 8 | No | Yes | No | No | No | Yes | Yes |
| 9 | No | No | Yes | No | Yes | No | No |

In this section, using set approximations, we discuss cognitive processes to learn one exact or two approximate cognitive concepts from a given object set, attribute set or pair of object and attribute sets based on the granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system. Before embarking on this issue, we introduce some basic notions related to rough set theory (e.g., lower and upper approximations, rough set, etc) in order to explicitly show where the idea of our set approximations comes from.

Formally, an *information system* can be considered as a pair $I = \langle U, AT\rangle$ [23], where.

- $U$ is a non-empty finite set of objects, called the universe.
- $AT$ is a non-empty finite set of attributes such that for any $a \in AT$, $V_a$ is the domain of attribute $a$.

For any $x \in U$, we denote by $a(x)$ the value of $x$ under the attribute $a \in AT$. Given $A \subseteq AT$, an indiscernibility relation $IND(A)$ can be defined as:

$$IND(A) = \{(x,y) \in U \times U | a(x) = a(y) \text{ forall } a \in A\}.$$

It is easy to verify that the relation $IND(A)$ is reflexive, symmetric and transitive. In other words, $IND(A)$ forms an equivalence relation which can partition $U$ into equivalence classes $[x]_A = \{y \in U | (x,y) \in IND(A)\}$. We denote this partition of $U$ by $U/IND(A)$. That is, $U/IND(A) = \{[x]_A | x \in U\}$. Then one can derive lower and upper approximations of an arbitrary subset $X$ of $U$ which are respectively defined as:

$$\underline{A}(X) = \bigcup_{Y \in U/IND(A), Y \subseteq X} Y \quad \text{and} \quad \overline{A}(X) = \bigcup_{Y \in U/IND(A), Y \cap X \neq \emptyset} Y.$$

In rough set theory [23], the pair $[\underline{A}(X), \overline{A}(X)]$ is referred to as the rough set of $X$ with respect to the attribute set $A$.

Note that the idea of lower and upper approximations in rough set theory was further extended by Saquer and Deogun [32], Yao and Chen [51], and Zhang and Qiu [55] to Wille's concept lattice, and by Shao et al. [34] to fuzzy concept lattice.

### 5.1. Concept learning from an object set

In this subsection, we investigate the problem of learning one exact or two approximate cognitive concepts from a given object set by means of set approximations. Firstly, we put forward an approach to approximate an object set.

Let $G_{\mathcal{L}_n\mathcal{H}_n}$ be the granular concepts of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$ and $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ be the corresponding cognitive concept lattice. Then, based on the discussion in Section 3, we know that $G_{\mathcal{L}_n\mathcal{H}_n}$ is the set of basic information granules of $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$.

Motivated by the existing set approximation ideas [23,32,34,51,55], we respectively define the lower and upper approximations of an object set $X_0$ in the cognitive concept lattice $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ as follows:

$$\begin{aligned} \underline{\mathrm{Apr}}(X_0) &= \mathrm{extent}\left(\bigvee_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X\subseteq X_0} (X,B)\right), \\ \overline{\mathrm{Apr}}(X_0) &= \mathrm{extent}\left(\bigwedge_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X_0\subseteq X} (X,B)\right), \end{aligned} \tag{24}$$

where extent($\bullet$) denotes the extent of a cognitive concept.

That is, the lower approximation $\underline{\mathrm{Apr}}(X_0)$ is the extent of the supremum of the cognitive concepts which are specializations of $(\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0))$, and the upper approximation $\overline{\mathrm{Apr}}(X_0)$ is the extent of the infimum of the cognitive concepts which are generalizations of $(\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0))$.

According to Eqs. (9) and (24), the lower and upper approximations of $X_0$ can also be represented as:

$$\begin{aligned} \underline{\mathrm{Apr}}(X_0) &= \mathcal{H}_n\mathcal{L}_n\left(\bigcup_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X\subseteq X_0} X\right), \\ \overline{\mathrm{Apr}}(X_0) &= \bigcap_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X_0\subseteq X} X. \end{aligned} \tag{25}$$

Note that

$$\mathcal{H}_n\mathcal{L}_n\left(\bigcup_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X\subseteq X_0} X\right) \subseteq \mathcal{H}_n\mathcal{L}_n(X_0) \subseteq \bigcap_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X_0\subseteq X} X.$$

Then

$$(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0))) \preceq (\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0)) \preceq (\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))). \tag{26}$$

Thus, we consider $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ as the result of learning cognitive concepts from the given object set $X_0$ by means of set approximations. Moreover, we define the (concept) learning accuracy as

$$\alpha(X_0) = 1 - \frac{|\overline{\mathrm{Apr}}(X_0)| - |\underline{\mathrm{Apr}}(X_0)|}{|U_n|} \tag{27}$$

which is used to measure the accuracy of learning cognitive concepts from $X_0$. Obviously, the learning accuracy $\alpha(X_0)$ equals 1 if and only if $\underline{\mathrm{Apr}}(X_0) = \overline{\mathrm{Apr}}(X_0)$.

Note that by Algorithm 1, we only obtain the granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$. Although, according to Eq. (10), we are able to induce all cognitive concepts $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ using $G_{\mathcal{L}_n\mathcal{H}_n}$ and then determine the lower and upper approximations of $X_0$. However, in practice, it is generally hard to implement such an action since the nodes of $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ increase exponentially in the worst case. To solve this problem, we need the following proposition.

For convenience, we denote

$$G^*_{\mathcal{L}_n\mathcal{H}_n} = \begin{cases} G_{\mathcal{L}_n\mathcal{H}_n} \cup \{(U_n, \emptyset)\}, & \text{if } (U_n, \emptyset) \in \mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n), \\ G_{\mathcal{L}_n\mathcal{H}_n}, & \text{otherwise}, \end{cases}$$

$$G^\#_{\mathcal{L}_n\mathcal{H}_n} = \begin{cases} G_{\mathcal{L}_n\mathcal{H}_n} \cup \{(\emptyset, A_n)\}, & \text{if } (\emptyset, A_n) \in \mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n), \\ G_{\mathcal{L}_n\mathcal{H}_n}, & \text{otherwise}. \end{cases}$$

**Proposition 8.** *Let $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ be the cognitive concept lattice of the cognitive operators $\mathcal{L}_n$ and $\mathcal{H}_n$, and $G_{\mathcal{L}_n\mathcal{H}_n}$ be the corresponding granular concepts. Then for any $X_0 \subseteq U$, we have*

$$\begin{aligned} \underline{\mathrm{Apr}}(X_0) &= \mathrm{extent}\left(\bigvee_{(X,B)\in G^\#_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} (X,B)\right), \\ \overline{\mathrm{Apr}}(X_0) &= \mathrm{extent}\left(\bigwedge_{(X,B)\in G^*_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} (X,B)\right). \end{aligned} \tag{28}$$

**Proof.** We denote by $G^{\#T}_{\mathcal{L}_n\mathcal{H}_n}$ the complement of $G^\#_{\mathcal{L}_n\mathcal{H}_n}$ with respect to $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ and by $G^{*T}_{\mathcal{L}_n\mathcal{H}_n}$ the complement of $G^*_{\mathcal{L}_n\mathcal{H}_n}$ with respect to $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$. That is, $G^{\#T}_{\mathcal{L}_n\mathcal{H}_n} = \mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n) - G^\#_{\mathcal{L}_n\mathcal{H}_n}$ and $G^{*T}_{\mathcal{L}_n\mathcal{H}_n} = \mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n) - G^*_{\mathcal{L}_n\mathcal{H}_n}$. Then by Eq. (10), we obtain

$$\begin{aligned} \bigvee_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X\subseteq X_0} (X,B) &= \left(\bigvee_{(X,B)\in G^\#_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} (X,B)\right) \bigvee \left(\bigvee_{(X,B)\in G^{\#T}_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} (X,B)\right) \\ &= \left(\bigvee_{(X,B)\in G^\#_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} (X,B)\right) \bigvee \left(\bigvee_{(X,B)\in G^{\#T}_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} \left(\bigvee_{x\in X} (\mathcal{H}_n\mathcal{L}_n(x), \mathcal{L}_n(x))\right)\right) \\ &= \left(\bigvee_{(X,B)\in G^\#_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} (X,B)\right) \end{aligned}$$

and

$$\begin{aligned} \bigwedge_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), X_0\subseteq X} (X,B) &= \left(\bigwedge_{(X,B)\in G^*_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} (X,B)\right) \bigwedge \left(\bigwedge_{(X,B)\in G^{*T}_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} (X,B)\right) \\ &= \left(\bigwedge_{(X,B)\in G^*_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} (X,B)\right) \bigwedge \left(\bigwedge_{(X,B)\in G^{*T}_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} \left(\bigwedge_{a\in B} (\mathcal{H}_n(a), \mathcal{L}_n\mathcal{H}_n(a))\right)\right) \\ &= \left(\bigwedge_{(X,B)\in G^*_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} (X,B)\right). \end{aligned}$$

As a result, Eq. (28) is at hand. □

Proposition 8 says that the granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ (exactly $G^*_{\mathcal{L}_n\mathcal{H}_n}$ and $G^\#_{\mathcal{L}_n\mathcal{H}_n}$), which can easily be computed by Algorithm 1 from a cognitive computing system, are able to approximate an object set $X_0$ with the same result as $\underline{\mathfrak{B}}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$. More precisely,

$$\begin{aligned} \underline{\mathrm{Apr}}(X_0) &= \mathcal{H}_n\mathcal{L}_n\left(\bigcup_{(X,B)\in G^\#_{\mathcal{L}_n\mathcal{H}_n}, X\subseteq X_0} X\right), \\ \overline{\mathrm{Apr}}(X_0) &= \bigcap_{(X,B)\in G^*_{\mathcal{L}_n\mathcal{H}_n}, X_0\subseteq X} X. \end{aligned} \tag{29}$$

This allows us to learn cognitive concept(s) from an object set $X_0$ using the granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ instead of the cognitive concept lattice $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$.

**Proposition 9.** *Let* $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ *and* $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ *be the learning cognitive concepts from an object set* $X_0$ *based on* $G_{\mathcal{L}_n\mathcal{H}_n}$. *If* $\underline{\mathrm{Apr}}(X_0) = \overline{\mathrm{Apr}}(X_0)$, *there is only one learning cognitive concept from* $X_0$ *which is* $(\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0))$.

**Proof.** It is immediate from Eq. (26). □

In summary, for a given object set $X_0$, we learn an exact cognitive concept $(\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0))$ with the learning accuracy $\alpha(X_0) = 1$ if $\underline{\mathrm{Apr}}(X_0) = \overline{\mathrm{Apr}}(X_0)$; otherwise, we can only learn two approximate cognitive concepts $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ with the learning accuracy $\alpha(X_0) = 1 - \frac{|\overline{\mathrm{Apr}}(X_0)| - |\underline{\mathrm{Apr}}(X_0)|}{|U_n|}$. Algorithm 2 describes how to compute them.

**Algorithm 2.** Concept learning from an object set

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**Require:** The granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$ and an object set $X_0$.

**Ensure:** An exact or two approximate cognitive concepts with learning accuracy for $X_0$.

Initialize $\Pi_x = \emptyset, \Pi_a = \emptyset, \Omega_x = \emptyset, \Omega_a = \emptyset, \alpha(X_0) = 1, m = 1$ and label the elements of $G_{\mathcal{L}_n\mathcal{H}_n}$ as
$(\mathcal{H}_n\mathcal{L}_n(x_1), \mathcal{L}_n(x_1)),\ (\mathcal{H}_n\mathcal{L}_n(x_2), \mathcal{L}_n(x_2)), \ldots, (\mathcal{H}_n\mathcal{L}_n(x_s), \mathcal{L}_n(x_s)),$
$(\mathcal{H}_n(a_1), \mathcal{L}_n\mathcal{H}_n(a_1)),\ (\mathcal{H}_n(a_2), \mathcal{L}_n\mathcal{H}_n(a_2)), \ldots, (\mathcal{H}_n(a_t), \mathcal{L}_n\mathcal{H}_n(a_t));$

**For** each $i \in \{1, 2, \ldots, s\}$
  **If** $\mathcal{H}_n\mathcal{L}_n(x_i) \subseteq X_0$
    $\Pi_x \leftarrow \Pi_x \cup \{(\mathcal{H}_n\mathcal{L}_n(x_i), \mathcal{L}_n(x_i))\}$;
  **End If**
  **If** $X_0 \subseteq \mathcal{H}_n\mathcal{L}_n(x_i)$
    $\Omega_x \leftarrow \Omega_x \cup \{(\mathcal{H}_n\mathcal{L}_n(x_i), \mathcal{L}_n(x_i))\}$;
  **End If**
**End For**
**For** each $j \in \{1, 2, \ldots, t\}$
  **If** $\mathcal{H}_n(a_j) \subseteq X_0$
    $\Pi_a \leftarrow \Pi_a \cup \{(\mathcal{H}_n(a_j), \mathcal{L}_n\mathcal{H}_n(a_j))\}$;
  **End If**
  **If** $X_0 \subseteq \mathcal{H}_n(a_j)$
    $\Omega_a \leftarrow \Omega_a \cup \{(\mathcal{H}_n(a_j), \mathcal{L}_n\mathcal{H}_n(a_j))\}$;
  **End If**
**End For**
Let $\Pi = \Pi_x \cup \Pi_a, \Omega = \Omega_x \cup \Omega_a$, add $(\emptyset, A_n)$ into $\Pi$ when $(\emptyset, A_n)$ is a cognitive concept and $(U_n, \emptyset)$ into $\Omega$ when $(U_n, \emptyset)$ is a cognitive concept, and compute $\underline{\mathrm{Apr}}(X_0) = \mathcal{H}_n\mathcal{L}_n\left(\bigcup_{(X,B)\in\Pi} X\right)$ and $\overline{\mathrm{Apr}}(X_0) = \bigcap_{(X,B)\in\Omega} X$;
**If** $\underline{\mathrm{Apr}}(X_0) = \overline{\mathrm{Apr}}(X_0)$
  $B_0 = \bigcap_{x\in\underline{\mathrm{Apr}}(X_0)} \mathcal{L}_n(x)$;
**Else**
  reset $m = 2$ and compute $\underline{B}_0 = \bigcap_{x\in\underline{\mathrm{Apr}}(X_0)} \mathcal{L}_n(x)$, $\overline{B}_0 = \bigcap_{x\in\overline{\mathrm{Apr}}(X_0)} \mathcal{L}_n(x)$ and $\alpha(X_0) = 1 - \frac{|\overline{\mathrm{Apr}}(X_0)| - |\underline{\mathrm{Apr}}(X_0)|}{|U_n|}$;
**End If**
**Return** $(\underline{\mathrm{Apr}}(X_0), B_0)$ and $\alpha(X_0)$ when $m = 1$; otherwise, $(\underline{\mathrm{Apr}}(X_0), \underline{B}_0)$, $(\overline{\mathrm{Apr}}(X_0), \overline{B}_0)$ and $\alpha(X_0)$.

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According to Eq. (29), Steps 2–18 in Algorithm 2 are to compute the lower and upper approximations of the object set $X_0$. Furthermore, Steps 19–24 are to find an exact or two approximate cognitive concepts for $X_0$ as well as the learning accuracy $\alpha(X_0)$. What is more, it is easy to check that the time complexity of Algorithm 2 is $O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$.

**Example 5.** Continued with Example 4, we suppose patients 3, 4 and 5 are children suffering from SARS, while patients 5 and 9 are old people suffering from SARS. Then which symptoms characterize these children (old people) exactly in terms of suffering from SARS? In order to answer this question, we need to learn cognitive concept(s) from the given object sets $X_1 = \{3,4,5\}$ and $X_2 = \{5,9\}$ based on the obtained granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$ which can be found at the end of Example 4. By Eq. (29), we have

$$\underline{\mathrm{Apr}}(X_1) = \mathcal{H}_2\mathcal{L}_2\left(\bigcup_{(X,B)\in G^{\#}_{\mathcal{L}_2\mathcal{H}_2}, X\subseteq X_1} X\right) = \mathcal{H}_2\mathcal{L}_2(\emptyset \cup \{5\} \cup \{4,5\}) = \{4,5\},$$

$$\overline{\mathrm{Apr}}(X_1) = \bigcap_{(X,B)\in G^{*}_{\mathcal{L}_2\mathcal{H}_2}, X_1\subseteq X} X = \{3,4,5,9\} \cap U_2 = \{3,4,5,9\},$$

$$\underline{\mathrm{Apr}}(X_2) = \mathcal{H}_2\mathcal{L}_2\left(\bigcup_{(X,B)\in G^{\#}_{\mathcal{L}_2\mathcal{H}_2}, X\subseteq X_2} X\right) = \mathcal{H}_2\mathcal{L}_2(\emptyset \cup \{5\} \cup \{5,9\}) = \{5,9\},$$

$$\overline{\mathrm{Apr}}(X_2) = \bigcap_{(X,B)\in G^{*}_{\mathcal{L}_2\mathcal{H}_2}, X_2\subseteq X} X = \{5,9\} \cap \{3,4,5,9\} \cap U_2 = \{5,9\}.$$

Thus, we learn two approximate cognitive concepts $(\{4,5\},\{a,c\})$ and $(\{3,4,5,9\},\{c\})$ from $X_1$ with the learning accuracy $\alpha(X_1) = \frac{7}{9}$. Then we know that there do not exist some symptoms characterizing these children exactly in terms of suffering from SARS, but *fever* and *headache* can characterize them approximately and so can *headache*. Moreover, we learn an exact cognitive concept $(\{5,9\},\{c,e\})$ from $X_2$ with $\alpha(X_2) = 1$, which means that *headache* and *diarrhea* can characterize the old people exactly in terms of suffering from SARS.

### *5.2. Concept learning from an attribute set*

In this subsection, we investigate the problem of learning one exact or two approximate cognitive concepts from a given attribute set by means of set approximations. Similar to the case in Section 5.1, we first propose an approach to approximate an attribute set in preparation for concept learning here.

We define the lower and upper approximations of an attribute set $B_0$ in $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ as follows:

$$\begin{aligned}\underline{\mathrm{Apr}}(B_0) &= \mathrm{intent}\left(\bigvee_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), B_0\subseteq B} (X,B)\right),\\ \overline{\mathrm{Apr}}(B_0) &= \mathrm{intent}\left(\bigwedge_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), B\subseteq B_0} (X,B)\right),\end{aligned} \tag{30}$$

where intent($\bullet$) denotes the intent of a cognitive concept.

That is, the lower approximation $\underline{\mathrm{Apr}}(B_0)$ is the intent of the supremum of the cognitive concepts which are specializations of $(\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0))$, and the upper approximation $\overline{\mathrm{Apr}}(B_0)$ is the intent of the infimum of the cognitive concepts which are generalizations of $(\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0))$.

According to Eqs. (9) and (30), the lower and upper approximations of an attribute set $B_0$ can also be represented as:

$$\begin{aligned}\underline{\mathrm{Apr}}(B_0) &= \bigcap_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), B_0\subseteq B} B,\\ \overline{\mathrm{Apr}}(B_0) &= \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), B\subseteq B_0} B\right).\end{aligned} \tag{31}$$

Note that

$$\mathcal{L}_n\mathcal{H}_n\left(\bigcup_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), B\subseteq B_0} B\right) \subseteq \mathcal{L}_n\mathcal{H}_n(B_0) \subseteq \bigcap_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n), B_0\subseteq B} B.$$

Then

$$(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)) \preceq (\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0)) \preceq (\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)). \tag{32}$$

Thus, we consider $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ as the result of learning cognitive concepts from the attribute set $B_0$ by means of set approximations. Moreover, we define the (concept) learning accuracy as

$$\beta(B_0) = 1 - \frac{|\underline{\mathrm{Apr}}(B_0)| - |\overline{\mathrm{Apr}}(B_0)|}{|A_n|}, \tag{33}$$

which is used to measure the accuracy of learning cognitive concepts from $B_0$. Obviously, the learning accuracy $\beta(B_0)$ equals 1 if and only if $\underline{\mathrm{Apr}}(B_0) = \overline{\mathrm{Apr}}(B_0)$.

Similarly, the granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system are able to approximate an attribute set with the same result as the cognitive concept lattice $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$. More precisely, we have the following proposition.

**Proposition 10.** *Let $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ be the cognitive concept lattice of the cognitive operators $\mathcal{L}_n$ and $\mathcal{H}_n$, and $G_{\mathcal{L}_n\mathcal{H}_n}$ be the corresponding granular concepts. Then for any $B_0 \subseteq A$, we have*

$$\begin{aligned} \underline{\mathrm{Apr}}(B_0) &= \mathrm{intent}\left(\bigvee_{(X,B)\in G^{\#}_{\mathcal{L}_n\mathcal{H}_n}, B_0\subseteq B} (X,B)\right), \\ \overline{\mathrm{Apr}}(B_0) &= \mathrm{intent}\left(\bigwedge_{(X,B)\in G^{*}_{\mathcal{L}_n\mathcal{H}_n}, B\subseteq B_0} (X,B)\right). \end{aligned} \tag{34}$$

*That is,*

$$\begin{aligned} \underline{\mathrm{Apr}}(B_0) &= \bigcap_{(X,B)\in G^{\#}_{\mathcal{L}_n\mathcal{H}_n}, B_0\subseteq B} B, \\ \overline{\mathrm{Apr}}(B_0) &= \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{(X,B)\in G^{*}_{\mathcal{L}_n\mathcal{H}_n}, B\subseteq B_0} B\right). \end{aligned} \tag{35}$$

**Proposition 11.** *Let $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ be the learning cognitive concepts from an attribute set $B_0$ based on $G_{\mathcal{L}_n\mathcal{H}_n}$. If $\underline{\mathrm{Apr}}(B_0) = \overline{\mathrm{Apr}}(B_0)$, there is only one learning cognitive concept from $B_0$ which is $(\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0))$.*

**Proof.** It is immediate from Eq. (32). □

In summary, for a given attribute set $B_0$, we learn one exact cognitive concept $(\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0))$ with the learning accuracy $\beta(B_0) = 1$ if $\underline{\mathrm{Apr}}(B_0) = \overline{\mathrm{Apr}}(B_0)$; otherwise, we can only learn two approximate cognitive concepts $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ with $\beta(B_0) = 1 - \frac{|\underline{\mathrm{Apr}}(B_0)| - |\overline{\mathrm{Apr}}(B_0)|}{|A_n|}$. Algorithm 3 describes how to compute them.

**Algorithm 3.** Concept learning from an attribute set

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**Require:** The granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n}\{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$ and an attribute set $B_0$.

**Ensure:** An exact or two approximate cognitive concepts with learning accuracy for $B_0$.

Initialize $\Pi_x = \emptyset, \Pi_a = \emptyset$, $\Omega_x = \emptyset, \Omega_a = \emptyset, \beta(B_0) = 1, m = 1$ and label the elements of $G_{\mathcal{L}_n\mathcal{H}_n}$ as
  $(\mathcal{H}_n\mathcal{L}_n(x_1), \mathcal{L}_n(x_1))$, $\mathcal{H}_n\mathcal{L}_n(x_2), \mathcal{L}_n(x_2)), \ldots, (\mathcal{H}_n\mathcal{L}_n(x_s), \mathcal{L}_n(x_s))$,
  $(\mathcal{H}_n(a_1), \mathcal{L}_n\mathcal{H}_n(a_1))$, $(\mathcal{H}_n(a_2), \mathcal{L}_n\mathcal{H}_n(a_2)), \ldots, (\mathcal{H}_n(a_t), \mathcal{L}_t\mathcal{H}_n(a_t))$;
**For** each $i \in \{1, 2, \ldots, s\}$
  **If** $B_0 \subseteq \mathcal{L}_n(x_i)$
    $\Pi_x \leftarrow \Pi_x \cup \{(\mathcal{H}_n\mathcal{L}_n(x_i), \mathcal{L}_n(x_i))\}$;
  **End If**
  **If** $\mathcal{L}_n(x_i) \subseteq B_0$
    $\Omega_x \leftarrow \Omega_x \cup \{(\mathcal{H}_n\mathcal{L}_n(x_i), \mathcal{L}_n(x_i))\}$;
  **End If**
**End For**
**For** each $j \in \{1, 2, \ldots, t\}$

(*continued on next page*)

11: **If** $B_0 \subseteq \mathcal{L}_n\mathcal{H}_n(a_j)$
12: $\Pi_a \leftarrow \Pi_a \cup \{(\mathcal{H}_n(a_j), \mathcal{L}_n\mathcal{H}_n(a_j))\}$;
13: **End If**
14: **If** $\mathcal{L}_n\mathcal{H}_n(a_j) \subseteq B_0$
15: $\Omega_a \leftarrow \Omega_a \cup \{(\mathcal{H}_n(a_j), \mathcal{L}_n\mathcal{H}_n(a_j))\}$;
16: **End If**
17: **End For**
18: Let $\Pi = \Pi_a \cup \Pi_x, \Omega = \Omega_a \cup \Omega_x$, add $(\emptyset, A_n)$ into $\Pi$ when $(\emptyset, A_n)$ is a cognitive concept and $(U_n, \emptyset)$ into $\Omega$ when $(U_n, \emptyset)$ is a cognitive concept, and compute $\underline{\mathrm{Apr}}(B_0) = \bigcap_{(X,B)\in\Pi} B$ and $\overline{\mathrm{Apr}}(B_0) = \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{(X,B)\in\Omega} B\right)$;
19: **If** $\underline{\mathrm{Apr}}(B_0) = \overline{\mathrm{Apr}}(B_0)$
20: $X_0 = \bigcap_{a\in\underline{\mathrm{Apr}}(B_0)} \mathcal{H}_n(a)$;
21: **Else**
22: reset $m = 2$ and compute $\underline{X}_0 = \bigcap_{a\in\underline{\mathrm{Apr}}(B_0)} \mathcal{H}_n(a)$, $\overline{X}_0 = \bigcap_{a\in\overline{\mathrm{Apr}}(B_0)} \mathcal{H}_n(a)$ and $\beta(B_0) = 1 - \frac{|\underline{\mathrm{Apr}}(B_0)|-|\overline{\mathrm{Apr}}(B_0)|}{|A_n|}$;
23: **End If**
24: **Return** $(X_0, \underline{\mathrm{Apr}}(B_0))$ and $\beta(B_0)$ when $m = 1$; otherwise, $(\underline{X}_0, \underline{\mathrm{Apr}}(B_0))$, $(\overline{X}_0, \overline{\mathrm{Apr}}(B_0))$ and $\beta(B_0)$.

According to Eq. (35), Steps 2–18 in Algorithm 3 are to compute the lower and upper approximations of the attribute set $B_0$. Moreover, Steps 19–24 are to learn one exact or two approximate cognitive concepts from $B_0$ as well as the learning accuracy $\beta(B_0)$. Moreover, it is easy to verify that the time complexity of Algorithm 3 is $O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$.

**Example 6.** Continued with Example 4, we suppose *fever* and *diarrhea* receive more attention from doctors and so do *headache* and *diarrhea*. Then which patients as a whole suffer and only suffer from *fever* and *diarrhea* (*headache* and *diarrhea*)? In order to answer this question, we need to learn cognitive concept(s) from the given attribute sets $B_1 = \{a, e\}$ and $B_2 = \{c, e\}$ based on the obtained granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$ which can be found at the end of Example 4. By Eq. (35), we have

$$\underline{\mathrm{Apr}}(B_1) = \bigcap_{(X,B)\in G^{\#}_{\mathcal{L}_2\mathcal{H}_2}, B_1\subseteq B} B = \{a, c, e\} \cap A_2 = \{a, c, e\},$$

$$\overline{\mathrm{Apr}}(B_1) = \mathcal{L}_2\mathcal{H}_2\left(\bigcup_{(X,B)\in G^{*}_{\mathcal{L}_2\mathcal{H}_2}, B\subseteq B_1} B\right) = \mathcal{L}_2\mathcal{H}_2(\emptyset \cup \{a\}) = \{a\},$$

$$\underline{\mathrm{Apr}}(B_2) = \bigcap_{(X,B)\in G^{\#}_{\mathcal{L}_2\mathcal{H}_2}, B_2\subseteq B} B = \{c, e\} \cap \{a, c, e\} \cap A_2 = \{c, e\},$$

$$\overline{\mathrm{Apr}}(B_2) = \mathcal{L}_2\mathcal{H}_2\left(\bigcup_{(X,B)\in G^{*}_{\mathcal{L}_2\mathcal{H}_2}, B\subseteq B_2} B\right) = \mathcal{L}_2\mathcal{H}_2(\{c\} \cup \{c, e\}) = \{c, e\}.$$

Thus, we learn two approximate cognitive concepts $(\{5\}, \{a, c, e\})$ and $(\{1, 4, 5, 7\}, \{a\})$ from $B_1$ with $\beta(B_1) = \frac{5}{7}$. Then we know that there do not exist some patients as a whole suffering and only suffering from *fever* and *diarrhea*, but patient 5 suffers from *fever*, *headache* and *diarrhea*, and patients 1, 4, 5, 7 as a whole suffer from *fever*. Moreover, we learn an exact cognitive concept $(\{5, 9\}, \{c, e\})$ from $B_2$ with $\beta(B_2) = 1$, which means that patients 5 and 9 as a whole suffer and only suffer from *headache* and *diarrhea*.

### *5.3. Concept learning from a pair of object and attribute sets*

In this subsection, we study the problem of learning one exact or two approximate cognitive concepts from a pair of object and attribute sets by means of set approximations.

For a pair $(X_0, B_0)$ of object and attribute sets, concept learning is quite different from that of an object set $X_0$ or attribute set $B_0$. More precisely, for a given object set $X_0$, we can consider it as $(X_0, \mathcal{L}_n(X_0))$ in which $\mathcal{L}_n(X_0)$ is an intent induced by $X_0$. Similarly, for a given attribute set $B_0$, we can consider it as $(\mathcal{H}_n(B_0), B_0)$ in which $\mathcal{H}_n(B_0)$ is an extent induced by $B_0$. However, for a given pair $(X_0, B_0)$, it is not known whether $B_0$ is an intent associated to $X_0$, or $X_0$ is an extent associated to $B_0$. Even sometimes, $X_0$ and $B_0$ are less associated with each other with respect to the extent-intent relationship, which means that in this case it may not be reasonable to learn cognitive concept(s) from the pair $(X_0, B_0)$. Motivated by this problem, we put forward the notion of a concept-inducible pair of object and attribute sets.

**Definition 9.** Let $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ be the cognitive concept lattice of the cognitive operators $\mathcal{L}_n$ and $\mathcal{H}_n$, and $G_{\mathcal{L}_n\mathcal{H}_n}$ be the corresponding granular concepts. For any $X_0 \subseteq U_n$ and $B_0 \subseteq A_n$, if $\underline{\mathrm{Apr}}(X_0) \subseteq \mathcal{H}_n(B_0) \subseteq \overline{\mathrm{Apr}}(X_0)$ and $\overline{\mathrm{Apr}}(B_0) \subseteq \mathcal{L}_n(X_0) \subseteq \underline{\mathrm{Apr}}(B_0)$, then $(X_0, B_0)$ is said to be concept-inducible; otherwise, it is said to be concept-uninducible.

**Example 7.** Continued with Examples 4–6, it can be known from Definition 9 that the pair $(\{3,4,5\}, \{a,e\})$ is concept-uninducible since $\underline{\mathrm{Apr}}(\{3,4,5\}) \subseteq \mathcal{H}_n(\{a,e\}) \subseteq \overline{\mathrm{Apr}}(\{3,4,5\})$ does not hold due to $\underline{\mathrm{Apr}}(\{3,4,5\}) = \{4,5\}$, $\mathcal{H}_n(\{a,e\}) = \{5\}$ and $\overline{\mathrm{Apr}}(\{3,4,5\}) = \{3,4,5,9\}$.

In what follows, we only focus on concept-inducible pairs for concept learning since concept-uninducible ones mean that the object set $X_0$ and the attribute set $B_0$ are less associated with each other with respect to the extent-intent relationship.

**Proposition 12.** *Let $\mathfrak{B}(U_n, A_n, \mathcal{L}_n, \mathcal{H}_n)$ be the cognitive concept lattice of the cognitive operators $\mathcal{L}_n$ and $\mathcal{H}_n$, and $G_{\mathcal{L}_n\mathcal{H}_n}$ be the corresponding granular concepts. Then, for any $X_0 \subseteq U_n$ and $B_0 \subseteq A_n$, $(X_0, \mathcal{L}_n(X_0))$ and $(\mathcal{H}_n(B_0), B_0)$ are concept-inducible pairs.*

**Proof.** Firstly, we prove that the pair $(X_0, \mathcal{L}_n(X_0))$ is concept-inducible. On one hand, by Eq. (26), we have $\underline{\mathrm{Apr}}(X_0) \subseteq \mathcal{H}_n(\mathcal{L}_n(X_0)) \subseteq \overline{\mathrm{Apr}}(X_0)$. On the other hand, by Eq. (31), we obtain

$$\underline{\mathrm{Apr}}(\mathcal{L}_n(X_0)) = \bigcap_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n),\mathcal{L}_n(X_0)\subseteq B} B,$$

$$\overline{\mathrm{Apr}}(\mathcal{L}_n(X_0)) = \mathcal{L}_n\mathcal{H}_n\left(\bigcup_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n),B\subseteq\mathcal{L}_n(X_0)} B\right).$$

Note that

$$\mathcal{L}_n\mathcal{H}_n\left(\bigcup_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n),B\subseteq\mathcal{L}_n(X_0)} B\right) \subseteq \mathcal{L}_n(X_0) \subseteq \bigcap_{(X,B)\in\mathfrak{B}(U_n,A_n,\mathcal{L}_n,\mathcal{H}_n),\mathcal{L}_n(X_0)\subseteq B} B.$$

Then it follows $\overline{\mathrm{Apr}}(\mathcal{L}_n(X_0)) \subseteq \mathcal{L}_n(X_0) \subseteq \underline{\mathrm{Apr}}(\mathcal{L}_n(X_0))$. To sum up, we conclude that $(X_0, \mathcal{L}_n(X_0))$ is concept-inducible.

In a similar manner, we can prove that the pair $(\mathcal{H}_n(B_0), B_0)$ is also concept-inducible. □

Combining Proposition 12 with the discussion in the front of Section 5.3, we know that it is concept-inducible for the case of an object set or attribute set. This is why we directly implement concept learning in Sections 5.1 and 5.2.

In what follows, we discuss how to learn cognitive concept(s) from a concept-inducible pair of object and attribute sets.

**Proposition 13.** *For a concept-inducible pair $(X_0, B_0)$, let $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ be the learning cognitive concepts from $X_0$, and $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ be the learning cognitive concepts from $B_0$. Then*

$$(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)) \preceq \lambda(X_0) \preceq (\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)) \tag{36}$$

*and*

$$(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0))) \preceq \mu(B_0) \preceq (\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))), \tag{37}$$

*where $\lambda(X_0) = (\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0))$ and $\mu(B_0) = (\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0))$.*

**Proof.** It follows directly from Definition 9. □

According to Eqs. (26), (32), (36) and (37), we have the following proposition.

**Proposition 14.** *For a concept-inducible pair $(X_0, B_0)$, let $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ be the learning cognitive concepts from $X_0$, and $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ be the learning cognitive concepts from $B_0$. Then*

$$(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0))) \vee (\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)) \preceq \lambda(X_0) \preceq (\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))) \wedge (\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)) \tag{38}$$

*and*

$$(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0))) \vee (\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)) \preceq \mu(B_0) \preceq (\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))) \wedge (\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)), \tag{39}$$

*where $\lambda(X_0) = (\mathcal{H}_n\mathcal{L}_n(X_0), \mathcal{L}_n(X_0))$ and $\mu(B_0) = (\mathcal{H}_n(B_0), \mathcal{L}_n\mathcal{H}_n(B_0))$.*

Based on the above discussion, if a pair $(X_0, B_0)$ is concept-inducible, we consider

$$(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0))) \bigvee (\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0)) \tag{40}$$

and

$$(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))) \bigwedge (\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0)) \tag{41}$$

as the learning cognitive concepts from $(X_0, B_0)$ by means of set approximations. Note that the formulas in Eqs. (40) and (41) are equal to $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$, respectively. Thus, we define the (concept) learning accuracy as

$$\gamma(X_0, B_0) = 1 - \left( \frac{|\overline{\mathrm{Apr}}(X_0)| - |\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))|}{2|U_n|} + \frac{|\underline{\mathrm{Apr}}(B_0)| - |\mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))|}{2|A_n|} \right), \tag{42}$$

which is used to measure the accuracy of learning cognitive concepts from $(X_0, B_0)$. Obviously, the learning accuracy $\gamma(X_0, B_0)$ equals 1 if and only if $\overline{\mathrm{Apr}}(X_0) = \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))$ and $\underline{\mathrm{Apr}}(B_0) = \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))$.

**Proposition 15.** *For a concept-inducible pair $(X_0, B_0)$, let $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ be the learning cognitive concepts from $X_0$, and $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ be the learning cognitive concepts from $B_0$. If $\overline{\mathrm{Apr}}(X_0) = \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))$ and $\underline{\mathrm{Apr}}(B_0) = \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))$, there is only one learning cognitive concept from $(X_0, B_0)$ which is $(\mathcal{H}_n(B_0), \mathcal{L}_n(X_0))$.*

**Proof.** It is immediate from Eqs. (38) and (39). □

In particular, if the concept-inducible pair $(X_0, B_0)$ is a cognitive concept, it is easy to verify that its learning cognitive concept is just itself.

In summary, for a given concept-inducible pair $(X_0, B_0)$, we learn an exact cognitive concept $(\mathcal{H}_n(B_0), \mathcal{L}_n(X_0))$ with the learning accuracy $\gamma(X_0, B_0) = 1$ if $\overline{\mathrm{Apr}}(X_0) = \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))$ and $\underline{\mathrm{Apr}}(B_0) = \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))$; otherwise, we can only learn two approximate cognitive concepts (see Eqs. (40) and (41) for details) with the learning accuracy $\gamma(X_0, B_0) < 1$ (see Eq. (42) for details).

**Algorithm 4.** Concept learning from a pair of object and attribute sets.

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**Require:** The granular concepts $G_{\mathcal{L}_n\mathcal{H}_n}$ of a cognitive computing system $\mathcal{S} = \bigcup_{i=2}^{n} \{\mathcal{S}_{\mathcal{L}_i\mathcal{H}_i}\}$ and the pair $(X_0, B_0)$.

**Ensure:** An exact or two approximate cognitive concepts with learning accuracy for concept-inducible pair $(X_0, B_0)$.

Initialize $m = 0$;

Call Algorithm 2 to learn the cognitive concepts $(\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0)))$ and $(\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0)))$ from $X_0$, and Algorithm 3 to learn the cognitive concepts $(\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$ and $(\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$ from $B_0$;

**If** $\underline{\mathrm{Apr}}(X_0) \subseteq \bigcap_{a \in B_0} \mathcal{H}_n(a) \subseteq \overline{\mathrm{Apr}}(X_0)$ or $\overline{\mathrm{Apr}}(B_0) \subseteq \bigcap_{x \in X_0} \mathcal{L}_n(x) \subseteq \underline{\mathrm{Apr}}(B_0)$ does not hold

  **Return** "the pair $(X_0, B_0)$ is concept-uninducible";

**Else**

  **If** $\overline{\mathrm{Apr}}(X_0) = \mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))$ and $\underline{\mathrm{Apr}}(B_0) = \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))$

    **Return** $(\mathcal{H}_n(B_0), \mathcal{L}_n(X_0))$ and $\gamma(X_0, B_0) = 1$;

**Else**

  do

$$(X_1, B_1) \leftarrow (\underline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\underline{\mathrm{Apr}}(X_0))) \vee (\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0)), \underline{\mathrm{Apr}}(B_0))$$
$$(X_2, B_2) \leftarrow (\overline{\mathrm{Apr}}(X_0), \mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))) \wedge (\mathcal{H}_n(\overline{\mathrm{Apr}}(B_0)), \overline{\mathrm{Apr}}(B_0))$$
$$\gamma(X_0, B_0) = 1 - \left( \frac{|\overline{\mathrm{Apr}}(X_0)| - |\mathcal{H}_n(\underline{\mathrm{Apr}}(B_0))|}{2|U_n|} + \frac{|\underline{\mathrm{Apr}}(B_0)| - |\mathcal{L}_n(\overline{\mathrm{Apr}}(X_0))|}{2|A_n|} \right)$$

  **Return** $(X_1, B_1), (X_2, B_2)$ and $\gamma(X_0, B_0)$;

  **End If**

**End If**

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It can be known from Definition 9 and Eqs. (40) and (41) that Algorithm 4 is designed to learn an exact or two approximate cognitive concepts (if any) from a pair of object and attribute sets. Moreover, it is easy to verify that the time complexity of Algorithm 4 is $O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$.

**Example 8.** Continued with Example 5, where patients 3, 4 and 5 were supposed to be children suffering from SARS. Moreover, it can be seen from Table 2 that these children have the common symptom *headache*. Then, can these children and the symptom *headache* characterize each other in terms of suffering from SARS? In order to answer this question, we need to learn cognitive concept(s) from the pair $(X_0, B_0)$ with $X_0 = \{3,4,5\}$ and $B_0 = \{c\}$ based on the obtained granular concepts $G_{\mathcal{L}_2\mathcal{H}_2}$ which can be found at the end of Example 4.

It is easy to verify that $\underline{\text{Apr}}(X_0) \subseteq \mathcal{H}_2(B_0) \subseteq \overline{\text{Apr}}(X_0)$ and $\overline{\text{Apr}}(B_0) \subseteq \mathcal{L}_2(X_0) \subseteq \underline{\text{Apr}}(B_0)$ can be satisfied simultaneously. By Definition 9, $(X_0, B_0)$ is concept-inducible. Furthermore, according to Algorithm 4, we learn an exact cognitive concept $(\{3,4,5,9\},\{c\})$ from $(X_0, B_0)$ with the learning accuracy $\gamma(X_0, B_0) = 1$. Thus, the children (i.e., patients 3, 4, 5) and the symptom *headache* cannot characterize each other in terms of suffering from SARS, but patients 3, 4, 5, 9 suffer and only suffer from *headache* and in the meanwhile the symptom *headache* characterizes patients 3, 4, 5 and 9 exactly.

## 6. Discussion

In this section, we discuss the differences and relations between the proposed method and the existing ones on concept learning, and give explanations on some obtained results in this paper.

Firstly, we analyze the differences from the following three aspects: (a) cognitive mechanism, (b) cognitive computing system, and (c) cognitive process.

- The existing literature mainly focused on concept learning via concept systems (e.g., [18,19,31,45]) which were established by axiomatic ways, regardless of the analysis on cognitive mechanism. However, the current study has carefully analyzed the cognitive mechanism of forming concepts based on the principles from philosophy and cognitive psychology, and then naturally found the constraints for cognitive operators to better simulate intelligence behaviors of the brain including perception, attention and learning.
- The existing work put forward cognitive computing systems (e.g., [18,19,31,45]) by means of different axiomatic ways, which cannot integrate past experiences into itself to deal with e.g., dynamic data. However, the current study has proposed such a cognitive computing system that is composed of a series of cognitive computing states and is able to integrate past experiences into itself through recursive thinking. In addition, granular computing has also been integrated into the proposed cognitive computing system to decrease the computation time sharply.
- The existing researches on cognitive process mainly investigated the problem of learning concept(s) from a given pair of object and attribute sets by means of iterative algorithms [31,45], regardless of discussing whether the pair of object and attribute sets is concept-inducible or not, let alone the learning accuracy. However, the current paper has put forward the notion of a concept-inducible pair of object and attribute sets and a simple way of learning concept(s) from the concept-inducible pair via set approximations. What is more, how to measure the concept learning accuracy has been studied as well.

Secondly, we analyze the relations between the proposed method and the existing ones from the following two aspects: (1) cognitive computing, and (2) application.

- The existing work (e.g., [18,19,31,45]) showed to some extent pieces of the idea of cognitive computing in studying concept learning. For example, the importance of applying cognitive viewpoint to concept learning was mentioned in [45]. Learning concept(s) from a given pair of object and attribute sets by iterative algorithm was considered as a natural reflection of cognition. Motivated by these work, in this paper we have explicitly applied the idea of cognitive computing to concept learning based on granular computing, including a detailed discussion of cognitive mechanism, cognitive computing system and cognitive process.
- Both the proposed method and the existing ones allow users to specify the operators in the cognitive computing system. In other words, users can remould the operators according to their certain requirements of data analysis before learning concepts through the cognitive computing system.

Finally, we give explanations on some obtained results in this paper.

- Some basic propositions (e.g., Propositions 1–4) of the proposed cognitive operators $\mathcal{L}$ and $\mathcal{H}$ are similar to those obtained by Zhang Wenxiu's research group [18,19,31] in formal concept analysis since $\mathcal{L}$ and $\mathcal{H}$ here form Galois connection as well. However, the difference is that the proposed cognitive operators are characterized by the axioms in Definition 1 which are from the principles in philosophy and cognitive psychology.
- Since the time complexities of Algorithms 1–4 are $O(n(|U_n| + |A_n|)|U_n||A_n|), O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$, $O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$ and $O(|U_n|^2 + |U_n||A_n| + |A_n|^2)$, respectively, the proposed concept learning method can be completed in polynomial time. However, the classical algorithms in [10] take exponential time in the worst case for concept learning. So, granular computing integrated into cognitive concept lattice can indeed improve the efficiency of concept learning.

## 7. Conclusions

In order to improve efficiency and flexibility of concept learning, this paper mainly focuses on concept learning via granular computing from the viewpoint of cognitive computing. To be more concrete, cognitive mechanism of forming concepts has been analyzed based on the principles from philosophy and cognitive psychology. Then granular computing has been combined with the cognitive concept lattice to improve the efficiency of concept learning. What is more, we have put forward a cognitive computing system which is the initial environment to learn composite concepts and can integrate past experiences into itself for enhancing the flexibility of concept learning. In addition, cognitive processes have also been investigated to deal with the problem of learning one exact or two approximate cognitive concepts from a given object set, attribute set or pair of object and attribute sets. The obtained results in this paper may be beneficial to simulating intelligence behaviors of the brain including perception, attention and learning, but it still needs to be further confirmed.

In fact, integrating the idea of cognitive computing into concept learning is a promising and challenging research direction. Although in this paper we have discussed it from the aspects of cognitive mechanism, cognitive computing system and cognitive process, an in-depth study of this issue still needs to be made in our future work. For instance, (i) how to deal with the problem of learning concept(s) from a concept-uninducible pair because in this paper we are only interested in concept-inducible pair in terms of concept learning from a pair of object and attribute sets; (ii) large databases in the real world should be conducted to show the effectiveness of the proposed concept learning method; (iii) motivated by the logical reasoning work [36] in rough set theory, logical reasoning should also be integrated (if possible) into the proposed concept learning method for better simulating intelligence behaviors of the brain including perception, attention and learning.

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