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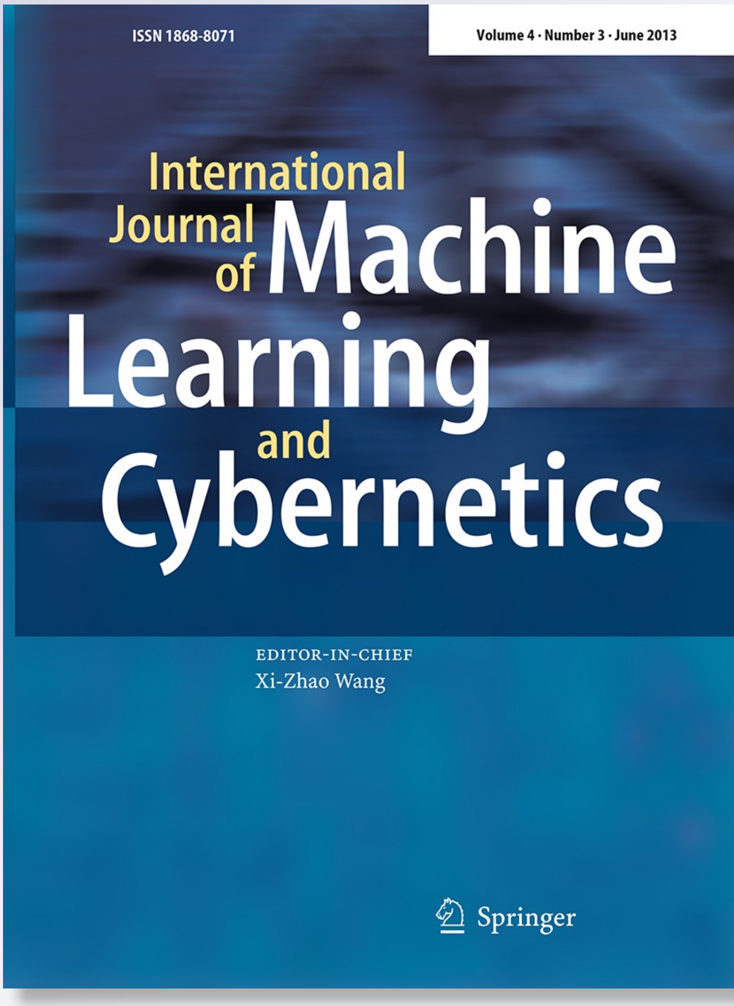

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ORIGINAL ARTICLE

Lattice-valued information systems based on dominance relation

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Abstract In this paper, as a naturally generalization of classical information systems, lattice-valued information systems based on dominance relation is proposed. An approach for ranking all objects in this system is constructed consequently, and decision makers can find objects with better property to make an useful and effective decision. In addition, the rough set approach to lattice-valued information systems based on dominance relation is established. And evidence theories in this system are formulated for the analysis of lattice-valued information systems based on dominance relation. What is more, in order to acquire concise knowledge representation and extract much simpler decision rules, the methods of attribute reductions based on discernibility matrix and evidence theory are investigated carefully. These results will be helpful for decision-making analysis in lattice-valued information systems based on dominance relation.

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1 Introduction

Rough set theory proposed by Pawlak [26, 27, 29] is an extension of the classical set theory and can be regarded as a soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. The theory has been found its successive applications in the fields of pattern recognition [39], medical diagnosis [40, 19], data mining [1, 5], conflict analysis [28], algebra [6, 7, 46], and so on [30]. Recently, the theory has generated a great deal of interest among more and more researchers.

It was well known that the rough set theory is based upon the classification mechanism, from which the classification can be viewed as an equivalence relation and knowledge granule induced by the equivalence relation can be viewed as a partition of the universe of discourse. In rough set theory, two classical sets, so-called lower and upper approximations or Pawlak' rough approximations, are constructed and any subset of universe of discourse can be expressed by them. It is a objective mathematical tool to deal with practical problems because prior knowledge but data base would not be applied in this processing.

Whereas the existence of uncertainty and complexity of particular problem, the problem would not be settled perfectly by means of classical rough set. Much works have been done in recent years to generalize the classical rough set model [54]. One is to substitute general binary relation for equivalence relation, such as tolerance relation [20, 21, 35], similarity relation [37, 49], dominance relation [12–17], neighborhood operators [50], and others [23, 41, 45, 47]. Another important contribution is to generalize the attribute value from real numbers to various forms, such as interval value [11, 22, 32, 38], set value [18, 31], etc. Especially, Dubois and Prade [9, 10] combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Banerjee and Pal [2] investigated the roughness of a fuzzy set making use of the concept of rough fuzzy sets. And moreover, many works about rough sets and fuzzy sets are researched carefully [3, 4, 24, 25, 42, 43].

Just as the rough set theory making use of lower and upper approximation operators to express the knowledge in the universe of discourse, a dual pair of belief and plausibility functions are also used to make numeric measurement in the Dempster–Shafer theory of evidence, which was first developed by Dempster's concept of lower and upper probabilities [8] and later extended as a theory by Shafer [33]. It has been demonstrated that various belief structures that is the basic representational structure in this theory are associated with various rough approximation spaces such that the different dual pairs of lower and upper approximation operators induced by rough approximation spaces may be used to interpret the corresponding dual pairs of belief and plausibility functions induced by belief structures [36, 44, 45, 51].

The purpose of this paper is to study a complex information system, which is a combination of the ordered information systems and the information systems with multiform valued field from objects to attributes. We call this new system the lattice-valued information system based on dominance relation. By redefining the dominance relation in this system, the method for ranking all objects is constructed, in which case decision makers could find objects with better property to make an useful and effective decision. Consequently, the rough set approach to lattice-valued information systems based on dominance relation is proposed and some properties are studied. Also, due to the strong connection between the Dempster–Shafer theory of evidence and rough set theory, the former theory is introduced to lattice-valued information systems based on dominance relation by applying relation partition function. Attribute reduction, as an important research problem of rough set theory, are investigated separately from the viewpoint of discernibility matrix and the Dempster–Shafer theory of evidence (to name evidence theory for short).

The rest of this paper is organized as follows. Some preliminary concepts required in our work are briefly recalled in Sect. 2. In Sect. 3 the lattice-valued information systems based on dominance relation is proposed and some of its properties are discussed carefully, and after that a method to rank all objects in lattice-valued information systems based on dominance relation is introduced. With introducing relation partition function, the believe and plausibility functions in lattice-valued information system based on dominance relation are defined in Sect. 4. Attribute reductions based on discernibility matrix and evidence theory are investigated carefully in Sect. 5. Finally, a summary and outlook for further research concluded our work in Sect. 6.

2 Rough sets and ordered information systems

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in the source papers [12–17], and a description has also been made in [52].

The notion of information system (sometimes called data tables, attribute valued systems, knowledge representation systems, etc.) provides a convenient basis for the representation of objects in terms of their attributes.

An information system is a quadruple $\mathcal{I} = (U, AT, V, f)$, where U is a non-empty finite set with n objects, $\{u_1, u_2, \ldots, u_n\}$, called the universe of discourse; $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with m attributes; $V = \bigcup_{a\in AT} V_a$ and V_a is the domain of attribute a;$f : U \times AT \longrightarrow V$ is an information function [27] such that $f(u, a) \in V_a$ for any $a \in AT, u \in U$. A decision table is a special case of an information system in which, among the attributes, we distinguish on called a decision attribute. The other attributes are called condition attributes. Therefore, $\mathcal{I} = (U, C\bigcup\{d\}, V, f)$ be a decision table, where set C and $\{d\}$ be condition attributes set and decision attribute set, respectively.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

Definition 2.1 (See [12–17]) An information system is called an ordered information system if all condition attributes are criterion.

Assumed that the domain of a criterion $a \in AT$ is complete pre-ordered by an outranking relation $\succcurlyeq_a$, then $u\succcurlyeq_{av}$ means that u is at least as good as (outranks) v with respect to the criterion a, and we can say that u dominates v or v is dominated by u. Being of type gain, that is $u\succcurlyeq_a v \Leftarrow\Rightarrow f(u, a) \geq f(v, a)$ (according to increasing preference) or $u\succcurlyeq_a v \Longleftrightarrow f(u, a) \leq f(v, a)$ (according to decreasing preference).Without any loss of generality and for simplicity, in the following we only consider condition attributes with increasing preference.

For a subset of attributes $A \subseteq AT$, we define $u\succcurlyeq_A v \Longleftrightarrow u\succcurlyeq_a v$ for any $a \in A$. That is, u dominates v with respect to all attributes in A. Generally speaking, we denote an ordered information system by $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$, and $\mathcal{I}^{\succcurlyeq}$ for short.

For a given ordered information system, we say that u dominates v with respect to $A \subseteq AT$ if $u \succcurlyeq_A v$, and denote by $uR_A^{\succcurlyeq}v$. That is

$$R_A^{\succcurlyeq} = \{(u,v) \in U \times U \mid u \succcurlyeq_A v\} = \{(u,v) \in U \times U \mid f(u,a) \geq f(v,a)\ \forall a \in A\},$$

and $R_A^{\succcurlyeq}$ are called a dominance relation of $\mathcal{I}^{\succcurlyeq}$. Let denote

$$[u_i]_A^{\succcurlyeq} = \{u_j \in U \mid (u_j, u_i) \in R_A^{\succcurlyeq}\},$$
$$U/R_A^{\succcurlyeq} = \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \ldots, [u_n]_A^{\succcurlyeq}\},$$

where $i \in \{1,2,\ldots,n\}$, then $[u_i]_A^{\succcurlyeq}$ will be called a dominance class or the granularity of information, and $U/R_A^{\succcurlyeq}$ be called a knowledge of U with respect to attribute set A.

From above description, the following properties of dominance relation in ordered information system are trivial.

Proposition 2.1 (See [12–17]) *Let $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$ be an ordered information system and $B, A \subseteq AT$, then we have that*

(1) *$R_A^{\succcurlyeq}$ is reflective, transitive, but not symmetric.*
(2) *If $B \subseteq A \subseteq AT$, then $R_{AT}^{\succcurlyeq} \subseteq R_A^{\succcurlyeq} \subseteq R_B^{\succcurlyeq}$.*

Similarly, for the dominance class induced by the dominance relation $R_A^{\succcurlyeq}$, the following properties are still correct.

Proposition 2.2 (See [12–17]) *Let $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$ be an ordered information system and $B, A \subseteq AT$, then we have that*

(1) *If $B \subseteq A \subseteq AT$, then $[u]_{AT}^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq} \subseteq [u]_B^{\succcurlyeq}$ for any $u \in U$;*
(2) *If $v \in [u]_A^{\succcurlyeq}$, then $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ and $[u]_A^{\succcurlyeq} = \bigcup\{[v]_A^{\succcurlyeq} \mid v \in [u]_A^{\succcurlyeq}\}$;*
(3) *$[u]_{AT}^{\succcurlyeq} = [v]_{AT}^{\succcurlyeq}$ if and only if $f(u,a) = f(v,a)$ for any $a \in AT$;*
(4) *$|[u]_{AT}^{\succcurlyeq}| \geq 1$ for any $u \in U$,*

where $|[u]_{AT}^{\succcurlyeq}|$ denotes the cardinality of the set $[u]_{AT}^{\succcurlyeq}$.

For any subset $X \subseteq U$ and $A \subseteq AT$ in $\mathcal{I}^{\succcurlyeq}$, the lower and upper approximation of X with respect to a dominance relation $R_A^{\succcurlyeq}$ could be defined as

$$\underline{R_A^{\succcurlyeq}}(X) = \{u_i \in U \mid [u_i]_A^{\succcurlyeq} \subseteq X\},$$
$$\overline{R_A^{\succcurlyeq}}(X) = \left\{u_i \in U \mid [u_i]_A^{\succcurlyeq} \bigcap X \neq \emptyset\right\}.$$

3 Lattice-valued information systems

In this section, we first introduce the dominance relation based lattice-valued information systems (LVIS-DR) and investigate the problem of approximation operators of it. After that, a method is proposed to rank all objects in this system.

3.1 Dominance relation based lattice-valued information systems

A lattice-valued information system $\mathcal{L} = (U, AT, V, f)$ is an information system, where U is a non-empty finite set with n objects, $\{u_1, u_2, \ldots, u_n\}$, called the universe of discourse; $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with m attributes; $V = \bigcup_{a \in AT} V_a$ and V_a is the domain of attribute $a; f: U \times AT \longrightarrow V$ is an information function such that $f(u,a) \in V_a$ for any $u \in U$, where $(V_a, \geqslant)$ is a finite lattice and "$\geqslant$" is the partial order relation on V_a for any $a \in AT$.

In practical decision making analysis, we always consider a dominance relation between objects that are possibly dominant in terms of values of attributes set in lattice-valued information systems. Generally speaking, an increasing preference or a decreasing preference is taken into consideration, in which case the attribute with such property is a criterion. Therefore, one can find that all condition attributes in a lattice-valued information system are criterion and from which a lattice-valued information system is still an ordered information system.

In the following, we introduce a dominance relation that identifies dominance classes to a lattice-valued information system. Assumed that the domain of a criterion $a \in AT$ is complete pre-ordered by an outranking relation $\succcurlyeq_a$, then $u \succcurlyeq_a v$ means that u is at least as good as v with respect to criterion a, and we can say that u dominates v or v is dominated by u and the notation $u \succcurlyeq_A v$ represents the meaning of $u \succcurlyeq_{av}$ for all $a \in A \subseteq AT$. Naturally, the dominance relation with respect to A(expressed as $\mathscr{R}_A^{\succcurlyeq}$) in lattice-valued information system can be denoted as

$$\mathscr{R}_A^{\succcurlyeq} = \{(u,v) \in U \times U \mid u \succcurlyeq_A v\}.$$

Moreover, the dominance class of every $u_i \in U$ with respect to A is

$$[u_i]_A^{\succcurlyeq} = \{u_j \in U \mid (u_j, u_i) \in \mathscr{R}_A^{\succcurlyeq}\}$$

and the knowledge of U with respect to A is

$$U/\mathscr{R}_A^{\succcurlyeq} = \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \ldots, [u_n]_A^{\succcurlyeq}\}.$$

Note that for simplicity and without any loss of generality, we still use notation "$\succcurlyeq$" to represent the partial relation on finite lattice V_a for any $a \in AT$, and hence the dominance relation based lattice-valued information system be expressed as $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$, and $\mathcal{L}^{\succcurlyeq}$ for short.

Example 3.1 A lattice-valued information system based on dominance relation is showed in Table 1, where $U = \{u_1, u_2, \ldots, u_{10}\}$ and $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

According to above expression, we can find $V_{a_1} = \{1,2,3\}$ is a finite lattice with real numbers, where the partial order relation on V_{a_1} is "$\geq$" between two real

Table 1 A LVIS-DR

U	a_1	a_2	a_3	a_4	a_5
u_1	2	0.7	[0.4, 0.7]	{1}	(0.1, 0.8)
u_2	3	0.8	[0.6, 0.8]	{1,2}	(0.1, 0.8)
u_3	2	0.6	[0.1, 0.6]	{1}	(0.1, 0.8)
u_4	2	0.7	[0.8, 0.9]	{1}	(0.4, 0.3)
u_5	1	0.6	[0.1, 0.6]	{1,2}	(0.4, 0.3)
u_6	1	0.6	[0.6, 0.8]	{1,2}	(1.0, 0.0)
u_7	3	0.7	[0.4, 0.7]	{1,2}	(0.4, 0.3)
u_8	1	0.6	[0.8, 0.9]	{1,2}	(1.0, 0.0)
u_9	1	0.6	[0.1, 0.6]	{1}	(0.4, 0.3)
u_{10}	3	0.7	[0.4, 0.7]	{1,2}	(1.0, 0.0)

numbers as its elements. So the dominance relation on U according to attribute a_1 can be defined as

$$\mathscr{R}^{\succcurlyeq}_{\{a_1\}} = \{(u,v) \in U \times U \mid f(u,a_1) \geq f(v,a_1)\}.$$

The domain $V_{a_2} = \{0.6, 0.7, 0.8\}$ is a finite lattice with numerical numbers located in interval [0, 1], from which the partial order relation on V_{a_2} is "$\geq$" between two fuzzy elements. And the dominance relation on U according to attribute a_2 can be defined as

$$\mathscr{R}^{\succcurlyeq}_{\{a_2\}} = \{(u,v) \in U \times U \mid f(u,a_2) \geq f(v,a_2)\}.$$

The domain $V_{a_3} = \{[0.4, 0.7], [0.6, 0.8], [0.1, 0.6], [0.8, 0.9]\}$ is a finite lattice with interval-valued elements, and the dominance relation on it can be defined as

$$\mathscr{R}^{\succcurlyeq}_{\{a_3\}} = \{(u,v) \in U \times U \mid f^{\pm}(u,a_3) \geq f^{\pm}(v,a_3)\},$$

where $f^{\pm}(u,a_3) \geq f^{\pm}(v,a_3)$ if and only if $f^{+}(u,a_3) \geq f^{+}(v,a_3)$ and $f^{-}(u,a_3) \geq f^{-}(v,a_3)$, $f^{+}(u,a_3)$ is the right endpoint of $f(u,a_3)$ and $f^{-}(u,a_3)$ is the left endpoint of $f(u,a_3)$, to name a couple for explanation.

The domain $V_{a_4} = \{\{1\}, \{1,2\}\}$ is a finite lattice with set-valued elements, where the partial order relation on V_{a_4} is "$\supseteq$" between two sets. Thus the dominance relation on U according to attribute a_4 can be defined as

$$\mathscr{R}^{\succcurlyeq}_{\{a_4\}} = \{(u,v) \in U \times U \mid f(u,a_4) \supseteq f(v,a_4)\}.$$

The domain $V_{a_5} = \{(0.1, 0.8), (0.4, 0.3), (1.0, 0.0)\}$ is a finite lattice with intuitionistic fuzzy sets formed elements, i.e., $f(u,a_5) = (\mu(u,a_5), \nu(u,a_5))$, where $\mu(u,a_5)$ is the membership function and $\nu(u,a_5)$ is the non-membership function of u respectively. Then the dominance relation on U according to attribute a_5 can be defined as

$$\mathscr{R}^{\succcurlyeq}_{\{a_5\}} = \{(u,v) \in U \times U \mid \mu(u,a_5) \geq \mu(v,a_5) \text{ and } \nu(u,a_5) \leq \nu(v,a_5)\}.$$

Definition 3.1 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $B, A \subseteq AT$.

(1) If $[u]^{\succcurlyeq}_{B} = [u]^{\succcurlyeq}_{A}$ for any $u \in U$, then we call that knowledge $U/\mathscr{R}^{\succcurlyeq}_{B}$ is equal to $U/\mathscr{R}^{\succcurlyeq}_{A}$, denoted by $U/\mathscr{R}^{\succcurlyeq}_{B} = U/\mathscr{R}^{\succcurlyeq}_{A}$.
(2) If $[u]^{\succcurlyeq}_{B} \subseteq [u]^{\succcurlyeq}_{A}$ for any $u \in U$, then we call that knowledge $U/\mathscr{R}^{\succcurlyeq}_{B}$ is finer than $U/\mathscr{R}^{\succcurlyeq}_{A}$, denoted by $U/\mathscr{R}^{\succcurlyeq}_{B} \subseteq U/\mathscr{R}^{\succcurlyeq}_{A}$.
(3) If $[u]^{\succcurlyeq}_{B} \subseteq [u]^{\succcurlyeq}_{A}$ for any $u \in U$ and $[u]^{\succcurlyeq}_{B} \neq [u]^{\succcurlyeq}_{A}$ for some $u \in U$, then we call that knowledge $U/\mathscr{R}^{\succcurlyeq}_{B}$ is properly finer than $U/\mathscr{R}^{\succcurlyeq}_{A}$, denoted by $U/\mathscr{R}^{\succcurlyeq}_{B} \subset U/\mathscr{R}^{\succcurlyeq}_{A}$.

From the definition of $\mathscr{R}^{\succcurlyeq}_{A}$ and $[u]^{\succcurlyeq}_{A}$, the following properties can be obtained directly.

Proposition 3.1 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A, B \subseteq AT$, then we can get*

(1) $\mathscr{R}^{\succcurlyeq}_{A} = \bigcap_{a \in A} \mathscr{R}^{\succcurlyeq}_{\{a\}}$;
(2) $\mathscr{R}^{\succcurlyeq}_{A}$ *is reflective, transitive, but not symmetric*;
(3) *If $B \subseteq A \subseteq AT$, then $\mathscr{R}^{\succcurlyeq}_{AT} \subseteq \mathscr{R}^{\succcurlyeq}_{A} \subseteq \mathscr{R}^{\succcurlyeq}_{B}$.*

Proposition 3.2 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $B, A \subseteq AT$, then we have that*

(1) *If $B \subseteq A \subseteq AT$, then $[u]^{\succcurlyeq}_{AT} \subseteq [u]^{\succcurlyeq}_{A} \subseteq [u]^{\succcurlyeq}_{B}$ for any $u \in U$.*
(2) *If $u \in [v]^{\succcurlyeq}_{A}$, then $[u]^{\succcurlyeq}_{A} \subseteq [v]^{\succcurlyeq}_{A}$ and $[v]^{\succcurlyeq}_{A} = \bigcup\{[u]^{\succcurlyeq}_{A} \mid u \in [v]^{\succcurlyeq}_{A}\}$.*
(3) *$[u]^{\succcurlyeq}_{AT} = [v]^{\succcurlyeq}_{AT}$ if and only if $f(u,a) = f(v,a)$ for any $a \in AT$.*
(4) *$|[u]^{\succcurlyeq}_{AT}| \geq 1$ for any $u \in U$.*

These properties mentioned above can be understood through the following example.

Example 3.2 (Continued From Example 3.1) By computing, One can obtain that

$$[u_1]_{AT}^{\succcurlyeq} = \{u_1, u_2, u_4, u_7, u_{10}\}, \quad [u_2]_{AT}^{\succcurlyeq} = \{u_2\},$$
$$[u_3]_{AT}^{\succcurlyeq} = \{u_1, u_2, u_3, u_4, u_7, u_{10}\}, \quad [u_4]_{AT}^{\succcurlyeq} = \{u_4\},$$
$$[u_5]_{AT}^{\succcurlyeq} = \{u_5, u_6, u_7, u_8, u_{10}\}, \quad [u_6]_{AT}^{\succcurlyeq} = \{u_6, u_8\},$$
$$[u_7]_{AT}^{\succcurlyeq} = \{u_7, u_{10}\}, \quad [u_8]_{AT}^{\succcurlyeq} = \{u_8\},$$
$$[u_9]_{AT}^{\succcurlyeq} = \{u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}, \quad [u_{10}]_{AT}^{\succcurlyeq} = \{u_{10}\}.$$

If take $A = \{a_2, a_3, a_5\}$, we can get that

$$[u_1]_{A}^{\succcurlyeq} = \{u_1, u_2, u_4, u_7, u_{10}\}, \quad [u_2]_{A}^{\succcurlyeq} = \{u_2\},$$
$$[u_3]_{A}^{\succcurlyeq} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}, \quad [u_4]_{A}^{\succcurlyeq} = \{u_4\},$$
$$[u_5]_{A}^{\succcurlyeq} = \{u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}, \quad [u_6]_{A}^{\succcurlyeq} = \{u_6, u_8\},$$
$$[u_7]_{A}^{\succcurlyeq} = \{u_4, u_7, u_{10}\}, \quad [u_8]_{A}^{\succcurlyeq} = \{u_8\},$$
$$[u_9]_{A}^{\succcurlyeq} = \{u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}, \quad [u_{10}]_{A}^{\succcurlyeq} = \{u_{10}\}.$$

Obviously, $[u]_{AT}^{\succcurlyeq} \subseteq [u]_{A}^{\succcurlyeq}$ for any $u \in U$ and $\mathscr{R}_{AT}^{\succcurlyeq} \subset \mathscr{R}_{A}^{\succcurlyeq}$. From this example, we can easily verify above propositions of lattice-valued information systems based on dominance relation.

Similar to classical information systems, the upper and lower approximation operators in LVIS-DR can be depicted as

$$\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) = \{u_i \in U \mid [u_i]_{A}^{\succcurlyeq} \subseteq X\};$$
$$\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) = \{u_i \in U \mid [u_i]_{A}^{\succcurlyeq} \bigcap X \neq \emptyset\},$$

where $\mathscr{R}_{A}^{\succcurlyeq}$ is the dominance relation with respect to $A \subseteq AT$.

Moreover, what follows are trivial but meaningful.

Proposition 3.3 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. For any $X \subseteq U$, we can get that*

(1) $\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \subseteq \underline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)$ *and* $\overline{\mathscr{R}_{AT}^{\succcurlyeq}}(X) \subseteq \overline{\mathscr{R}_{A}^{\succcurlyeq}}(X)$;
(2) *If* $\mathscr{R}_{A}^{\succcurlyeq} = \mathscr{R}_{AT}^{\succcurlyeq}$, *then* $\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) = \underline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)$ *and* $\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) = \overline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)$.

Proposition 3.4 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. For any $X, Y \subseteq U$, the following results hold.*

$$(1L)\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \subseteq X \quad \text{(Contraction)}$$
$$(1U)\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \supseteq X \quad \text{(Extension)}$$
$$(2)\ \underline{\mathscr{R}_{A}^{\succcurlyeq}}(\sim X) = \sim \overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \quad \text{(Duality)}$$
$$\overline{\mathscr{R}_{A}^{\succcurlyeq}}(\sim X) = \sim \underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \quad \text{(Duality)}$$
$$(3L)\underline{\mathscr{R}_{A}^{\succcurlyeq}}(\emptyset) = \emptyset \quad \text{(Normality)}$$
$$(3U)\overline{\mathscr{R}_{A}^{\succcurlyeq}}(\emptyset) = \emptyset \quad \text{(Normality)}$$
$$(4L)\underline{\mathscr{R}_{A}^{\succcurlyeq}}(U) = U \quad \text{(Co-normality)}$$
$$(4U)\overline{\mathscr{R}_{A}^{\succcurlyeq}}(U) = U \quad \text{(Co-normality)}$$
$$(5L)\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X \bigcap Y) = \underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \bigcap \underline{\mathscr{R}_{A}^{\succcurlyeq}}(Y) \quad \text{(Multiplication)}$$
$$(5U)\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X \bigcup Y) = \overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \bigcup \overline{\mathscr{R}_{A}^{\succcurlyeq}}(Y) \quad \text{(Addition)}$$
$$(6L)\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X \bigcup Y) \supseteq \underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \bigcup \underline{\mathscr{R}_{A}^{\succcurlyeq}}(Y) \quad \text{(F-Multiplication)}$$
$$(6U)\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X \bigcap Y) \subseteq \overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \bigcap \underline{\mathscr{R}_{A}^{\succcurlyeq}}(Y) \quad \text{(F-Addition)}$$
$$(7L)X \subseteq Y \Longrightarrow \underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \subseteq \underline{\mathscr{R}_{A}^{\succcurlyeq}}(Y) \quad \text{(Monotone)}$$
$$(7U)X \subseteq Y \Longrightarrow \overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \subseteq \overline{\mathscr{R}_{A}^{\succcurlyeq}}(Y) \quad \text{(Monotone)}$$
$$(8L)\underline{\mathscr{R}_{A}^{\succcurlyeq}}(\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X)) = \underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \quad \text{(Idempotency)}$$
$$(8U)\overline{\mathscr{R}_{A}^{\succcurlyeq}}(\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X)) = \overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) \quad \text{(Idempotency)}$$

Note that the proof of them are similar to the case of Properties in [34] and hence we omitted it here.

Another topic in rough set theory is uncertainty measure of a rough set. Uncertainty of a rough set is due to the existence of the boundary region. The greater the boundary region of a rough set is, the lower the accuracy of the rough set is, and vice versa. In order to measure the imprecision of a rough set induced by dominance relation $\mathscr{R}_{A}^{\succcurlyeq}$ in LVIS-DR, the concept of precision degree is introduced in the following.

Definition 3.2 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. The precision degree of any $X \subseteq U$ with respect to $A \subseteq AT$ is defined as

$$\alpha_{A}^{\succcurlyeq}(X) = \frac{|\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X)|}{|\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X)|}.$$

By the definition, we can easily find that the precision degree $\alpha_{A}^{\succcurlyeq}(X)$ reflects our acquaintance with the knowledge about X under the dominance relation $\mathscr{R}_{A}^{\succcurlyeq}$ in LVIS-DR. Obviously, for any $A \subseteq AT$ and $X \subseteq U$, $0 \leq \alpha_{A}^{\succcurlyeq}(X) \leq 1$. If $\alpha_{A}^{\succcurlyeq}(X) = 1$, then X is a definable set with respect to A, and if $0 < \alpha_{A}^{\succcurlyeq}(X) < 1$, then X is an rough set with respect to A.

Corollary 3.1 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. For $X \subseteq U$ and $A \subseteq AT$, we have that*

$$\alpha_{A}^{\succcurlyeq}(X) = \frac{|\underline{\mathscr{R}_{A}^{\succcurlyeq}}(X)|}{|U| - |\underline{\mathscr{R}_{A}^{\succcurlyeq}}(\sim X)|}.$$

Proposition 3.5 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation, $X \subseteq U$ and $A \subseteq AT$. If $\mathscr{R}_{A}^{\succcurlyeq} = \mathscr{R}_{AT}^{\succcurlyeq}$, then $\alpha_{A}^{\succcurlyeq}(X) = \alpha_{AT}^{\succcurlyeq}(X)$.*

Proof Since $\mathscr{R}_{A}^{\succcurlyeq} = \mathscr{R}_{AT}^{\succcurlyeq}$, we can get $[u]_{A}^{\succcurlyeq} = [u]_{AT}^{\succcurlyeq}$ for any $u \in U$. Hence, for any $X \subseteq U, \underline{\mathscr{R}_{A}^{\succcurlyeq}}(X) = \underline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)$ and $\overline{\mathscr{R}_{A}^{\succcurlyeq}}(X) = \overline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)$. That is,

$$\alpha_A^{\succcurlyeq}(X) = \frac{|\underline{\mathscr{R}_A^{\succcurlyeq}}(X)|}{|\overline{\mathscr{R}_A^{\succcurlyeq}}(X)|} = \frac{|\underline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)|}{|\overline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)|} = \alpha_{AT}^{\succcurlyeq}(X).$$

The proposition was proved.

Example 3.3 (Continued from Examples 3.1 and 3.2) Given that $X = \{u_1, u_2, u_3, u_4, u_7, u_9, u_{10}\}$. By computing we can get that

$$\underline{\mathscr{R}_A^{\succcurlyeq}}(X) = \{u_1, u_2, u_4, u_7, u_{10}\},$$
$$\overline{\mathscr{R}_A^{\succcurlyeq}}(X) = \{u_1, u_2, u_3, u_4, u_5, u_7, u_9, u_{10}\};$$

and

$$\underline{\mathscr{R}_{AT}^{\succcurlyeq}}(X) = \{u_1, u_2, u_3, u_4, u_7, u_{10}\},$$
$$\overline{\mathscr{R}_{AT}^{\succcurlyeq}}(X) = \{u_1, u_2, u_3, u_4, u_5, u_7, u_9, u_{10}\}.$$

So we have that

$$\alpha_A^{\succcurlyeq}(X) = \frac{|\underline{\mathscr{R}_A^{\succcurlyeq}}(X)|}{|\overline{\mathscr{R}_A^{\succcurlyeq}}(X)|} = \frac{5}{8}, \quad \alpha_{AT}^{\succcurlyeq}(X) = \frac{|\underline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)|}{|\overline{\mathscr{R}_{AT}^{\succcurlyeq}}(X)|} = \frac{6}{8}.$$

Hence, one can get

$$\alpha_A^{\succcurlyeq}(X) \leq \alpha_{AT}^{\succcurlyeq}(X).$$

3.2 Ranking for objects in LVIS-DR

Zhang et al. [53] defined a concept of dominance degree between any two objects in classical ordered information system in, from which decision makers could find objects with better property to make an useful and effective decision. To rank all objects in LVIS-DR, we first introduce the concept of dominance degree.

Definition 3.3 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. The dominance degree of u to v with respect to $\mathscr{R}_A^{\succcurlyeq}$ can be defined as

$$d_A(u, v) = 1 - \frac{|[u]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})|}{|U|}.$$

From above definition, we can obtain the following properties.

Proposition 3.6 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$.*

(1) $\frac{1}{|U|} \leq d_A(u, v) \leq 1$ *for any* $u, v \in U$;
(2) *If* $u \in [v]_A^{\succcurlyeq}$, *then* $d_A(u, v) = 1$.

Proof

(1) It is directly obtained by Definition 3.3.
(2) Since $u \in [v]_A^{\succcurlyeq}$, we can get $[u]_A^{\succcurlyeq} \subseteq [v]_A^{\succcurlyeq}$ by Proposition 3.2. So, $[u]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq}) = \emptyset$. Hence one can obtain that

$$d_A(u, v) = 1 - \frac{|[u]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})|}{|U|} = 1 - \frac{|\emptyset|}{|U|} = 1.$$

The proposition was proved.

Proposition 3.7 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. If $v \in [u]_A^{\succcurlyeq}$, then $d_A(w, v) \leq d_A(w, u)$ and $d_A(v, w) \geq d_A(u, w)$ for any $w \in U$.*

Proof Since $v \in [u]_A^{\succcurlyeq}$, one can have $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ by Proposition 3.2. That is, $[w]_A^{\succcurlyeq} \bigcap (\sim [u]_A^{\succcurlyeq}) \subseteq [w]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})$ and $[v]_A^{\succcurlyeq} \bigcap (\sim [w]_A^{\succcurlyeq}) \subseteq [u]_A^{\succcurlyeq} \bigcap (\sim [w]_A^{\succcurlyeq})$ for any $w \in U$. So, we have

$$\begin{aligned} d_A(w, v) - d_A(w, u) &= 1 - \frac{[w]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})}{|U|} - \left(1 - \frac{[w]_A^{\succcurlyeq} \bigcap (\sim [u]_A^{\succcurlyeq})}{|U|}\right) \\ &= \frac{|[w]_A^{\succcurlyeq} \bigcap (\sim [u]_A^{\succcurlyeq})| - |[w]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})|}{|U|} \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} d_A(v, w) - d_A(u, w) &= 1 - \frac{[v]_A^{\succcurlyeq} \bigcap (\sim [w]_A^{\succcurlyeq})}{|U|} - \left(1 - \frac{[u]_A^{\succcurlyeq} \bigcap (\sim [w]_A^{\succcurlyeq})}{|U|}\right) \\ &= \frac{|[u]_A^{\succcurlyeq} \bigcap (\sim [w]_A^{\succcurlyeq})| - |[v]_A^{\succcurlyeq} \bigcap (\sim [w]_A^{\succcurlyeq})|}{|U|} \\ &\geq 0. \end{aligned}$$

The proposition was proved.

Proposition 3.8 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. If $w \in [v]_A^{\succcurlyeq}$ and $v \in [u]_A^{\succcurlyeq}$, then we can get*

(1) $d_A(u, w) \leq d_A(v, w)$ and$d_A(w, u) \geq d_A(w, v)$.
(2) $d_A(u, w) \leq d_A(u, v)$ and$d_A(w, u) \geq d_A(v, u)$.

Proof It is similar to the proof of Proposition 3.7.

From Definition 3.3, we can use a matrix to show the dominance degree of any two objects.

Definition 3.4 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. Denote

$$M_A^{\succcurlyeq} = \begin{pmatrix} d_A(u_1,u_1) & \ldots & d_A(u_1,u_n) \\ \vdots & \ddots & \vdots \\ d_A(u_n,u_1) & \ldots & d_A(u_n,u_n) \end{pmatrix},$$

then we call the matrix $M_A^{\succcurlyeq}$ to be a dominance degree matrix with respect to A induced by the dominance relation $\mathscr{R}_A^{\succcurlyeq}$.

Moreover, if denote

$$d_A(u) = \frac{1}{|U|}\sum_{v \in U} d_A(u,v),$$

then we call $d_A(u)$ to be the whole dominance degree of u with respect to $\mathscr{R}_A^{\succcurlyeq}$ for every $u \in U$.

The whole dominance degree reflect the measure of every object, the lager the value of $d_A(u)$ is, the better the properties of u are, and vice versa. As a result of above discussions, we come to the following corollary.

Corollary 3.2 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. If $\mathscr{R}_A^{\succcurlyeq} = \mathscr{R}_{AT}^{\succcurlyeq}$, then for any $u, v \in U$, we can get*

(1) $d_A(u,v) = d_{AT}(u,v)$;
(2) $d_A(u) = d_{AT}(u)$;
(3) $M_A^{\succcurlyeq} = M_{AT}^{\succcurlyeq}$.

Proposition 3.9 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$.*

(1) $\frac{1}{|U|} \le d_A(u) \le 1$;
(2) If $v \in [u]_A^{\succcurlyeq}$, *then* $d_A(u) \le d_A(v)$.

Proof

(1) It follows directly from Proposition 3.6 and the definition of whole dominance degree of each object.
(2) Since $v \in [u]_A^{\succcurlyeq}$, we can get $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ by Proposition 3.2, that is, $[v]_A^{\succcurlyeq} \bigcap (\sim [u]_A^{\succcurlyeq}) \subseteq [u]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})$.

According to the definition of whole dominance degree, one have that

$$\begin{aligned} d_A(u, av) &= 1 - \frac{|[u]_A^{\succcurlyeq} \bigcap (\sim [v]_A^{\succcurlyeq})|}{|U|} \\ &\le 1 - \frac{|[v]_A^{\succcurlyeq} \bigcap (\sim [u]_A^{\succcurlyeq})|}{|U|} \\ &= d_A(v,u). \end{aligned}$$

According to Proposition 3.7, we have that

$$\begin{aligned} d_A(u) &= \frac{1}{|U|}\sum_{\substack{w \neq u \\ w \neq v}} d_A(u,w) + \frac{1}{|U|}d_A(u,u) + \frac{1}{|U|}d_A(u,v) \\ &\le \frac{1}{|U|}\sum_{\substack{w \neq u \\ w \neq v}} d_A(v,w) + \frac{1}{|U|}d_A(u,u) + \frac{1}{|U|}d_A(u,v) \\ &= \frac{1}{|U|}\sum_{\substack{w \neq u \\ w \neq v}} d_A(v,w) + \frac{1}{|U|}d_A(v,v) + \frac{1}{|U|}d_A(u,v) \\ &\le \frac{1}{|U|}\sum_{\substack{w \neq u \\ w \neq v}} d_A(v,w) + \frac{1}{|U|}d_A(v,v) + \frac{1}{|U|}d_A(v,u) = d_A(v). \end{aligned}$$

The proposition was proved.

Note that $d_{AT}(u) \ge d_{AT}(v)$ does not mean that $u \in [v]_{AT}^{\succcurlyeq}$ holds for ever and this can be explained by the following compositive example.

Example 3.4 (Continued from Example 3.2) By the definition of dominance degree, we can have the dominance degree matrix as

$$M_{AT}^{\succcurlyeq} = \begin{pmatrix} 1.0 & 0.6 & 1.0 & 0.6 & 0.7 & 0.5 & 0.7 & 0.5 & 0.8 & 0.6 \\ 1.0 & 1.0 & 1.0 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.5 & 1.0 & 0.5 & 0.6 & 0.4 & 0.6 & 0.4 & 0.7 & 0.5 \\ 1.0 & 0.9 & 1.0 & 1.0 & 0.9 & 0.9 & 0.9 & 0.9 & 1.0 & 0.9 \\ 0.7 & 0.5 & 0.7 & 0.5 & 1.0 & 0.7 & 0.7 & 0.6 & 1.0 & 0.6 \\ 0.8 & 0.8 & 0.8 & 0.8 & 1.0 & 1.0 & 0.8 & 0.9 & 1.0 & 0.8 \\ 1.0 & 0.8 & 1.0 & 0.8 & 1.0 & 0.8 & 1.0 & 0.8 & 1.0 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.9 & 1.0 & 1.0 & 0.9 & 1.0 & 1.0 & 0.9 \\ 0.6 & 0.3 & 0.6 & 0.4 & 0.8 & 0.5 & 0.5 & 0.5 & 1.0 & 0.4 \\ 1.0 & 0.9 & 1.0 & 0.9 & 1.0 & 0.9 & 1.0 & 0.9 & 1.0 & 1.0 \end{pmatrix}$$

and

$$\begin{aligned} &d_{AT}(u_1) = 0.70, \quad d_{AT}(u_2) = 0.93, \\ &d_{AT}(u_3) = 0.61, \quad d_{AT}(u_4) = 0.94, \\ &d_{AT}(u_5) = 0.70, \quad d_{AT}(u_6) = 0.87, \\ &d_{AT}(u_7) = 0.91, \quad d_{AT}(u_8) = 0.94, \\ &d_{AT}(u_9) = 0.56, \quad d_{AT}(u_{10}) = 0.96. \end{aligned}$$

Therefore, according to the number of $d_{AT}(u_i)$, we can get the ranking of all objects as

$u_{10} \succcurlyeq u_8 = u_4 \succcurlyeq u_2 \succcurlyeq u_7 \succcurlyeq u_6 \succcurlyeq u_5 = u_1 \succcurlyeq u_3 \succcurlyeq u_9$.

4 Evidence theory in LVIS-DR

In Dempster–Shafer theory of evidence [8, 33], for an universe U a mass function can be defined by a map

$m : 2^U \longrightarrow [0,1]$,

which is called a basic probability assignment and satisfies two axioms:

$(M1)\, m(\emptyset) = 0$

$(M2) \sum_{X \subseteq U} m(X) = 1.$

The value $m(X)$ represents the degree of belief that a specific element of U belongs to set X, but not to any particular subset of X. A subset $X \subseteq U$ with $m(X) > 0$ is called a focal element, and $\mathbb{M} = \{X \subseteq U \mid m(X) > 0\}$ is the family of all focal elements of m.

The pair $(\mathbb{M}, m)$ is called a belief structure. Associated with each belief structure in information systems based on classical equivalence relation, a pair of belief and plausibility functions can be derived.

Definition 4.1 (See [8, 33]) Let $(\mathbb{M}, m)$ be a belief structure. A set function $Bel : 2^U \longrightarrow [0,1]$ is referred to as a belief function on U, if

$$Bel(X) = \sum_{Y \subseteq U} m(Y), \quad \forall X \in 2^U.$$

A set function $Pl : 2^U \longrightarrow [0,1]$ is referred to as a plausibility function on U, if

$$Pl(X) = \sum_{Y \bigcap U \neq \emptyset} m(Y), \quad \forall X \in 2^U.$$

From above, we can shown that a mass function of classical information systems is a basic probability assignment. However, relations which are induced by attributes sets are not equivalence relations in LVIS-DR. Based on the observation, a mass function is defined in LVIS-DR as follows.

Definition 4.2 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. If we denote

$$h(X) = \{u \in U \mid [u]_A^{\succcurlyeq} = X\}$$

for any $X \in U/\mathscr{R}_A^{\succcurlyeq}$, then a mass function of $\mathcal{L}^{\succcurlyeq}$ can be defined by a map $m : U/\mathscr{R}_A^{\succcurlyeq} \longrightarrow [0,1]$, where

$$m(X) = \frac{|h(X)|}{|U|}.$$

By above definition, we can easily find that a mass function of LVIS-DR still satisfies two basic axioms. In other words, for any $X \in U/\mathscr{R}_A^{\succcurlyeq}$ in LVIS-DR, the following properties still hold directly:

$(M1) \quad m(\emptyset) = 0,$

$(M2) \quad \sum_{X \in U/\mathscr{R}_A^{\succcurlyeq}} m(X) = 1.$

As same as classical information systems, we denote by $\mathbb{M}$ the family of all focal elements of m in LVIS-DR. The pair $(\mathbb{M}, m)$ is called a belief structure of LVIS-DR, and a pair of belief and plausibility functions in lattice ordered information systems can be constructed immediately.

Definition 4.3 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$, and $(\mathbb{M}, m)$ be a belief structure. A set function $Bel : 2^U \longrightarrow [0,1]$ is referred to as a belief function on U, if

$$Bel(X) = \sum_{\substack{Y \subseteq X \\ Y \in U/\mathscr{R}_A^{\succcurlyeq}}} m(Y), \quad \forall X \in 2^U.$$

A set function $Pl : 2^U \longrightarrow [0,1]$ is referred to as a plausibility function on U, if

$$Pl(X) = \sum_{\substack{Y \bigcap X \neq \emptyset \\ Y \in U/\mathscr{R}_A^{\succcurlyeq}}} m(Y), \quad \forall X \in 2^U.$$

Belief and plausibility functions based on the same belief structure are connected by the dual property:

$$Pl(X) = 1 - Bel(\sim X)$$

and furthermore, $Bel(X) \leq Pl(X)$ for all $X \in 2^U$.

The following theorems shows that the classical belief and plausibility functions can be interpreted in terms of the Pawlaks lower and upper approximations of sets [51].

Proposition 4.1 (See [51]) *Let (U, AT, F) be an information system and $A \subseteq AT$. For any $X \subseteq U$, denote*

$$Bel_A(X) = \frac{|\underline{R_A^{\succcurlyeq}}(X)|}{|U|}$$

$$Pl_A(X) = \frac{|\overline{R_A^{\succcurlyeq}}(X)|}{|U|}$$

Then $Bel_B(X)$ is the belief function and $Pl_B(X)$ is the plausibility function of U, where the corresponding mass distribution is

$$m_B(X) = \begin{cases} P(Y) & \text{if } Y \in U/R_B \\ 0 & \text{otherwise} \end{cases}$$

There are strong connections between rough set theory and Dempster–Shafer theory of evidence. It has been demonstrated that various belief structures are associated with various rough approximation spaces such that the different dual pairs of lower and upper approximation operators induced by rough approximation spaces may be used to interpret the corresponding dual pairs of belief and plausibility functions induced by belief structures [36, 44].

Hence, we can acquire the following results which show that the pair of lower and upper approximation operators in

LVIS-DR generates a pair of belief and plausibility functions respectively.

Proposition 4.2 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. For $X \subseteq U$, denote*

$$Bel_A^{\succcurlyeq(X)} = \frac{|\underline{\mathscr{R}^{\succcurlyeq_A}}(X)|}{|U|}, \quad Pl_A^{\succcurlyeq}(X) = \frac{|\overline{\mathscr{R}_A^{\succcurlyeq}}(X)|}{|U|}.$$

Then $Bel_A^{\succcurlyeq}(X)$ is the belief function and $Pl_A^{\succcurlyeq}(X)$ is the plausibility function of U, where the corresponding mass distribution is

$$m(X) = \begin{cases} \frac{|h(Y)|}{|U|} & \text{if } Y \in U/\mathscr{R}_A^{\succcurlyeq} \\ 0 & \text{otherwise} \end{cases}$$

Proof It is similar to the proof of Theorem 3.2 in [48].

As for the above discussion, we have the following corollary.

Corollary 4.1 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $B \subseteq A \subseteq AT$ then for any $X \subseteq U$, we can get that*

$$Bel_A^{\succcurlyeq}(X) \leq Bel_{AT}^{\succcurlyeq}(X) \leq \frac{|X|}{|U|} \leq Pl_{AT}^{\succcurlyeq}(X) \leq Pl_A^{\succcurlyeq}(X).$$

Example 4.1 (Continued from Examples 3.2 and 3.3) By computing, we can get that

$$Bel_A^{\succcurlyeq} = \frac{|\underline{\mathscr{R}_A^{\succcurlyeq}}(X)|}{|U|} = \frac{5}{10}, \quad Bel_A^{\succcurlyeq}T = \frac{|\underline{\mathscr{R}_A^{\succcurlyeq}T}(X)|}{|U|} = \frac{6}{10},$$

and

$$Pl_A^{\succcurlyeq} = \frac{|\overline{\mathscr{R}_A^{\succcurlyeq}}(X)|}{|U|} = \frac{8}{10}, \quad Pl_{AT}^{\succcurlyeq} = \frac{|\overline{\mathscr{R}_A^{\succcurlyeq}T}(X)|}{|U|} = \frac{8}{10}.$$

Therefore,

$$Bel_A^{\succcurlyeq}(X) \leq Bel_{AT}^{\succcurlyeq}(X) \leq \frac{|X|}{|U|} \leq Pl_{AT}^{\succcurlyeq}(X) \leq Pl_A^{\succcurlyeq}(X).$$

5 Attribute reduction in LVIS-DR

One fundamental aspect of rough set theory involves the search for particular subsets of attributes which provide the same information for classification purposes as the full set of available attributes. Such subsets are called attribute reductions. In the context of dominance relations, to simplify knowledge representation in LVIS-DR, attribute reduction is, thus, necessary. So, in this section approaches to attribute reduction in LVIS-DR will be established and illustrative examples are employed to show its validity.

5.1 Attribute reduction based on discernibility matrix

In the first part of this section, an approach of attribute reduction based on discernibility matrix in LVIS-DR is proposed and an illustrative example is employed to show its validity.

Definition 5.1 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. $A \subseteq AT$ is regarded as a classical consistent set of $\mathcal{L}^{\succcurlyeq}$ if $\mathscr{R}_A^{\succcurlyeq} = \mathscr{R}_{AT}^{\succcurlyeq}$. Moreover, if A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$ and any proper subset of A is not a classical consistent set of $\mathcal{L}^{\succcurlyeq}$, then A is referred to as a classical reduction of $\mathcal{L}^{\succcurlyeq}$.

Proposition 5.1 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. A is a classical reduction of $\mathcal{L}^{\succcurlyeq}$, then $d_A(u, v) = d_{AT}(u, v)$ for any $u, v \in U$.*

Proof It follows directly from the depiction of approximation operators and the definition of dominance degree.

It is obvious that a reduction of $\mathcal{L}^{\succcurlyeq}$ is a minimal attribute subset satisfying $\mathscr{R}_A^{\succcurlyeq} = \mathscr{R}_{AT}^{\succcurlyeq}$. An attribute $a \in AT$ is called dispensable with respect to $\mathscr{R}_{AT}^{\succcurlyeq}$ if $\mathscr{R}_{AT}^{\succcurlyeq} = \mathscr{R}_{AT-\{a\}}^{\succcurlyeq}$; otherwise a is called indispensable. The set of all indispensable attributes is called a core with respect to $\mathscr{R}_{AT}^{\succcurlyeq}$ and is denoted by $Core(AT)$. An attribute in the core must be in every attribute reduction. In other words, $Core(AT)$ is the intersection of all classical reduction of the system, in which case the Core may be empty set.

Definition 5.2 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. For any $u, v \in U$, if we denote

$$Dis(u, v) = \{a \in AT \mid (u, v) \notin \mathscr{R}_a^{\succcurlyeq}\},$$

then we call $Dis(u, v)$ a discernibility attribute set between u and v, and

$$\mathscr{M}_{Dis}^{\succcurlyeq} = \begin{pmatrix} Dis(u_1, u_1) & \dots & Dis(u_1, u_n) \\ \vdots & \ddots & \vdots \\ Dis(u_n, u_1) & \dots & Dis(u_n, u_n) \end{pmatrix}$$

a discernibility matrix of this lattice-valued information system based on dominance relation.

Proposition 5.2 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. For any $u, v \in U$, we can get that*

(1) $Dis(u, u) = \emptyset$
(2) $Dis(u, v) \bigcap Dis(v, u) = \emptyset$.

To investigate the attribute reduction of $\mathcal{L}^{\succcurlyeq}$ from the viewpoint of discernibility matrix, the judgement method for a classical reduction is proposed in LVIS-DR.

Theorem 5.1 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation, $A \subseteq AT$ and $Dis(u, v)$ is the discernibility attribute set for $u, v \in U$ with respect to $\mathscr{R}^{\succcurlyeq}_{AT}$, then the following two items are equivalent.*

(1) *A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$.*
(2) *If $Dis(u, v) \neq \emptyset$, then $A \bigcap Dis(u, v) \neq \emptyset$ for $u, v \in U$.*

Proof $(1) \Longrightarrow (2)$: Since A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$, we can get $\mathscr{R}^{\succcurlyeq}_{A} = \mathscr{R}^{\succcurlyeq}_{AT}$, that is, $[u]^{\succcurlyeq}_{A} = [u]^{\succcurlyeq}_{AT}$ for any $u \in U$. On account of $Dis(u, v) \neq \emptyset$, there must exist $u \notin [v]^{\succcurlyeq}_{AT}$.Thus, $\exists a \in A$, s.t. $(u, v) \notin \mathscr{R}^{\succcurlyeq}_{A}$. So one has $a \in Dis(u, v)$. Thus, when $Dis(u, v) \neq \emptyset$, then $A \bigcap Dis(u, v) \neq \emptyset$ for $u, v \in U$.

$(1) \Longleftarrow (2)$: For $u, v \in U$, if $Dis(u, v) \neq \emptyset$, then $u \notin [v]^{\succcurlyeq}_{AT}$. Moreover, we know that $A \bigcap D(u, v) \neq \emptyset$ when $Dis(u, v) \neq \emptyset$. Hence there exist $a \in A$ such that $(u, v) \notin \mathscr{R}^{\succcurlyeq}_{\{a\}}$, that is, $u \notin [v]^{\succcurlyeq}_{A}$. Therefore, we can find that if $u \notin [v]^{\succcurlyeq}_{AT}$ then $u \notin [v]^{\succcurlyeq}_{A}$. In other words, if $u \in [v]^{\succcurlyeq}_{A}$ then $u \in [v]^{\succcurlyeq}_{AT}$. That is, $[u]^{\succcurlyeq}_{A} \subseteq [u]^{\succcurlyeq}_{AT}$, i.e. $\mathscr{R}^{\succcurlyeq}_{A} \subseteq \mathscr{R}^{\succcurlyeq}_{AT}$. According to Proposition 3.1, we can get that $\mathscr{R}^{\succcurlyeq}_{A} = \mathscr{R}^{\succcurlyeq}_{AT}$. This completes the proof.

According to above Theorem, we can see that $A \subseteq AT$ is an classical reduction in $\mathcal{L}^{\succcurlyeq}$ if and only if A is the minimal set satisfying $A \bigcap Dis(u, v) \neq \emptyset$ for any $Dis(u, v) \neq \emptyset$.

Proposition 5.3 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. Then $a \in Core(AT)$ if and only if there exists $Dis(u, v) \neq \emptyset$, s.t. $Dis(u, v) = \{a\}$.*

Proof $\Longrightarrow$: Since $a \in Core(AT)$, we can provide $K_a = \{Dis(u, v) \neq \emptyset \mid a \in Dis(u, v)\}$. If $|Dis(u, v)| \geq 2$ for any $Dis(u, v) \in K_a$, then it is easy to see that $K \bigcap Dis(u, v) \neq \emptyset$ for all $Dis(u, v) \neq \emptyset$, where $K = \bigcup_{Dis(u,v)\neq\emptyset}(Dis(u, v) - \{a\})$. By Theorem 5.1, we can get that K is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$. Then there exists $K' \subseteq K$ such that K' is an classical reduction of $\mathcal{L}^{\succcurlyeq}$. Clearly, $a \notin K'$, this is a contradiction with $a \in Core(AT)$.

$\Longleftarrow$: Suppose there exists $Dis(u, v) \neq \emptyset$, s.t. $Dis(u, v) = \{a\}$, thus, existing $u, v \in U$ with $u \neq v$ such that $Dis(u, v) = \{a\}$. By the definition of discernibility attribute set, we can get that $(u, v) \notin \mathscr{R}^{\succcurlyeq}_{\{a\}}$ and $(u, v) \in \mathscr{R}^{\succcurlyeq}_{AT-\{a\}}$. It follows that $\mathscr{R}^{\succcurlyeq}_{AT} \neq \mathscr{R}^{\succcurlyeq}_{AT-\{a\}}$. Not that $a \in AT$ is an element of $Core(AT)$ if and only if $\mathscr{R}^{\succcurlyeq}_{AT} \neq \mathscr{R}^{\succcurlyeq}_{AT-\{a\}}$. Therefore, $a \in Core(AT)$. This completes the proof.

Definition 5.3 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation, $A \subseteq AT$ and $Dis(u, v)$ discernibility attribute set between u and v. Denoted by

$$F^{\succcurlyeq} = \bigwedge\Big\{\bigvee\{a \mid a \in Dis(u, v),\ \forall u, v \in U\}\Big\},$$

then $F^{\succcurlyeq}$ is referred to as a discernibility formula.

Based on the discernibility formula, a practical approach to attribute reduction in LVIS-DR is constructed as follows.

Theorem 5.2 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation. The minimal disjunctive normal form of discernibility formula $F^{\succcurlyeq}$ is*

$$F^{\succcurlyeq} = \bigvee_{k=1}^{p}\left(\bigwedge_{l=1}^{q_k} a_l\right).$$

Proof It follows directly from Theorem 5.1 and the definition of minimal disjunctive normal of the discernibility formula.

From above, we have find that it provides a useful approach to attribute reduction in LVIS-DR, which can be illustrated by the following example.

Example 5.1 (Continued from Example 3.1) By the definition of discernibility matrix, one can obtain the discernibility matrix of this system as

$$\begin{pmatrix}
a_0 & a_{1234} & a_0 & a_{35} & a_{45} & a_{345} & a_{145} & a_{345} & a_5 & a_{145} \\
a_0 & a_0 & a_0 & a_{35} & a_5 & a_5 & a_5 & a_{35} & a_5 & a_5 \\
a_{23} & a_{1234} & a_0 & a_{235} & a_{45} & a_{345} & a_{12345} & a_{345} & a_5 & a_{12345} \\
a_0 & a_{124} & a_0 & a_0 & a_4 & a_{45} & a_{14} & a_{45} & a_0 & a_{145} \\
a_{123} & a_{123} & a_1 & a_{123} & a_0 & a_{35} & a_{123} & a_{35} & a_0 & a_{1235} \\
a_{12} & a_{12} & a_1 & a_{123} & a_0 & a_0 & a_{12} & a_3 & a_0 & a_{12} \\
a_0 & a_{23} & a_0 & a_3 & a_0 & a_{35} & a_0 & a_{35} & a_0 & a_5 \\
a_{12} & a_{12} & a_1 & a_{12} & a_0 & a_0 & a_{12} & a_0 & a_0 & a_{12} \\
a_{123} & a_{1234} & a_1 & a_{123} & a_4 & a_{345} & a_{1234} & a_{35} & a_0 & a_{12345} \\
a_0 & a_{23} & a_0 & a_3 & a_3 & a_0 & a_0 & a_3 & a_0 & a_0
\end{pmatrix}$$

Connected with Definition 5.3 and Theorem 5.2, we can gave that

$$F^{\succcurlyeq} = a_1 \wedge a_3 \wedge a_4 \wedge a_5.$$

So, there is only one classical reduction for the system, which is $\{a_1, a_3, a_4, a_5\}$.

Note that for simplicity, we use the notation "a_0" to express $\emptyset$ and "a_{145}" for $\{a_1, a_4, a_5\}$, to name a couple for explanation.

5.2 Attribute reduction based on evidence theory

In this section, we discuss the attribute reduction in LVIS-DR by proposing the concepts of belief and plausibility reductions in this information systems, and compare them with the existing classical reduction.

Definition 5.4 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$.

(1) A is referred to as a belief consistent set of $\mathcal{L}^{\succcurlyeq}$ if $Bel^{\succcurlyeq}_A(X) = Bel^{\succcurlyeq}_{AT}(X)$ for any $X \in U/\mathscr{R}^{\succcurlyeq}_{AT}$. Moreover, if A is a belief consistent set of $\mathcal{L}^{\succcurlyeq}$ and no proper subset of A is a belief consistent set of $\mathcal{L}^{\succcurlyeq}$, then A is referred to as a belief reduction of $\mathcal{L}^{\succcurlyeq}$.
(2) A is referred to as a plausibility consistent set of $\mathcal{L}^{\succcurlyeq}$ if $Pl^{\succcurlyeq}_A(X) = Pl^{\succcurlyeq}_{AT}(X)$ for any $X \in U/\mathscr{R}^{\succcurlyeq}_{AT}$. Moreover, if A is a plausibility consistent set of $\mathcal{L}^{\succcurlyeq}$ and no proper subset of A is a plausibility consistent set of $\mathcal{L}^{\succcurlyeq}$, then A is referred to as a plausibility reduction of $\mathcal{L}^{\succcurlyeq}$.

The following proposition will reflect the relation between belief reduction, plausibility reduction and classical reduction.

Theorem 5.3 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. Then A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$ if and only if A is a belief consistent set of $\mathcal{L}^{\succcurlyeq}$.*

Proof $\Longrightarrow$: Since A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$, we can get $\mathscr{R}^{\succcurlyeq}_A = \mathscr{R}^{\succcurlyeq}_{AT}$ by Definition 5.1. According to Definition 3.1, we can get that $[u]^{\succcurlyeq}_A = [u]^{\succcurlyeq}_{AT}$ for all $u \in U$. Thus,

$$[u]^{\succcurlyeq}_A \subseteq X \Longleftrightarrow [u]^{\succcurlyeq}_{AT} \subseteq X, \ \ X \in U/\mathscr{R}^{\succcurlyeq}_{AT}.$$

Then, by Definition 5.1, one obtain

$$u \in \underline{\mathscr{R}^{\succcurlyeq}_A} \Longleftrightarrow u \in \underline{\mathscr{R}^{\succcurlyeq}_{AT}}, \quad \forall u \in U.$$

According to Proposition 4.2, it is easy to see that

$$Bel^{\succcurlyeq}_A(X) = Bel^{\succcurlyeq}_{AT}, \ \ \forall X \in U/\mathscr{R}^{\succcurlyeq}_{AT}.$$

Therefore, A is a belief consistent set of $\mathcal{L}^{\succcurlyeq}$.

$\Longleftarrow$: Since A is a belief consistent set, we can get

$$Bel^{\succcurlyeq}_A([u]^{\succcurlyeq}_{AT}) = Bel^{\succcurlyeq}_{AT}([u]^{\succcurlyeq}_{AT}), \quad \forall u \in U.$$

That is,

$$\frac{|\underline{\mathscr{R}^{\succcurlyeq}_A}([u]^{\succcurlyeq}_{AT})|}{|U|} = \frac{|\underline{\mathscr{R}^{\succcurlyeq}_{AT}}([u]^{\succcurlyeq}_{AT})|}{|U|}.$$

By Proposition 3.3, we obtain $\underline{\mathscr{R}^{\succcurlyeq}_A}([u]^{\succcurlyeq}_{AT}) = \underline{\mathscr{R}^{\succcurlyeq}_{AT}}([u]^{\succcurlyeq}_{AT})$ for any $u \in U$. That is to say,

$$[v]^{\succcurlyeq}_A \subseteq [u]^{\succcurlyeq}_{AT} \Longleftrightarrow [v]^{\succcurlyeq}_{AT} \subseteq [u]^{\succcurlyeq}_{AT}, \ \ \forall u, v \in U.$$

If take $u = v$, then

$$[u]^{\succcurlyeq}_A \subseteq [u]^{\succcurlyeq}_{AT} \Longleftrightarrow [u]^{\succcurlyeq}_{AT} \subseteq [u]^{\succcurlyeq}_{AT}.$$

Hence, we have $[u]^{\succcurlyeq}_A \subseteq [u]^{\succcurlyeq}_{AT}$ for all $u \in U$. So we have that $[u]^{\succcurlyeq}_A = [u]^{\succcurlyeq}_{AT}$ for all $u \in U$. Therefore, according to Definitions 3.1 and 5.1, A is a classical consistent set. This completes the proof.

From above we can easily obtain the following corollary.

Corollary 5.1 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. Then A is a classical reduction of $\mathcal{L}^{\succcurlyeq}$ if and only if A is a belief reduction set of $\mathcal{L}^{\succcurlyeq}$.*

Definition 5.5 Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $U/\mathscr{R}^{\succcurlyeq}_{AT} = \{X_1, X_2, \ldots, X_t\}$, denote

$$\mathcal{BS} = \sum_{i=1}^{t} Bel^{\succcurlyeq}_{AT}(X_i),$$

then $\mathcal{BS}$ is referred to as belief sum of $\mathcal{L}^{\succcurlyeq}$. By above definition, we can have the following results.

Proposition 5.4 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. Then we have that*

(1) *A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$ iff $\sum_{i=1}^{t} Bel^{\succcurlyeq}_A(X_i) = \mathcal{BS}$.*
(2) *A is a classical reduction of $\mathcal{L}^{\succcurlyeq}$ iff $\sum_{i=1}^{t} Bel^{\succcurlyeq}_A(X_i) = \mathcal{BS}$, and for any nonempty proper subset $B \subset A$, $\sum_{i=1}^{t} Bel^{\succcurlyeq}_B(X_i) < \mathcal{BS}$ is true.*

Proof

(1) By the Theorem 5.3, we know that A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$ if and only if A is a belief consistent set of $\mathcal{L}^{\succcurlyeq}$. Thus A is classical consistent set of $\mathcal{L}^{\succcurlyeq}$ if and only if $\sum_{i=1}^{t} Bel^{\succcurlyeq}_A(X_i) = \sum_{i=1}^{t} Bel^{\succcurlyeq}_{AT}(X_i)$. That is, A is classical consistent set of $\mathcal{L}^{\succcurlyeq}$ if and only if $\sum_{i=1}^{t} Bel^{\succcurlyeq}_A(X_i) = \mathcal{BS}$.
(2) It follows directly from (1) of this Proposition and Definition 5.4.

The proof was completed.

Example 5.2 (Continued from Examples 3.1 and 3.2) Given that $X_i = [u_i]^{\succcurlyeq}_{AT}$ for every $u_i \in U$, by computing we have that

$$\mathcal{BS} = \sum_{i=1}^{10} Bel^{\succcurlyeq}_{AT}(X_i) = \frac{31}{10}.$$

Moreover, one can have that the belief sum with respect to $B = \{a_1, a_3, a_4, a_5\}$ is equal to $\mathcal{BS}$ obtained above, which means that $B = \{a_1, a_3, a_4, a_5\}$ is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$. In fact, the subset B is also a belief reduction, which is consistent with Example 5.1.

Proposition 5.5 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. If A is a classical consistent set of $\mathcal{L}^{\succcurlyeq}$, then A is a plausibility consistent set of $\mathcal{L}^{\succcurlyeq}$.*

Proof It is straightforward from Definitions 5.1 and 5.4.

Corollary 5.2 *Let $\mathcal{L}^{\succcurlyeq} = (U, AT, V, f)$ be a lattice-valued information system based on dominance relation and $A \subseteq AT$. If A is a classical reduction of $\mathcal{L}^{\succcurlyeq}$, then A is a plausibility reduction set of $\mathcal{L}^{\succcurlyeq}$.*

Note that the converse proposition of Proposition 5.5 does not hold, which can be illustrative by the following example.

Example 5.3 Consider another lattice-valued information system based on dominance relation (Table 2).

By computing we can get that

$$X_1 = [u_1]^{\succcurlyeq}_{AT} = \{u_1\}, \quad X_2 = [u_2]^{\succcurlyeq}_{AT} = \{u_2, u_3\},$$
$$X_3 = [u_3]^{\succcurlyeq}_{AT} = \{u_3\}, \quad X_4 = [u_4]^{\succcurlyeq}_{AT} = \{u_3, u_4\}.$$

According to Proposition 4.2, one obtain that

$$Pl^{\succcurlyeq}_{AT}(X_1) = \frac{1}{4}, \quad Pl^{\succcurlyeq}_{AT}(X_2) = \frac{3}{4},$$
$$Pl^{\succcurlyeq}_{AT}(X_3) = \frac{3}{4}, \quad Pl^{\succcurlyeq}_{AT}(X_4) = \frac{3}{4}.$$

If take $A = \{a_2, a_3\}$, it easy to find that

$$[u_1]^{\succcurlyeq}_{A} = \{u_1\}, \quad [u_2]^{\succcurlyeq}_{A} = \{u_2, u_3, u_4\},$$
$$[u_3]^{\succcurlyeq}_{A} = \{u_3\}, \quad [u_4]^{\succcurlyeq}_{A} = \{u_3, u_4\},$$

and

$$Pl^{\succcurlyeq}_{A}(X_1) = \frac{1}{4}, \quad Pl^{\succcurlyeq}_{A}(X_2) = \frac{3}{4},$$
$$Pl^{\succcurlyeq}_{A}(X_3) = \frac{3}{4}, \quad Pl^{\succcurlyeq}_{A}(X_4) = \frac{3}{4}.$$

Therefore, A is a plausibility consistent set of $\mathcal{L}^{\succcurlyeq}$. But it is not a classical consistent set according to $[u_2]^{\succcurlyeq}_{AT} \neq [u_2]^{\succcurlyeq}_{A}$.

Table 2 Another LVIS-DR

U	a_1	a_2	a_3
u_1	0.70	[0.6, 1.0]	(0.0, 1.0)
u_2	0.80	[0.4, 0.5]	(0.8, 0.1)
u_3	0.85	[0.5, 0.7]	(1.0, 0.0)
u_4	0.70	[0.5, 0.7]	(0.8, 0.1)

6 Conclusion

As a suitable mathematical model to handle partial knowledge in data bases, rough set theory is emerging as a powerful theory and has been found its successive applications in many fields. However, because there exist some complexities of practical problem, one of the most important research tasks is to generalize the classical rough set model.

In this paper, our focus was to consider the lattice-valued information systems based on dominance relation. By proposing two approximation operators, the rough set approach to lattice-valued information system based on dominance relation have been established. According to making a fully analysis for this system, evidence theory was also been discussed. And in order to extract much simpler rules from this system, the problem of attribute reduction was researched carefully from the viewpoint of discernibility matrix and evidence theory. It was easy to find that these two approaches to attribute reduction have strong connections. As space is limited, some problems, such as lattice-valued decision systems based on dominance relation, rules extraction, etc., will be studied in our future works.

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