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# KNOWLEDGE REDUCTION IN LATTICE-VALUED INFORMATION SYSTEMS WITH INTERVAL-VALUED INTUITIONISTIC FUZZY DECISION

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In this paper, associated with dominance relation, lattice theory and intuitionistic fuzzy sets theory, the lattice-valued information systems with interval-valued intuitionistic fuzzy decision are proposed and some of its properties are investigated carefully. And, an approach to knowledge reduction based on discernibility matrix in consistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision is constructed and an illustrative example is applied to show its validity. Moreover, extended from the idea of knowledge reduction in consistent information systems, four types of reductions and approaches to obtaining the knowledge reductions of the inconsistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision are formulated via the use of discernibility matrix. Furthermore, examples are considered to show that the approaches are useful and effective. One can obtain that the research is meaningful both in theory and in application for the issue of knowledge reduction in complex information systems.

*Keywords*: Dominance relation; interval-valued intuitionistic fuzzy sets; knowledge reduction; lattice-valued information systems; rough set.

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## 1. Introduction

Rough set theory proposed by Pawlak[1–3] is an extension of the classical set theory and can be regarded as a soft computing tool to handle imprecision, vagueness and uncertainty in data analysis. The theory has been found its successive applications in the fields of pattern recognition,[4] medical diagnosis,[5] data mining,[6–9] conflict analysis,[10] algebra,[11–13] and so on. Recently, the theory has generated a great deal of interest among more and more researchers.

However, in practice, due to the existence of uncertainty and complexity of particular problem, the problem would not be settled perfectly by means of classical rough set. Therefore, it is vital to generalize the classical rough set model. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful general models in recent years by proposing other binary relations, such as tolerance relations,[14] neighborhood operators,[15] and others.[3,10,13,16–23] However, the original rough set theory does not consider attributes with preference ordered domain. Particularly, in many real situations, we often face the problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco, Matarazzo and Slowinski *et al.*[24–28] proposed the extension of rough set theory, the dominance-based rough set approach(DRSA), to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, condition attributes are criteria and classes are preference ordered, the knowledge approximated is a collection of upward and downward unions of classes and the dominance classed are sets of objects defined by using a dominance relation. In recent years, several studies have been made about properties and algorithmic implementations of DRSA.[4,20–22,29,30] Nevertheless, only a limited number of methods using DRSA to acquire knowledge from the inconsistent ordered information systems have been proposed. Pioneering work on inconsistent ordered information systems with the DRSA has been proposed by Greco, Matarazzo and Slowinski,[24–28] but they did not clearly point out the semantic explanation of unknown values. Shao and Zhang[31] further proposed an extension of the dominance relation in an inconsistent ordered information systems.

The classical rough set theory is based upon the classification mechanism, from which the classification can be viewed as an equivalence relation and knowledge granule induced by the equivalence relation can be viewed as a partition of the universe of discourse. In rough set theory, two classical sets, so-called lower and upper approximations or Pawlak's rough approximations, are constructed and any subset of universe of discourse can be expressed by them. A primary use of rough set theory is to reduce the number of attributes in databases, thereby improving the performance of applications in a number of aspects including speed, storage, and accuracy. For a data set with discrete attribute values, this can be done by reducing the number of redundant attributes and finding a subset of the original attributes that are the most informative. As is well known, an information system may usually have more than one reduction. This means the set of rules deriving from knowledge reduction is not unique. In practice, it is always hoped to obtain the set of the most concise rules. Therefore, people have been attempting to find the minimal reduction of information systems, which means that the number of attributes contained in the reduction is minimal. Unfortunately, it has been proven that finding the minimal reduction of an information system is a NP-hard problem. However, many types of knowledge reduction have been proposed in the area of rough sets.[2,8,32–37] Possible rules and possible reductions have been proposed as a means to deal with inconsistence in an inconsistent decision table.[38–40] Approximation rules[41] are also used as an alternative to possible rules. On the other hand, generalized decision rules and generalized decision reductions provide a decision maker with more flexible selection of decision behavior. Komorowski *et al.*[42] proposed the notions of $\alpha$-reduct and $\alpha$-relative reduction for decision tables. The $\alpha$-reduction allows occurrence of additional inconsistency that is controlled by means of a parameter. Slezak[43] presented a new concept of attribute reduction that keeps the class membership distribution unchanging for all objects in the information system. It was shown by Slezak[44] that the knowledge reduction preserving the membership distribution is equivalent to the knowledge reduction preserving the value of generalized inference measure function.

In Ref. 44, Slezak introduced a generalized knowledge reduction that allows the value of generalized inference measure function after the attribute reduction to be different from the original one by user-specified threshold. By eliminating the rigorous conditions required by distribution reduction, maximum distribution reduction was introduced by Zhang *et al.*[45]

The purpose of this paper is to study a complex information system, which is a combination of the ordered information systems and the information systems with lattice theory and intuitionistic fuzzy sets theory. We call this new system the lattice-valued information systems with interval-valued intuitionistic fuzzy decision. By discussing the important properties in this system, the method for knowledge reduction will be constructed in consistent and inconsistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision, in which case decision makers could find objects with better property to make an useful and effective decision.

The rest of this paper is organized as follows. Some preliminary concepts required in our work are briefly recalled in Section 2. In Section 3, the lattice-valued information systems with interval-valued intuitionistic fuzzy decision is proposed and some of its properties are carefully discussed. Section 4 is devoted to knowledge reduction in consistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision. In Section 5, approaches to knowledge reduction in inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision are constructed and examples are applied to investigate its validity. And finally, the paper is concluded by a summary and outlook for further research in Section 6.

## 2. Preliminaries

In this section, we make a brief overview of some necessary concepts and preliminaries required in the sequel of our work. Detailed description of the intuitionistic fuzzy sets theory can be found in the source papers,[46–48] and rough sets theory in the source papers.[1–3]

### 2.1. *Interval-valued intuitionistic fuzzy sets*

In this paper, we use the notation $I([0,1])$ to express the family of closed interval contained in interval $[0,1]$, that is, $I([0,1]) = \{[\alpha, \beta] \mid \alpha, \beta \in [0,1]\}$.

Let $U$ be a non-empty finite set called the universe of discourse, then the interval-valued intuitionistic fuzzy set $A$ in $U$ proposed by Atanassov *et al.*[49] can be expressed as

$$A = \{(x, [\mu_A^-(x),\ \mu_A^+(x)], [\nu_A^-(x),\ \nu_A^+(x)]) \mid x \in U\}$$

with the condition

$$0 \leq \mu_A^+(x) + \nu_A^+(x) \leq 1, \ \ \forall x \in U,$$

where

$$\mu_A : U \longrightarrow I([0,1])$$

is the membership function and $\mu_A(x) = [\mu_A^-(x),\ \mu_A^+(x)]$ is the degree of membership of $x$ to $A$, and

$$\nu_A : U \longrightarrow I([0,1])$$

is the non-membership function and $\nu_A(x) = [\nu_A^-(x),\ \nu_A^+(x)]$ is the degree of non-membership of $x$ to $A$, respectively.

If $\mu_A^-(x) = \mu_A^+(x)$ and $\nu_A^-(x) = \nu_A^+(x)$, i.e., $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ for any $x \in U$, then the interval-valued intuitionistic fuzzy set $A$ is called a classical intuitionistic fuzzy set. In addition, if $\mu_A(x) \in [0,1]$, $\nu_A(x) \in [0,1]$ and $\mu_A(x) + \nu_A(x) \equiv 1$ for any $x \in U$, then the intuitionistic fuzzy set degenerates into an original fuzzy set proposed by Zadeh,[51] and in which case it can be denoted by

$$A = \{(x, \mu_A(x)) \mid x \in U\}.$$

To discuss conveniently the problem of our paper, some basic operations between any two interval-valued intuitionistic fuzzy sets can be redefined as follows. Let $A, B$ be two interval-valued intuitionistic fuzzy sets and then

$$A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x) \ \text{ and } \ \nu_A(x) \geq \nu_B(x) \ \text{ for any } x \in U,$$

$$A = B \Longleftrightarrow \mu_A(x) = \mu_B(x) \ \text{ and } \ \nu_A(x) = \nu_B(x) \ \text{ for any } x \in U,$$

where

$$\begin{aligned}
\mu_A(x) \le \mu_B(x) &\Longleftrightarrow [\mu_A^-(x),\ \mu_A^+(x)] \le [\mu_B^-(x),\ \mu_B^+(x)] \\
&\Longleftrightarrow \mu_A^-(x) \le \mu_B^-(x) \text{ and } \mu_A^+(x) \le \mu_B^+(x), \\
\mu_A(x) = \mu_B(x) &\Longleftrightarrow [\mu_A^-(x),\ \mu_A^+(x)] = [\mu_B^-(x),\ \mu_B^+(x)] \\
&\Longleftrightarrow \mu_A^-(x) = \mu_B^-(x) \text{ and } \mu_A^+(x) = \mu_B^+(x), \\
\nu_A(x) \ge \nu_B(x) &\Longleftrightarrow [\nu_A^-(x),\ \nu_A^+(x)] \ge [\nu_B^-(x),\ \nu_B^+(x)] \\
&\Longleftrightarrow \nu_A^-(x) \ge \nu_B^-(x) \text{ and } \nu_A^+(x) \ge \nu_B^+(x), \\
\nu_A(x) = \nu_B(x) &\Longleftrightarrow [\nu_A^-(x),\ \nu_A^+(x)] = [\nu_B^-(x),\ \nu_B^+(x)] \\
&\Longleftrightarrow \nu_A^-(x) = \nu_B^-(x) \text{ and } \nu_A^+(x) = \nu_B^+(x).
\end{aligned}$$



### 2.2. *Rough sets and ordered information systems*

The notion of information system(sometimes called data tables, attribute valued systems, knowledge representation systems, etc.) provides a convenient basis for the representation of objects in terms of their attributes.

An information system is a quadruple $\mathcal{I} = (U, AT, V, f)$, where $U$ is a non-empty finite set with $n$ objects, $\{u_1, u_2, \ldots, u_n\}$, called the universe of discourse; $AT = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with $m$ attributes; $V = \bigcup\limits_{a \in AT} V_a$ and $V_a$ is the domain of attribute $a$; $f : U \times AT \longrightarrow V$ is an information function such that $f(u, a) \in V_a$ for any $u \in U$.

An information system with decision is a special case of an information system, in which case the attributes set $AT$ is divided into two disjoint sets, $C$ and $\{d\}$ with the condition $C \bigcap \{d\} = \emptyset$ and $C \bigcup \{d\} = AT$. Therefore, an information system with decision can be expressed as $\mathcal{I}_d = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$, where sets $C$ and $\{d\}$ be condition attributes set and decision attribute set respectively, and $V_C = \bigcup\limits_{a \in C} V_a$.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

**Definition 2.1.**[24–28] An information system is called an ordered information system if all condition attributes are criterion.

Assumed that the domain of the criterion $a \in C \bigcup \{d\}$ is completely pre-ordered by an outranking relation $\succcurlyeq_a$, then $u \succcurlyeq_a v$ means that $u$ is at least as good as (outranks) $v$ with respect to $a$, and we say that $u$ dominates $v$ or $v$ is dominated by $u$. Being of type gain, that is, $u \succcurlyeq_a v \Longleftrightarrow f(u, a) \ge f(v, a)$(according to increasing preference) or $u \succcurlyeq_a v \Longleftrightarrow f(u, a) \le f(v, a)$ (according to decreasing preference). Without any loss of generality and for simplicity, in the following we only consider attributes with increasing preference.

**Definition 2.2.** Let $\mathcal{I}_d = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$ be an information system with decision and $A \subseteq C$. Given

$$R_A^{\succcurlyeq} = \{(u,v) \mid f(u,a) \geq f(v,a) \ \forall u,v \in U, a \in A)\}$$

and

$$R_d^{\succcurlyeq} = \{(u,v) \mid f(u,d) \geq f(v,d), \forall u,v \in U\},$$

then $R_A^{\succcurlyeq}$ and $R_d^{\succcurlyeq}$ are called the dominance relation with respect to (w.r.t.) condition attributes set $A$ and decision attributes set $\{d\}$ respectively, and in which case the information system $\mathcal{I}_d = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$ is called an ordered information system with decision and denoted by $\mathcal{I}_d^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$, or $\mathcal{I}_d^{\succcurlyeq}$ for simplicity.

Let us denote

$$[u_i]_A^{\succcurlyeq} = \{u_j \mid f(u_j,a) \geq f(u_i,a), \forall a \in A\},$$

$$U/R_A^{\succcurlyeq} = \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \cdots, [u_n]_A^{\succcurlyeq}\},$$

where $i \in \{1, 2, \ldots, n\}$, then $[u_i]_A^{\succcurlyeq}$ will be called the dominance class of $u_i \in U$ and $U/R_A^{\succcurlyeq}$ be classification of $U$ w.r.t. condition attributes set $A$, respectively. Just like the condition attributes set, $[u_i]_d^{\succcurlyeq}$ and $U/R_d^{\succcurlyeq}$ have the analogous interpretation.

**Definition 2.3.** Let $\mathcal{I}_d^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$ be an ordered information system with decision, if $R_C^{\succcurlyeq} \subseteq R_d^{\succcurlyeq}$, then the ordered information system with decision is called a consistent ordered information system with decision; otherwise, it is called an inconsistent ordered information system with decision.

From above description, one can get that the following properties of dominance relation in ordered information systems are trivial.

**Proposition 2.1.**[24–28] *Let $\mathcal{I}_d^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$ be an ordered information system with decision and $B, A \subseteq C$, then we have that*

(1) *$R_C^{\succcurlyeq}$ is reflective, transitive, but not symmetric, so it is not an equivalence relation;*
(2) *If $B \subseteq A \subseteq C$, then $R_C^{\succcurlyeq} \subseteq R_A^{\succcurlyeq} \subseteq R_B^{\succcurlyeq}$.*

Similarly, for the dominance class induced by dominance relation $R_A^{\succcurlyeq}$, the following properties are still true.

**Proposition 2.2.**[24–28] *Let $\mathcal{I}_d^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d, f)$ be an ordered information system with decision and $B, A \subseteq C$, then we have that*

(1) *If $B \subseteq A \subseteq C$, then $[u]_C^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq} \subseteq [u]_B^{\succcurlyeq}$ for any $u \in U$;*
(2) *If $v \in [u]_A^{\succcurlyeq}$, then $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ and $[u]_A^{\succcurlyeq} = \bigcup\{[v]_A^{\succcurlyeq} \mid v \in [u]_A^{\succcurlyeq}\}$;*
(3) *$[u]_C^{\succcurlyeq} = [v]_C^{\succcurlyeq}$ if and only if $f(u,a) = f(v,a)$ for any $a \in C$;*
(4) *$|[u]_C^{\succcurlyeq}| \geq 1$ for any $u \in U$,*

*where $|X|$ denotes the cardinality of the set $X$.*

For any subset $X \subseteq U$ and $A \subseteq C$ in $\mathcal{I}^{\succcurlyeq}$, if we denote

$$\underline{R_A^{\succcurlyeq}}(X) = \{u_i \mid [u_i]_A^{\succcurlyeq} \subseteq X, u_i \in U\},$$
$$\overline{R_A^{\succcurlyeq}}(X) = \{u_i \mid [u_i]_A^{\succcurlyeq} \bigcap X \neq \emptyset, u_i \in U\},$$

then $\underline{R_A^{\succcurlyeq}}(X)$ and $\overline{R_A^{\succcurlyeq}}(X)$ are the lower and upper approximation of $X$ w.r.t. $R_A^{\succcurlyeq}$, respectively.

## 3. Lattice-Valued Information Systems with Interval-Valued Intuitionistic Fuzzy Decision

In this section, we first propose an new extension of information system with decision, which is lattice-valued information system with interval-valued intuitionistic fuzzy decision (For lattice-valued information systems based on dominance relation, please refer Ref. 52). Then, the approximation operators w.r.t. a dominance relation are introduced and some of its properties are investigated.

**Definition 3.1.** A lattice-valued information system with interval-valued intuitionistic fuzzy decision is an information system $\mathcal{L}_{id} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$, where

(1) $U$ is a non-empty finite set with $n$ objects, $\{u_1, u_2, \ldots, u_n\}$, called the universe of discourse;

(2) $C = \{a_1, a_2, \ldots, a_m\}$ is a non-empty finite set with $m$ condition attributes and $\{d\}$ is the decision attributes set;

(3) $V_C = \bigcup\limits_{a \in C} V_a$ and $V_a$ is the domain of attribute $a$ such that $(V_a, \succcurlyeq_a)$ is a finite lattice;

(4) $V_d = \{(u, \mu_d(u), \nu_d(u)) \mid u \in U, \mu_d(u), \nu_d(u) \in I([0,1])\}$ is interval-valued intuitionistic fuzzy set.

(5) $f : U \times C \bigcup \{d\} \longrightarrow V_C \bigcup V_d^i$ is an information function such that for any $u \in U$, $f(u, a) \in V_a$ when $a \in C$ and $f(u, d) \in V_d^i$.

From above definition, we can find that the domain of every attribute can be ordered according to a decreasing or increasing preference, that is, every attribute is a criterion. Thus, lattice-valued information system with interval-valued intuitionistic fuzzy decision(LS-IFD) is an ordered information system. In general, it can be denoted by $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$, or $\mathcal{L}_{id}^{\succcurlyeq}$ for simplicity.

In the following, just like the description of dominance relation in Section 2, the dominance relation in lattice-valued information systems with interval-valued intuitionistic fuzzy decision can de redefined as follows.

**Definition 3.2.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. Given

$$\mathcal{R}_A^{\succcurlyeq} = \{(u, v) \mid f(u, a) \succcurlyeq_a f(v, a),\ \forall a \in A\}$$

and

$$\mathcal{R}_d^{\succcurlyeq} = \{(u,v) \mid f(u,d) \succcurlyeq_d f(v,d)\},$$

then $\mathcal{R}_A^{\succcurlyeq}$ and $\mathcal{R}_d^{\succcurlyeq}$ are called the dominance relations w.r.t. condition attributes set $A$ and decision attributes set $\{d\}$, respectively.

Let us denote

$$[u_i]_A^{\succcurlyeq} = \{u_j \mid f(u_j,a) \succcurlyeq_a f(u_i,a),\ \forall a \in A\},$$
$$U/\mathcal{R}_A^{\succcurlyeq} = \{[u_1]_A^{\succcurlyeq}, [u_2]_A^{\succcurlyeq}, \ldots, [u_n]_A^{\succcurlyeq}\},$$

where $i \in \{1,2,\ldots,n\}$, then $[u_i]_A^{\succcurlyeq}$ will be called a dominance class and $U/\mathcal{R}_A^{\succcurlyeq}$ be a classification of $U$ w.r.t. $A$ in lattice-valued information system with interval-valued intuitionistic fuzzy decision.

Similarly, in a LS-IFD, one can get that

$$[u_i]_d^{\succcurlyeq} = \{u_j \mid f(u_j,d) \succcurlyeq_d f(u_i,d)\}$$

is the dominance class of $u_i$ and

$$U/\mathcal{R}_d^{\succcurlyeq} = \{[u_1]_d^{\succcurlyeq}, [u_2]_d^{\succcurlyeq}, \ldots, [u_n]_d^{\succcurlyeq}\}$$

is a classification of $U$ w.r.t decision attributes set $\{d\}$, respectively.

**Remark 3.1.** In the following, we use $\mathcal{R}_a^{\succcurlyeq}$ instead of $\mathcal{R}_{\{a\}}^{\succcurlyeq}$ and $[u_i]_a^{\succcurlyeq}$ instead of $[u_i]_{\{a\}}^{\succcurlyeq}$ for any $a \in C$.

**Definition 3.3.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision. If $\mathcal{R}_C^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$, then the lattice-valued information system with interval-valued intuitionistic fuzzy decision is called a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision; otherwise, it is called an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision.

Notice that for simplification, the consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision is abbreviated to CLS-IFD and inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision is abbreviated to ILS-IFD, respectively.

**Example 3.1.** Consider a lattice-valued information system with interval-valued intuitionistic fuzzy decision in Table 1, where $U = \{u_1, u_2, \ldots, u_6\}$ and $C = \{a_1, a_2, a_3, a_4, a_5\}$.

According to above expression, we can find $V_{a_1} = \{1,2,3\}$ is a finite lattice with real numbers, where the partial order relation on $V_{a_1}$ is "$\geq$" between two real

Table 1. LS-IFD.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $d$ |
|---|---|---|---|---|---|---|
| $u_1$ | 2 | 0.6 | [0.3, 0.8] | {0,1,2} | (0.8, 0.1) | ([0.25, 0.25], [0.30, 0.40]) |
| $u_2$ | 3 | 0.7 | [0.1, 0.5] | {0,1,2} | (0.0, 1.0) | ([0.55, 0.65], [0.05, 0.30]) |
| $u_3$ | 2 | 0.6 | [0.3, 0.8] | {0} | (0.5, 0.3) | ([0.25, 0.25], [0.30, 0.40]) |
| $u_4$ | 2 | 0.7 | [0.3, 0.8] | {0} | (0.3, 0.6) | ([0.55, 0.65], [0.05, 0.30]) |
| $u_5$ | 1 | 0.6 | [0.4, 0.9] | {0,1,2} | (0.5, 0.3) | ([0.60, 0.80], [0.00, 0.00]) |
| $u_6$ | 1 | 0.6 | [0.2, 0.6] | {0,1} | (0.3, 0.6) | ([0.60, 0.80], [0.00, 0.00]) |

numbers. So the dominance relation on $U$ according to attribute $a_1$ can be defined as

$$\mathcal{R}_{a_1}^{\succcurlyeq} = \{(u,v) \mid f(u,a_1) \geq f(v,a_1)\}.$$

The domain $V_{a_2} = \{0.6, 0.7\}$ is a finite lattice with fuzzy elements where the partial order relation on $V_{a_2}$ is "$\geq$" between two fuzzy elements. And the dominance relation on $U$ according to attribute $a_2$ can be defined as

$$\mathcal{R}_{a_2}^{\succcurlyeq} = \{(u,v) \mid f(u,a_2) \geq f(v,a_2)\}.$$

The domain $V_{a_3} = \{[0.1,0.5],[0.3,0.8],[0.2,0.6],[0.4,0.9]\}$ is a finite lattice with interval-valued elements, and the dominance relation on it can be defined as

$$\mathcal{R}_{a_3}^{\succcurlyeq} = \{(u,v) \mid f^{\pm}(u,a_3) \geq f^{\pm}(v,a_3)\},$$

where $f^{\pm}(u,a_3) \geq f^{\pm}(v,a_3)$ if and only if $f^{+}(u,a_3) \geq f^{+}(v,a_3)$ and $f^{-}(u,a_3) \geq f^{-}(v,a_3)$, $f^{+}(u,a_3)$ is the right endpoint of $f(u,a_3)$ and $f^{-}(u,a_3)$ is the left endpoint of $f(u,a_3)$, to name a couple for explanation.

The domain $V_{a_4} = \{\{0\},\{0,1\},\{0,1,2\}\}$ is a finite lattice with set-valued elements, where the partial order relation on $V_{a_4}$ is "$\supseteq$" between two sets. Thus the dominance relation on $U$ according to attribute $a_4$ can be defined as

$$\mathcal{R}_{a_4}^{\succcurlyeq} = \{(u,v) \mid f(u,a_4) \supseteq f(v,a_4)\}.$$

The domain $V_{a_5} = \{(0,1),(0.3,0.6),(0.5,0.3),(0.8,0.1)\}$ is a finite lattice, every element of which is a classical intuitionistic fuzzy set. Thus the dominance relation on $U$ according to attribute $a_5$ can be defined as

$$\mathcal{R}_{a_5}^{\succcurlyeq} = \{(u,v) \mid \mu_{a_5}(u) \geq \mu_{a_5}(v) \text{ and } \nu_{a_5}(u) \leq \nu_{a_5}(v)\}.$$

Just like the dominance relations on $U$ w.r.t. $a_3$ and $a_5$, the dominance relation on $U$ according to decision attribute $d$ can be defined as

$$\begin{aligned}\mathcal{R}_{d}^{\succcurlyeq} &= \{(u,v) \mid \mu_d(u) \geq \mu_d(v) \text{ and } \nu_d(u) \leq \nu_d(v)\} \\ &= \left\{(u,v) \mid \begin{cases}\mu_d^{+}(u) \geq \mu_d^{+}(v) \\ \mu_d^{-}(u) \geq \mu_d^{-}(v)\end{cases} \text{ and } \begin{cases}\nu_d^{+}(u) \leq \nu_d^{+}(v) \\ \nu_d^{-}(u) \leq \nu_d^{-}(v)\end{cases}\right\}.\end{aligned}$$

By computing one can get that

$$[u_1]_C^{\succcurlyeq} = \{u_1\},$$
$$[u_2]_C^{\succcurlyeq} = \{u_2\},$$
$$[u_3]_C^{\succcurlyeq} = \{u_1, u_3\},$$
$$[u_4]_C^{\succcurlyeq} = \{u_4\},$$
$$[u_5]_C^{\succcurlyeq} = \{u_5\},$$
$$[u_6]_C^{\succcurlyeq} = \{u_1, u_5, u_6\};$$

and

$$[u_1]_d^{\succcurlyeq} = \{u_1, u_2, u_3, u_4, u_5, u_6\},$$
$$[u_2]_d^{\succcurlyeq} = \{u_2, u_4, u_5, u_6\},$$
$$[u_3]_d^{\succcurlyeq} = \{u_1, u_2, u_3, u_4, u_5, u_6\},$$
$$[u_4]_d^{\succcurlyeq} = \{u_2, u_4, u_5, u_6\},$$
$$[u_5]_d^{\succcurlyeq} = \{u_5, u_6\},$$
$$[u_6]_d^{\succcurlyeq} = \{u_5, u_6\}.$$

Since $\mathcal{R}_C^{\succcurlyeq} \nsubseteq \mathcal{R}_d^{\succcurlyeq}$, the lattice-valued information system with interval-valued intuitionistic fuzzy decision is not a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. That is, it is an ILS-IFD.

One can get that the lattice-valued information systems with interval-valued intuitionistic fuzzy decision are the most comprehensive lattice-valued information systems with fuzzy decision, because interval-valued intuitionistic fuzzy sets can be seen as a generalization of intuitionistic fuzzy sets, or even fuzzy sets. In fact, we have the following results.

**Corollary 3.1.** *A lattice-valued information system with interval-valued intuitionistic fuzzy decision is called a lattice-valued information systems with intuitionistic fuzzy decision if and only if* $V_d = \{(u, \mu_d(u), \nu_d(u)) \mid u \in U\}$ *is an intuitionistic fuzzy set, i.e.,* $\mu_d(u) \in [0, 1]$,$\nu_d(u) \in [0, 1]$ *with the condition* $\mu_d(u) + \nu_d(u) \leq 1$ *for any* $u \in U$.

**Corollary 3.2.** *A lattice-valued information system with interval-valued intuitionistic fuzzy decision is called a lattice-valued information system with fuzzy decision if and only if* $V_d = \{(u, \mu_d(u), \nu_d(u)) \mid u \in U\}$ *is an intuitionistic fuzzy set, i.e.,* $\mu_d(u) \in [0, 1]$ *and* $\nu_d(u) \in [0, 1]$ *with the condition* $\mu_d(u) + \nu_d(u) \equiv 1$ *for any* $u \in U$.

As can be seen from above, all results in lattice-valued information systems with interval-valued intuitionistic fuzzy decision can be successfully applied to lattice-valued information systems with intuitionistic fuzzy decision and lattice-valued information systems with fuzzy decision. Without any loss of generality, in the following we only consider lattice-valued information systems with interval-valued intuitionistic fuzzy decision.

**Definition 3.4.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision, and $B, A \subseteq C$.

(1) If $[u]_B^{\succcurlyeq} = [u]_A^{\succcurlyeq}$ for any $u \in U$, then we have that classification $U/\mathcal{R}_B^{\succcurlyeq}$ is equal to $U/\mathcal{R}_A^{\succcurlyeq}$, denoted by $U/\mathcal{R}_B^{\succcurlyeq} = U/\mathcal{R}_A^{\succcurlyeq}$.

(2) If $[u]_B^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ for any $u \in U$, then we have that classification $U/\mathcal{R}_B^{\succcurlyeq}$ is finer than $U/\mathcal{R}_A^{\succcurlyeq}$, denoted by $U/\mathcal{R}_B^{\succcurlyeq} \subseteq U/\mathcal{R}_A^{\succcurlyeq}$.

(3) If $[u]_B^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ for any $u \in U$ and $[v]_B^{\succcurlyeq} \neq [v]_A^{\succcurlyeq}$ for some $v \in U$, then we have that classification $U/\mathcal{R}_B^{\succcurlyeq}$ is proper finer than $U/\mathcal{R}_A^{\succcurlyeq}$, denoted by $U/\mathcal{R}_B^{\succcurlyeq} \subset U/\mathcal{R}_A^{\succcurlyeq}$.

From the definition of $\mathcal{R}_A^{\succcurlyeq}$ and $[u]_A^{\succcurlyeq}$, the following properties can be obtained directly.

**Proposition 3.1.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision, and $B, A \subseteq C$, then we can get*

(1) $\mathcal{R}_A^{\succcurlyeq} = \bigcap_{a \in A} \mathcal{R}_a^{\succcurlyeq}$;

(2) $\mathcal{R}_A^{\succcurlyeq}$ *is reflective, transitive, but not symmetric, so it is not an equivalence relation;*

(3) *If $B \subseteq A \subseteq C$, then $\mathcal{R}_C^{\succcurlyeq} \subseteq \mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_B^{\succcurlyeq}$.*

**Proposition 3.2.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision, and $B, A \subseteq C$, then we have that*

(1) *If $B \subseteq A \subseteq C$, then $[u]_C^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq} \subseteq [u]_B^{\succcurlyeq}$ for any $u \in U$.*
(2) *If $u \in [v]_A^{\succcurlyeq}$, then $[u]_A^{\succcurlyeq} \subseteq [v]_A^{\succcurlyeq}$ and $[v]_A^{\succcurlyeq} = \bigcup\{[u]_A^{\succcurlyeq} \mid u \in [v]_A^{\succcurlyeq}\}$.*
(3) *$[u]_C^{\succcurlyeq} = [v]_C^{\succcurlyeq}$ if and only if $f(u, a) = f(v, a)$ for any $a \in C$.*
(4) *$|[u]_C^{\succcurlyeq}| \geq 1$ for any $u \in U$.*

In the following, we will investigate the problem of approximation operators w.r.t. a dominance relation $\mathcal{R}_A^{\succcurlyeq}$ in lattice-valued information systems with intuitionistic fuzzy decision.

To investigate the issue of approximation operators in LS-IFD, here we formally introduce the maximum ($inf$) and minimum ($sup$) of an interval-valued intuitionistic fuzzy set $A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}$ as

$$supA = \max\{(y, \mu_A(y), \nu_A(y)) \mid \mu_A(y) \geq \mu_A(x) \text{ and } \nu_A(y) \leq \nu_A(x), \forall\, x, y \in U\},$$

$$infA = \min\{(y, \mu_A(y), \nu_A(y)) \mid \mu_A(y) \leq \mu_A(x) \text{ and } \nu_A(y) \geq \nu_A(x), \forall\, x, y \in U\}.$$

In other words, the "$supA$" is the maximum value contained in set $A$ and "$infA$" is the minimum value contained in set $A$. For example, if $A = \{2, 3, 1, 5, 7\}$, then $supA = 7$ and $infA = 1$. Therefore, just as the lower and upper approximation operators in $\mathcal{I}_{id}^{\succcurlyeq}$, the upper and lower approximation operators in lattice-valued information systems with interval-valued intuitionistic fuzzy decision can be described as follows.

**Definition 3.5.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. The lower and upper approximation operators of $\{d\}$ with respect to $A$ is denoted by $\underline{A_d^{\succcurlyeq}}$ and $\overline{A_d^{\succcurlyeq}}$, respectively. And their membership function are defined by

$$\underline{A_d^{\succcurlyeq}}(u) = inf\{f(v, d) \mid v \in [u]_A^{\succcurlyeq}\},$$

$$\overline{A_d^{\succcurlyeq}}(u) = sup\{f(v, d) \mid v \in [u]_A^{\succcurlyeq}\}.$$

From above definition, one can easily obtain the following results.

**Proposition 3.3.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be a lattice-valued information system with interval-valued intuitionistic fuzzy decision, and $B, A \subseteq C$, then we have that*

(1) *If $B \subseteq A$, then $\underline{B_d^{\succcurlyeq}} \subseteq \underline{A_d^{\succcurlyeq}}$ and $\overline{A_d^{\succcurlyeq}} \subseteq \overline{B_d^{\succcurlyeq}}$.*

(2) *If $\mathcal{R}_A^{\succcurlyeq} = \mathcal{R}_B^{\succcurlyeq}$, then $\underline{B_d^{\succcurlyeq}} = \underline{A_d^{\succcurlyeq}}$ and $\overline{A_d^{\succcurlyeq}} = \overline{B_d^{\succcurlyeq}}$.*

These properties mentioned above can be understood through the following example.

**Example 3.2** (*Continued from Example 3.1*). If take $A = \{a_1, a_2, a_3\}$, by calculating we have that

$$[u_1]_A^{\succcurlyeq} = \{u_1, u_3, u_4\},$$

$$[u_2]_A^{\succcurlyeq} = \{u_2\},$$

$$[u_3]_A^{\succcurlyeq} = \{u_1, u_3, u_4\},$$

$$[u_4]_A^{\succcurlyeq} = \{u_4\},$$

$$[u_5]_A^{\succcurlyeq} = \{u_5\},$$

$$[u_6]_A^{\succcurlyeq} = \{u_1, u_3, u_4, u_5, u_6\}.$$

Obviously, $U/\mathcal{R}_A^{\succcurlyeq}$ is a covering of $U$ and $U/\mathcal{R}_C^{\succcurlyeq} \subset U/\mathcal{R}_A^{\succcurlyeq}$.

Moreover, one can obtain that

$$\underline{A_d^{\succcurlyeq}} = \frac{f_1^d}{u_1} + \frac{f_2^d}{u_2} + \frac{f_1^d}{u_3} + \frac{f_2^d}{u_4} + \frac{f_5^d}{u_5} + \frac{f_1^d}{u_6},$$

$$\overline{A_d^{\succcurlyeq}} = \frac{f_2^d}{u_1} + \frac{f_2^d}{u_2} + \frac{f_2^d}{u_3} + \frac{f_2^d}{u_4} + \frac{f_5^d}{u_5} + \frac{f_5^d}{u_6};$$

and

$$\underline{C_d^{\succcurlyeq}} = \frac{f_1^d}{u_1} + \frac{f_2^d}{u_2} + \frac{f_1^d}{u_3} + \frac{f_2^d}{u_4} + \frac{f_5^d}{u_5} + \frac{f_1^d}{u_6},$$

$$\overline{C_d^{\succcurlyeq}} = \frac{f_1^d}{u_1} + \frac{f_2^d}{u_2} + \frac{f_1^d}{u_3} + \frac{f_2^d}{u_4} + \frac{f_5^d}{u_5} + \frac{f_5^d}{u_6},$$

where $f_1^d = f(u_1, d) = ([0.25, 0.25], [0.30, 0.40])$, and the same as $f_i^d$ with $i = 2, 3, \ldots, 6$. According to the definition of equivalence between two interval-valued intuitionistic fuzzy set and the dominance relation "$\succcurlyeq_d$" w.r.t. decision attributes set $\{d\}$, we have that

$$f_5^d = f_6^d \succcurlyeq_d f_2^d = f_4^d \succcurlyeq_d f_1^d = f_3^d.$$

Thus, we have that $\underline{A_d^{\succcurlyeq}} \subseteq \underline{C_d^{\succcurlyeq}}$ and $\overline{C_d^{\succcurlyeq}} \subseteq \overline{A_d^{\succcurlyeq}}$.

## 4. Knowledge Reduction in CLS-IFD

One fundamental aspect of rough set theory involves the search for particular subsets of attributes which provide the same information for classification purposes as the full set of available attributes. Such subsets are called reductions. In the context of dominance relation, to simplify knowledge representation in lattice-valued information systems with intuitionistic fuzzy decision, attribute reduction, thus, is necessary.

In this section, our main work is to investigate the problem of attribute reduction in consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. An approach of attribute reduction based on discernibility matrix in CLS-IFD is constructed and an illustrative example is applied to show its validity.

**Definition 4.1.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. $A$ is referred to as a consistent attributes set if $\mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$. Moreover, if $A$ is a consistent attributes set and any proper subset of $A$ is not a consistent attributes set, then $A$ is referred to as a consistent reduction of $\mathcal{L}_{id}^{\succcurlyeq}$.

Naturally, a consistent reduction of $\mathcal{L}_{id}^{\succcurlyeq}$ is a minimal attribute subset satisfying $\mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$. An attribute $a \in C$ is called dispensable w.r.t. $\mathcal{R}_C^{\succcurlyeq}$ if $\mathcal{R}_C^{\succcurlyeq} = \mathcal{R}_{C-\{a\}}^{\succcurlyeq}$, or else $a$ is called indispensable. The set of all indispensable attributes is called a core with respect to $\mathcal{R}_C^{\succcurlyeq}$ and is denoted by $Core(C)$. An attribute in the core must be in every consistent reduction. In other words, $Core(C)$ is the intersection of all classical reductions of the system, in which case the core may be empty set.

Obviously, we have that it is a complicated work to find all consistent reductions of $\mathcal{L}_{id}^{\succcurlyeq}$ because there exists $2^m - 1$ non-empty attributes set if the cordiality of condition attributes set is $m$. However, the following property is true forever.

**Proposition 4.1.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision, then there exist as least one consistent reduction of $\mathcal{L}_{id}^{\succcurlyeq}$.*

**Proof.** It can be proved directly by Definition 4.1. □

Next we propose an approach to attributes reduction which can be applied to find all consistent reductions of the consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. At first, the notation of discernibility attributes set, as a foundation of the approach, is constructed and some of its properties are discussed.

**Definition 4.2.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. For any $u, v \in U$, if we denote

$$\mathcal{D}(u,v) = \begin{cases} \{a \mid (u,v) \notin \mathcal{R}_a^{\succcurlyeq}, a \in C\} & (u,v) \notin \mathcal{R}_d^{\succcurlyeq} \\ \emptyset & (u,v) \in \mathcal{R}_d^{\succcurlyeq} \end{cases}$$

then $\mathcal{D}(u,v)$ is called a discernibility attributes set between $u$ and $v$, and

$$\mathcal{M} = \begin{pmatrix} \mathcal{D}(u_1,u_1) & \cdots & \mathcal{D}(u_1,u_n) \\ \vdots & \ddots & \vdots \\ \mathcal{D}(u_n,u_1) & \cdots & \mathcal{D}(u_n,u_n) \end{pmatrix}$$

is called a discernibility matrix of this consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision.

Let us denote $\mathcal{D}_\emptyset = \{\mathcal{D}(u,v) \mid \mathcal{D}(u,v) \neq \emptyset\}$, then we have the following property.

**Proposition 4.2.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$, then the following claims are equivalent.*

(1) $\mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$;
(2) *For any* $\mathcal{D} \in \mathcal{D}_\emptyset$, $A \bigcap \mathcal{D} \neq \emptyset$;
(3) *For any* $B \subseteq C$, *if* $B \bigcap A = \emptyset$, *then* $B \notin \mathcal{D}_\emptyset$.

**Proof.** (1)$\Longleftrightarrow$(2): Suppose that $\mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$, if $(u,v) \in \mathcal{R}_A^{\succcurlyeq}$, then $(u,v) \in \mathcal{R}_d^{\succcurlyeq}$. Hence, $\mathcal{D}(u,v) = \emptyset$, i.e., $\mathcal{D}(u,v) \notin \mathcal{D}_\emptyset$. So we have that for any $\mathcal{D} \in \mathcal{D}_\emptyset$, there must exist $u, v \in U$ s.t. $(u,v) \notin \mathcal{R}_d^{\succcurlyeq}$. According to the hypothesis, $(u,v) \notin \mathcal{R}_A^{\succcurlyeq}$ is true. By

Proposition 3.1, we have that there exist $a' \in A$ s.t. $(u,v) \notin \mathcal{R}_{a'}^{\succcurlyeq}$. Thus, $a' \in \mathcal{D}(u,v)$, that is, $A \bigcap \mathcal{D} \neq \emptyset$. Conversely, for any $v \in U$, if $(u_0, v) \notin \mathcal{R}_d^{\succcurlyeq}$ for $u_0 \in U$, then by condition $A \bigcap \mathcal{D}(u_0, v) \neq \emptyset$ we have that there exists $a \in A$ s.t. $(u_0, v) \notin \mathcal{R}_a^{\succcurlyeq}$. So, $(u_0, v) \notin \mathcal{R}_A^{\succcurlyeq}$. Therefore, if $(u_0, v) \in \mathcal{R}_A^{\succcurlyeq}$, then $(u_0, v) \in \mathcal{R}_d^{\succcurlyeq}$, i.e., $\mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$.

(1)$\Longleftrightarrow$(3): By the equivalence of claims (1) and (2) we have that $\mathcal{R}_A^{\succcurlyeq} \subseteq \mathcal{R}_d^{\succcurlyeq}$ if and only if $A \bigcap \mathcal{D} \neq \emptyset$ with the condition $\mathcal{D} \in \mathcal{D}_{\emptyset}$. So, if $B \bigcap A = \emptyset$ for any $B \subseteq C$, then $B \notin \mathcal{D}_{\emptyset}$.

The proof is completed. □

**Proposition 4.3.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. Then $a \in Core(C)$ if and only if there exists $\mathcal{D} \in \mathcal{D}_{\emptyset}$ such that $\mathcal{D} = \{a\}$.*

**Proof.** Suppose that $K_a = \{\mathcal{D} \mid a \in \mathcal{D} \& \mathcal{D} \in \mathcal{D}_{\emptyset}\}$. If $|\mathcal{D}| \geq 2$ for any $\mathcal{D} \in K_a$, then it is easy to see that $K \bigcap \mathcal{D} \neq \emptyset$ for all $\mathcal{D} \in \mathcal{D}_{\emptyset}$, where $K = \bigcup_{\mathcal{D} \in \mathcal{D}_{\emptyset}} (\mathcal{D} - \{a\})$. By Proposition 4.2, we can get that $K$ is a consistent attributes set of $\mathcal{L}_{id}^{\succcurlyeq}$. Then there exists $K' \subseteq K$ s.t. $K'$ is a consistent reduction of $\mathcal{L}^{\succcurlyeq}$. Clearly, $a \notin K'$, this is a contradiction with $a \in Core(C)$.

Conversely, Suppose that there exists $\mathcal{D} \in \mathcal{D}_{\emptyset}$, s.t. $\mathcal{D} = \{a\}$, thus, existing $u, v \in U$ with $u \neq v$ s.t. $\mathcal{D} = \{a\}$. By the definition of discernibility attribute set, we can get that $(u,v) \notin \mathcal{R}_a^{\succcurlyeq}$ and $(u,v) \in \mathcal{R}_{C-\{a\}}^{\succcurlyeq}$. It follows that $\mathcal{R}_C^{\succcurlyeq} \neq \mathcal{R}_{C-\{a\}}^{\succcurlyeq}$. Notice that $a \in C$ is an element of $Core(C)$ if and only if $\mathcal{R}_C^{\succcurlyeq} \neq \mathcal{R}_{C-\{a\}}^{\succcurlyeq}$. Therefore, $a \in Core(C)$.

The proof is completed. □

To investigate the attribute reduction of $\mathcal{L}_{id}^{\succcurlyeq}$ from the viewpoint of discernibility matrix, the judgement method for a consistent reduction of the consistent lattice-valued information system intuitionistic fuzzy decision is proposed in the following.

**Proposition 4.4.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$, then $A$ is a consistent reduction if and only it satisfies the following assertions.*

(1) *For any $\mathcal{D} \in \mathcal{D}_{\emptyset}$, $A \bigcap \mathcal{D} \neq \emptyset$;*
(2) *For any $a \in A$, $\exists\ \mathcal{D} \in \mathcal{D}_{\emptyset}$, s.t. $(A - \{a\}) \bigcap \mathcal{D} = \emptyset$.*

**Proof.** It can be proved by Proposition 4.2 and the definition of consistent reduction. □

According to above Proposition, we can see that $A \subseteq C$ is a consistent reduction in $\mathcal{L}_{id}^{\succcurlyeq}$ if and only if $A$ is the minimal set satisfying $A \bigcap \mathcal{D}(u,v) \neq \emptyset$ for any $\mathcal{D} \in \mathcal{D}_{\emptyset}$.

**Definition 4.3.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and if we denote

$$\mathcal{F} = \bigwedge_{i,j=1}^{n} \left( \bigvee \mathcal{D}(u_i, u_j) \right),$$

then $\mathcal{F}$ is referred to as a discernibility formula.

**Definition 4.4.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. Let

$$\mathcal{F} = \bigwedge_{i,j=1}^{n} \left( \bigvee \mathcal{D}(u_i, u_j) \right) = \bigvee_{k=1}^{p} \left( \bigwedge_{l=1}^{q_k} a_{i_l} \right),$$

then $\mathcal{F}$ is referred to as a minimal disjunctive normal form of the discernibility formula if the cardinal number of any $\mathcal{B}_k = \{a_{i_l} \mid l \leq q_k\}$ is equal to $q_k$.

Based on the discernibility formula, a practical approach to consistent reduction in consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision can be introduced as follows.

**Proposition 4.5.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be a consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. Then we have that*

$$\mathcal{F} = \bigvee_{k=1}^{p} \left( \bigwedge_{l=1}^{q_k} a_{i_l} \right) = \bigvee_{k=1}^{p} \mathcal{B}_k$$

*contains all consistent reductions of this consistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and each $\mathcal{B}_k$ is a consistent reduction.*

**Proof.** In order to prove the conclusion, at first we let

$$\mathcal{Q}(C) = \left\{ \bigvee_{l=1}^{h} a_{i_l} \mid a_{i_l} \in C, i_l \leq h \leq m \right\}.$$

And for

$$\bigvee_{l=1}^{k} a_{i_l} \in \mathcal{Q}(C), \quad \bigvee_{l=1}^{p} a_{j_l} \in \mathcal{Q}(C),$$

the partial order relation "$\lhd$" on $\mathcal{Q}(C)$ can be defined as

$$\bigvee_{l=1}^{k} a_{i_l} \lhd \bigvee_{l=1}^{p} a_{j_l} \Longleftrightarrow \{a_{i_l} \mid l \leq k\} \subseteq \{a_{j_l} \mid l \leq q\}.$$

In the following, we will prove the proposition.

Table 2. CLS-IFD.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $d$ |
|---|---|---|---|---|
| $u_1$ | 0.5 | [0.6, 0.8] | 1 | ([0.10, 0.20], [0.40, 0.75]) |
| $u_2$ | 0.7 | [0.6, 0.8] | 2 | ([0.35, 0.60], [0.20, 0.25]) |
| $u_3$ | 0.5 | [0.4, 0.7] | 2 | ([0.10, 0.20], [0.40, 0.75]) |
| $u_4$ | 0.6 | [0.4, 0.7] | 3 | ([0.30, 0.40], [0.30, 0.40]) |
| $u_5$ | 0.7 | [0.7, 0.9] | 2 | ([0.35, 0.60], [0.20, 0.25]) |
| $u_6$ | 0.7 | [0.6, 0.8] | 3 | ([0.35, 0.60], [0.20, 0.25]) |

One can obtain that $\mathcal{B}_k \bigcap \mathcal{D} \neq \emptyset$ for $\mathcal{D} \in \mathcal{D} \neq \emptyset$, because we have that

$$\mathcal{B}_k = \bigwedge_{l=1}^{q_k} a_{i_l} \lhd \mathcal{F}.$$

With the condition $\mathcal{F} = \bigvee_{k=1}^{p} \mathcal{B}_k$, it is easy to obtain that there must exist $\mathcal{D} \in \mathcal{D}_\emptyset$ s.t. $\mathcal{B}_k^* \bigcap \mathcal{D} = \emptyset$, where $\mathcal{B}_k^* = \mathcal{B}_k - \{a^*\}$ for any $a^* \in \mathcal{B}_k$. According to Proposition 4.4, we have that $\mathcal{B}_k$ is a consistent reduction of CLS-IFD. Since discernibility formula contain all of the discernibility attributes sets, there would not exist another consistent reduction except $\mathcal{B}_k$ for $k = 1, 2, \ldots, p$.

The proof is completed. □

**Example 4.1.** Consider a CLS-IFD in Table 2, where $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $C = \{a_1, a_2, a_3\}$.

According to Definition 4.2, one can obtain that discernibility matrix $\mathcal{M}$ of above information system is

$$\begin{pmatrix} \emptyset & \{a_1, a_3\} & \emptyset & \{a_1, a_3\} & \{a_1, a_2, a_3\} & \{a_1, a_3\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \{a_1, a_2\} & \emptyset & \{a_1, a_3\} & \{a_1, a_2\} & \{a_1, a_2, a_3\} \\ \emptyset & \{a_1, a_2\} & \emptyset & \emptyset & \{a_1, a_2\} & \{a_1, a_2\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}$$

Hence, we can get that

$$\begin{aligned} F^{\succcurlyeq} &= (a_1 \vee a_3) \wedge (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_2) \\ &= a_1 \vee (a_2 \wedge a_3). \end{aligned}$$

Therefore, there are two consistent reductions for the system, which are $\{a_1\}$ and $\{a_2, a_3\}$.

## 5. Knowledge Reduction in ILS-IFD

Since knowledge reduction in inconsistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision is more complicated than that in consistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision, and in which case the approach to attributes reduction in CLS-IFD is not applicable to ILS-IFD any more, the definition of knowledge reductions, such as upper approximation reduction, lower approximation reduction, etc., are therefore proposed and the approaches to obtain all reductions are constructed respectively. Also, examples are used to illustrate its validity.

### 5.1. *Upper and lower approximation reductions in ILS-IFD*

In this subsection, we introduce the definition of upper approximation reduction and lower approximation reduction in inconsistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision, and then present approaches to obtain all the upper and lower reductions of ILS-IFD.

**Definition 5.1.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$.

(1) If $\overline{A_d^{\succcurlyeq}} = \overline{C_d^{\succcurlyeq}}$, then $A$ is referred to as an upper approximation consistent set of $\mathcal{L}_{id}^{\succcurlyeq}$. Moreover, if $A$ is an upper approximation consistent set and $\overline{B_d^{\succcurlyeq}} \neq \overline{C_d^{\succcurlyeq}}$ for any $B \subset A$, then $A$ is referred to as an upper approximation reduction of $\mathcal{L}_{id}^{\succcurlyeq}$.

(2) If $\underline{A_d^{\succcurlyeq}} = \underline{C_d^{\succcurlyeq}}$, then $A$ is referred to as a lower approximation consistent set of $\mathcal{L}_{id}^{\succcurlyeq}$. Moreover, if $A$ is a lower approximation consistent set and $\underline{B_d^{\succcurlyeq}} \neq \underline{C_d^{\succcurlyeq}}$ for any $B \subset A$, then $A$ is referred to as a lower approximation reduction of $\mathcal{L}_{id}^{\succcurlyeq}$.

From above definition, we have the following properties.

**Proposition 5.1.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. Then $A$ is an upper approximation consistent set if and only if there exists $a_k \in A$ such that $f(u, a_k) > f(v, a_k)$ when $\overline{C_d^{\succcurlyeq}}(u) < \overline{C_d^{\succcurlyeq}}(v)$ for $u, v \in U$.*

**Proof.** "$\Longrightarrow$": Suppose $f(u, a_k) \leq f(v, a_k)$ for any $a_k \in A$ when $\overline{C_d^{\succcurlyeq}}(v) < \overline{C_d^{\succcurlyeq}}(u)$, then we have that $v \in [u]_A^{\succcurlyeq}$. According to the Proposition 3.2, one can get that $[v]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$. By Definition 3.5, $\overline{A_d^{\succcurlyeq}}(v) \leq \overline{A_d^{\succcurlyeq}}(u)$ is true. With the condition that $A$ is an upper approximation consistent set, we have that $\overline{A_d^{\succcurlyeq}}(u) = \overline{C_d^{\succcurlyeq}}(u)$ and $\overline{A_d^{\succcurlyeq}}(v) = \overline{C_d^{\succcurlyeq}}(v)$. That is, $\overline{C_d^{\succcurlyeq}}(u) \geq \overline{C_d^{\succcurlyeq}}(v)$, which is a contradiction with $\overline{C_d^{\succcurlyeq}}(u) < \overline{C_d^{\succcurlyeq}}(v)$. Therefore, there exists $a_k \in A$ s.t. $f(u, a_k) > f(v, a_k)$ when $\overline{C_d^{\succcurlyeq}}(u) < \overline{C_d^{\succcurlyeq}}(v)$ for any $u, v \in U$.

"$\Longleftarrow$": Suppose $A$ is not an upper approximation consistent set, that is, $\overline{A_d^{\succcurlyeq}} \neq \overline{C_d^{\succcurlyeq}}$. Thus, there exists $u_0 \in U$ s.t. $\overline{A_d^{\succcurlyeq}}(u_0) \neq \overline{C_d^{\succcurlyeq}}(u_0)$. By Definition 3.5 and Proposition 3.3, we have that $\overline{A_d^{\succcurlyeq}}(u_0) > \overline{C_d^{\succcurlyeq}}(u_0)$. Moreover, let $v_0 \in [u_0]_A^{\succcurlyeq}$ s.t. $\overline{A_d^{\succcurlyeq}}(u_0) = f(v_0, d)$. And we have that $v_0 \in [v_0]_C^{\succcurlyeq}$, so one can obtain that $\max\{f(u,d) \mid u \in [v_0]_C^{\succcurlyeq}\} \geq f(v_0, d)$. Hence, $\overline{C_d^{\succcurlyeq}}(v_0) > \overline{C_d^{\succcurlyeq}}(u_0)$. So, there exists $a_k \in A$ s.t. $f(u_0, a_k) > f(v_0, a_k)$, that is, $v_0 \notin [u_0]_A^{\succcurlyeq}$. And this is a contradiction with $v_0 \in [u_0]_A^{\succcurlyeq}$.

This proposition is proved. □

**Proposition 5.2.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. Then $A$ is a lower approximation consistent set if and only if there exists $a_k \in A$ s.t. $f(u, a_k) < f(v, a_k)$ when $\underline{C_d^{\succcurlyeq}}(u) < \underline{C_d^{\succcurlyeq}}(v)$ for any $u, v \in U$.*

**Proof.** It is similar to the proof of Proposition 5.1. □

From above one can get that Proposition 5.1 and Proposition 5.2 are just an equivalent description of the upper and lower approximation consistent sets respectively. To realize the purpose of obtaining the knowledge reduction in ILS-IFD, the notion of discernibility matrix will be proposed and, then, the detailed methods for researching upper and lower approximation reductions are constructed.

Let us take

$$\mathcal{D}_f^+ = \{(u,v) \mid \overline{C_d^{\succcurlyeq}}(u) < \overline{C_d^{\succcurlyeq}}(v),\ u, v \in U\},$$

$$\mathcal{D}_f^- = \{(u,v) \mid \underline{C_d^{\succcurlyeq}}(u) < \underline{C_d^{\succcurlyeq}}(v),\ u, v \in U\},$$

and then the notion of discernibility matrix can be defined as follows.

**Definition 5.2.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. For any $u, v \in U$, if we denote

$$\mathcal{D}_f^+(u,v) = \begin{cases} \{a \mid f(u,a) > f(v,a),\ a \in C\} & (u,v) \in \mathcal{D}_f^+ \\ \emptyset & (u,v) \notin \mathcal{D}_f^+ \end{cases}$$

and

$$\mathcal{D}_f^-(u,v) = \begin{cases} \{a \mid f(u,a) < f(v,a),\ a \in C\} & (u,v) \in \mathcal{D}_f^- \\ \emptyset & (u,v) \notin \mathcal{D}_f^- \end{cases},$$

then we call $\mathcal{D}_f^+(u,v)$ an upper approximation discernibility attributes set and $\mathcal{D}_f^-(u,v)$ a lower approximation discernibility attributes set between $u$ and $v$, respectively.

The matrix $\mathcal{M}^+ = (\mathcal{D}_f^+(u,v))_{|U|\times|U|}$ and $\mathcal{M}^- = (\mathcal{D}_f^-(u,v))_{|U|\times|U|}$ are called upper approximation discernibility matrix and lower approximation discernibility matrix, respectively.

**Proposition 5.3.** *Let* $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ *be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and* $A \subseteq C$.

(1) *$A$ is an upper approximation consistent set if and only if* $A \bigcap \mathcal{D}_f^+(u,v) \neq \emptyset$ *for all* $(u,v) \in \mathcal{D}_f^+$.

(2) *$A$ is a lower approximation consistent set if and only if* $A \bigcap \mathcal{D}_f^-(u,v) \neq \emptyset$ *for all* $(u,v) \in \mathcal{D}_f^-$.

**Proof.** Without any loss of generality, we only prove claim (1).

"$\Longrightarrow$": By Proposition 5.1 we have that there exists $a \in A$ s.t. $f(u,a) > f(v,a)$ for any $(u,v) \in \mathcal{D}_f^+$. It is easy to obtain that $a \in \mathcal{D}_f^+(u,v)$. Therefore, we have that $A \bigcap \mathcal{D}_f^+(u,v) \neq \emptyset$.

"$\Longleftarrow$": If $A \bigcap \mathcal{D}_f^+(u,v) \neq \emptyset$ for all $(u,v) \in \mathcal{D}_f^+$, then there exists $a \in A \bigcap \mathcal{D}_f^+(u,v)$, i.e., $a \in \mathcal{D}_f^+(u,v)$. With the definition of $\mathcal{D}_f^+(u,v)$, we have that $f(u,a) > f(v,a)$. According to Proposition 5.1, one can get that $A$ is an upper approximation consistent set.

This proposition is proved. □

**Definition 5.3.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. If we denote

$$\mathcal{F}^+ = \bigwedge_{i,j=1}^{n} \left( \bigvee \mathcal{D}_f^+(u_i,u_j) \right)$$

$$\mathcal{F}^- = \bigwedge_{i,j=1}^{n} \left( \bigvee \mathcal{D}_f^-(u_i,u_j) \right),$$

then $\mathcal{F}^+$ is referred to as an upper approximation discernibility formula and $\mathcal{F}^-$ is referred to as a lower approximation discernibility formula.

Moreover, if $\mathcal{F}^+$ and $\mathcal{F}^-$ can be expressed as

$$\mathcal{F}^+ = \bigvee_{k=1}^{p} \left( \bigwedge_{l=1}^{q_k} a_{i_l} \right) = \bigvee_{k=1}^{p} \mathcal{B}_k$$

$$\mathcal{F}^- = \bigvee_{k=1}^{t} \left( \bigwedge_{l=1}^{s_k} a_{j_l} \right) = \bigvee_{k=1}^{t} \mathcal{A}_k$$

then $\mathcal{F}^+$ is referred to as an upper approximation minimal disjunctive normal form of the discernibility formula and $\mathcal{F}^-$ is referred to as a lower approximation minimal disjunctive normal form if $|\mathcal{B}_k| = a_k$ and $|\mathcal{A}_k| = s_k$.

Based on above description, it is easy to obtain that practical approaches to knowledge reduction in inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision can be constructed as follows.

**Proposition 5.4.** *Let* $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ *be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. Then we have that*

(1) $\mathcal{B}_k$ *is an upper approximation reduction and* $\mathcal{F}^+ = \bigvee_{k=1}^{p} \mathcal{B}_k$ *contains all upper approximation reductions.*

(2) $\mathcal{A}_k$ *is a lower approximation reduction and* $\mathcal{F}^- = \bigvee_{k=1}^{p} \mathcal{A}_k$ *contains all lower approximation reductions.*

**Proof.** The proof is similar to Proposition 4.5. □

**Example 5.1** (*Continued from Example 3.1*). By computing we have that $\mathcal{M}^+$ is

$$\begin{pmatrix} \emptyset & a_3a_5 & \emptyset & a_4a_5 & a_1a_5 & a_1a_3a_4a_5 \\ \emptyset & \emptyset & \emptyset & \emptyset & a_1a_2 & a_1a_2a_4 \\ \emptyset & a_3a_5 & \emptyset & a_5 & a_1 & a_1a_3a_5 \\ \emptyset & \emptyset & \emptyset & \emptyset & a_1a_2 & a_1a_2a_3 \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}$$

According to Definition 5.3, one can get that

$$\begin{aligned} \mathcal{F}^+ &= (a_3 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_5) \wedge (a_1 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2) \\ &\quad \wedge a_1 \wedge (a_1 \vee a_2 \vee a_4) \wedge a_5 \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3) \\ &= a_1 \wedge a_5 \end{aligned}$$

Therefore, there is only one upper approximation reduction of this system, which is $\{a_1, a_5\}$.

Similarly, one can get that $\mathcal{M}^-$ is

$$\begin{pmatrix} \emptyset & a_1a_2 & \emptyset & a_2 & a_3 & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & a_3a_5 & \emptyset \\ \emptyset & a_1a_2a_4 & \emptyset & a_2 & a_3a_4 & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & a_3a_4a_5 & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & a_1a_2a_4 & \emptyset & a_1a_2a_3 & a_3a_4a_5 & \emptyset \end{pmatrix}$$

And by Definition 5.3, we have that

$$\begin{aligned}\mathcal{F}^- &= (a_1 \vee a_2) \wedge a_2 \wedge a_3 \wedge (a_3 \vee a_5) \wedge (a_1 \vee a_2 \vee a_4) \\ &\quad \wedge (a_3 \vee a_4) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3) \\ &= a_2 \wedge a_3\end{aligned}$$

So, $\{a_2, a_3\}$ is the only one lower approximation reduction of this system.

### 5.2. *Assignment and partial uniform reductions*

In this subsection, our main work is to discuss the knowledge reduction methods, which are called assignment reduction and partial uniform reductions. At first, assignment function and partial uniform function is constructed. Given an ILS-IFD and $A \subseteq C$, let us denote

$$\sigma_A(u) = \{[v]_d^{\succcurlyeq} \mid [v]_d^{\succcurlyeq} \bigcap [u]_A^{\succcurlyeq} \neq \emptyset\}$$

$$\delta_A(u) = \{[v]_d^{\succcurlyeq} \mid [u]_A^{\succcurlyeq} \subseteq [v]_d^{\succcurlyeq}\},$$

then $\sigma_A(u)$ and $\delta_A(u)$ be called assignment function and partial uniform function of $u \in U$ w.r.t. attributes set $A$, respectively.

**Proposition 5.5.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $B, A \subseteq C$.*

(1) *If $B \subseteq A$, then $\sigma_A(u) \subseteq \sigma_B(u)$ and $\delta_B(u) \subseteq \delta_A(u)$ for any $u \in U$;*
(2) *If $[u]_A^{\succcurlyeq} \subseteq [v]_A^{\succcurlyeq}$ for any $u, v \in U$, then $\sigma_A(u) \subseteq \sigma_A(v)$ and $\delta_A(v) \subseteq \delta_A(u)$.*

**Definition 5.4.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$.

(1) If $\sigma_A(u) = \sigma_C(u)$ for any $u \in U$, then $A$ is referred to as an assignment consistent set of $\mathcal{L}_{id}^{\succcurlyeq}$. Moreover, if $A$ is a distribution consistent set and $\sigma_B(u) \neq \sigma_C(u)$ for any $B \subset A$, then $A$ is referred to as an assignment reduction of $\mathcal{L}_{id}^{\succcurlyeq}$.

(2) If $\delta_A(u) = \delta_C(u)$ for any $u \in U$, then $A$ is referred to as a partial uniform consistent set of $\mathcal{L}_{id}^{\succcurlyeq}$. Moreover, if $A$ is a partial uniform consistent set and $\delta_B(u) \neq \delta_C(u)$ for any $B \subset A$, then $A$ is referred to as a partial uniform reduction of $\mathcal{L}_{id}^{\succcurlyeq}$.

From above definition, one can get that the assignment consistent set is to keep the invariability of possible decision rules for all objects and the partial uniform consistent set is to preserve the invariability of certain decision rules for some objects.

**Proposition 5.6.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$.*

(1) *A is an assignment consistent set iff* $[u]_A^{\succcurlyeq} \not\subseteq [v]_A^{\succcurlyeq}$ *when* $\sigma_C(u) \not\subseteq \sigma_C(v)$ *for any* $u, v \in U$.

(2) *A is an partial uniform consistent set iff* $[u]_A^{\succcurlyeq} \not\subseteq [v]_A^{\succcurlyeq}$ *when* $\sigma_C(v) \not\subseteq \sigma_C(u)$ *for any* $u, v \in U$.

**Proof.** (1) Suppose that $[u]_A^{\succcurlyeq} \subseteq [v]_A^{\succcurlyeq}$ for any $u, v \in U$, then $\sigma_A(u) \subseteq \sigma_A(v)$ by Proposition 5.5. Hence, $\sigma_C(u) \subseteq \sigma_C(v)$ is true because $A$ is an assignment consistent set, and in which case one can get that $[u]_A^{\succcurlyeq} \not\subseteq [v]_A^{\succcurlyeq}$ when $\sigma_C(u) \not\subseteq \sigma_C(v)$ for any $u, v \in U$.

Conversely, by Proposition 5.5, one only needs to obtain $\sigma_A(u) \subseteq \sigma_C(u)$ for any $u \in U$. For any $[u_j]_d^{\succcurlyeq} \in \sigma_A(u)$, we have that $[u_j]_d^{\succcurlyeq} \bigcap [u]_A^{\succcurlyeq} \neq \emptyset$. Let $v_i \in [u_j]_d^{\succcurlyeq} \bigcap [u]_A^{\succcurlyeq}$, then $v_i \in [u_j]_d^{\succcurlyeq}$ and $v_i \in [u]_A^{\succcurlyeq}$, from which $[v_i]_A^{\succcurlyeq} \subseteq [u]_A^{\succcurlyeq}$ is true. With the given condition we have that $\sigma_C(v_i) \subseteq \sigma_C(u)$. What's more, $v_i \in [u_j]_d^{\succcurlyeq} \bigcap [v_i]_C^{\succcurlyeq}$ because $v_i \in [v_i]_C^{\succcurlyeq}$. Therefore, $\sigma_A(u) \subseteq \sigma_C(v_i)$, in which case $\sigma_A(u) \subseteq \sigma_C(u)$.

The proof of claim (2) is similar to claim (1).

The proof is completed. □

From above one can get that Proposition 5.6 is just an equivalent description of the assignment and partial uniform consistent sets. To realize the purpose of obtaining the knowledge reduction in ILS-IFD, the notion of discernibility matrix will be proposed and, then, the detailed methods for researching assignment and partial uniform reductions are constructed.

Take

$$\mathcal{D}_f^{\sigma} = \{(u, v) \mid \sigma_C(v) \not\subseteq \sigma_C(u),\ u, v \in U\}$$
$$\mathcal{D}_f^{\delta} = \{(u, v) \mid \delta_C(u) \not\subseteq \delta_C(v),\ u, v \in U\}$$

for simplicity.

**Definition 5.5.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. For any $u, v \in U$, if we denote

$$\mathcal{D}_f^{\sigma}(u, v) = \begin{cases} \{a \mid (v, u) \notin \mathcal{R}_a^{\succcurlyeq},\ a \in C\} & (u, v) \in \mathcal{D}_f^{\sigma} \\ \emptyset & (u, v) \notin \mathcal{D}_f^{\sigma} \end{cases}$$

and

$$\mathcal{D}_f^{\delta}(u, v) = \begin{cases} \{a \mid (v, u) \notin \mathcal{R}_a^{\succcurlyeq},\ a \in C\} & (u, v) \in \mathcal{D}_f^{\delta} \\ \emptyset & (u, v) \notin \mathcal{D}_f^{\delta} \end{cases},$$

then we call $\mathcal{D}_f^{\sigma}(u, v)$ a assignment discernibility attributes set and $\mathcal{D}_f^{\delta}(u, v)$ a partial uniform discernibility attributes set between $u$ and $v$, respectively.

The matrix $\mathcal{M}^\sigma = (\mathcal{D}_f^\sigma(u,v))_{|U|\times|U|}$ and $\mathcal{M}^\delta = (\mathcal{D}_f^\delta(u,v))_{|U|\times|U|}$ be assignment discernibility matrix and partial uniform discernibility matrix, respectively.

**Proposition 5.7.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. Then $A$ is an assignment consistent set if and only if $A\bigcap\mathcal{D}_f^\sigma(u,v) \neq \emptyset$ for all $(u,v) \in \mathcal{D}_f^\sigma$.*

**Proof.** "$\Longrightarrow$": By the definition of $\mathcal{D}_f^\sigma$, we have that $\sigma_C(v) \nsubseteq \sigma_C(u)$ for any $(u,v) \in \mathcal{D}_f^\xi$. By Proposition 5.6, we only need to prove that (1) $[u]_A^{\succcurlyeq} \subset [v]_A^{\succcurlyeq}$, or (2) $[u]_A^{\succcurlyeq}\bigcap[v]_A^{\succcurlyeq} = \emptyset$, or (3) $[u]_A^{\succcurlyeq}\bigcap[v]_A^{\succcurlyeq} \subset [u]_A^{\succcurlyeq}$ and $[u]_A^{\succcurlyeq}\bigcap[v]_A^{\succcurlyeq} \subset [v]_A^{\succcurlyeq}$.

(1) If $[u]_A^{\succcurlyeq} \subset [v]_A^{\succcurlyeq}$, then we have that there exists at least one $u_i \in [v]_A^{\succcurlyeq}$ s.t. $u_i \notin [u]_A^{\succcurlyeq}$. Thus, there exists $a \in A$ s.t. $f(u_i,a) \not\succcurlyeq_a f(u,a)$ and $f(u_i,a) \succcurlyeq_a f(v,a)$, that is $f(v,a) < f(u,a)$. According to Definition 5.5, one can get that $a \in \mathcal{D}_f^\sigma(u,v)$. Therefore, $A\bigcap\mathcal{D}_f^\sigma(u,v) \neq \emptyset$.

(2) If $[u]_A^{\succcurlyeq}\bigcap[v]_A^{\succcurlyeq} = \emptyset$, there must exist $a_k \in A$ s.t. $f(u,a_k) > f(v,a_k)$, i.e. $A\bigcap\mathcal{D}_f^\sigma(u,v) \neq \emptyset$. Otherwise, we have that $f(u,a_l) \leq f(v,a_l)$ for any $a_l \in A$, that is, $v \in [u]_A^{\succcurlyeq}$, which is a contradiction with $[u]_A^{\succcurlyeq}\bigcap[v]_A^{\succcurlyeq} = \emptyset$.

(3) It is similar to the proof of claim (1).

"$\Longleftarrow$": If $A\bigcap\mathcal{D}_f^\sigma(u,v) \neq \emptyset$ for all $(u,v) \in \mathcal{D}_f^\sigma$, then there exists $a_k \in A$ s.t. $a_k \in \mathcal{D}_f^\sigma(u,v)$. Hence, one can get $f(u,a_k) > f(v,a_k)$, i.e. $v \notin [u]_A^{\succcurlyeq}$. Therefore, $[u]_A^{\succcurlyeq}\bigcap[v]_A^{\succcurlyeq} \neq [v]_A^{\succcurlyeq}$ because $v \in [v]_A^{\succcurlyeq}$. On the other hand, we have that $\sigma_C(u) \subset \sigma_C(v)$ for any $(u,v) \in \mathcal{D}_f^\sigma$. According to Proposition 5.6, we get that $A$ is a distribution consistent set.

This proposition is proved. □

**Proposition 5.8.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision and $A \subseteq C$. Then $A$ is a partial uniform consistent set if and only if $A\bigcap\mathcal{D}_f^\delta(u,v) \neq \emptyset$ for all $(u,v) \in \mathcal{D}_f^\delta$.*

**Proof.** It is similar to the proof of Proposition 5.7. □

**Definition 5.6.** Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C\bigcup\{d\}, V_C\bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. If we denote

$$\mathcal{F}^\sigma = \bigwedge_{i,j=1}^{n}\left(\bigvee\mathcal{D}_f^\sigma(u_i,u_j)\right)$$

$$\mathcal{F}^\delta = \bigwedge_{i,j=1}^{n}\left(\bigvee\mathcal{D}_f^\delta(u_i,u_j)\right),$$

then $\mathcal{F}^\sigma$ is referred to as an assignment discernibility formula and $\mathcal{F}^\delta$ is referred to as a partial uniform discernibility formula.

Moreover, if $\mathcal{F}^\sigma$ and $\mathcal{F}^\delta$ can be expressed as

$$\mathcal{F}^\sigma = \bigvee_{k=1}^{p} \left( \bigwedge_{l=1}^{q_k} a_{i_l} \right) = \bigvee_{k=1}^{p} \mathcal{B}_k^\sigma$$

$$\mathcal{F}^\delta = \bigvee_{k=1}^{t} \left( \bigwedge_{l=1}^{s_k} a_{j_l} \right) = \bigvee_{k=1}^{t} \mathcal{A}_k^\delta$$

then $\mathcal{F}^\sigma$ is referred to as an assignment minimal disjunctive normal form of the discernibility formula and $\mathcal{F}^\delta$ is referred to as a partial uniform minimal disjunctive normal form if $|\mathcal{B}_k^\sigma| = a_k$ and $|\mathcal{A}_k^\delta| = s_k$.



**Proposition 5.9.** *Let $\mathcal{L}_{id}^{\succcurlyeq} = (U, C \bigcup \{d\}, V_C \bigcup V_d^i, f)$ be an inconsistent lattice-valued information system with interval-valued intuitionistic fuzzy decision. Then we have that*

(1) *$\mathcal{B}_k^\sigma$ is an assignment reduction and $\mathcal{F}^\sigma = \bigvee_{k=1}^{p} \mathcal{B}_k^\sigma$ contains all assignment reductions.*

(2) *$\mathcal{A}_k^\delta$ is a partial uniform reduction and $\mathcal{F}^\delta = \bigvee_{k=1}^{p} \mathcal{A}_k^\delta$ contains all partial uniform reductions.*

**Proof.** The proof is similar to Proposition 5.4. □

**Example 5.2** (*Continued from Example 3.1*). By computing we have that $\mathcal{M}^\sigma$ is

$$\begin{pmatrix}
\emptyset & a_3a_5 & \emptyset & a_4a_5 & a_1a_5 & a_1a_3a_4a_5 \\
\emptyset & \emptyset & \emptyset & \emptyset & a_1a_2 & a_1a_2a_4 \\
\emptyset & a_3a_5 & \emptyset & a_5 & a_1 & a_1a_3a_5 \\
\emptyset & \emptyset & \emptyset & \emptyset & a_1a_2 & a_1a_2a_3 \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset
\end{pmatrix}$$

According to Definition 5.6, one can get that

$$\begin{aligned}
\mathcal{F}^\sigma &= (a_3 \vee a_5) \wedge (a_4 \vee a_5) \wedge (a_1 \vee a_5) \wedge (a_1 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2) \\
&\quad \wedge (a_1 \vee a_2 \vee a_4) \wedge a_5 \wedge a_1 \wedge (a_1 \vee a_3 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3) \\
&= a_1 \wedge a_5
\end{aligned}$$

So, there is only assignment reduction of this system, which is $\{a_1, a_5\}$.

Similarly, one can get that $\mathcal{M}^\delta$ is

$$\begin{pmatrix} \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ a_1a_2 & \emptyset & a_1a_2a_4 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ a_2 & \emptyset & a_2 & \emptyset & \emptyset & \emptyset \\ a_3 & a_3a_5 & a_3a_4 & a_3a_4a_5 & \emptyset & \emptyset \\ \emptyset & a_3a_5 & a_4 & a_4 & \emptyset & \emptyset \end{pmatrix}$$

And by Definition 5.3, we have that

$$\begin{aligned} \mathcal{F}^\delta &= (a_1 \vee a_2) \wedge (a_1 \vee a_2 \vee a_4) \wedge a_2 \wedge a_4 \wedge a_3 \\ &\wedge (a_3 \vee a_5) \wedge (a_3 \vee a_4 \vee a_5) \wedge (a_3 \vee a_4) \\ &= a_2 \wedge a_3 \wedge a_4 \end{aligned}$$

So, $\{a_2, a_3, a_4\}$ is the only one partial uniform reduction of this system.

## 6. Conclusion

The original rough set model cannot be used to deal with the information systems with complicated context. Nevertheless, by relaxing the indiscernibility relation to more general binary relations, many improved rough set models have been successfully applied into the information systems with complicated context for knowledge acquisition. Associated with dominance relation, lattice theory and intuitionistic fuzzy sets theory, the lattice-valued information systems with interval-valued intuitionistic fuzzy decision are proposed in this paper. Firstly, an approach of attribute reduction based on discernibility matrix in consistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision is constructed. Inspired by the idea of knowledge reduction in consistent information systems, four notions of knowledge reduction in the inconsistent lattice-valued information systems with interval-valued intuitionistic fuzzy decision are formulated, and the approaches to obtaining all reductions are constructed via the use of discernibility matrix, and examples show that the approaches are useful and effective. One can get that this paper provides a qualitative theoretical framework, which may be important both in theory and in application for analysis of knowledge acquisition in complex information systems.

Based on the equivalence relation, Zhang *et al.*[50] present the notion of distribution reduction in inconsistent information systems with decision, and the approach to judge whether a subset of attributes is consistent or not was provided and practical knowledge reduction method was constructed via introduction discernibility attribute set.

Naturally, the issue of extending some conclusions of distribution reduction in the sense of equivalence to general binary relation is taken into consideration, such as replacing $\mu_B(x)$ by

$$\mu_B^{\succcurlyeq}(x) = \left(\frac{|D_1 \bigcap [x]_B|}{|U|}, \frac{|D_2 \bigcap [x]_B|}{|U|}, \ldots, \frac{|D_r \bigcap [x]_B|}{|U|}\right), \qquad x \in U.$$

However, it is easy to obtain that

$$\sum_{j=1}^{r} \frac{|D_j \bigcap [x]_B|}{|U|} = 1$$

is not true any more, i.e., $\mu_B^{\succcurlyeq}(x)$ is not a probability distribution on $U/R_D$. Hence, how to redefine the distribution function and provide an practical approach to distribution reduction in the sense of dominance relation via discernibility matrix is interesting and vital.Also, this is an open subject and we yearn for everyone's active participation.

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