

Research Article

Ranking for Objects and Attribute Reductions in Intuitionistic Fuzzy Ordered Information Systems

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Received 27 November 2011; Accepted 12 March 2012

Academic Editor: Hung Nguyen-Xuan

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We aim to investigate intuitionistic fuzzy ordered information systems. The concept of intuitionistic fuzzy ordered information systems is proposed firstly by introducing an intuitionistic fuzzy relation to ordered information systems. And a ranking approach for all objects is constructed in this system. In order to simplify knowledge representation, it is necessary to reduce some dispensable attributes in the system. Theories of rough set are investigated in intuitionistic fuzzy ordered information systems by defining two approximation operators. Moreover, judgement theorems and methods of attribute reduction are discussed based on discernibility matrix in the systems, and an illustrative example is employed to show its validity. These results will be helpful for decisionmaking analysis in intuitionistic fuzzy ordered information systems.

1. Introduction

Rough set theory was first proposed by Pawlak in the early 1980s [1]. The theory is an extension of the classical set theory for modeling uncertainty or imprecision information. The research has recently roused great interest in the theoretical and application fronts, such as machine learning, pattern recognition, and data analysis. It is a new mathematical approach to uncertain and vague data analysis and plays an important role in many fields of data mining and knowledge discovery.

Partition or equivalence (indiscernibility relation) is an important and primitive concept in Pawlak's original rough set theory. However, partition or equivalence relation is still restrictive for many applications. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful general models in recent years by proposing other binary relations, such as tolerance relations [2], neighborhood operators [3], and others [4–11]. However, the original rough set theory does not consider attributes with preference ordered domain, that is criteria. Particularly, in many real situations, we are often

faced with the problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco et al. [12–17] proposed an extension rough set theory, called the dominance-based rough set approach (DRSA) to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, condition attributes are criteria and classes that are preference ordered; the knowledge approximated is a collection of upward and downward unions of classes and the dominance classed are sets of objects defined by using a dominance relation. In recent years, several studies have been made about properties and algorithmic implementations of DRSA [18–22].

Another important mathematical structure to cope with imperfect and/or imprecise information is called the “intuitionistic fuzzy (IF, for short) set” initiated by Atanassov [23, 24] on the basis of orthopairs of fuzzy sets. (Though the term of intuitionistic fuzzy set has been the argument of a large debate [25–27], we still use this notion due to its underlying mathematical structure, and because it is becoming increasing popular topic of investigation in the fuzzy set community.) An IF set is naturally considered as an extension of Zadeh’s fuzzy sets [28] defined by a pair of membership functions: while a fuzzy set gives a degree to which an element belongs to a set, an IF set gives both a membership degree and a non-membership degree. The membership and nonmembership values induce an indeterminacy index, which models the hesitancy of deciding the degree to which an object satisfies a particular property. Recently, IF set theory has been successfully applied in decision analysis and pattern recognition (see, e.g., [27, 29–31]).

Combining IF set theory and rough set theory may result in a new hybrid mathematical structure for the requirement of knowledge-handling systems. Research on this topic has been investigated by a number of authors. Çoker [32] first revealed the relationship between IF set theory and rough set theory and showed that a fuzzy rough set was in fact an intuitionistic fuzzy set. Various tentative definitions of IF rough sets were explored to extend rough set theory to the IF environment [33–38]. For example, according to fuzzy rough sets in the sense of Nanda and Majumdar [39], Jena et al. [35] and Chakrabarty et al. [33] independently proposed the concept of an IF rough set in which the lower and upper approximations are both IF sets.

In this paper, the intuitionistic fuzzy relation is introduced to DRSA. Actually, in real life, the intuitionistic fuzzy relation is an important type of data tables in ordered information systems. We aim to introduce dominance relation to ordered information systems with intuitionistic fuzzy relation and establish a rough set approach and evidence theory in this system.

The rest of this paper is organized as follows. Some preliminary concepts of rough sets and ordered information systems are briefly recalled in Section 2. In Section 3, the intuitionistic fuzzy ordered information system is introduced and some important properties are discussed. In Section 4, a rank approach with dominance class is considered by proposing the concept of dominance degree in intuitionistic fuzzy ordered information system. In Section 5, a rough set approach is investigated by establishing the upper and lower approximation operators in intuitionistic fuzzy ordered information system. In Section 6, attribute reductions are discussed in this system. Finally, we conclude the paper with a summary and outlook for further research in Section 7.

2. Preliminaries

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in the source papers [12–17]. A description has also been made in [40].

The notion of information system (sometimes called data tables, attribute-value systems, knowledge representation systems, etc.) provides a convenient tool for the representation of objects in terms of their attribute values.

An information system is an ordered quadruple $\mathcal{O} = (U, AT, V, f)$, where $U = \{x_1, x_2, \dots, x_n\}$ is a nonempty finite set of objects, and $AT = \{a_1, a_2, \dots, a_p\}$ is a nonempty finite set of attributes, $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a , $f : U \times AT \rightarrow V$ is a function such that $f(x, a) \in V_a$, for every $a \in AT$, $x \in U$, called an information function. A decision table is a special case of information systems in which, among the attributes, we distinguish one called a decision attribute. The other attributes are called condition attributes. Therefore, $\mathcal{O} = (U, C \cup \{d\}, V, f)$ and $C \cap \{d\} = \emptyset$, where set C and $\{d\}$ are condition attributes and the decision attribute, respectively.

In information systems, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

Definition 2.1 (see [12–17]). An information system is called an ordered information system (OIS) if all condition attributes are criteria.

Assuming that the domain of a criterion $a \in AT$ is completely preordered by an out-ranking relation \succeq_a , then $x \succeq_a y$ means that x is at least as good as y with respect to criterion a . And we can say that x dominates y . In the following, without any loss of generality, we consider criteria having a numerical domain, that is, $V_a \subseteq \mathcal{R}$ (\mathcal{R} denotes the set of real numbers). Being of type gain, that is, $x \succeq_a y \Leftrightarrow f(x, a) \geq f(y, a)$ (according to increasing preference) or $x \succeq_a y \Leftrightarrow f(x, a) \leq f(y, a)$ (according to decreasing preference), where $a \in AT$, $x, y \in U$. Without any loss of generality and for simplicity, in the following we only consider condition attributes with increasing preference.

For a subset of attributes $B \subseteq AT$, we define $x \succeq_B y \Leftrightarrow$ for all $a \in B$, $f(x, a) \geq f(y, a)$, and that is to say x dominates y with respect to all attributes in B . In general, we denote an ordered information system by $\mathcal{O}^\succeq = (U, AT, V, f)$.

For a given OIS, we say that x dominates y with respect to $B \subseteq AT$ if $x \succeq_B y$ and denote by $xR_B^\succeq y$. That is

$$\begin{aligned} R_B^\succeq &= \{(x, y) \in U \times U \mid x \succeq_B y\} \\ &= \{(x, y) \in U \times U \mid f(x, a) \geq f(y, a), \forall a \in B\}. \end{aligned} \quad (2.1)$$

R_B^\succeq are called dominance relations of ordered information system \mathcal{O}^\succeq .

Let

$$\begin{aligned} [x_i]_B^\succeq &= \{x_j \in U \mid (x_j, x_i) \in R_B^\succeq\} \\ &= \{x_j \in U \mid f(x_j, a) \geq f(x_i, a), \forall a \in B\}, \\ \frac{U}{R_B^\succeq} &= \{[x_i]_B^\succeq \mid x_i \in U\}, \end{aligned} \quad (2.2)$$

where $i \in \{1, 2, \dots, |U|\}$, then $[x_i]_B^\succeq$ will be called a dominance class or the granularity of information, and U/R_B^\succeq be called a classification of U about attribute set B .

The following properties of a dominance relation are trivial by the above definition.

Table 1: An ordered information system.

U	a_1	a_2	a_3
x_1	1	2	1
x_2	3	2	2
x_3	1	1	2
x_4	2	1	3
x_5	3	3	2
x_6	3	2	3

Proposition 2.2 (see [12–17]). Let R_A^\succeq be a dominance relation.

- (1) R_A^\succeq is reflexive, transitive, but not symmetric, so it is not an equivalence relation.
- (2) If $B \subseteq A$, then $R_A^\succeq \subseteq R_B^\succeq$.
- (3) If $B \subseteq A$, then $[x_i]_A^\succeq \subseteq [x_i]_B^\succeq$.
- (4) If $x_j \in [x_i]_A^\succeq$, then $[x_j]_A^\succeq \subseteq [x_i]_A^\succeq$ and $[x_i]_A^\succeq = \cup\{[x_j]_A^\succeq \mid x_j \in [x_i]_A^\succeq\}$.
- (5) $[x_j]_A^\succeq = [x_i]_A^\succeq$ if and only if $f(x_i, a) = f(x_j, a)$ for all $a \in A$.
- (6) $|[x_i]_B^\succeq| \geq 1$ for any $x_i \in U$.
- (7) U/R_B^\succeq constitute a covering of U , that is, for every $x \in U$ we have that $[x]_B^\succeq \neq \phi$ and $\bigcup_{x \in U} [x]_B^\succeq = U$,

where $|\cdot|$ denotes cardinality of the set.

For any subset $X \subseteq U$ and $A \subseteq AT$ in \mathcal{D}^\succeq , the lower and upper approximation of X with respect to a dominance relation R_A^\succeq could be defined as follows (see [12–17]):

$$\begin{aligned} \underline{R}_A^\succeq(X) &= \{x \in U \mid [x]_A^\succeq \subseteq X\}; \\ \overline{R}_A^\succeq(X) &= \{x \in U \mid [x]_A^\succeq \cap X \neq \phi\}. \end{aligned} \quad (2.3)$$

Unlike classical rough set theory, one can find easily that $\underline{R}_A^\succeq(X) = \cup\{[x]_A^\succeq \mid [x]_A^\succeq \subseteq X\}$ and $\overline{R}_A^\succeq(X) = \cup\{[x]_A^\succeq \mid [x]_A^\succeq \cap X \neq \phi\}$ do not hold.

Example 2.3. An OIS is presented in Table 1, where $U = \{x_1, x_2, \dots, x_6\}$, $AT = \{a_1, a_2, a_3\}$.

From the table we can have

$$\begin{aligned} [x_1]_{AT}^\succeq &= \{x_1, x_2, x_5, x_6\}, \\ [x_2]_{AT}^\succeq &= \{x_2, x_5, x_6\}, \\ [x_3]_{AT}^\succeq &= \{x_2, x_3, x_4, x_5, x_6\}, \\ [x_4]_{AT}^\succeq &= \{x_4, x_6\}, \\ [x_5]_{AT}^\succeq &= \{x_5\}, \\ [x_6]_{AT}^\succeq &= \{x_6\}. \end{aligned} \quad (2.4)$$

Table 2: An IS based on intuitionistic fuzzy relation.

U	a_1	a_2	a_3	a_4	a_5
x_1	(0.3, 0.5)	(0.6, 0.4)	(0.5, 0.2)	(0.7, 0.1)	(0.5, 0.4)
x_2	(0.2, 0.7)	(0.1, 0.8)	(0.4, 0.5)	(0.1, 0.8)	(0.2, 0.8)
x_3	(0.2, 0.7)	(0.1, 0.8)	(0.4, 0.5)	(0.7, 0.1)	(0.2, 0.8)
x_4	(0.1, 0.8)	(0.1, 0.8)	(0.2, 0.7)	(0.1, 0.8)	(0.2, 0.8)
x_5	(0.9, 0.1)	(0.8, 0.1)	(0.8, 0.1)	(0.9, 0.0)	(0.7, 0.1)
x_6	(0.4, 0.6)	(0.8, 0.1)	(0.6, 0.3)	(0.9, 0.0)	(0.7, 0.1)
x_7	(0.3, 0.5)	(0.7, 0.3)	(0.5, 0.1)	(0.7, 0.1)	(0.6, 0.3)
x_8	(0.8, 0.3)	(0.8, 0.1)	(0.7, 0.1)	(1.0, 0.0)	(0.7, 0.1)
x_9	(0.8, 0.3)	(0.9, 0.0)	(0.7, 0.1)	(0.8, 0.2)	(1.0, 0.0)
x_{10}	(0.9, 0.1)	(0.9, 0.0)	(0.8, 0.1)	(0.6, 0.3)	(1.0, 0.0)

If $X = \{x_2, x_3, x_5\}$, then

$$\underline{R}_{AT}^{\geq}(X) = \{x_5\}, \quad \overline{R}_{AT}^{\geq}(X) = \{x_1, x_2, x_3, x_5\}. \quad (2.5)$$

3. Intuitionistic Fuzzy Ordered Information Systems

An intuitionistic fuzzy information system is an ordered quadruple $\mathcal{D}^i = (U, AT, V, f)$, where $U = \{x_1, x_2, \dots, x_n\}$ is a nonempty finite set of objects, and $AT = \{a_1, a_2, \dots, a_p\}$ is a nonempty finite set of attributes, $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a , $f : U \times AT \rightarrow V$ is a function such that $f(x, a) \in V_a$, for every $a \in AT$, $x \in U$, called an information function, where V_a is a intuitionistic fuzzy set of universe U . That is

$$f(x, a) = (\mu_a(x), \nu_a(x)), \quad \forall a \in AT, \quad (3.1)$$

where $\mu_a : U \rightarrow [0, 1]$ and $\nu_a : U \rightarrow [0, 1]$ satisfy $0 \leq \mu_a(x) + \nu_a(x) \leq 1$, for all $x \in U$. And $\mu_a(x)$ and $\nu_a(x)$ are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to attribute a . We denote $\tilde{a}(x) = (\mu_a(x), \nu_a(x))$, then it is clear that \tilde{a} is an intuitionistic fuzzy set of U .

In other words, an intuitionistic fuzzy information system is an information system in which the relation between universe U and attributes set AT is an intuitionistic fuzzy relation.

Example 3.1. An intuitionistic fuzzy information system is presented in Table 2, where $U = \{x_1, x_2, \dots, x_{10}\}$, $AT = \{a_1, a_2, a_3, a_4, a_5\}$.

In practical decision-making analysis, we always consider a binary dominance relation between objects that are possibly dominant in terms of values of attributes set in information systems based on intuitionistic fuzzy relation, in general, an increasing preference and a decreasing preference, then the attribute is a criterion.

Definition 3.2. An intuitionistic fuzzy information system is called an intuitionistic fuzzy ordered information system (IFOIS) if all condition attributes are criteria.

In general, we denote an intuitionistic fuzzy ordered information system by $\mathcal{D}^{\geq} = (U, AT, V, f)$.

Assuming that the domain of a criterion $a \in AT$ is completely preordered by an out-ranking relation \succeq_a , then $x \succeq_a y$ means that x is at least as good as y with respect to criterion a . And we can say that x dominates y . For a subset of attributes $A \subseteq AT$, we define $x \succeq_A y \Leftrightarrow$ for all $a \in A$, $x \succeq_a y$. In other words, x is at least as good as y with respect to all attributes in A .

In the following, we introduce a dominance relation that identifies dominance classes to an intuitionistic fuzzy ordered information system. In a given IFOIS, we say that x dominates y with respect to $A \subseteq AT$ if $x \succeq_A y$, and is denoted by $xR_A^{\succeq}y$. That is

$$R_A^{\succeq} = \{(x, y) \in U \times U \mid x \succeq_A y\}. \quad (3.2)$$

Obviously, if $(x, y) \in R_A^{\succeq}$, then x dominates y with respect to A . R_A^{\succeq} are called a dominant relations of IFOIS.

Similarly, the relation R_A^{\preceq} , which is called a dominated relation, can be defined as follows:

$$R_A^{\preceq} = \{(x, y) \in U \times U \mid y \succeq_A x\}. \quad (3.3)$$

For simplicity and without any loss of generality, in the following we only consider condition attributes with increasing preference. Let us define this dominant relation in intuitionistic fuzzy ordered information systems as follows:

$$R_A^{\succeq} = \{(x, y) \in U \times U \mid \mu_a(x) \geq \mu_a(y), \nu_a(x) \leq \nu_a(y), \forall a \in A\}. \quad (3.4)$$

That is to say that R_A^{\succeq} are called dominance relations of IFOIS \mathcal{D}^{\succeq} .

Let

$$\begin{aligned} [x_i]_A^{\succeq} &= \{x_j \in U \mid (x_j, x_i) \in R_A^{\succeq}\} \\ &= \{x_j \in U \mid \mu_a(x_j) \geq \mu_a(x_i), \nu_a(x_j) \leq \nu_a(x_i), \forall a \in A\}, \\ \frac{U}{R_A^{\succeq}} &= \{[x_i]_A^{\succeq} \mid x_i \in U\}, \end{aligned} \quad (3.5)$$

where $i \in \{1, 2, \dots, |U|\}$, then $[x_i]_A^{\succeq}$ describe the set of objects that may dominate x_i in terms of A in IFOIS \mathcal{D}^{\succeq} and will be called a dominance class of IFOIS \mathcal{D}^{\succeq} , and U/R_A^{\succeq} be called a classification of U about attribute set A in IFOIS \mathcal{D}^{\succeq} .

Definition 3.3. Let $\mathcal{D}^{\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $B, A \subseteq AT$.

- (1) If $[x]_B^{\succeq} = [x]_A^{\succeq}$ for any $x \in U$, then we call that classification U/R_B^{\succeq} is equal to U/R_A^{\succeq} , denoted by $U/R_B^{\succeq} = U/R_A^{\succeq}$.

- (2) If $[x]_B^{\succeq} \subseteq [x]_A^{\succeq}$ for any $x \in U$, then we call that classification U/R_B^{\succeq} is finer than U/R_A^{\succeq} , denoted by $U/R_B^{\succeq} \subseteq U/R_A^{\succeq}$.
- (3) If $[x]_B^{\succeq} \subseteq [x]_A^{\succeq}$ for any $x \in U$ and $[x]_B^{\succeq} \neq [x]_A^{\succeq}$ for some $x \in U$, then we call that classification U/R_B^{\succeq} is properly finer than U/R_A^{\succeq} , denoted by $U/R_B^{\succeq} \subset U/R_A^{\succeq}$.

From the definitions of R_A^{\succeq} and $[x]_A^{\succeq}$, the following properties can easily be obtained.

Proposition 3.4. *Letting $\mathcal{D}^{\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, and $A \subseteq AT$, one can have*

$$R_A^{\succeq} = \bigcap_{a \in A} R_{\{a\}}^{\succeq}. \quad (3.6)$$

Proposition 3.5. *Let $\mathcal{D}^{\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, and $A \subseteq AT$. Then*

- (1) R_A^{\succeq} is reflexive,
- (2) R_A^{\succeq} is unsymmetric,
- (3) R_A^{\succeq} is transitive.

Proposition 3.6. *Letting $\mathcal{D}^{\succeq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, and $A, B \subseteq AT$, one has the following results.*

- (1) If $B \subseteq A$, then $R_A^{\succeq} \subseteq R_B^{\succeq}$.
- (2) If $B \subseteq A$, then $[x_i]_A^{\succeq} \subseteq [x_i]_B^{\succeq}$.
- (3) If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \cup \{ [x_j]_A^{\succeq} \mid x_j \in [x_i]_A^{\succeq} \}$.
- (4) $[x_j]_A^{\succeq} = [x_i]_A^{\succeq}$ iff $\mu_a(x_i) = \mu_a(x_j)$ and $\nu_a(x_i) = \nu_a(x_j)$ for all $a \in A$.

These properties mentioned above can be understood through the following example.

Example 3.7 (continued from Example 3.1). Computing the classification induced by the dominance relation R_{AT}^{\succeq} in Table 2.

From the table, one can obtain that

$$\begin{aligned} [x_1]_{AT}^{\succeq} &= \{x_1, x_5, x_7, x_8\}, \\ [x_2]_{AT}^{\succeq} &= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \\ [x_3]_{AT}^{\succeq} &= \{x_1, x_3, x_5, x_6, x_7, x_8\}, \\ [x_4]_{AT}^{\succeq} &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \end{aligned}$$

$$\begin{aligned}
[x_5]_{AT}^{\succeq} &= \{x_5\}, \\
[x_6]_{AT}^{\succeq} &= \{x_5, x_6, x_8\}, \\
[x_7]_{AT}^{\succeq} &= \{x_5, x_7, x_8\}, \\
[x_8]_{AT}^{\succeq} &= \{x_8\}, \\
[x_9]_{AT}^{\succeq} &= \{x_9\}, \\
[x_{10}]_{AT}^{\succeq} &= \{x_{10}\}.
\end{aligned} \tag{3.7}$$

If $A = \{a_1, a_2, a_3, a_5\} \subseteq AT$, we can get that

$$\begin{aligned}
[x_1]_A^{\succeq} &= \{x_1, x_5, x_7, x_8, x_9, x_{10}\}, \\
[x_2]_A^{\succeq} &= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \\
[x_3]_A^{\succeq} &= \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \\
[x_4]_A^{\succeq} &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}, \\
[x_5]_A^{\succeq} &= \{x_5, x_{10}\}, \\
[x_6]_A^{\succeq} &= \{x_5, x_6, x_8, x_9, x_{10}\}, \\
[x_7]_A^{\succeq} &= \{x_5, x_7, x_8, x_9, x_{10}\}, \\
[x_8]_A^{\succeq} &= \{x_5, x_8, x_9, x_{10}\}, \\
[x_9]_A^{\succeq} &= \{x_9, x_{10}\}, \\
[x_{10}]_A^{\succeq} &= \{x_{10}\}.
\end{aligned} \tag{3.8}$$

Obviously, $[x_i]_{AT}^{\succeq} \subseteq [x_i]_A^{\succeq}$ and

$$\frac{U}{R_{AT}^{\succeq}} = \frac{U}{R_A^{\succeq}} = \left\{ [x_i]_A^{\succeq} \mid i = 1, 2, \dots, 10 \right\}. \tag{3.9}$$

From this example, we can easily verify above propositions of information systems based on intuitionistic fuzzy relation.

4. Ranking for Objects in IFOIS

In general, there are two classes of problems in intelligent decision making. One is to find satisfactory results through ranking with information aggregation. And the other is to find dominance rules through relations.

In this section, we mainly investigate how to rank all objects by the dominance relation in an intuitionistic fuzzy ordered information system.

Definition 4.1. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $A \subseteq AT$. Dominance degree between two objects $x_i, x_j \in U$ with respect to the dominance relation R_A^{\geq} is defined as

$$d_A(x_i, x_j) = 1 - \frac{|[x_i]_A^{\geq} \cap (\sim [x_j]_A^{\geq})|}{|U|}. \quad (4.1)$$

We say that dominance degree of x_i to x_j is $d_A(x_i, x_j)$.

From the definition, the dominance degree $d_A(x_i, x_j)$ depict the proportion of some objects which are at least as good as x_j in dominance class $[x_i]_A^{\geq}$. Moreover, we can obtain the following properties.

Proposition 4.2. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, $A \subseteq AT$ and dominance degree between two objects x_j and x_i be $d_A(x_i, x_j)$ with respect to the dominance relation R_A^{\geq} , then the following holds.

- (1) $0 \leq d_A(x_i, x_j) \leq 1$ and $d_A(x_i, x_i) = 1$.
- (2) If $x_i \in [x_j]_A^{\geq}$, then $d_A(x_i, x_j) = 1$.
- (3) If $x_j \in [x_k]_A^{\geq}$, then $d_A(x_i, x_j) \leq d_A(x_i, x_k)$.
- (4) If $x_j \in [x_k]_A^{\geq}$ and $x_k \in [x_i]_A^{\geq}$, then $d_A(x_i, x_j) \leq d_A(x_k, x_j)$ and $d_A(x_i, x_j) \leq d_A(x_i, x_k)$.

Proof. (1) is directly obtained by the definition.

(2) Since $x_i \in [x_j]_A^{\geq}$, one can have $[x_i]_A^{\geq} \subseteq [x_j]_A^{\geq}$ by Proposition 3.6. So, we have $[x_i]_A^{\geq} \cap (\sim [x_j]_A^{\geq}) = \phi$. That is to say

$$d_A(x_i, x_j) = 1 - \frac{|[x_i]_A^{\geq} \cap (\sim [x_j]_A^{\geq})|}{|U|} = 1. \quad (4.2)$$

(3) If $x_j \in [x_k]_A^{\geq}$, then we can obtain $[x_j]_A^{\geq} \subseteq [x_k]_A^{\geq}$. So we have $(\sim [x_j]_A^{\geq}) \supseteq (\sim [x_k]_A^{\geq})$. Thus

$$\frac{|[x_i]_A^{\geq} \cap (\sim [x_j]_A^{\geq})|}{|U|} \geq \frac{|[x_i]_A^{\geq} \cap (\sim [x_k]_A^{\geq})|}{|U|}. \quad (4.3)$$

Thus

$$d_A(x_i, x_j) \leq d_A(x_i, x_k). \quad (4.4)$$

(4) If $x_j \in [x_k]_A^{\geq}$ and $x_k \in [x_i]_A^{\geq}$, then we can obtain $[x_j]_A^{\geq} \subseteq [x_k]_A^{\geq} \subseteq [x_i]_A^{\geq}$. That is $(\sim [x_j]_A^{\geq}) \supseteq (\sim [x_k]_A^{\geq}) \supseteq (\sim [x_i]_A^{\geq})$ hold. So we have

$$\begin{aligned} \frac{|[x_i]_A^{\geq} \cap (\sim [x_j]_A^{\geq})|}{|U|} &\geq \frac{|[x_k]_A^{\geq} \cap (\sim [x_j]_A^{\geq})|}{|U|}, \\ \frac{|[x_i]_A^{\geq} \cap (\sim [x_j]_A^{\geq})|}{|U|} &\geq \frac{|[x_i]_A^{\geq} \cap (\sim [x_k]_A^{\geq})|}{|U|}. \end{aligned} \quad (4.5)$$

Thus

$$d_A(x_i, x_j) \leq d_A(x_k, x_j), \quad d_A(x_i, x_j) \leq d_A(x_i, x_k). \quad (4.6)$$

The proposition was proved. \square

Definition 4.3. Let $\mathcal{J}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $A \subseteq AT$. Denote

$$M_A^{\geq} = (r_{ij})_{|U| \times |U|}, \quad \text{where } r_{ij} = d_A(x_i, x_j). \quad (4.7)$$

Then, we call the matrix M_A^{\geq} to be a dominance matrix with respect to A induced by the intuitionistic fuzzy dominance relation R_A^{\geq} .

Moreover, if

$$d_A(x_i) = \frac{1}{|U|} \sum_{x_j \in U} d_A(x_i, x_j), \quad (4.8)$$

then we call $d_A(x_i)$ to be dominance degree of x_i with respect to relation R_A^{\geq} , for every $x_i \in U$.

By definition of dominance matrix and dominance degree of the object with respect to relation R_A^{\geq} , we can directly receive the following properties. For all $x_i \in U$, the degree can be calculated according to the following formula:

$$d_A(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} r_{ij}. \quad (4.9)$$

As a result of the above discussions, we come to the following two corollaries.

Corollary 4.4. Let $\mathcal{J}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $A \subseteq AT$. If $R_A^{\geq} = R_{AT}^{\geq}$, then $d_A(x_i, x_j) = d_{AT}(x_i, x_j)$, $d_A(x_i) = d_{AT}(x_i)$ and $M_A^{\geq} = M_{AT}^{\geq}$, for $x_i, x_j \in U$.

From the dominance degree of each object on the universe, we can rank all objects according to the number of d_A . A larger number implies a better object. This idea can be understood by the following example.

Example 4.5 (continued from Example 3.1). Rank all objects in U according to the dominance relation R_{AT}^{\geq} in the system of Example 3.1.

By Example 3.7, we can easily obtain the dominance degree of two objects and dominance matrix in the system as follows:

$$M_{AT}^{\geq} = \begin{bmatrix} 1 & 1 & 1 & 1 & \frac{7}{10} & \frac{8}{10} & \frac{9}{10} & \frac{7}{10} & \frac{6}{10} & \frac{6}{10} \\ \frac{5}{10} & 1 & \frac{7}{10} & 1 & \frac{2}{10} & \frac{4}{10} & \frac{4}{10} & \frac{2}{10} & \frac{2}{10} & \frac{2}{10} \\ \frac{8}{10} & 1 & 1 & 1 & \frac{5}{10} & \frac{7}{10} & \frac{7}{10} & \frac{5}{10} & \frac{4}{10} & \frac{4}{10} \\ \frac{4}{10} & \frac{9}{10} & \frac{6}{10} & 1 & \frac{1}{10} & \frac{3}{10} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} \\ \frac{9}{10} & 1 & 1 & 1 & \frac{8}{10} & 1 & \frac{9}{10} & \frac{8}{10} & \frac{7}{10} & \frac{7}{10} \\ 1 & 1 & 1 & 1 & \frac{8}{10} & \frac{9}{10} & 1 & \frac{9}{10} & \frac{7}{10} & \frac{7}{10} \\ 1 & 1 & 1 & 1 & \frac{9}{10} & 1 & 1 & 1 & \frac{9}{10} & \frac{9}{10} \\ \frac{9}{10} & 1 & \frac{9}{10} & 1 & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & 1 & \frac{9}{10} \\ \frac{9}{10} & 1 & \frac{9}{10} & 1 & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & 1 \end{bmatrix}. \quad (4.10)$$

So, we can have

$$\begin{aligned} d_{AT}(x_1) &= 0.83, & d_{AT}(x_2) &= 0.48, & d_{AT}(x_3) &= 0.7, \\ d_{AT}(x_4) &= 0.39, & d_{AT}(x_5) &= 0.97, & d_{AT}(x_6) &= 0.88, \\ d_{AT}(x_7) &= 0.90, & d_{AT}(x_8) &= 0.97, & d_{AT}(x_9) &= 0.93, \\ d_{AT}(x_{10}) &= 0.93. \end{aligned} \quad (4.11)$$

Therefore, according the above we rank all objects in the following:

$$x_5 = x_8 \geq x_9 = x_{10} \geq x_7 \geq x_6 \geq x_1 \geq x_3 \geq x_2 \geq x_4. \quad (4.12)$$

5. Rough Set Approach to IFOIS

In this section, we investigate the problem of set approximation with respect to a dominance relation R_A^{\geq} in intuitionistic fuzzy ordered information systems.

Similar to ordered information systems, we can define the upper and lower approximation sets in IFOIS.

Definition 5.1. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system. For any $X \subseteq U$ and $A \subseteq AT$, the upper and lower approximations of X with respect to the dominance relation R_A^{\geq} are defined as follows:

$$\begin{aligned}\overline{R}_A^{\geq}(X) &= \{x \in U \mid [x]_A^{\geq} \cap X \neq \emptyset\}, \\ \underline{R}_A^{\geq}(X) &= \{x \in U \mid [x]_A^{\geq} \subseteq X\}.\end{aligned}\tag{5.1}$$

From above definition, one can briefly notice that $\underline{R}_A^{\geq}(X)$ is a set of objects that belong to X with certainty and $\overline{R}_A^{\geq}(X)$ is a set of objects that probably belong to X . If $\underline{R}_A^{\geq}(X) \neq \overline{R}_A^{\geq}(X)$, we say the subset X of U is rough, otherwise X is precise. $Bn_A(X) = \overline{R}_A^{\geq}(X) - \underline{R}_A^{\geq}(X)$ is called to a boundary of the rough set.

Moreover, we can directly obtain the following results.

Proposition 5.2. *Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $A \subseteq AT$. For any $X \subseteq U$, the following always holds:*

- (1) $\underline{R}_A^{\geq}(X) \subseteq \underline{R}_{AT}^{\geq}(X)$ and $\overline{R}_A^{\geq}(X) \supseteq \overline{R}_{AT}^{\geq}(X)$.
- (2) If $R_A^{\geq} = R_{AT}^{\geq}$, then $\underline{R}_A^{\geq}(X) = \underline{R}_{AT}^{\geq}(X)$ and $\overline{R}_A^{\geq}(X) = \overline{R}_{AT}^{\geq}(X)$.

Proposition 5.3. *Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system. For any $X, Y \subseteq U$ and $A \subseteq AT$, then the properties in Table 3 hold.*

Proof. The proof are similar to the case of Properties in [41]. □

Another topic is uncertainty of a rough set in rough set theory. Uncertainty of a rough set is due to the existence of a borderline region. The greater the borderline region of a rough set, the lower the accuracy of the rough set. In order to measure the imprecision of a rough set induced by intuitionistic dominance relation in ordered information systems, definition of accuracy measure is introduced in the following.

Definition 5.4. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, $X \subseteq U$ and $A \subseteq AT$. Roughness measure of X with respect to the dominance relation R_A^{\geq} is defined as

$$\rho(R_A^{\geq}, X) = 1 - \frac{|\underline{R}_A^{\geq}(X)|}{|\overline{R}_A^{\geq}(X)|}.\tag{5.2}$$

By the definition, we can easily find that the roughness measure expresses the rough degree of the knowledge about X , given the knowledge U/R_A^{\geq} . It is clear that $0 \leq \rho \leq 1$. We can directly get that X is definable in IFOIS when $\rho = 0$, in which $\underline{R}_A^{\geq}(X) = \overline{R}_A^{\geq}(X)$.

We have the following properties about roughness $\rho(R_A^{\geq}, X)$.

Table 3

(1L)	$\overline{R_A^{\geq}}(X) \subseteq X$	(Contraction)
(1U)	$X \subseteq \overline{R_A^{\geq}}(X)$	(Extension)
(2)	$\overline{R_A^{\geq}}(\sim X) = \sim \overline{R_A^{\geq}}(X)$	(Duality)
	$\overline{R_A^{\geq}}(\sim X) = \sim \overline{R_A^{\geq}}(X)$	(Duality)
(3L)	$\overline{R_A^{\geq}}(\phi) = \phi$	(Normality)
(3U)	$\overline{R_A^{\geq}}(\phi) = \phi$	(Normality)
(4L)	$\overline{R_A^{\geq}}(U) = U$	(Co-normality)
(4U)	$\overline{R_A^{\geq}}(U) = U$	(Co-normality)
(5L)	$\overline{R_A^{\geq}}(X \cap Y) = \overline{R_A^{\geq}}(X) \cap \overline{R_A^{\geq}}(Y)$	(Multiplication)
(5U)	$\overline{R_A^{\geq}}(X \cup Y) = \overline{R_A^{\geq}}(X) \cup \overline{R_A^{\geq}}(Y)$	(Addition)
(5L')	$\overline{R_A^{\geq}}(X \cup Y) \supseteq \overline{R_A^{\geq}}(X) \cup \overline{R_A^{\geq}}(Y)$	(F-Multiplication)
(5U')	$\overline{R_A^{\geq}}(X \cap Y) \subseteq \overline{R_A^{\geq}}(X) \cap \overline{R_A^{\geq}}(Y)$	(F-Addition)
(6L)	$X \subseteq Y \Rightarrow \overline{R_A^{\geq}}(X) \subseteq \overline{R_A^{\geq}}(Y)$	(Monotone)
(6U)	$X \subseteq Y \Rightarrow \overline{R_A^{\geq}}(X) \subseteq \overline{R_A^{\geq}}(Y)$	(Monotone)
(7L)	$\overline{R_A^{\geq}}(\overline{R_A^{\geq}}(X)) = \overline{R_A^{\geq}}(X)$	(Idempotency)
(7U)	$\overline{R_A^{\geq}}(\overline{R_A^{\geq}}(X)) = \overline{R_A^{\geq}}(X)$	(Idempotency)

Proposition 5.5. Letting $\mathcal{O}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, $X \subseteq U$ and $A \subseteq AT$, the following results hold:

- (1) $\rho(R_A^{\geq}, X) = 1 - \frac{|\overline{R_A^{\geq}}(X)|}{|R_A^{\geq}(X)|} = 1 - \frac{|\overline{R_A^{\geq}}(X)|}{(|U| - |\overline{R_A^{\geq}}(\sim X)|)}$;
- (2) if $R_A^{\geq} = R_{AT}^{\geq}$, then $\rho(R_A^{\geq}, X) = \rho(R_{AT}^{\geq}, X)$;
- (3) if $B \subseteq A \subseteq AT$, then $\rho(R_{AT}^{\geq}, X) \leq \rho(R_A^{\geq}, X) \leq \rho(R_B^{\geq}, X)$.

Proof. (1) and (2) can be directly obtained by Definitions 5.1 and 5.4 and Proposition 5.3.

- (3) Since $B \subseteq A$, we can get that $\overline{R_B^{\geq}}(X) \subseteq \overline{R_A^{\geq}}(X)$ and $\overline{R_B^{\geq}}(X) \supseteq \overline{R_A^{\geq}}(X)$. So we have

$$\frac{|\overline{R_B^{\geq}}(X)|}{|\overline{R_B^{\geq}}(X)|} \leq \frac{|\overline{R_A^{\geq}}(X)|}{|\overline{R_A^{\geq}}(X)|}. \quad (5.3)$$

That is

$$\rho(R_A^{\geq}, X) \leq \rho(R_B^{\geq}, X). \quad (5.4)$$

Similarly, we can obtain $\rho(R_{AT}^{\geq}, X) \leq \rho(R_A^{\geq}, X)$.

The proof is completed. \square

Example 5.6 (continued from Examples 3.1 and 3.7). Consider the ordered information system based on intuitionistic fuzzy relation in Example 3.1.

Letting $A = \{a_1, a_2, a_3, a_5\} \subseteq AT$ and $X = \{x_1, x_5, x_6, x_8\}$, compute approximation operators of set X approximated by intuitionistic fuzzy relation R_A^{\geq} and R_{AT}^{\geq} , respectively.

According to Definition 5.1 and Example 3.7, the lower and upper approximations of X with respect to relation $R_{AT}^{\geq}(X)$ can be received as follows:

$$\begin{aligned} \overline{R_{AT}^{\geq}}(X) &= \{x_5, x_6, x_8\}, \\ \underline{R_{AT}^{\geq}}(X) &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}. \end{aligned} \quad (5.5)$$

Similarly, the lower and upper approximations of X with respect to relation $R_A^{\geq}(X)$ can be received as follows:

$$\begin{aligned} \overline{R_A^{\geq}}(X) &= \phi, \\ \underline{R_A^{\geq}}(X) &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}. \end{aligned} \quad (5.6)$$

Thus, we have that

$$\begin{aligned} \rho(R_A^{\geq}, X) &= 1 - \frac{|\overline{R_A^{\geq}}(X)|}{|\underline{R_A^{\geq}}(X)|} = 1, \\ \rho(R_{AT}^{\geq}, X) &= 1 - \frac{|\overline{R_{AT}^{\geq}}(X)|}{|\underline{R_{AT}^{\geq}}(X)|} = \frac{5}{8}. \end{aligned} \quad (5.7)$$

Hence, one can get

$$\rho(R_{AT}^{\geq}, X) \leq \rho(R_A^{\geq}, X). \quad (5.8)$$

From the above, we can easily find that the subset $X \subseteq U$ is more rough in system based on intuitionistic fuzzy relation R_A^{\geq} than relation R_{AT}^{\geq} , which is consistent with the fact.

6. Attribute Reduction Based on Discernibility Matrix in IFOIS

In order to simplify knowledge representation, it is necessary to reduce some dispensable attributes in a given intuitionistic fuzzy ordered information system. In this section, an approach to attribute reduction in ordered information systems based intuitionistic fuzzy relation will be established and an illustrative example is employed to show its validity.

Definition 6.1. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $X \subseteq U$. An attribute subset $A \subseteq AT$ is referred to as a classical consistent set of \mathcal{D}^{\geq} if $R_A^{\geq} = R_{AT}^{\geq}$. Moreover, if A is a classical consistent set of \mathcal{D}^{\geq} and no proper subset of A is a classical consistent set of \mathcal{D}^{\geq} , then A is referred to as a classical reduction of \mathcal{D}^{\geq} .

From the definition, we can directly obtain the following property.

Proposition 6.2. *Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $A \subseteq AT$. If A is a classical reduction, then $d_A(x_i, x_j) = d_{AT}(x_i, x_j)$ for any $x_i, x_j \in U$.*

It is obvious that a classical reduction of an IFOIS is a minimal attribute subset satisfying $R_A^{\geq} = R_{AT}^{\geq}$. An attribute $a \in AT$ is called dispensable with respect to R_{AT}^{\geq} if $R_{AT}^{\geq} = R_{AT-\{a\}}^{\geq}$; otherwise a is called indispensable. The set of all indispensable attributes is called a *core* with respect to the dominance relation R_{AT}^{\geq} and is denoted by $core(AT)$. An attribute in the core must be in every attribute reduction. In other words, $core(AT)$ is the intersection of all classical reductions of the system. Thus the core may be empty set.

Definition 6.3. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system. For any $x_i, x_j \in U$, if we denote

$$\begin{aligned} \text{Dis}(x_i, x_j) &= \{a \in AT \mid \mu_a(x_j) < \mu_a(x_i), \text{ or } \nu_a(x_j) > \nu_a(x_i)\}, \\ \mathcal{M}_{\text{Dis}} &= (l_{ij})_{|U| \times |U|}, \quad \text{where } l_{ij} = \text{Dis}(x_i, x_j), \end{aligned} \quad (6.1)$$

then we call $\text{Dis}(x_i, x_j)$ to be a discernibility attribute set between objects x_i and x_j , and matrix \mathcal{M}_{Dis} to be a discernibility matrix of an IFOIS.

One can easily get the following property by the above definition.

Proposition 6.4. *Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system and $x_i, x_j \in U$, then $\text{Dis}(x_i, x_i) = \emptyset$ and $\text{Dis}(x_i, x_j) \cap \text{Dis}(x_j, x_i) = \emptyset$.*

The following property provides a judgement method of a classical reduction of an IFOIS.

Proposition 6.5. *Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, $A \subseteq AT$ and $\text{Dis}(x_i, x_j)$ the discernibility attributes set of \mathcal{D}^{\geq} with respect to R_{AT}^{\geq} . Then the following two proposition are equivalent.*

- (1) A is a classical consistent set of \mathcal{D}^{\geq} .
- (2) If $\text{Dis}(x_i, x_j) \neq \emptyset$, then $A \cap \text{Dis}(x_i, x_j) \neq \emptyset$ for $x_i, x_j \in U$.

Proof. “(1) \Rightarrow (2)” If A is a classical consistent set of \mathcal{D}^{\geq} , then we have $R_A^{\geq} = R_{AT}^{\geq}$. By the definition of the dominance relation, one can know $[x]_A^{\geq} = [x]_{AT}^{\geq}$ for any $x \in U$. On the other hand, since $\text{Dis}(x_i, x_j) \neq \emptyset$, then $x_j \notin [x_i]_{AT}^{\geq}$. So $x_j \notin [x_i]_A^{\geq}$. That is to say that there exists $a \in A$ such that $f(a, x_j) < f(a, x_i)$. Therefore $a \in \text{Dis}(x_i, x_j)$. Thus $A \cap \text{Dis}(x_i, x_j) \neq \emptyset$.

“(2) \Rightarrow (1)” Because we have known $R_{AT}^{\geq} \subseteq R_A^{\geq}$ by the Proposition 3.6, we need only prove $R_A^{\geq} \subseteq R_{AT}^{\geq}$.

For $x_i, x_j \in U$, if $\text{Dis}(x_i, x_j) \neq \phi$, then $x_j \notin [x_i]_{AT}^{\geq}$. Moreover, we know that $A \cap \text{Dis}(x_i, x_j) \neq \phi$ when $\text{Dis}(x_i, x_j) \neq \phi$. So there exists $a \in A$ such that $\mu_a(x_j) < \mu_a(x_i)$ or $\nu_a(x_j) > \nu_a(x_i)$, that is to say $x_j \notin [x_i]_A^{\geq}$. Hence, we can find that if $x_j \notin [x_i]_{AT}^{\geq}$ then $x_j \notin [x_i]_A^{\geq}$. In other words, if $x_j \in [x_i]_A^{\geq}$ then $x_j \in [x_i]_{AT}^{\geq}$. That is $[x_i]_A^{\geq} \subseteq [x_i]_{AT}^{\geq}$, that is, $R_A^{\geq} \subseteq R_{AT}^{\geq}$.

This completes the proof. \square

Definition 6.6. Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system, $A \subseteq AT$ and \mathcal{M}_{Dis} be discernibility matrix. Denoted by

$$F^{\geq} = \bigwedge \left\{ \bigvee \{a_k \mid a_k \in \text{Dis}(x_i, x_j)\}, x_i, x_j \in U \right\}, \quad (6.2)$$

then F^{\geq} is called discernibility formula.

Based on the discernibility formula, we can design a practical approach to classical reduction in an IFOIS as follows.

Proposition 6.7. *Let $\mathcal{D}^{\geq} = (U, AT, V, f)$ be an intuitionistic fuzzy ordered information system. The minimal disjunctive normal form of discernibility formula of distribution is*

$$F^{\geq} = \bigvee_{k=1}^p \left(\bigwedge_{s=1}^{q_k} \right) a_s. \quad (6.3)$$

Denote $B_k^{\geq} = \{a_s \mid s = 1, 2, \dots, q_k\}$, then $\{B_k^{\geq} \mid k = 1, 2, \dots, p\}$ are just sets of all distribution reductions of \mathcal{D}^{\geq} .

Proof. It follows directly from Proposition 6.5 and the definition of minimal disjunctive normal of the discernibility formula. \square

In the following, we analyze how to obtain classical reductions from all attributes in an IFOIS by an illustrative example.

Example 6.8 (continued from Example 3.1). Calculate all classical reductions of the IFOIS in Example 3.1.

By the definition of discernibility matrix, one can obtain the discernibility of the system in Table 4.

Hence, we can have that

$$\begin{aligned} F^{\geq} &= (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_2 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_5) \\ &\quad \wedge (a_4) \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_3 \vee a_4) \\ &= (a_1 \vee \vee a_3) \wedge (\vee a_2 \vee a_5) \wedge a_4 \\ &= (a_1 \wedge a_2 \wedge a_4) \vee (a_1 \wedge a_4 \wedge a_5) \vee (a_2 \wedge a_3 \wedge a_4) \vee (a_3 \wedge a_4 \wedge a_5). \end{aligned} \quad (6.4)$$

So, there are four classical reductions for the system, which are $\{a_1, a_2, a_4\}$, $\{a_1, a_4, a_5\}$, $\{a_2, a_3, a_4\}$, and $\{a_3, a_4, a_5\}$. And it is clear that the core of the system is $\{a_4\}$.

Table 4: The discernibility matrix of the system in Example 3.1.

x_i/x_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	ϕ	ϕ	ϕ	ϕ	$a_1 a_2 a_3$	AT	$a_2 a_5$	AT	AT	$a_1 a_2 a_3 a_5$
x_2	AT	ϕ	ϕ	AT	AT	AT	AT	AT	AT	AT
x_3	$a_1 a_2 a_3 a_5$	ϕ	ϕ	ϕ	AT	AT	$a_1 a_2 a_3 a_5$	AT	AT	$a_1 a_2 a_3 a_5$
x_4	AT	$a_1 a_3$	$a_1 a_3 a_4$	ϕ	AT	AT	AT	AT	AT	AT
x_5	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	a_4	$a_2 a_5$	$a_2 a_5$
x_6	$a_1 a_3$	ϕ	ϕ	ϕ	$a_1 a_3$	ϕ	$a_1 a_3 a_5$	$a_1 a_3 a_4$	$a_1 a_2 a_3 a_5$	$a_1 a_2 a_3 a_5$
x_7	ϕ	ϕ	ϕ	ϕ	AT	AT	ϕ	AT	AT	$a_1 a_2 a_3 a_5$
x_8	ϕ	ϕ	ϕ	ϕ	$a_1 a_3$	ϕ	ϕ	ϕ	$a_2 a_5$	$a_1 a_2 a_3 a_5$
x_9	a_4	ϕ	a_4	ϕ	$a_1 a_3 a_4$	a_4	a_4	a_4	ϕ	$a_1 a_3$
x_{10}	a_4	ϕ	a_4	ϕ	a_4	a_4	a_4	a_4	a_4	ϕ

7. Conclusions

Rough set theory is a new mathematical tool to deal with vagueness and uncertainty. Development of a rough computational method is one of the most important research tasks. While, in practise, intuitionistic fuzzy ordered information system confines the applications of classical rough set theory. In this paper, we mainly considered some important concepts and properties in this system. We defined two approximation operators and established the rough set approach to intuitionistic fuzzy ordered information systems. Moreover, we also investigated the problem of attribute reductions based on rough sets and presented method of the reduction. However, extracted dominance rules from the system is another important class of problem in decision-making analysis. So, our further works discuss ordered decision table based on intuitionistic fuzzy relation and dominance rules extracted from this type of decision tables.

Acknowledgments

This work is supported by National Natural Science Foundation of China (nos. 61105041, 71071124, and 11001227), Postdoctoral Science Foundation of China (no. 20100481331), and Natural Science Foundation Project of CQ CSTC (no. cstc2011jjA40037).

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