Journal of Intelligent & Fuzzy Systems xx (20xx) x–xx
DOI:10.3233/IFS-130818
IOS Press

Multi-granulation fuzzy rough sets

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Abstract. Based on analysis of Pawlak's rough set model in the view of single equivalence relation and the theory of fuzzy set, associated with multi-granulation rough set models proposed by Qian, two types of new rough set models are constructed, which are multi-granulation fuzzy rough sets. It follows the research on the properties of the lower and upper approximations of the new multi-granulation fuzzy rough set models. Then it can be found that the Pawlak rough set model, fuzzy rough set model and multi-granulation rough set models are special cases of the new one from the perspective of the considered concepts and granular computing. The notion of rough measure and $(\alpha,\ \beta)$-rough measure which are used to measure uncertainty in multi-granulation fuzzy rough sets are introduced and some basic properties of the measures are examined. The construction of the multi-granulation fuzzy rough set model is a meaningful contribution in the view of the generalization of the classical rough set model.

Keywords: Approximation operators, fuzzy rough set, multi-granulation, rough measure

1. Introduction

Rough set theory, proposed by Pawlak [15–17], has become a well-established mechanism for uncertainty management in a wide variety of applications related to artificial intelligence [3, 4, 12]. The theory has been applied successfully in the fields of pattern recognition, medical diagnosis, data mining, conflict analysis, algebra [1, 18, 24], which are related to an amount of imprecise, vague and uncertain information. In recent years, the rough set theory has generated a great deal of interest among more and more researchers. The generalization of the rough set model is one of the most important research directions.

On the one hand, rough set theory is generalized by combining with other theories that deal with uncertain knowledge such as fuzzy set. It has been acknowledged by different studies that fuzzy set theory and rough set theory are complementary in terms of handling different kinds of uncertainty. The fuzzy set theory deals with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences [5]. Rough sets, in turn, deal with uncertainty following from ambiguity of information [15, 16]. The two types of uncertainty can be encountered together in real-life problems. For this reason, many approaches have been proposed to combine fuzzy set theory with rough set theory. Dubois and Prade proposed concepts of rough fuzzy sets and fuzzy rough sets based on approximations of fuzzy sets by crisp approximations spaces, and crisp sets by fuzzy approximation spaces, respectively [6]. A fuzzy rough set is a pair of fuzzy sets resulting from the approximation of a fuzzy set in a crisp approximation space, and a rough fuzzy set is a pair of fuzzy sets resulting from the approximation of a crisp set in a fuzzy approximation space. Besides, some other researches about fuzzy rough set and rough fuzzy set from other directions have been discussed [2, 7–9, 13, 23, 25, 26, 32, 35, 36].

On the other hand, rough set theory was discussed with the point view of granular computing. Information granules refer to pieces, classes and groups divided in accordance with characteristics and performances of complex information in the process of human understanding, reasoning and decision-making. Zadeh firstly

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proposed the concept of granular computing and discussed issues of fuzzy information granulation in 1979 [39]. Then the basic idea of information granulation has been applied to many fields including rough set [15, 16]. In 1985, Hobbs proposed the concept of granularity [10]. And granular computing played a more and more important role gradually in soft computing, knowledge discovery, data mining and many excellent results were achieved [14, 21, 22, 27–31, 33, 34, 37, 38]. In the point view of granulation computing, the classical Pawlak rough set is based on a single granulation induced from an indiscernibility relation. And an equivalence relation on the universe can be regarded as a granulation. For convenience, single granulation fuzzy rough set, denoted by SGFRS. This approach to describing a concept is mainly based on the following assumption:

If R_A and R_B are two relations induced by the attributes subsets A and B and $X \subseteq U$ is a target concept, then the rough set of X is derived from the quotient set $U/(R_A \cup R_B) = \{[x]_{R_A} \cap [x]_{R_B} | [x]_{R_A} \in U/R_A, [x]_{R_B} \in U/R_B, [x]_{R_A} \cap [x]_{R_B} \neq \emptyset\}$, which suggests that we can perform an intersection operation between $[x]_{R_A}$ and $[x]_{R_B}$ and the target concept is approximately described by using the quotient set $U/(R_A \cup R_B)$. Then the target concept is described by a finer granulation (partitions) formed through combining two known granulations (partitions) induced from two-attribute subsets. However, the combination that generates a much finer granulation and more knowledge destroys the original granulation structure.

In fact, the above assumption cannot always be satisfied or required generally. In some data analysis issues, for the same object, there is a contradiction or inconsistent relationship between its values under one attribute set A and those under another attribute set B. In other words, we can not perform the intersection operations between their quotient sets and the target concept cannot be approximated by using $U/(R_A \cup R_B)$. For the solution of the above contradition, Qian, Xu and M. Khan extended the Pawlak rough set to multi-granulation rough set models in which the approximation operators were defined by multiple equivalence relations on the universe [11, 19–21, 29, 30].

Associated fuzzy rough set with granulation computing, we will propose two types of multi-granulation fuzzy rough set models. The main objective of this paper is to extend Pawlak's rough set model determined by single binary relation to multi-granulation fuzzy rough sets in which set approximations are defined by multiple equivalence relations. The rest of this paper is organized as follows. Some preliminary concepts of Pawlak's rough set theory and fuzzy rough sets theory are proposed [5] in Section 2. In Section 3, based on multiple ordinary equivalence relations, two types of multi-granulation fuzzy rough approximation operators of a fuzzy concept in a fuzzy target information system, are constructed and a number of important properties of them are discussed in detail. Then it follows the comparison and relations among the properties of the two types of multi-granulation fuzzy rough sets and single-granulation fuzzy rough set in Section 4. In Section 5, a notion of rough measure and rough measure with respect to parameters α and β of the multi-granulation fuzzy rough sets are defined and illustrative examples are used to show its rationality and essence. And finally, the paper is concluded by a summary and outlook for further research in Section 6.

2. Preliminaries

In this section, we will first review some basic concepts and notions in the theory of Pawlak rough set and fuzzy rough set and the models of the multi-granulation rough set. More details can be seen in references [15, 40].

2.1. Pawlak rough set

The notion of information system provides a convenient tool for the representation of objects in terms of their attribute values.

An information system is an ordered triple $\mathcal{I} = (U, AT, F)$, where

$U = \{u_1, u_2, ..., u_n\}$ is a non-empty finite set of objects;

$AT = \{a_1, a_2, ..., a_m\}$ is a non-empty finite set of attributes;

$F = \{f_j \mid j \leq m\}$ is a set of relationship between U and AT, where $f_j : U \to V_j (j \leq m)$, V_j is the domain of attribute a_j and m is the number of the attributes.

Let $\mathcal{I} = (U, AT, F)$ be an information system. For $A \subseteq AT$, denote

$$R_A = \{(x, y) \mid f_j(x) = f_j(y), \forall a_j \in A\}$$

then R_A is reflexive, symmetric and transitive. So it is an equivalence relation on U.

Moreover, denote

$$[x]_A = \{x \mid (x, y) \in R_A\},$$

$$U/A = \{[x]_A | \forall x \in U\},$$

then $[x]_A$ is called the equivalence class of x, and the quotient set U/A is called the equivalence class set of U.

For any subset $X \subseteq U$ and $A \subseteq AT$ in the information system $\mathcal{I} = (U, AT, F)$, the Pawlak's lower and upper approximations of X with respect to equivalence relation R_A could be defined as following.

$$\underline{R_A}(X) = \{x \mid [x]_A \subseteq X\},$$

$$\overline{R_A}(X) = \{x \mid [x]_A \cap X \neq \emptyset\},$$

The set $Bn_A(X) = \overline{R_A}(X) - \underline{R_A}(X)$ is called the boundary of X.

To measure the imprecision and roughness of a rough set, Pawlak defined the rough measure of $X \neq \emptyset$ as

$$\rho_A(X) = 1 - \frac{|\underline{R_A}(X)|}{|\overline{R_A}(X)|}.$$

2.2. Fuzzy rough set

Let U is still a finite and non-empty set called universe. A fuzzy set X is a mapping from U into the unit interval $[0, 1]$, $\mu : U \to [0, 1]$, where each $\mu(x)$ is the membership degree of x in X. The set of all the fuzzy sets defined on U is denoted by $F(U)$.

Let U be the universe, R be an equivalence relation. For a fuzzy set $X \in F(U)$, if denote

$$\underline{R}(X)(x) = \wedge\{A(y) | y \in [x]_R\},$$

$$\overline{R}(X)(x) = \vee\{A(y) | y \in [x]_R\},$$

then $\underline{R}(X)$ and $\overline{R}(X)$ are called the lower and upper approximation of the fuzzy set X with respect to the relation R, where " $\wedge$ " means "min" and " $\vee$ " means "max". X is a fuzzy definable set if and only if X satisfies $\underline{R}(X) = \overline{R}(X)$. Otherwise, X is called a fuzzy rough set.

Let $\mathcal{I} = (U, AT, F)$ be an information system. $F = \{f_j \mid j \leq n\}$ is a set of relationship between U and AT. $D_j : U \to [0, 1] (j \leq r)$, r is the number of the decision attributes. If denote

$$\mathbf{D} = \{D_j \mid j \leq r\},$$

then $(U, AT, F, \mathbf{D})$ is a fuzzy target information system. In a fuzzy target information system, we can define the approximation operators with respect to the decision attribute D similarly.

Let U be the universe, R be an equivalence relation, $X, Y \in F(U)$. The fuzzy lower and upper approximation with respect to relation R have the following properties.

(1) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$.
(2) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$, $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$.
(3) $\underline{R}(X) = \sim \overline{R}(\sim X)$, $\overline{R}(X) = \sim \underline{R}(\sim X)$.
(4) $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$, $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$.
(5) $\overline{R}(\overline{R}(X)) = \underline{R}(\overline{R}(X)) = \overline{R}(X)$.
(6) $\overline{R}(\underline{R}(X)) = \underline{R}(\underline{R}(X)) = \underline{R}(X)$.
(7) $\underline{R}(U) = U$, $\overline{R}(\emptyset) = \emptyset$.
(8) $X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$ and $\overline{R}(X) \subseteq \overline{R}(Y)$.

To measure the imprecision and roughness of a fuzzy rough set, the rough measure of $X \neq \emptyset$ is defined as

$$\rho_A(X) = 1 - \frac{|\underline{R_A}(X)|}{|\overline{R_A}(X)|}.$$

where $|\underline{R_A}(X)| = \sum_{x \in U} \underline{R_A}(X)(x)$ and $|\overline{R_A}(X)| = \sum_{x \in U} \overline{R_A}(X)(x)$. If $\overline{R_A}(X) = 0$, we prescribe $\rho_A(X) = 0$.

What is more, for any $0 < \beta \leq \alpha \leq 1$, the α, β rough measure of fuzzy set is defined as

$$\rho_A(X)_{\alpha,\beta} = 1 - \frac{|\underline{R_A}(X)_\alpha|}{|\overline{R_A}(X)_\beta|}.$$

where $|\underline{R_A}(X)_\alpha|$ is the cardinality of the α-cut set of $\underline{R_A}(X)$, and $|\overline{R_A}(X)_\beta|$ is the cardinality of the β-cut set of $\overline{R_A}(X)$.

More details about the properties of above measures can be found in reference [40].

2.3. Multi-granulation rough sets

For simplicity, we just recall the models of multi-granulation rough sets and details can be seen in references [20, 21, 29].

Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT$, $1 \leq i \leq m$, m is the number of the considered attribute sets. The optimistic lower and upper approximations of the set $X \in U$ with respect to $A_i \subseteq AT$, $(1 \leq i \leq m)$ are

$$\underline{OR_{\sum_{i=1}^{m} A_i}}(X) = \{x \mid \bigvee_{i=1}^{m} [x]_{A_i} \subseteq X, 1 \leq i \leq m\},$$

$$\overline{OR_{\sum_{i=1}^{m} A_i}}(X) = \{x \mid \bigwedge_{i=1}^{m} [x]_{A_i} \cap X \neq \emptyset, 1 \leq i \leq m\},$$

where $[x]_{A_i} = \{y | (x, y) \in R_{A_i}\}$, and R_{A_i} is an equivalent relation with respect to the attributes set A_i.

Table 1

U	Transportation	Population density	Consumption level
x_1	Dood	Big	High
x_2	Dood	Big	Midium
x_3	Bad	Small	Low
x_4	Bad	Small	High
x_5	Dood	Small	High
x_6	Common	Big	High

Moreover, $\underline{OR_{\sum_{i=1}^{m} A_i}}(X) \neq \overline{OR_{\sum_{i=1}^{m} A_i}}(X)$, we say that X is the optimistic rough set with respect to multiple equivalence relations or multiple granulations. Otherwise, we say that X is the optimistic definable set with respect to multiple equivalence relations or multiple granulations.

Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT, 1 \leq i \leq m$, m is the number of the considered attribute sets. The pessimistic lower and upper approximations of the set $X \in U$ with respect to $A_i \subseteq AT, 1 \leq i \leq m$ are

$$\underline{PR_{\sum_{i=1}^{m} A_i}}(X) = \{x \mid \bigwedge_{i=1}^{m} [x]_{A_i} \subseteq X, 1 \leq i \leq m\},$$

$$\overline{PR_{\sum_{i=1}^{m} A_i}}(X) = \{x \mid \bigvee_{i=1}^{m} [x]_{A_i} \cap X \neq \emptyset, 1 \leq i \leq m\},$$

Moreover, $\underline{PR_{\sum_{i=1}^{m} A_i}}(X) \neq \overline{PR_{\sum_{i=1}^{m} A_i}}(X)$, we say that X is the pessimistic rough set with respect to multiple equivalence relations or multiple granulations. Otherwise, we say that X is pessimistic definable set with respect to multiple equivalence relations or multiple granulations.

Example 2.1. An information system about six cities' condition are given in table 1. The universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ stands for six cities, the set of condition attributes $AT = \{Transportation, \ Population\ density, \ Consumption\ level\}$. Now, denote $A_1 = \{Transportation, \ Population\ density, \}$ and $A_2 = \{Population\ density, \ Consumption\ level\}$. Let $X = \{x_2, x_4, x_5, x_6\}$.

By computing, we have that

$$U/A_1 = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}, \{x_6\}\}$$

$$U/A_2 = \{\{x_1, x_6\}, \{x_2\}, \{x_3\}, \{x_4, x_5\}\}$$

According to the above equivalence class, we can obtain the lower and upper approximation of X based on optimistic multi-granulation rough sets model as follows:

$$\underline{OR_{A_1+A_2}}(X) = \{x_2, x_4, x_5, x_6\}$$

$$\overline{OR_{A_1+A_2}}(X) = \{x_1, x_2, x_4, x_5, x_6\}$$

If we compute the lower and upper approximation of X based on the pessimistic multi-granulation rough sets model, the result can been seen as follows:

$$\underline{PR_{A_1+A_2}}(X) = \{x_5\}$$

$$\overline{PR_{A_1+A_2}}(X) = U$$

Form the two types of rough sets models, we can see that the optimistic boundary region is more small and the pessmistic boundary region is more big compared the classical rough sets model. In some cases, it can deal with uncertain problems easily.

3. Optimistic and pessimistic multi-granulation fuzzy rough sets

In this section, we will research about multi-granulation fuzzy rough sets which are the problems of the rough approximations of a fuzzy set based on multiple classical equivalence relations.

3.1. The optimistic multi-granulation fuzzy rough set

First, the optimistic two-granulation fuzzy rough set (in brief OTGFRS) of a fuzzy set is defined.

Definition 3.1. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\underline{OR_{A+B}}(X)(x) = \{\wedge\{X(y) \mid y \in [x]_A\}\} \vee \{\wedge\{X(y) \mid y \in [x]_B\}\},$$

$$\overline{OR_{A+B}}(X)(x) = \{\vee\{X(y) \mid y \in [x]_A\}\} \wedge \{\vee\{X(y) \mid y \in [x]_B\}\},$$

where " $\vee$ " means "max" and " $\wedge$ " means "min", then $\underline{OR_{A+B}}(X)$ and $\overline{OR_{A+B}}(X)$ are respectively called the optimistic two-granulation lower approximation and upper approximation of X with respect to the subsets of attributes A and B. X is a two-granulation fuzzy rough set if and only if $\underline{OR_{A+B}}(X) \neq \overline{OR_{A+B}}(X)$. Otherwise, X is a two-granulation fuzzy definable set. The boundary of the fuzzy rough set X is defined as

$$Bnd^O_{R_{A+B}}(X) = \overline{OR_{A+B}}(X) \cap (\sim \underline{OR_{A+B}}(X)).$$

From the above definition, it can be seen that the approximations in the OTGFRS are defined through using the equivalence classes induced by multiple independent equivalence relations, whereas the standard fuzzy rough approximations are represented via those derived by only one equivalence relation. In fact, the OTGFRS will be degenerated into a fuzzy rough set when $A = B$. That is to say, the fuzzy rough set model is a special instance of the OTGFRS. What's more, the OTGFRS will be degenerated into Pawlak rough set if $A = B$ and the considered concept X is a crisp set.

In the following, we employ an example to illustrate the above concepts.

Example 3.1. A fuzzy target information system about ten colledge students' performance are given in Table 1. The universe $U = \{x_1, x_2, \cdots, x_{10}\}$ which consists of ten students in a colledge; the set of condition attributes $AT = \{CP, RP, MP\}$, in which CP means "Course Performance", RP means "Research Performance", and MP means "Morality Performance", and the bigger the value of the condition attribute is, the better the students' performance is; the set of decision attribute $D = \{CA\}$ in which CA represents a fuzzy concept and means "Student's Comprehensive Accomplishment is good", and the value of the decision attribute is the membership degree of "good". We evaluate the students' comprehensive performance by the following cases:

Case 1: we evaluate the student by "Course Performance" and "Research Performance", that is, the first granulation is $A = \{CP, RP\}$;
Case 2: we evaluate the student by "Course Performance" and "Morality Performance", that is, the second granulation is $B = \{CP, MP\}$.

And the equivalence relation is defined as $R_A(R_B) = \{(x_i, x_j) \mid f_l(x_i) = f_l(x_j), a_l \in A(B)\}$ which means the students' comprehensive accomplishments is definitely indiscernible. Then under the equivalence relation $R_A(R_B)$, the students whose performance are the same belong to the same classification. We consider the optimistic two-granulation lower and upper approximation of D with respect to A and B. The optimistic two-granulation lower approximation here represents that the students' comprehensive performance is good at least at some degree if we consider either case, while the optimistic two-granulation upper approximation here represents that the students' comprehensive performance is good at most at another bigger degree if we consider both two cases.From the table, we can easily obtain

Table 2

U	CP	RP	MP	CA
x_1	2	1	3	0.6
x_2	3	2	1	0.7
x_3	2	1	3	0.7
x_4	2	2	3	0.9
x_5	1	1	4	0.5
x_6	1	1	2	0.4
x_7	3	2	1	0.7
x_8	1	1	4	0.7
x_9	2	1	3	0.8
x_{10}	3	2	1	0.7

$$U/A = \{\{x_1, x_3, x_9\}, \{x_2, x_7, x_{10}\}, \{x_4\}, \{x_5, x_6, x_8\}\},$$
$$U/B = \{\{x_1, x_3, x_4, x_9\}, \{x_2, x_7, x_{10}\}, \{x_5, x_6, x_8\}, \{x_6\}\},$$
$$U/(A \cup B) = \{\{x_1, x_3, x_9\}, \{x_2, x_7, x_{10}\}, \{x_4\}, \{x_5, x_8\}, \{x_6\}\}.$$

Then the single granulation lower and upper approximation of D are

$$\underline{R_A}(D) = (0.6, 0.7, 0.6, 0.9, 0.4, 0.4, 0.7, 0.4, 0.6, 0.7),$$
$$\overline{R_A}(D) = (0.8, 0.7, 0.8, 0.9, 0.7, 0.7, 0.7, 0.7, 0.6, 0.7);$$
$$\underline{R_B}(D) = (0.6, 0.7, 0.6, 0.6, 0.5, 0.4, 0.7, 0.5, 0.6, 0.7),$$
$$\overline{R_B}(D) = (0.9, 0.7, 0.9, 0.9, 0.7, 0.4, 0.7, 0.8, 0.9, 0.7);$$
$$\underline{R_{A\cup B}}(D) = (0.6, 0.7, 0.6, 0.9, 0.5, 0.4, 0.7, 0.5, 0.6, 0.7),$$
$$\overline{R_{A\cup B}}(D) = (0.8, 0.7, 0.8, 0.9, 0.7, 0.4, 0.7, 0.7, 0.8, 0.7);$$
$$\underline{R_A}(D) \cup \underline{R_B}(D) = (0.6, 0.7, 0.6, 0.9, 0.5, 0.4, 0.7, 0.5, 0.6, 0.7),$$
$$\overline{R_A}(D) \cap \overline{R_B}(D) = (0.8, 0.7, 0.8, 0.9, 0.7, 0.4, 0.7, 0.7, 0.8, 0.7).$$

From the Definition 3.1, we can compute optimistic two-granulation lower and upper approximation of D is

$$\underline{OR_{A+B}}(D) = (0.6, 0.7, 0.6, 0.9, 0.5, 0.4, 0.7, 0.5, 0.6, 0.7),$$

$$\overline{OR_{A+B}}(D) = (0.8, 0.7, 0.8, 0.9, 0.7, 0.4, 0.7, 0.7, 0.8, 0.7).$$

We can find that the ten students are good at least at the degree 0.6, 0.7, 0.6, 0.9, 0.5, 0.4, 0.7, 0.5, 0.6, 0.7, respectively, if we only evaluate the students by either A or B; and the ten students are good at most at the degree 0.8, 0.7, 0.8, 0.9, 0.7, 0.4, 0.7, 0.7, 0.8, 0.7, respectively, if we evaluate the students by both A and B.

Obviously, the following can be found

$$\underline{OR_{A+B}}(D) = \underline{R_A}(D) \cup \underline{R_B}(D),$$

$$\overline{OR_{A+B}}(D) = \overline{R_A}(D) \cap \overline{R_B}(D),$$

$$\underline{OR_{A+B}}(D) \subseteq \underline{R_{A\cup B}}(D) \subseteq D \subseteq \overline{R_{A\cup B}}(D) \subseteq \overline{OR_{A+B}}(D).$$

Just from Definition 3.1, we can obtain some properties of the OGFRS in an information system.

Proposition 3.1. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{OR_{A+B}}(X) \subseteq X$,
(2) $\overline{OR_{A+B}}(X) \supseteq X$;
(3) $\underline{OR_{A+B}}(\sim X) = \sim \overline{OR_{A+B}}(X)$,
(4) $\overline{OR_{A+B}}(\sim X) = \sim \underline{OR_{A+B}}(X)$;
(5) $\underline{OR_{A+B}}(U) = \overline{OR_{A+B}}(U) = U$,
(6) $\underline{OR_{A+B}}(\emptyset) = \overline{OR_{A+B}}(\emptyset) = \emptyset$.

Proof. It is obvious that all terms hold when $A = B$, since OGFRS degenerates into Pawlak fuzzy rough set. When $A \neq B$, the proposition can be proved as follows.

(1) For any $x \in U$ and $A,\ B \subseteq AT$, since $\underline{R_A}(X) \subseteq X$, we know

$$\wedge\{X(y) \mid y \in [x]_A\} \leq X(y)$$

and

$$\wedge\{X(y) \mid y \in [x]_B\} \leq X(y)$$

Therefore,

$$\{\wedge\{X(y) \mid y \in [x]_A\}\} \vee \{\wedge\{X(y) \mid y \in [x]_B\}\} \leq X(y).$$

i.e., $\underline{OR_{A+B}}(X) \subseteq X$.

(2) For any $x \in U$ and $A,\ B \subseteq AT$, since $X \subseteq \overline{R_A}(X)$, we know

$$X(y) \leq \vee\{X(y) \mid y \in [x]_A\}$$

and

$$X(y) \leq \vee\{X(y) \mid y \in [x]_B\}.$$

Therefore,

$$X(y) \leq \{\vee\{X(y) \mid y \in [x]_A\}\} \wedge \{\vee\{X(y) \mid y \in [x]_B\}\}.$$

i.e., $X \subseteq \overline{OR_{A+B}}(X)$.

(3) For any $x \in U$ and $A,\ B \subseteq AT$, since $\underline{R_A}(\sim X) = \sim \overline{R_A}(X)$ and $\underline{R_B}(\sim X) = \sim \overline{R_B}(X)$, then we have

$$\begin{aligned}\underline{OR_{A+B}}(\sim X)(x) &= \{\wedge\{1 - X(y) \mid y \in [x]_A\}\} \vee \\ &\quad \{\wedge\{1 - X(y) \mid y \in [x]_B\}\} \\ &= \{1 - \vee\{X(y) \mid y \in [x]_A\}\} \vee \\ &\quad \{1 - \vee\{X(y) \mid y \in [x]_B\}\} \\ &= 1 - \{\vee\{X(y) \mid y \in [x]_A\}\} \wedge \\ &\quad \{\vee\{X(y) \mid y \in [x]_B\}\} \\ &= \sim \overline{OR_{A+B}}(X)(x).\end{aligned}$$

(4) By $\underline{OR_{A+B}}(\sim X) = \sim \overline{OR_{A+B}}(X)$, we have $\underline{OR_{A+B}}(X) = \sim \overline{OR_{A+B}}(\sim X)$. So it can be found that $\overline{OR_{A+B}}(\sim X) = \sim \underline{OR_{A+B}}(X)$.

(5) Since for any $x \in U$, $U(x) = 1$, then for any $A, B \subseteq U$,

$$\underline{OR_{A+B}}(U)(x) = \{\wedge\{U(y) \mid y \in [x]_A\}\} \vee \{\wedge\{U(y) \mid y \in [x]_B\}\} = 1 = U(x)$$

and

$$\overline{OR_{A+B}}(U)(x) = \{\vee\{U(y) \mid y \in [x]_A\}\} \wedge \{\vee\{U(y) \mid y \in [x]_B\}\} = 1 = U(x).$$

So $\underline{OR_{A+B}}(U) = \overline{OR_{A+B}}(U) = U$.

(6) From the duality of the approximation operators in (3) and (4), it is easy to prove $\underline{OR_{A+B}}(\emptyset) = \overline{OR_{A+B}}(\emptyset) = \emptyset$ by property (5). □

Proposition 3.2. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT$, $X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{OR_{A+B}}(X \cap Y) \subseteq \underline{OR_{A+B}}(X) \cap \underline{OR_{A+B}}(Y)$,
(2) $\overline{OR_{A+B}}(X \cup Y) \supseteq \overline{OR_{A+B}}(X) \cup \overline{OR_{A+B}}(Y)$;
(3) $X \subseteq Y \Rightarrow \underline{OR_{A+B}}(X) \subseteq \underline{OR_{A+B}}(Y)$,
(4) $X \subseteq Y \Rightarrow \overline{OR_{A+B}}(X) \subseteq \overline{OR_{A+B}}(Y)$;
(5) $\underline{OR_{A+B}}(X \cup Y) \supseteq \underline{OR_{A+B}}(X) \cup \underline{OR_{A+B}}(Y)$,
(6) $\overline{OR_{A+B}}(X \cap Y) \subseteq \overline{OR_{A+B}}(X) \cap \overline{OR_{A+B}}(Y)$.

Proof. All terms hold when $A = B$ or $X = Y$ as they will degenerate into single granulation fuzzy rough set. If $A \neq B$ and $X \neq Y$, the proposition can be proved as follows.

(1) For any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
&\underline{OR_{A+B}}(X \cap Y)(x)\\
&= \{\wedge\{(X \cap Y)(y) \mid y \in [x]_A\}\} \vee \\
&\quad \{\wedge\{(X \cap Y)(y) \mid y \in [x]_B\}\}\\
&= \{\wedge\{X(y) \wedge Y(y) \mid y \in [x]_A\}\} \vee \\
&\quad \{\wedge\{X(y) \wedge Y(y) \mid y \in [x]_B\}\}\\
&= \{\underline{R_A}(X)(x) \wedge \underline{R_A}(Y)(x)\} \vee \{\underline{R_B}(X)(x) \wedge \underline{R_B}(Y)(x)\}\\
&\leq \{\underline{R_A}(X)(x) \vee \underline{R_B}(X)(x)\} \wedge \{\underline{R_A}(Y)(x)\\
&\quad \vee \underline{R_B}(Y)(x)\}\\
&= \underline{OR_{A+B}}(X)(x) \wedge \underline{OR_{A+B}}(Y)(x).
\end{aligned}$$

Then $\underline{OR_{A+B}}(X \cap Y) \subseteq \underline{OR_{A+B}}(X) \cap \underline{OR_{A+B}}(Y)$.

(2) Similarly, for any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
&\overline{OR_{A+B}}(X \cup Y)(x)\\
&= \{\vee\{(X \cup Y)(y) \mid y \in [x]_A\}\} \wedge \\
&\quad \{\vee\{(X \cup Y)(y) \mid y \in [x]_B\}\}\\
&= \{\vee\{X(y) \vee Y(y) \mid y \in [x]_A\}\} \wedge \\
&\quad \{\vee\{X(y) \vee Y(y) \mid y \in [x]_B\}\}\\
&= \{\overline{R_A}(X)(x) \vee \overline{R_A}(Y)(x)\} \wedge \{\overline{R_B}(X)(x) \vee \overline{R_B}(Y)(x)\}\\
&\geq \{\overline{R_A}(X)(x) \wedge \overline{R_B}(X)(x)\} \vee \{\overline{R_A}(Y)(x)\\
&\quad \wedge \overline{R_B}(Y)(x)\}\\
&= \overline{OR_{A+B}}(X)(x) \vee \overline{OR_{A+B}}(Y)(x).
\end{aligned}$$

Then $\overline{OR_{A+B}}(X \cup Y) \supseteq \overline{OR_{A+B}}(X) \cup \overline{OR_{A+B}}(Y)$.

(3) Since for any $x \in U$, we have $X(y) \leq Y(y)$. Then the properties hold obviously by Definition 3.1.
(4) The properties can be proved similarly to (3).
(5) Since $X \subseteq X \cup Y$, and $Y \subseteq X \cup Y$, then $\underline{OR_{A+B}}(X) \subseteq \underline{OR_{A+B}}(X \cup Y)$ and $\underline{OR_{A+B}}(Y) \subseteq \underline{OR_{A+B}}(X \cup Y)$. So the property $\underline{OR_{A+B}}(X \cup Y) \supseteq \underline{OR_{A+B}}(X) \cup \underline{OR_{A+B}}(Y)$ obviously holds.
(6) This item can be proved similarly to (5) by (4).

The proposition was proved. □

The lower and upper approximation in Definition 3.1 are a pair of fuzzy sets. If we associate the cut set of a fuzzy set, we can make a description of a fuzzy set X by a classical set in an information system.

Definition 3.2. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$ and $X \in F(U)$. For any $0 < \beta \leq \alpha \leq 1$, the lower approximation $\underline{OR_{A+B}}(X)$ and upper approximation $\overline{OR_{A+B}}(X)$ of X about the α, β cut sets are defined, respectively, as follows

$$\underline{OR_{A+B}}(X)_\alpha = \{x \mid \underline{OR_{A+B}}(X)(x) \geq \alpha\},$$
$$\overline{OR_{A+B}}(X)_\beta = \{x \mid \overline{OR_{A+B}}(X)(x) \geq \beta\}.$$

$\underline{OR_{A+B}}(X)_\alpha$ can be explained as the set of objects in U which possibly belong to X and the memberships of which are more than α, while $\overline{OR_{A+B}}(X)_\beta$ is the set of objects in U which possibly belong to X and the memberships of which are more than β.

Proposition 3.3. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$ and $X, Y \in F(U)$. For any $0 < \beta \leq \alpha \leq 1$, we have

(1) $\underline{OR_{A+B}}(X \cap Y)_\alpha \subseteq \underline{OR_{A+B}}(X)_\alpha \cap \underline{OR_{A+B}}(Y)_\alpha$,
(2) $\overline{OR_{A+B}}(X \cup Y)_\beta \supseteq \overline{OR_{A+B}}(X)_\beta \cup \overline{OR_{A+B}}(Y)_\beta$;
(3) $X \subseteq Y \Rightarrow \underline{OR_{A+B}}(X)_\alpha \subseteq \underline{OR_{A+B}}(Y)_\alpha$,
(4) $X \subseteq Y \Rightarrow \overline{OR_{A+B}}(X)_\beta \subseteq \overline{OR_{A+B}}(Y)_\beta$;
(5) $\underline{OR_{A+B}}(X \cup Y)_\alpha \supseteq \underline{OR_{A+B}}(X)_\alpha \cup \underline{OR_{A+B}}(Y)_\alpha$,
(6) $\overline{OR_{A+B}}(X \cap Y)_\beta \subseteq \overline{OR_{A+B}}(X)_\beta \cap \overline{OR_{A+B}}(Y)_\beta$.

Proof. It is easy to prove by Definition 3.2 and Proposition 3.2. □

In the following, we will introduce the optimistic multi-granulation fuzzy rough set (in brief OMGFRS) and its corresponding properties by extending the optimistic two-granulation fuzzy rough set.

Definition 3.3. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\underline{OR}_{\sum_{i=1}^{m} A_i}(X)(x) = \bigvee_{i=1}^{m}\{\bigwedge\{X(y) \mid y \in [x]_{A_i}\}\},$$

$$\overline{OR}_{\sum_{i=1}^{m} A_i}(X)(x) = \bigwedge_{i=1}^{m}\{\bigvee\{X(y) \mid y \in [x]_{A_i}\}\},$$

where " $\bigvee$ " means "max" and " $\bigwedge$ " means "min", then $\underline{FR}_{\sum_{i=1}^{m} A_i}(X)$ and $\overline{OR}_{\sum_{i=1}^{m} A_i}(X)$ are respectively called the optimistic multi-granulation lower approximation and upper approximation of X with respect to the subsets of attributes A_i, $1 \leq i \leq m$. X is a multi-granulation fuzzy rough set if and only if $\underline{OR}_{\sum_{i=1}^{m} A_i}(X) \neq \overline{OR}_{\sum_{i=1}^{m} A_i}(X)$. Otherwise, X is a multi-granulation fuzzy definable set. The boundary of the fuzzy rough set X is defined as

$$Bnd^{O}_{R_{\sum_{i=1}^{m} A_i}}(X) = \overline{OR}_{\sum_{i=1}^{m} A_i}(X) \cap (\sim \underline{OR}_{\sum_{i=1}^{m} A_i}(X)).$$

It can be found that the OMGFRS will be degenerated into fuzzy rough set when $A_i = A_j,\ i \neq j$. That is to say, a fuzzy rough set is a special instance of OMGFRS. Besides, this model can also been turned the OMGRS if the considered set is a crisp one. What's more, the OMGFRS will be degenerated into Pawlak rough set if $A_i = A_j,\ i \neq j$ and the considered concept X is a crisp set.

The properties about OMGFRS are listed in the following which can be extended from the OTGFRS model.

Proposition 3.4. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{OR}_{\sum_{i=1}^{m} A_i}(X) \subseteq X$,

(2) $\overline{OR}_{\sum_{i=1}^{m} A_i}(X) \supseteq X$;

(3) $\underline{OR}_{\sum_{i=1}^{m} A_i}(\sim X) = \sim \overline{R}_{\sum_{i=1}^{m} A_i}(X)$,

(4) $\overline{OR}_{\sum_{i=1}^{m} A_i}(\sim X) = \sim \underline{OR}_{\sum_{i=1}^{m} A_i}(X)$;

(5) $\underline{OR}_{\sum_{i=1}^{m} A_i}(U) = \overline{OR}_{\sum_{i=1}^{m} A_i}(U) = U$,

(6) $\underline{OR}_{\sum_{i=1}^{m} A_i}(\emptyset) = \overline{OR}_{\sum_{i=1}^{m} A_i}(\emptyset) = \emptyset$.

Proof. The proof of this proposition is similar to Proposition 3.1. □

Proposition 3.5. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m,\ X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{OR}_{\sum_{i=1}^{m} A_i}(X \cap Y) \subseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(X) \cap \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)$,

(2) $\overline{OR}_{\sum_{i=1}^{m} A_i}(X \cup Y) \supseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(X) \cup \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)$;

(3) $X \subseteq Y \Rightarrow \underline{OR}_{\sum_{i=1}^{m} A_i}(X) \subseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)$,

(4) $X \subseteq Y \Rightarrow \overline{OR}_{\sum_{i=1}^{m} A_i}(X) \subseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)$;

(5) $\underline{OR}_{\sum_{i=1}^{m} A_i}(X \cup Y) \supseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(X) \cup \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)$;

(6) $\overline{OR}_{\sum_{i=1}^{m} A_i}(X \cap Y) \subseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(X) \cap \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)$.

Proof. The proof of this proposition is similar to Proposition 3.2. □

Definition 3.4. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m$, and $X \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, the lower approximation $\underline{OR}_{\sum_{i=1}^{m} A_i}(X)$ and upper approximation $\overline{OR}_{\sum_{i=1}^{m} A_i}(X)$ of X about the α, β cut sets are defined, respectively, as follows

$$\underline{OR}_{\sum_{i=1}^{m} A_i}(X)_{\alpha} = \{x \mid \underline{OR}_{\sum_{i=1}^{m} A_i}(X)(x) \geq \alpha\},$$

$$\overline{OR}_{\sum_{i=1}^{m} A_i}(X)_{\beta} = \{x \mid \overline{OR}_{\sum_{i=1}^{m} A_i}(X)(x) \geq \beta\}.$$

$\underline{OR}_{\sum_{i=1}^{m} A_i}(X)_{\alpha}$ can be explained as the set of objects in U which surely belong to X and the memberships of which are more than α, while $\overline{OR}_{\sum_{i=1}^{m} A_i}(X)_{\beta}$ is the set of objects in U which possibly belong to X and the memberships of which are more than β.

Proposition 3.6. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \le i \le m$, and $X, Y \subseteq U$. For any $0 < \beta \le \alpha \le 1$, we have

(1) $\underline{OR}_{\sum_{i=1}^{m} A_i}(X \cap Y)_\alpha \subseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \cap \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$,
(2) $\overline{OR}_{\sum_{i=1}^{m} A_i}(X \cup Y)_\beta \supseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(X)_\beta \cup \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$;
(3) $X \subseteq Y \Rightarrow \underline{OR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \subseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$,
(4) $X \subseteq Y \Rightarrow \overline{OR}_{\sum_{i=1}^{m} A_i}(X)_\beta \subseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$;
(5) $\underline{OR}_{\sum_{i=1}^{m} A_i}(X \cup Y)_\alpha \supseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \cup \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$,
(6) $\overline{OR}_{\sum_{i=1}^{m} A_i}(X \cap Y)_\beta \subseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(X)_\beta \cap \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$.

Proof. It is easy to prove by Definition 3.4 and Proposition 3.5. □

3.2. The pessimistic multi-granulation fuzzy rough set

In this subsection, we will propose another type of MGFRS. We first define the pessimistic two-granulation fuzzy rough set (in brief the PTGFRS).

Definition 3.5. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\underline{PR_{A+B}}(X)(x) = \{\wedge\{X(y) \mid y \in [x]_A\}\} \wedge \{\wedge\{X(y) \mid y \in [x]_B\}\},$$
$$\overline{PR_{A+B}}(X)(x) = \{\vee\{X(y) \mid y \in [x]_A\}\} \vee \{\vee\{X(y) \mid y \in [x]_B\}\},$$

then $\underline{PR_{A+B}}(X)$ and $\overline{PR_{A+B}}(X)$ are respectively called the pessimistic two-granulation lower approximation and upper approximation of X with respect to the subsets of attributes A and B. X is the pessimistic two-granulation fuzzy rough set if and only if $\underline{PR_{A+B}}(X) \neq \overline{PR_{A+B}}(X)$. Otherwise, X is the pessimistic two-granulation fuzzy definable set. The boundary of the fuzzy rough set X is defined as

$$Bnd^P_{R_{A+B}}(X) = \overline{PR_{A+B}}(X) \cap (\sim \underline{PR_{A+B}}(X)).$$

It can be found that the PTGFRS will be degenerated into a fuzzy rough set when $A = B$. That is to say, a fuzzy rough set is also a special instance of the PTGFRS. What's more, the PTGFRS will be degenerated into Pawlak rough set if $A = B$ and the considered concept X is a crisp set.

In the following, we employ an example to illustrate the above concepts.

Example 3.2. (Continued from Example 3.1) From Definition 3.2, we can compute the pessimistic two-granulation lower and upper approximation of D is

$$\underline{PR_{A+B}}(D) = (0.6, 0.7, 0.6, 0.6, 0.4, 0.4, 0.7, 0.4, 0.6, 0.7),$$
$$\overline{PR_{A+B}}(D) = (0.8, 0.7, 0.9, 0.9, 0.7, 0.7, 0.7, 0.7, 0.9, 0.7).$$

We can find that the ten students are good at most at the degree 0.6, 0.7, 0.6, 0.6, 0.4, 0.4, 0.7, 0.4, 0.6, 0.7, respectively, if we evaluate the students by both A and B; and the ten students are good at least at the degree 0.8, 0.7, 0.9, 0.9, 0.7, 0.7, 0.7, 0.7, 0.9, 0.7, respectively, if we evaluate the students only by either A or B.

Obviously, the following can be found

$$\underline{PR_{A+B}}(D) = \underline{R_A}(D) \cap \underline{R_B}(D),$$
$$\overline{PR_{A+B}}(D) = \overline{R_A}(D) \cup \overline{R_B}(D),$$
$$\underline{PR_{A+B}}(D) \subseteq \underline{R_{A\cup B}}(D) \subseteq D \subseteq \overline{R_{A\cup B}}(D) \subseteq \overline{PR_{A+B}}(D).$$

Proposition 3.7. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{A+B}}(X) \subseteq X$,
(2) $\overline{PR_{A+B}}(X) \supseteq X$;
(3) $\underline{PR_{A+B}}(\sim X) = \sim \overline{PR_{A+B}}(X)$,
(4) $\overline{PR_{A+B}}(\sim X) = \sim \underline{PR_{A+B}}(X)$;
(5) $\underline{PR_{A+B}}(U) = \overline{PR_{A+B}}(U) = U$,
(6) $\underline{PR_{A+B}}(\emptyset) = \overline{PR_{A+B}}(\emptyset) = \emptyset$.

Proof. It is obvious that all terms hold when $A = B$. When $A \neq B$, the proposition can be proved as follows.

(1) For any $x \in U$ and $A,\ B \subseteq AT$, since $\underline{R_A}(X) \subseteq X$, we know

$$\wedge\{X(y) \mid y \in [x]_A\} \leq X(y)$$

and

$$\wedge\{X(y) \mid y \in [x]_B\} \leq X(y)$$

Therefore,

$$\{\wedge\{X(y) \mid y \in [x]_A\}\} \wedge \{\wedge\{X(y) \mid y \in [x]_B\}\} \leq X(y).$$

i.e., $\underline{PR_{A+B}}(X) \subseteq X$.

(2) For any $x \in U$ and $A, B \subseteq AT$, since $X \subseteq \overline{R_A}(X)$, we know

$$X(y) \leq \vee\{X(y) \mid y \in [x]_A\}$$

and

$$X(y) \leq \vee\{X(y) \mid y \in [x]_B\}$$

Therefore,

$$X(y) \leq \{\vee\{X(y) \mid y \in [x]_A\}\} \vee \{\vee\{X(y) \mid y \in [x]_B\}\}.$$

i.e., $X \subseteq \overline{PR_{A+B}}(X)$.

(3) For any $x \in U$ and $A, B \subseteq AT$, since $\underline{R_A}(\sim X) =\sim \overline{R_A}(X)$ and $\underline{R_B}(\sim X) =\sim \overline{R_B}(X)$, then we have

$$\begin{aligned}
\underline{PR_{A+B}}(\sim X)(x) &= \{\wedge\{1 - X(y) \mid y \in [x]_A\}\} \\
&\quad \wedge\{\wedge\{1 - X(y) \mid y \in [x]_B\}\} \\
&= \{1 - \vee\{X(y) \mid y \in [x]_A\}\} \\
&\quad \wedge\{1 - \vee\{X(y) \mid y \in [x]_B\}\} \\
&= 1 - \{\vee\{X(y) \mid y \in [x]_A\}\} \\
&\quad \vee\{\vee\{X(y) \mid y \in [x]_B\}\} \\
&= \sim \overline{PR_{A+B}}(X)(x).
\end{aligned}$$

(4) By $\underline{PR_{A+B}}(\sim X) =\sim \overline{PR_{A+B}}(X)$, we have $\underline{PR_{A+B}}(X) =\sim \overline{PR_{A+B}}(\sim X)$. So it can be found that $\overline{PR_{A+B}}(\sim X) =\sim \underline{PR_{A+B}}(X)$.

(5) Since for any $x \in U$, $U(x) = 1$, then for any $A, B \subseteq U$, we have

$$\begin{aligned}
\underline{PR_{A+B}}(U)(x) &= \{\wedge\{U(y) \mid y \in [x]_A\}\} \wedge \\
&\quad \{\wedge\{U(y) \mid y \in [x]_B\}\} = 1 = U(x), \\
&\quad \overline{PR_{A+B}}(U)(x) \\
&= \{\vee\{U(y) \mid y \in [x]_A\}\} \vee \\
&\quad \{\vee\{U(y) \mid y \in [x]_B\}\} = 1 = U(x).
\end{aligned}$$

So $\underline{PR_{A+B}}(U) = \overline{PR_{A+B}}(U) = U$.

(6) From the duality of the approximation operators in (6), it is easy to prove $\underline{PR_{A+B}}(\emptyset) = \overline{PR_{A+B}}(\emptyset) = \emptyset$. □

Proposition 3.8. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT$, $X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{A+B}}(X \cap Y) = \underline{PR_{A+B}}(X) \cap \underline{PR_{A+B}}(Y)$,
(2) $\overline{PR_{A+B}}(X \cup Y) = \overline{PR_{A+B}}(X) \cup \overline{PR_{A+B}}(Y)$;
(3) $X \subseteq Y \Rightarrow \underline{PR_{A+B}}(X) \subseteq \underline{PR_{A+B}}(Y)$,
(4) $X \subseteq Y \Rightarrow \overline{PR_{A+B}}(X) \subseteq \overline{PR_{A+B}}(Y)$;
(5) $\underline{PR_{A+B}}(X \cup Y) \supseteq \underline{PR_{A+B}}(X) \cup \underline{PR_{A+B}}(Y)$;
(6) $\overline{PR_{A+B}}(X \cap Y) \subseteq \overline{PR_{A+B}}(X) \cap \overline{PR_{A+B}}(Y)$.

Proof. All terms hold when $A = B$ or $X = Y$ as they will degenerate into single granulation fuzzy rough set. If $A \neq B$ and $X \neq Y$, the proposition can be proved as follows.

(1) For any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
\underline{PR_{A+B}}(X \cap Y)(x) &= \{\wedge\{(X \cap Y)(y) \mid y \in [x]_A\}\} \wedge \\
&\quad \{\wedge\{(X \cap Y)(y) \mid y \in [x]_B\}\} \\
&= \{\wedge\{X(y) \wedge Y(y) \mid y \in [x]_A\}\} \wedge \\
&\quad \{\wedge\{X(y) \wedge Y(y) \mid y \in [x]_B\}\} \\
&= \{\underline{R_A}(X)(x) \wedge \underline{R_A}(Y)(x)\} \wedge \\
&\quad \{\underline{R_B}(X)(x) \wedge \underline{R_B}(Y)(x)\} \\
&= \{\underline{R_A}(X)(x) \wedge \underline{R_B}(X)(x)\} \wedge \\
&\quad \{\underline{R_A}(Y)(x) \wedge \underline{R_B}(Y)(x)\} \\
&= \underline{R_{A+B}}(X)(x) \wedge \underline{R_{A+B}}(Y)(x).
\end{aligned}$$

Then $\underline{PR_{A+B}}(X \cap Y) = \underline{PR_{A+B}}(X) \cap \underline{PR_{A+B}}(Y)$.

(2) Similarly, for any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
\overline{PR_{A+B}}(X \cup Y)(x) &= \{\vee\{(X \cup Y)(y) \mid y \in [x]_A\}\} \vee \\
&\quad \{\vee\{(X \cup Y)(y) \mid y \in [x]_B\}\} \\
&= \{\vee\{X(y) \vee Y(y) \mid y \in [x]_A\}\} \vee \\
&\quad \{\vee\{X(y) \vee Y(y) \mid y \in [x]_B\}\} \\
&= \{\overline{R_A}(X)(x) \vee \overline{R_A}(Y)(x)\} \vee \\
&\quad \{\overline{R_B}(X)(x) \vee \overline{R_B}(Y)(x)\} \\
&= \{\overline{R_A}(X)(x) \vee \overline{R_B}(X)(x)\} \vee \\
&\quad \{\overline{R_A}(Y)(x) \vee \overline{R_B}(Y)(x)\} \\
&= \overline{PR_{A+B}}(X)(x) \vee \overline{PR_{A+B}}(Y)(x).
\end{aligned}$$

Then $\overline{PR_{A+B}}(X \cup Y) = \overline{PR_{A+B}}(X) \cup \overline{PR_{A+B}}(Y)$.

(3) Since for any $x \in U$, we have $X(y) \leq Y(y)$. Then the properties hold obviously by Definition 3.5.
(4) The properties can be proved as (3).
(5) Since $X \subseteq X \cup Y$, and $Y \subseteq X \cup Y$, then $\underline{PR_{A+B}}(X) \subseteq \underline{PR_{A+B}}(X \cup Y)$ and $\underline{PR_{A+B}}(Y) \subseteq \underline{PR_{A+B}}(X \cup Y)$. So the property $\underline{PR_{A+B}}(X \cup Y) \supseteq \underline{PR_{A+B}}(X) \cup \underline{PR_{A+B}}(Y)$ obviously holds.
(6) This item can be proved similarly to (5) by (4).

The proposition was proved.

Definition 3.6. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$ and $X \in F(U)$. For any $0 < \beta \leq \alpha \leq 1$, the lower approximation $\underline{PR_{A+B}}(X)$ and upper approximation $\overline{PR_{A+B}}(X)$ of X about the α, β cut sets are defined, respectively, as follows

$$\underline{PR_{A+B}}(X)_\alpha = \{x \mid \underline{PR_{A+B}}(X)(x) \geq \alpha\},$$

$$\overline{PR_{A+B}}(X)_\beta = \{x \mid \overline{PR_{A+B}}(X)(x) \geq \beta\}.$$

$\underline{PR_{A+B}}(X)_\alpha$ can be explained as the set of objects in U which possibly belong to X and the memberships of which are more than α, while $\overline{PR_{A+B}}(X)_\beta$ is the set of objects in U which possibly belong to X and the memberships of which are more than β.

Proposition 3.9. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$ and $X, Y \in F(U)$. For any $0 < \beta \leq \alpha \leq 1$, we have

(1) $\underline{\underline{PR_{A+B}}}(X \cap Y)_\alpha = \underline{\underline{PR_{A+B}}}(X)_\alpha \cap \underline{\underline{PR_{A+B}}}(Y)_\alpha$,
(2) $\overline{\overline{PR_{A+B}}}(X \cup Y)_\beta = \overline{\overline{PR_{A+B}}}(X)_\beta \cup \overline{\overline{PR_{A+B}}}(Y)_\beta$;
(3) $X \subseteq Y \Rightarrow \underline{\underline{PR_{A+B}}}(X)_\alpha \subseteq \underline{\underline{PR_{A+B}}}(Y)_\alpha$,
(4) $X \subseteq Y \Rightarrow \overline{\overline{PR_{A+B}}}(X)_\beta \subseteq \overline{\overline{PR_{A+B}}}(Y)_\beta$;
(5) $\underline{\underline{PR_{A+B}}}(X \cup Y)_\alpha \supseteq \underline{\underline{PR_{A+B}}}(X)_\alpha \cup \underline{\underline{PR_{A+B}}}(Y)_\alpha$,
(6) $\overline{\overline{PR_{A+B}}}(X \cap Y)_\beta \subseteq \overline{\overline{PR_{A+B}}}(X)_\beta \cap \overline{\overline{PR_{A+B}}}(Y)_\beta$.

Proof. It is easy to prove by Definition 3.6 and Proposition 3.8. □

In the following, we will introduce the pessimistic multi-granulation fuzzy rough set (in brief the PMGFRS) and its corresponding properties by extending the pessimistic two-granulation fuzzy rough set.

Definition 3.7. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A, B \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\underline{PR_{\sum_{i=1}^m A_i}}(X)(x) = \bigwedge_{i=1}^m \{\bigwedge\{X(y) \mid y \in [x]_{A_i}\}\},$$

$$\overline{PR_{\sum_{i=1}^m A_i}}(X)(x) = \bigvee_{i=1}^m \{\bigvee\{X(y) \mid y \in [x]_{A_i}\}\},$$

where "$\bigvee$" means "max" and "$\bigwedge$" means "min", then $\underline{PR_{\sum_{i=1}^m A_i}}(X)$ and $\overline{PR_{\sum_{i=1}^m A_i}}(X)$ are respectively called the pessimistic multi-granulation lower approximation and upper approximation of X with respect to the subsets of attributes $A_i (1 \leq i \leq m)$. X is the pessimistic multi-granulation fuzzy rough set if and only if $\underline{PR_{\sum_{i=1}^m A_i}}(X) \neq \overline{PR_{\sum_{i=1}^m A_i}}(X)$. Otherwise, X is the pessimistic multi-granulation fuzzy definable set. The boundary of the fuzzy rough set X is defined as

$$Bnd^P_{R_{\sum_{i=1}^m A_i}}(X) = \overline{PR_{\sum_{i=1}^m A_i}}(X) \cap (\sim \underline{PR_{\sum_{i=1}^m A_i}}(X)).$$

It can be found that the PMGFRS will be degenerated into fuzzy rough set when $A_i = A_j,\ i \neq j$. That is to say, a fuzzy rough set is also a special instance of the PMGFRS. Besides, this model can also been turned the pessimistic MGRS if the considered set is a crisp one. What's more, the MGFRS will be degenerated into Pawlak rough set if $A_i = A_j,\ i \neq j$ and the considered concept X is a crisp set.

The properties about the PMGFRS are listed in the following which can be extended from the PTGFRS model.

Proposition 3.10. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{\sum_{i=1}^m A_i}}(X) \subseteq X$,
(2) $\overline{\overline{PR_{\sum_{i=1}^m A_i}}}(X) \supseteq X$;
(3) $\underline{PR_{\sum_{i=1}^m A_i}}(\sim X) = \sim \overline{PR_{\sum_{i=1}^m A_i}}(X)$,
(4) $\overline{\overline{PR_{\sum_{i=1}^m A_i}}}(\sim X) = \sim \underline{PR_{\sum_{i=1}^m A_i}}(X)$;
(5) $\underline{PR_{\sum_{i=1}^m A_i}}(U) = \overline{PR_{\sum_{i=1}^m A_i}}(U) = U$,

(6) $\underline{PR}_{\sum_{i=1}^{m} A_i}(\emptyset) = \overline{PR}_{\sum_{i=1}^{m} A_i}(\emptyset) = \emptyset$.

Proof. The proof of this proposition is similar to Proposition 3.7. □

Proposition 3.11. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \le i \le m$, $X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{PR}_{\sum_{i=1}^{m} A_i}(X \cap Y) = \underline{PR}_{\sum_{i=1}^{m} A_i}(X) \cap \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)$,
(2) $\overline{PR}_{\sum_{i=1}^{m} A_i}(X \cup Y) = \overline{PR}_{\sum_{i=1}^{m} A_i}(X) \cup \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)$;
(3) $X \subseteq Y \Rightarrow \underline{PR}_{\sum_{i=1}^{m} A_i}(X) \subseteq \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)$,
(4) $X \subseteq Y \Rightarrow \overline{PR}_{\sum_{i=1}^{m} A_i}(X) \subseteq \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)$;
(5) $\underline{PR}_{\sum_{i=1}^{m} A_i}(X \cup Y) \supseteq \underline{PR}_{\sum_{i=1}^{m} A_i}(X) \cup \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)$,
(6) $\overline{PR}_{\sum_{i=1}^{m} A_i}(X \cap Y) \subseteq \overline{PR}_{\sum_{i=1}^{m} A_i}(X) \cap \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)$.

Proof. The proof of this proposition is similar to Proposition 3.8.

Definition 3.8. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \le i \le m$, and $X \in F(U)$. For any $0 < \beta \le \alpha \le 1$, the lower approximation $\underline{PR}_{\sum_{i=1}^{m} A_i}(X)$ and upper approximation $\overline{PR}_{\sum_{i=1}^{m} A_i}(X)$ of X about the α, β cut sets are defined, respectively, as follows

$$\underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha = \{x \mid \underline{PR}_{\sum_{i=1}^{m} A_i}(X)(x) \ge \alpha\},$$

$$\overline{PR}_{\sum_{i=1}^{m} A_i}(X)_\beta = \{x \mid \overline{PR}_{\sum_{i=1}^{m} A_i}(X)(x) \ge \beta\}.$$

$\underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha$ can be explained as the set of objects in U which surely belong to X and the memberships of which are more than α, while $\overline{PR}_{\sum_{i=1}^{m} A_i}(X)_\beta$ is the set of objects in U which possibly belong to X and the memberships of which are more than β.

Proposition 3.12. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \le i \le m$, and $X, Y \in F(U)$. For any $0 < \beta \le \alpha \le 1$, we have

(1) $\underline{PR}_{\sum_{i=1}^{m} A_i}(X \cap Y)_\alpha = \underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \cap \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$,
(2) $\overline{PR}_{\sum_{i=1}^{m} A_i}(X \cup Y)_\beta = \overline{PR}_{\sum_{i=1}^{m} A_i}(X)_\beta \cup \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$;
(3) $X \subseteq Y \Rightarrow \underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \subseteq \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$,
(4) $X \subseteq Y \Rightarrow \overline{PR}_{\sum_{i=1}^{m} A_i}(X)_\beta \subseteq \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$;
(5) $\underline{PR}_{\sum_{i=1}^{m} A_i}(X \cup Y)_\alpha \supseteq \underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \cup \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$,
(6) $\overline{PR}_{\sum_{i=1}^{m} A_i}(X \cap Y)_\beta \subseteq \overline{SR}_{\sum_{i=1}^{m} A_i}(X)_\beta \cap \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$.

Proof. It is easy to prove by Definition 3.8 and Proposition 3.11. □

4. The interrelationship among SGFRS, the OMGFRS and the PMGFRS

After the discussion of the properties of the OMGFRS and the PMGFRS, we will investigate the interrelationship among SGFRS, the OMGFRS and the PMGFRS in this section.

Proposition 4.1. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT$, $X \in F(U)$. Then the following properties hold.

(1) $\underline{OR}_{A+B}(X) = \underline{R_A}(X) \cup \underline{R_B}(X)$,
(2) $\overline{OR}_{A+B}(X) = \overline{R_A}(X) \cap \overline{R_B}(X)$;
(3) $\underline{OR}_{A+B}(X) \subseteq \underline{R_{A \cup B}}(X)$,
(4) $\overline{OR}_{A+B}(X) \supseteq \overline{R_{A \cup B}}(X)$.

Proof. (1) For any $x \in U$, $A, B \subseteq AT$ and $X \in F(U)$,

$$\begin{aligned}\underline{OR}_{A+B}(X)(x) &= \{\wedge\{X(y) \mid y \in [x]_A\}\} \vee \\ &\quad \{\wedge\{X(y) \mid y \in [x]_B\}\} \\ &= \underline{R_A}(X)(x) \vee \underline{R_B}(X)(x).\end{aligned}$$

That is to say $\underline{OR}_{A+B}(X) = \underline{R_A}(X) \cup \underline{R_B}(X)$ is true.

(2) For any $x \in U$, $A, B \subseteq AT$ and $X \in F(U)$,

$$\overline{OR_{A+B}}(X)(x) = \{\vee\{X(y) \mid y \in [x]_A\}\}\wedge \{\vee\{X(y) \mid y \in [x]_B\}\} = \overline{R_A}(X)(x) \wedge \overline{R_B}(X)(x).$$

So $\overline{OR_{A+B}}(X) = \overline{R_A}(X) \cap \overline{R_B}(X)$ holds.

(3) Since $[x]_{A\cup B} \subseteq [x]_A$ and $[x]_{A\cup B} \subseteq [x]_B$, then we have

$$\wedge\{X(y) \mid y \in [x]_A\} \leq \wedge\{X(y) \mid y \in [x]_{A\cup B}\}$$

and

$$\wedge\{X(y) \mid y \in [x]_B\} \leq \wedge\{X(y) \mid y \in [x]_{A\cup B}\}.$$

Therefore, we have $\{\wedge\{X(y) \mid y \in [x]_A\}\} \vee \{\wedge\{X(y) \mid y \in [x]_B\}\} \leq \wedge\{X(y) \mid y \in [x]_{A\cup B}\}$. That is to say, $\underline{OR_{A+B}}(X) \subseteq \underline{R_{A\cup B}}(X)$ holds.

(4) This item can be proved similarly as (3).

Proposition 4.2. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m, X \in F(U)$. Then the following properties hold.

(1) $\underline{OR_{\sum_{i=1}^{m} A_i}}(X) = \bigcup_{i=1}^{m} \underline{R_{A_i}}(X)$,
(2) $\overline{OR_{\sum_{i=1}^{m} A_i}}(X) = \bigcap_{i=1}^{m} \overline{R_{A_i}}(X)$;
(3) $\underline{OR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{R_{\bigcup_{i=1}^{m} A_i}}(X)$,
(4) $\overline{OR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{R_{\bigcup_{i=1}^{m} A_i}}(X)$.

Proof The proof of this proposition is similar to Proposition 4.1. □

Proposition 4.3. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT, X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{A+B}}(X) = \underline{R_A}(X) \cap \underline{R_B}(X)$,
(2) $\overline{PR_{A+B}}(X) = \overline{R_A}(X) \cup \overline{R_B}(X)$;
(3) $\underline{PR_{A+B}}(X) \subseteq \underline{R_{A\cup B}}(X)$,
(4) $\overline{PR_{A+B}}(X) \supseteq \overline{R_{A\cup B}}(X)$.

Proof. (1) For any $x \in U$, A, $B \subseteq AT$ and $X \in F(U)$,

$$\underline{PR_{A+B}}(X)(x) = \{\wedge\{X(y) \mid y \in [x]_A\}\}\wedge \{\wedge\{X(y) \mid y \in [x]_B\}\} = \underline{R_A}(X)(x) \wedge \underline{R_B}(X)(x).$$

That is to say, $\underline{PR_{A+B}}(X) = \underline{R_A}(X) \cap \underline{R_B}(X)$ is true.

(2) For any $x \in U$, A, $B \subseteq AT$ and $X \in F(U)$,

$$\overline{PR_{A+B}}(X)(x) = \{\vee\{X(y) \mid y \in [x]_A\}\}\vee \{\vee\{X(y) \mid y \in [x]_B\}\} = \overline{R_A}(X)(x) \vee \overline{R_B}(X)(x).$$

So $\overline{PR_{A+B}}(X) = \overline{R_A}(X) \cup \overline{R_B}(X)$ holds.

(3) Since $[x]_{A\cup B} \subseteq [x]_A$ and $[x]_{A\cup B} \subseteq [x]_B$, then we have

$$\wedge\{X(y) \mid y \in [x]_A\} \leq \wedge\{X(y) \mid y \in [x]_{A\cup B}\}$$

and

$$\wedge\{X(y) \mid y \in [x]_B\} \leq \wedge\{X(y) \mid y \in [x]_{A\cup B}\}.$$

Therefore, we have$\{\wedge\{X(y) \mid y \in [x]_A\}\} \wedge \{\wedge\{X(y) \mid y \in [x]_B\}\} \leq \wedge\{X(y) \mid y \in [x]_{A\cup B}\}$. That is to say, $\underline{PR_{A+B}}(X) \subseteq \underline{R_{A\cup B}}(X)$ holds.

(4) This item can be proved similarly to (3).

Proposition 4.4. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m, X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{\sum_{i=1}^{m} A_i}}(X) = \bigcap_{i=1}^{m} \underline{R_{A_i}}(X)$,
(2) $\overline{PR_{\sum_{i=1}^{m} A_i}}(X) = \bigcup_{i=1}^{m} \overline{R_{A_i}}(X)$;
(3) $\underline{PR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{R_{\bigcup_{i=1}^{m} A_i}}(X)$,
(4) $\overline{PR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{R_{\bigcup_{i=1}^{m} A_i}}(X)$.

Proof. The proof of this proposition is similar to Proposition 4.3. □

Proposition 4.5. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT, X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{A+B}}(X) \subseteq \underline{OR_{A+B}}(X) \subseteq \underline{R_{A\cup B}}(X)$;
(2) $\overline{PR_{A+B}}(X) \supseteq \overline{OR_{A+B}}(X) \supseteq \overline{R_{A\cup B}}(X)$.

Proof. It can be obtained by Definition 3.1, 3.3 and Proposition 4.1. □

Proposition 4.6. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m, X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{OR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{R_{\bigcup_{i=1}^{m} A_i}}(X)$;

(2) $\overline{PR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{OR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{R_{\bigcup_{i=1}^{m} A_i}}(X)$.

Proof. It can be obtained easily by Proposition 4.5. □

Proposition 4.7. Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT, X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{A+B}}(X) \subseteq \underline{R_A}(X)($ or $\underline{R_B}(X)) \subseteq \underline{OR_{A+B}}(X)$;
(2) $\overline{PR_{A+B}}(X) \supseteq \overline{R_A}(X)($ or $\overline{R_B}(X)) \supseteq \overline{OR_{A+B}}(X)$.

Proof. It can be obtained by the former two terms in Proposition 4.1, 4.3. □

Proposition 4.8. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m, X \in F(U)$. Then the following properties hold.

(1) $\underline{PR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{R_{A_i}}(X) \subseteq \underline{OR_{\sum_{i=1}^{m} A_i}}(X)$;

(2) $\overline{PR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{R_{A_i}}(X) \supseteq \overline{OR_{\sum_{i=1}^{m} A_i}}(X)$.

Proof. It can be obtained directly by Proposition 4.7.□

5. Measures of the OMGFRS and PMGFRS

The uncertainty of a set is due to the existence of the borderline region. The wider the borderline region of a set is, the lower the accuracy of the set is. To express this idea more precisely, some elementary measures are usually defined to describe the accuracy of a set. For the above discussed MGFRS, we introduce the accuracy measure of them in the following.

Definition 5.1. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m$. The optimistic and the pessimistic rough measure of the fuzzy set X by $\sum_{i=1}^{m} A_i$ are defined as

$$\rho^{F}_{\sum_{i=1}^{m} A_i}(X) = 1 - \frac{\left|\underline{OR_{\sum_{i=1}^{m} A_i}}(X)\right|}{\left|\overline{OR_{\sum_{i=1}^{m} A_i}}(X)\right|},$$

$$\rho^{S}_{\sum_{i=1}^{m} A_i}(X) = 1 - \frac{\left|\underline{PR_{\sum_{i=1}^{m} A_i}}(X)\right|}{\left|\overline{PR_{\sum_{i=1}^{m} A_i}}(X)\right|},$$

where $|\ .\ |$ means the cardinality of fuzzy set. If $\left|\overline{OR_{\sum_{i=1}^{m} A_i}}(X)\right| = 0$ or $\left|\overline{PR_{\sum_{i=1}^{m} A_i}}(X)\right| = 0$, we prescribe $\rho^{O}_{\sum_{i=1}^{m} A_i}(X) = 0$ or $\rho^{P}_{\sum_{i=1}^{m} A_i}(X) = 0$.

It is obvious that $0 \leq \rho^{O}_{\sum_{i=1}^{m} A_i}(X) \leq 1$ and $0 \leq \rho^{P}_{\sum_{i=1}^{m} A_i}(X) \leq 1$. If the fuzzy set X is the optimistic or the pessimistic multi-granulation definable, then $\rho^{O}_{\sum_{i=1}^{m} A_i}(X) = 0$ or $\rho^{P}_{\sum_{i=1}^{m} A_i}(X) = 0$.

Definition 5.2. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT,\ 1 \leq i \leq m$. For any $0 < \beta \leq \alpha \leq 1$, the optimistic $\alpha,\ \beta$ rough measure and the pessimistic $\alpha,\ \beta$ rough measure of the fuzzy set X by $\sum_{i=1}^{m} A_i$ are defined respectively as

$$\rho^{O}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} = 1 - \frac{\left|\underline{OR_{\sum_{i=1}^{m} A_i}}(X)_{\alpha}\right|}{\left|\overline{OR_{\sum_{i=1}^{m} A_i}}(X)_{\beta}\right|},$$

$$\rho^{P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} = 1 - \frac{\left|\underline{PR_{\sum_{i=1}^{m} A_i}}(X)_{\alpha}\right|}{\left|\overline{PR_{\sum_{i=1}^{m} A_i}}(X)_{\beta}\right|}.$$

If $\left|\overline{OR_{\sum_{i=1}^{m} A_i}}(X)_{\beta}\right| = 0$ or $\left|\overline{PR_{\sum_{i=1}^{m} A_i}}(X)_{\beta}\right| = 0$, we prescribe $\rho^{O}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} = 0$ or $\rho^{P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} = 0$.

To describe conveniently in the following context, we express the optimistic $\alpha,\ \beta$ rough measure and the pessimistic $\alpha,\ \beta$ rough measure of the fuzzy set X by $\sum_{i=1}^{m} A_i$ by using $\rho^{O,P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)}$.

For the information system $\mathcal{I} = (U, AT, F)$, denote

$$U/AT = \{X_1,\ X_2,\ \cdots,X_r\}.$$

Proposition 5.1. For any $0 < \beta \leq \alpha \leq 1$, the optimistic $\alpha,\ \beta$ rough measure and the pessimistic $\alpha,\ \beta$ rough measure of the fuzzy set X by $\sum_{i=1}^{m} A_i$ satisfy the following properties.

(1) $0 \leq \rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} \leq 1$;

(2) $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)}$ is non-decreasing for α and non-increasing for β;

(3) If $\bigvee_{i=1}^{r} \bigwedge_{x\in X_i} X(x) < \alpha$, then $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = 1$;

(4) If $\alpha = \beta$, $X(x) = c_i\ (\forall x \in X_i,\ i \leq r)$, i.e., if X is a constant fuzzy set in every equivalence class of U/AT, then $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = 0$.

Proof. (1) Since $0 < \beta \leq \alpha \leq 1$, then $\underline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha \subseteq \overline{OR}_{\sum_{i=1}^m A_i}(X)_\beta$ and $\underline{PR}_{\sum_{i=1}^m A_i}(X)_\alpha \subseteq \overline{PR}_{\sum_{i=1}^m A_i}(X)_\beta$. It is easy to obtain that $0 \leq \rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} \leq 1$.

(2) If $\alpha_1 < \alpha_2$, then $\underline{OR}_{\sum_{i=1}^m A_i}(X)_{\alpha_2} \subseteq \underline{OR}_{\sum_{i=1}^m A_i}(X)_{\alpha_1}$.

So we have

$$\left|\underline{OR}_{\sum_{i=1}^m A_i}(X)_{\alpha_2}\right| \leq \left|\underline{OR}_{\sum_{i=1}^m A_i}(X)_{\alpha_1}\right|.$$

And so is for the pessimistic multi-granulation fuzzy rough lower approximations. Therefore, $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha_1,\ \beta)} \leq \rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha_2,\ \beta)}$. When $\beta_1 < \beta_2$, we have $\overline{OR}_{\sum_{i=1}^m A_i}(X)_{\beta_2} \subseteq \overline{OR}_{\sum_{i=1}^m A_i}(X)_{\beta_1}$. Then

$$\left|\overline{OR}_{\sum_{i=1}^m A_i}(X)_{\beta_2}\right| \leq \left|\overline{OR}_{\sum_{i=1}^m A_i}(X)_{\beta_1}\right|.$$

And so is for the pessimistic multi-granulation fuzzy rough upper approximations. So $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\ \beta_1)} \geq \rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\ \beta_2)}$.

(3) When $\bigvee_{i=1}^{r} \bigwedge_{x\in X_i} X(x) < \alpha$, we have $\underline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha = \emptyset$ and $\underline{PR}_{\sum_{i=1}^m A_i}(X)_\alpha = \emptyset$. Then $\left|\underline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha\right| = 0$ and $\left|\underline{PR}_{\sum_{i=1}^m A_i}(X)_\alpha\right| = 0$. So $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = 1$.

(4) If $\alpha = \beta$ and $X(x) = c_i\ (\forall x \in X_i,\ i \leq r)$, then $\underline{OR}_{\sum_{i=1}^m A_i}(X) \equiv \overline{OR}_{\sum_{i=1}^m A_i}(X)$. Thus $\underline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha \equiv \overline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha$. That is, $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = 0$. □

Proposition 5.2. For any $0 < \beta \leq \alpha \leq 1$, X is a constant fuzzy set on U, i.e., $X(x) = \delta(\forall x \in U)$, then

$$\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = \begin{cases} 1,\ \beta < \delta < \alpha, \\ 0,\ \text{otherwise}. \end{cases}$$

Proof. When $\beta < \delta < \alpha$, we have $\underline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha$, $\underline{PR}_{\sum_{i=1}^m A_i}(X)_\alpha = \emptyset$, and $\overline{OR}_{\sum_{i=1}^m A_i}(X)_\beta$, $\overline{PR}_{\sum_{i=1}^m A_i}(X)_\beta = U$. Thus $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = 1$.

If $\delta < \beta \leq \alpha$, then $\underline{OR}_{\sum_{i=1}^m A_i}(X)_\alpha = \overline{OR}_{\sum_{i=1}^m A_i}(X)_\beta = \emptyset$ and $\underline{PR}_{\sum_{i=1}^m A_i}(X)_\alpha = \overline{PR}_{\sum_{i=1}^m A_i}(X)_\beta = \emptyset$. Thus $\rho^{O,P}_{\sum_{i=1}^m A_i}(X)_{(\alpha,\beta)} = 0$ from the prescript.

If $\beta \leq \alpha \leq \delta$, then $\underline{OR}_{\sum_{i=1}^{m} A_i}(X)_\alpha = \overline{OR}_{\sum_{i=1}^{m} A_i}(X)_\beta = U$ and $\underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha = \overline{PR}_{\sum_{i=1}^{m} A_i}(X)_\beta = U$. Thus $\rho^{O,P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} = 0$. □

Proposition 5.3. Let $X, Y \in F(U)$. If $X \subseteq Y$, $\overline{OR}_{\sum_{i=1}^{m} A_i}(X)_\beta = \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$ and $\overline{PR}_{\sum_{i=1}^{m} A_i}(X)_\beta = \overline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$, then

$$\rho^{O,P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \leq \rho^{O,P}_{\sum_{i=1}^{m} A_i}(Y)_{(\alpha,\beta)}.$$

Proof. For $X \subseteq Y$, we have $\underline{OR}_{\sum_{i=1}^{m} A_i}(X)_\alpha \subseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$ and $\overline{OR}_{\sum_{i=1}^{m} A_i}(X)_\beta = \overline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\beta$. And so is for the pessimistic multi-granulation fuzzy rough approximations. Thus the proposition holds. □

Proposition 5.4. Let $X, Y \in F(U)$. If $X \subseteq Y$, $\underline{OR}_{\sum_{i=1}^{m} A_i}(X)_\alpha = \underline{OR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$ and $\underline{PR}_{\sum_{i=1}^{m} A_i}(X)_\alpha = \underline{PR}_{\sum_{i=1}^{m} A_i}(Y)_\alpha$, then $\rho^{O,P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \leq \rho^{O,P}_{\sum_{i=1}^{m} A_i}(Y)_{(\alpha,\beta)}$.

Proof. The proof is similar to Proposition 5.3. □

Proposition 5.5. Let $\mathcal{I} = (U, AT, F)$ be an information system, $A_i \subseteq AT$, $1 \leq i \leq m$. The optimistic rough measure, the pessimistic rough measure of the fuzzy set X by $\sum_{i=1}^{m} A_i$ and the rough measure of the fuzzy set X by A_i have the following relations.

$$\rho^{P}_{\sum_{i=1}^{m} A_i}(X) \geq \rho_{A_i}(X) \geq \rho^{O}_{\sum_{i=1}^{m} A_i}(X) \geq \rho_{\bigcup_{i=1}^{m} A_i}(X).$$

Proof. It is easy to prove by Proposition 4.8 and Definition 5.1. □

Example 5.1. (Continued from Example 3.1 and 3.2) We can compute the optimistic rough measure, the pessimistic rough measure of D by A and B and compare with the rough measure of D by A or B. It follows that

$$\rho^O_{A+B}(D) = 1 - \frac{|\underline{OR_{A+B}}(D)|}{|\overline{OR_{A+B}}(D)|} = 1 - \frac{6.2}{7.2} \approx 0.139,$$

$$\rho^P_{A+B}(D) = 1 - \frac{|\underline{PR_{A+B}}(D)|}{|\overline{PR_{A+B}}(D)|} = 1 - \frac{5.7}{7.7} \approx 0.260,$$

$$\rho_A(D) = 1 - \frac{|\underline{R_A}(D)|}{|\overline{R_A}(D)|} = 1 - \frac{6}{7.3} \approx 0.178,$$

$$\rho_B(D) = 1 - \frac{|\underline{R_B}(D)|}{|\overline{R_B}(D)|} = 1 - \frac{5.9}{7.6} \approx 0.223,$$

$$\rho_{A\cup B}(D) = 1 - \frac{|\underline{R_{A\cup B}}(D)|}{|\overline{R_{A\cup B}}(D)|} = 1 - \frac{6.2}{6.9} \approx 0.101.$$

Clearly, we have

$$\rho^P_{A+B}(D) \geq \rho_A(D) \geq \rho^O_{A+B}(D) \geq \rho_{A\cup B}(D)$$

and

$$\rho^P_{A+B}(D) \geq \rho_B(D) \geq \rho^O_{A+B}(D) \geq \rho_{A\cup B}(D).$$

Proposition 5.6. For any $0 < \beta \leq \alpha \leq 1$, the optimistic $\alpha,\ \beta$ rough measure, pessimistic $\alpha,\ \beta$ rough measure of the fuzzy set X by $\sum_{i=1}^{m} A_i$ and the $\alpha,\ \beta$ rough measure of the fuzzy set X by A_i have the following relations. $\rho^{P}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \geq \rho_{A_i}(X)_{(\alpha,\beta)} \geq \rho^{O}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \geq \rho_{\bigcup_{i=1}^{m} A_i}(X)_{(\alpha,\beta)}$. *Proof.* From Proposition 4.6, 4.8 and

Definition 3.4, 3.8, we can obtain that

$$\underline{PR}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \subseteq \underline{R_{A_i}}(X)_{(\alpha,\beta)} \subseteq \underline{OR}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \subseteq \underline{R}_{\bigcup_{i=1}^{m} A_i}(X)_{(\alpha,\beta)}$$

and

$$\overline{PR}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \supseteq \overline{R_A}(X)_{(\alpha,\beta)} \supseteq \overline{OR}_{\sum_{i=1}^{m} A_i}(X)_{(\alpha,\beta)} \supseteq \overline{R}_{\bigcup_{i=1}^{m} A_i}(X)_{(\alpha,\beta)}.$$

Then we have

$$\frac{\left|\underline{PR_{\sum_{i=1}^{m} A_i}}(X)_\alpha\right|}{\left|\overline{PR_{\sum_{i=1}^{m} A_i}}(X)_\beta\right|} \leq \frac{\left|\underline{R_{A_i}}(X)_\alpha\right|}{\left|\overline{R_{A_i}}(X)_\beta\right|} \leq \frac{\left|\underline{OR_{\sum_{i=1}^{m} A_i}}(X)_\alpha\right|}{\left|\overline{OR_{\sum_{i=1}^{m} A_i}}(X)_\beta\right|}$$

$$\leq \frac{\left|\underline{R_{\bigcup_{i=1}^{m} A_i}}(X)_{(\alpha,\beta)}\right|}{\left|\overline{R_{\bigcup_{i=1}^{m} A_i}}(X)_{(\alpha,\beta)}\right|}.$$

Thus the proposition hold. □

Example 5.2. (Continued from Example 3.1 and 3.2) Let $\alpha = 0.7,\ \beta = 0.6$, we can compute the optimistic $\alpha,\ \beta$ rough measure, the pessimistic $\alpha,\ \beta$ rough measure of D by A and B and compare with the $\alpha,\ \beta$ rough measure of D by A or B. It follows that

$$\begin{aligned}
\rho^O_{A+B}(D)_{(0.7,0.6)} &= 1 - \frac{|\underline{\underline{FR_{A+B}}}(D)_{(0.7,0.6)}|}{|\overline{\overline{FR_{A+B}}}(D)_{(0.7,0.6)}|} \\
&= 1 - \frac{4}{9} = \frac{5}{9}, \\
\rho^P_{A+B}(D)_{(0.7,0.6)} &= 1 - \frac{|\underline{\underline{SR_{A+B}}}(D)_{(0.7,0.6)}|}{|\overline{\overline{SR_{A+B}}}(D)_{(0.7,0.6)}|} \\
&= 1 - \frac{3}{10} = \frac{3}{10}, \\
\rho_A(D)_{(0.7,0.6)} &= 1 - \frac{|\underline{R_A}(D)_{(0.7,0.6)}|}{|\overline{R_A}(D)_{(0.7,0.6)}|} = 1 - \frac{4}{10} \\
&= \frac{6}{10}, \\
\rho_B(D)_{(0.7,0.6)} &= 1 - \frac{|\underline{R_B}(D)_{(0.7,0.6)}|}{|\overline{R_B}(D)_{(0.7,0.6)}|} = 1 - \frac{3}{9} \\
&= \frac{6}{9}, \\
\rho_{A\cup B}(D)_{(0.7,0.6)} &= 1 - \frac{|\underline{R_{A\cup B}}(D)_{(0.7,0.6)}|}{|\overline{R_{A\cup B}}(D)_{(0.7,0.6)}|} = 1 - \frac{4}{9} \\
&= \frac{5}{9}.
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
\rho^P_{A+B}(D)_{(0.7,0.6)} \geq \rho_A(D)_{(0.7,0.6)} &\geq \rho^O_{A+B}(D)_{(0.7,0.6)} \\
&\geq \rho_{A\cup B}(D)_{(0.7,0.6)}
\end{aligned}$$

and

$$\begin{aligned}
\rho^P_{A+B}(D)_{(0.7,0.6)} \geq \rho_B(D)_{(0.7,0.6)} &\geq \rho^O_{A+B}(D)_{(0.7,0.6)} \\
&\geq \rho_{A\cup B}(D)_{(0.7,0.6)}.
\end{aligned}$$

6. Conclusions

In this paper, we combined multi-granulation rough sets theory and fuzzy sets theory in order to dealing with problems of uncertainty and imprecision easily. The theory of fuzzy set mainly focuses on the fuzziness of knowledge while the theory of rough set on the roughness of knowledge. Because of the complement of the two types of theory, fuzzy rough set models are investigated to solve practical problem. Besides, multi-granulation rough sets models have been proposed by Professor Qian which also are studied from the perspective of granular computing. The contribution of this paper have constructed two different types of multi-granulation fuzzy rough set associated with granular computing, in which the approximation operators are defined based on multiple equivalence relations. What's more, we make conclusions that rough sets, fuzzy rough set models and multi-granulation rough set models are special cases of the two types of multi-granulation fuzzy rough set by analyzing the definitions of them. More properties of the two types of fuzzy rough set are discussed and comparison are made with single-granulation fuzzy rough set(SGFRS). Finally, we make a description of the accuracy of a set by defining the rough measure and $(\alpha,\ \beta)$-rough measure and discussing the corresponding properties. The construction of the new types of fuzzy rough set models is an extension in view of granular computing and is meaningful compared with the generalization of rough set theory.

Acknowledgments

This work is supported by National Natural Science Foundation of China (No. 61105041, 71071124 and 11001227), Postdoctoral Science Foundation of China (No. 20100481331) and Natural Science Foundation Project of CQ CSTC(No. cstc2011jjA40037).

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