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## Multiple granulation rough set approach to ordered information systems

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With granular computing point of view, the classical dominance-based rough set model is based on a single granulation. For an ordered information system, this paper proposes two new types of multiple granulation rough set (MGRS) models, where a target concept is approximated from different kinds of views by using dominant classes induced by multiple granulations. And a number of important properties of the two types of MGRS are investigated in an ordered information system. From the properties, it can be found that Greco's rough set model is a special instance compared to our MGRS model. Moreover, the relationships and differences are discussed carefully among Greco's rough set and two new types of MGRS. Furthermore, several important measures are presented in two types of MGRS models, such as rough measure, quality of approximation in an ordered information system. In order to illustrate our MGRS models in an ordered information system, a real life example is considered, which is helpful for applying this theory in practical issues. One can see get that the research is meaningful in applications for the issue of knowledge reduction in complex ordered information systems.

**Keywords:** rough set; dominance relation; information systems; multiple granulation

### 1. Introduction

Rough set theory proposed by Pawlak (1982, 1991) is an extension of the classical set theory and can be regarded as a soft computing tool to handle imprecision, vagueness, and uncertainty in data analysis. The theory has been found successful in applications in the fields of pattern recognition (Swininarski and Skowron 2003), medical diagnosis (Tsumoto 1998), data mining (Chan 1998; Kim 2001; Ananthanarayana, Narasimha, and Subramanian 2003), conflict analysis (Pawlak 2005), algebra (Davvaz and Mahdavi pour 2006; Xiao and Zhang 2006; Cheng, Mo, and Wang 2007) and so on. Recently, the theory has generated a great deal of interest among more and more researchers.

However, in practice, due to the existence of uncertainty and complexity of particular problems, the problem would not be settled perfectly by means of classical rough sets. Therefore, it is vital to generalize the classical rough set model. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful general models in recent years by proposing other binary relations, such as tolerance relations (Skowron and Stepaniuk 1996), neighbourhood operators (Yao 1998), and others

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(Wu, Leung, and Zhang 2002; Pawlak 2005; Wu, Zhang, Li, and Mi 2005; Leung, Wu, and Zhang 2006; Pawlak and Skowron 2007a, 2007b; Xu and Zhang 2007; Yu, Hu, and Wu 2007; Xu, Zhang, and Zhang 2009; Xu, Zhong, Zhang, and Zhang 2010). However, the original rough set theory which does not consider attributes with preference ordered domain, that is criteria. Particularly, in many real situations, we often face the problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco, Matarazzo, and Slowinski (1998, 1999, 2001, 2002, 2007) proposed an extension rough set theory called the dominance-based rough set approach (DRSA) to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, condition attributes are criteria and classes are preference ordered, the knowledge approximated is a collection of upward and downward unions of classes and the dominance classes are sets of objects defined by using a dominance relation. In recent years, several studies have been made on properties and algorithmic implementations of DRSA (Susmaga, Slowinski, Greco, and Matarazzo 2000; Dembczynski, Pindur, and Susmaga 2003a, 2003b; Xu and Zhang 2007; Xu et al. 2009, 2010). Nevertheless, only a limited number of methods using DRSA to acquire knowledge from inconsistent ordered information systems have been proposed. Pioneering work on inconsistent ordered information systems with the DRSA has been proposed by Greco et al. (1998, 1999, 2001, 2002, 2007), but they did not clearly point out the semantic explanation of unknown values. Shao and Zhang (2005) further proposed an extension of the dominance relation in an inconsistent ordered information system.

On the other hand, information granules have played a significant role in human cognitive processes. Information granules refer to pieces, classes, and groups divided in accordance with characteristics and performances of complex information in the process of human understanding, reasoning, and decision making. Such information processing is called information granulation. Zadeh (1979) firstly proposed and discussed the issue of fuzzy information granulation in 1979. Then, the basic idea of information granulation has been applied to many fields, such as theory of rough sets (Pawlak 1982, 1991; Pawlak and Skowron 2007a), fuzzy sets (Zadeh 1985, 1997), evidence theories (Shafer 1976), etc., and a growing number of scholars are concerned about the discipline. In 1985, Hobbs proposed the concept of granularity. And Zadeh (1997) firstly presented the concept of granular computing during the period 1996–1997. At this time, granular computing has played a more and more important role in soft computing, knowledge discovery, data mining, and some studies have achieved a large amount of excellent results (Lin 1997, 2005; Klir 1998; Yao 2000, 2001, 2004, 2006; Zhang et al. 2001; Liang and Qian 2006; Liang, Shi, and Li 2006; Qian, Liang, Yao, and Dang 2010).

With granular computing point of view, an equivalence relation on the universe can be regarded as a granulation, and a partition on the universe can be regarded as a granulation space (Yao 2000). Hence, the classical rough set theory is based on a single granulation (only one equivalence relation). Note that any attribute set can induce a certain equivalence relation in an information system. However, when the rough set is based on many granulations induced from several equivalence relations, we can have some cases as follows:

- Case 1. There exists a granulation at least such that the elements surely belong to the concept.
- Case 2. There are some granulations such that the elements surely belong to the concept.
- Case 3. All of the granulations such that the elements surely belong to the concept.

Case 4. There exists a granulation at least such that the elements possibly belong to the concept.

Case 5. There are some granulations such that the elements possibly belong to the concept.

Case 6. All of the granulations such that the elements possibly belong to the concept.

For these above cases, many researchers have extended the rough set to the multi-granulation rough sets (Rasiowa 1988, 1990, 1991; Rasiowa and Marek 1993; Rauszer 1994; Leuang et al. 2006; Khan and Ma 2011). To more widely apply the rough set theory in practical applications, Qian et al. (2010) extended Pawlak's single-granulation rough set model to a multiple granulation rough set (MGRS) model, where set approximations are defined by multiple equivalence relations on the universe. Based on this, many researchers have extended the multi-granulation rough set to the generalized multi-granulation rough sets (Xu, Wang, and Zhang 2011; Xu, Zhang, and Wang 2012). Similarly, in essence, the approximations in Greco's DRSA, Shao's and some approaches are still based on a singleton granulation induced from a dominant relation in an ordered information system, which can be applied to knowledge representation in distributive systems and groups of intelligent agents.

The main objective of this paper is to extend Greco's single-granulation rough set model based on dominant relation in ordered information systems to two new types of MGRS models where the set approximations are defined by using multiple equivalence relations on the universe. The rest of the paper is organized as follows. Some preliminary concepts in Pawlak's and Greco's rough set theory are briefly reviewed in Section 2. In Sections 3 and 4, for an ordered information system, based on multiple dominant relations, two new types of MGRS models are obtained, respectively, where a target concept is approximated from different kinds of views by using the dominant classes induced by multiple dominant relations. And a number of important properties of the two types of MGRS models are investigated in an ordered information system. It is shown that some of the properties of Greco's rough set theory based on dominant relations are special instances of those of our MGRS in an ordered information system. In Section 5, the relationships and differences are discussed among Greco's rough set and two new types of MGRS models in an ordered information system. In Section 6, several important measures are presented in two types of MGRS models, such as rough measure and quality of approximation. Finally, the paper is concluded by a summary and outlook for further research in Section 7.

## 2. Rough set theory and ordered information systems

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in the source papers (Pawlak 1982, 1991; Greco et al. 1998, 1999, 2001, 2002, 2007; Pawlak and Skowron 2007a). A detailed description has also been made by Zhang et al. (2001).

A notion of information system (sometimes called data tables, attribute valued systems, knowledge representation systems, etc.) provides a convenient basis for the representation of objects in terms of their attributes.

An information system is a quadruple  $\mathcal{I} = (U, AT, V, f)$ , where  $U$  is a non-empty finite set with  $n$  objects,  $\{u_1, u_2, \dots, u_n\}$ , called the universe of discourse;  $AT = \{a_1, a_2, \dots, a_m\}$  is a non-empty finite set with  $m$  attributes;  $V = \cup_{a \in AT} V_a$  and  $V_a$  is the domain of attribute  $a$ ;  $f : U \times AT \rightarrow V$  is a function such that  $f(u, a) \in V_a$  for any  $a \in AT, u \in U$ , called an information function (Pawlak 1982, 1991; Lin 2005). A decision table is a special case of an information system in which, among the attributes, we distinguish a decision attribute. The other attributes are called condition attributes.

Therefore,  $\mathcal{I} = (U, C \cup \{d\}, V, f)$  is a decision table, where sets  $C$  and  $\{d\}$  are condition attribute set and decision attribute set, respectively.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

An information system is called an ordered information system if all condition attributes are criterion (see Greco et al. 1998, 1999, 2001, 2002, 2007).

Assumed that the domain of a criterion  $a \in AT$  is completely pre-ordered by an outranking relation  $\succsim_a$ , then  $u \succsim_a v$  means that  $u$  is at least as good as (outranks)  $v$  with respect to the criterion  $a$ , and we can say that  $u$  dominates  $v$  or  $v$  is dominated by  $u$ . Being of type gain, that is  $u \succsim_a v \Leftrightarrow f(u, a) \geq f(v, a)$  (according to increasing preference) or  $u \succsim_a v \Leftrightarrow f(u, a) \leq f(v, a)$  (according to decreasing preference). Without any loss of generality and for simplicity, in the following, we only consider condition attributes with increasing preference.

For a subset of attributes  $A \subseteq AT$ , we define  $u \succsim_A v \Leftrightarrow u \succsim_a v$  for  $\forall a \in A$ . It is that  $u$  dominates  $v$  with respect to all attributes in  $A$ . In general, we denote an ordered information system by  $\mathcal{I}^\succsim = (U, AT, V, f)$ .

For a given ordered information system, we say that  $u$  dominates  $v$  with respect to  $A \subseteq AT$  if  $u \succsim_A v$ , and denote by  $uR_A^\succsim v$ . That is,

$$R_A^\succsim = \{(u, v) \in U \times U \mid u \succsim_A v\} = \{(u, v) \in U \times U \mid f(u, a) \geq f(v, a) \forall a \in A\},$$

where  $R_A^\succsim$  is called a dominance relation of ordered information system  $\mathcal{I}^\succsim$ .

Let us denote

$$\begin{aligned} [u_i]_A^\succsim &= \{u_j \in U \mid (u_j, u_i) \in R_A^\succsim\} \\ &= \{u_j \in U \mid f(u_j, a) \geq f(u_i, a) \forall a \in A\}, \\ \frac{U}{R_A^\succsim} &= \{[u_1]_A^\succsim, [u_2]_A^\succsim, \dots, [u_n]_A^\succsim\}, \end{aligned}$$

where  $i \in \{1, 2, \dots, n\}$ , then  $[u_i]_A^\succsim$  is called a dominance class or the granularity of information, and  $U/R_A^\succsim$  is called a classification of  $U$  about attribute set  $A$ .

From above description, the following properties of dominance relation in ordered information system are trivial.

**PROPOSITION 2.1.** Let  $\mathcal{I}^\succsim = (U, AT, V, f)$  be an ordered information system and  $B, A \subseteq AT$  (see Greco et al. 1998, 1999, 2001, 2002, 2007), then we have that

- (1)  $R_A^\succsim$  is reflective, transitive, but not symmetric, so it is not an equivalence relation.
- (2) If  $B \subseteq A \subseteq AT$ , then  $R_{AT}^\succsim \subseteq R_A^\succsim \subseteq R_B^\succsim$ .

Similarly, for the dominance class induced by the dominance relation  $R_A^\succsim$ , the following properties are still correct.

**PROPOSITION 2.2.** Let  $\mathcal{I}^\succsim = (U, AT, V, f)$  be an ordered information system and  $B, A \subseteq AT$  (see Greco et al. 1998, 1999, 2001, 2002, 2007), then we have that

- (1) if  $B \subseteq A \subseteq AT$ , then  $[u]_{AT}^\succsim \subseteq [u]_A^\succsim \subseteq [u]_B^\succsim$  for any  $u \in U$ ;
- (2) if  $v \in [u]_A^\succsim$ , then  $[v]_A^\succsim \subseteq [u]_A^\succsim$  and  $[u]_A^\succsim = \cup \{[v]_A^\succsim \mid v \in [u]_A^\succsim\}$ ;
- (3)  $[u]_{AT}^\succsim = [v]_{AT}^\succsim$  if and only if  $f(u, a) = f(v, a)$  for any  $a \in AT$ ; and
- (4)  $|[u]_{AT}^\succsim| \geq 1$  for any  $u \in U$ ;

where  $|X|$  denotes the cardinality of set  $X$ .

For any subset  $X \subseteq U$  and  $A \subseteq AT$  in  $\mathcal{I}^{\succsim}$ , the lower and upper approximation of  $X$  with respect to a dominance relation  $R_A^{\succsim}$  could be defined as

$$\underline{R}_A^{\succsim}(X) = \{u \in U \mid [u]_A^{\succsim} \subseteq X\}, \overline{R}_A^{\succsim}(X) = \{u \in U \mid [u]_A^{\succsim} \cap X \neq \emptyset\}.$$

PROPOSITION 2.3. Let  $\mathcal{I}^{\succsim} = (U, AT, V, f)$  be an ordered information system and  $B, A \subseteq AT$  (see Greco et al. 1998, 1999, 2001, 2002, 2007), then we have that

- (L<sub>1</sub>)  $\underline{R}_A^{\succsim}(X) \subseteq X$  (Contraction),
- (U<sub>1</sub>)  $\overline{R}_A^{\succsim}(X) \supseteq X$  (Extension),
- (L<sub>2</sub>)  $\underline{R}_A^{\succsim}(\sim X) = \sim \overline{R}_A^{\succsim}(X)$  (Duality),
- (U<sub>2</sub>)  $\overline{R}_A^{\succsim}(\sim X) = \sim \underline{R}_A^{\succsim}(X)$  (Duality),
- (L<sub>3</sub>)  $\underline{R}_A^{\succsim}(\emptyset) = \emptyset$  (Normality),
- (U<sub>3</sub>)  $\overline{R}_A^{\succsim}(\emptyset) = \emptyset$  (Normality),
- (L<sub>4</sub>)  $\underline{R}_A^{\succsim}(U) = U$  (Co-normality),
- (U<sub>4</sub>)  $\overline{R}_A^{\succsim}(U) = U$  (Co-normality),
- (L<sub>5</sub>)  $\underline{R}_A^{\succsim}(X \cap Y) = \underline{R}_A^{\succsim}(X) \cap \underline{R}_A^{\succsim}(Y)$  (Multiplication),
- (U<sub>5</sub>)  $\overline{R}_A^{\succsim}(X \cup Y) = \overline{R}_A^{\succsim}(X) \cup \overline{R}_A^{\succsim}(Y)$  (Addition),
- (L<sub>6</sub>)  $\underline{R}_A^{\succsim}(X \cup Y) \supseteq \underline{R}_A^{\succsim}(X) \cup \underline{R}(Y)$  ( $F$ -multiplication),
- (U<sub>6</sub>)  $\overline{R}_A^{\succsim}(X \cap Y) \subseteq \overline{R}_A^{\succsim}(X) \cap \overline{R}(Y)$  ( $F$ -addition),
- (L<sub>7</sub>)  $\underline{R}_A^{\succsim}(\underline{R}_A^{\succsim}(X)) = \underline{R}_A^{\succsim}(X)$  (Idempotency),
- (U<sub>7</sub>)  $\overline{R}_A^{\succsim}(\overline{R}_A^{\succsim}(X)) = \overline{R}_A^{\succsim}(X)$  (Idempotency).

To measure the imprecision and roughness of a rough set, Pawlak recommended  $X \neq \emptyset$  the ratio

$$\rho_A(X) = 1 - \frac{|\underline{R}_A^{\succsim}(X)|}{|\overline{R}_A^{\succsim}(X)|},$$

which is called the rough measure of  $X$  by equivalence relation  $R_A^{\succsim}$ .

Furthermore, for an information system with the decision  $\mathcal{I} = (U, C \cup \{d\}, V, f)$  and  $A \subseteq C$ , a frequently applied measure for the situation is the quality of approximation of  $R_d^{\succsim}$  by  $R_A^{\succsim}$ , also called the degree of dependency. It is defined as

$$\gamma(A, d) = \frac{1}{|U|} \sum_{j=1}^k \underline{R}_A^{\succsim}(D_j),$$

where  $R_d^{\succsim} = \{(u, v) \in U \times U \mid g(u, d) = g(v, d)\}$  and  $U/d = \{[u]_d, \forall u \in U\} = \{D_1, D_2, \dots, D_k\}$ .

### 3. The optimistic MGRS in ordered information systems (OIS)

In the section, we will consider the optimistic multiple granulation approximations of a target set by using multiple dominant relations in an ordered information system.

DEFINITION 3.1. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_1, A_2, \dots, A_s \subseteq AT$  be attribute subsets ( $s \leq 2^{|AT|}$ ), and  $R_{A_1}^{\geq}, R_{A_2}^{\geq}, \dots, R_{A_s}^{\geq}$  be dominant relations, respectively. The operators  $\underline{OM}_{\sum_{i=1}^s A_i}^{\geq}$  and  $\overline{OM}_{\sum_{i=1}^s A_i}^{\geq} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$  are defined as follows: for  $\forall X \in \mathcal{P}(U)$ ,

$$\underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) = \left\{ u \mid \bigvee_{i=1}^m ([u]_{A_i}^{\geq} \subseteq X) \right\}, \quad \overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) = \left\{ u \mid \bigwedge_{i=1}^m ([u]_{A_i}^{\geq} \cap X \neq \emptyset) \right\},$$

where ‘ $\vee$ ’ means ‘or’ and ‘ $\wedge$ ’ means ‘and’. We call them the optimistic multiple granulation lower and upper approximation operators, and call  $\underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X)$  and  $\overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X)$  the optimistic multiple granulation lower approximation set and upper approximation set of  $X$  in the ordered information system, respectively.

Moreover, if

$$\underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) \neq \overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X),$$

we say that  $X$  is the optimistic rough set with respect to multiple granulation spaces  $A_1, A_2, \dots, A_s$  in the ordered information system. Otherwise, we say that  $X$  is the optimistic definable set with respect to these multiple granulation spaces in the ordered information system.

Similarly, the area of uncertainty or boundary region of this rough set in an ordered information system is defined as

$$Bn_{\sum_{i=1}^s A_i}^O(X) = \overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) - \underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X).$$

Here, we employ an example to illustrate the above concepts with respect to the optimistic multiple granulation rough set (OMGRS) in an ordered information system.

Example 3.1. Suppose Table 1 is an ordered information system about the achievements of some students,  $U = \{x_1, x_2, \dots, x_{10}\}$  is a universe which consists of 10 students in some college; *Mathematic*, *English*, *Morality*, *Physical* are the conditional attributes of the system, and the dominant preference are as follows:  $A \geq B \geq C \geq D$ . *Decision* is the result of excellent students by the experts according to the achievements of these students,  $Y$  expresses that the student is excellent, and  $N$  expresses the student is not excellent.

However, we often face the phenomenon that some universities may give some conditions of excellent students as follows:

Condition 1: Not only the marks are higher, but also the morality is better.

Condition 2: Not only the marks are higher, but also the health is better.



Table 1. An ordered information system about the achievements of some students.

Course	Mathematics	English	Morality	Physical	Decision
$x_1$	85	90	B	A	Y
$x_2$	86	90	B	B	Y
$x_3$	90	87	A	C	Y
$x_4$	88	86	C	D	N
$x_5$	87	85	D	B	Y
$x_6$	86	87	A	C	Y
$x_7$	84	83	B	A	N
$x_8$	88	88	A	C	Y
$x_9$	87	85	C	B	N
$x_{10}$	89	88	B	D	Y

We can obtain that the decision has two decision classes  $D(Y), D(N)$  from Table 1, and it is easy to find out that  $X = \{x_1, x_2, x_3, x_5, x_6, x_8, x_{10}\}$  is a set which consists of excellent students.

If we only consider one of these conditions, who is an excellent student and who may be an excellent student.

According to Conditions 1 and 2, we can obtain two dominant relations denoted as  $R_1$  and  $R_2$ .

$$R_1 = \begin{pmatrix} 1000001000 \\ 1100001000 \\ 0011111010 \\ 0001100010 \\ 0000100000 \\ 0000011000 \\ 0000001000 \\ 0001111110 \\ 0000100010 \\ 0001101011 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1000001000 \\ 0100010000 \\ 0011010010 \\ 0001000000 \\ 0000100010 \\ 0000010000 \\ 0000001000 \\ 0001010100 \\ 0000100010 \\ 0001000001 \end{pmatrix}.$$

If we consider only Condition 1, we can obtain

$$\begin{aligned} \underline{R_1}^{\supseteq}(X) &= \{x_5\}, \\ \overline{R_1}^{\supseteq}(X) &= \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}\}. \end{aligned}$$

It is easy to find out that  $x_5$  must be an excellent student, and  $x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}$  may be excellent students, if we consider only Condition 1.

If we consider only Condition 2, we can obtain

$$\begin{aligned} \underline{R_2}^{\supseteq}(X) &= \{x_2, x_6\}, \\ \overline{R_2}^{\supseteq}(X) &= \{x_1, x_2, x_3, x_5, x_6, x_8, x_9, x_{10}\}. \end{aligned}$$



It is easy to find out that  $x_2, x_6$  must be excellent students, and  $x_1, x_2, x_3, x_5, x_6, x_8, x_9, x_{10}$  may be excellent students, if we consider only Condition 2.

According to these conditions, we can also raise some questions now.

Question 1: If we consider one of these conditions at least, who is an excellent student?

Question 2: When we consider both of these conditions, which one may be an excellent student?

We cannot solve these questions according to the definition of Greco's rough set, but we can use the definition of OMGRS to solve the above questions. So we have

$$\begin{aligned} \underline{\underline{OM}}_{1+2}^{\geq}(X) &= \{x_2, x_5, x_6\}, \\ \underline{\underline{OM}}_{1+2}^{\geq}(X) &= \{x_1, x_2, x_3, x_5, x_6, x_8, x_9, x_{10}\}. \end{aligned}$$

We can find out that the students  $x_2, x_5, x_6$  must be excellent, when we consider at least one of these conditions, and the students  $x_1, x_2, x_3, x_5, x_6, x_8, x_9, x_{10}$  may be excellent, if we consider both of these conditions.

Furthermore, one can check the following properties:

$$\begin{aligned} \underline{R}_1^{\geq}(X) \cup \underline{R}_2^{\geq}(X) &= \underline{\underline{OM}}_{1+2}^{\geq}(X), \\ \underline{R}_1^{\geq}(X) \cap \underline{R}_2^{\geq}(X) &= \underline{\underline{OM}}_{1+2}^{\geq}(X). \end{aligned}$$

To describe conveniently in our context, we express the optimistic MGRS by OMGRS in ordered information systems. Moreover, one can obtain the following properties of the OMGRS approximations in ordered information systems.

PROPOSITION 3.1. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then the following properties hold:

- (OL<sub>1</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq X$  (Contraction),
- (OU<sub>1</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \supseteq X$  (Extension),
- (OL<sub>2</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(\sim X) = \sim \underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X)$  (Duality),
- (OU<sub>2</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(\sim X) = \sim \underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X)$  (Duality),
- (OL<sub>3</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(\emptyset) = \emptyset$  (Normality),
- (OU<sub>3</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(\emptyset) = \emptyset$  (Normality),
- (OL<sub>4</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(U) = U$  (Co-normality),
- (OU<sub>4</sub>)  $\underline{\underline{OM}}_{\sum_{i=1}^s A_i}^{\geq}(U) = U$  (Co-normality).

*Proof.* Since the number of the granulations is finite, we prove that the results are true only when the ordered information system has two dominance relations ( $A, B \subseteq AT$ ) for convenience. It is obvious that all terms hold when  $A = B$ . When  $A \neq B$ , the proposition can be proved as follows.

(OL<sub>1</sub>) For any  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X)$ , it can be known that  $[u]_A^{\geq} \subseteq X$  or  $[u]_B^{\geq} \subseteq X$  by Definition 3.1. However,  $u \in [u]_A^{\geq}$  and  $u \in [u]_B^{\geq}$ . So we can have that  $u \in X$ . Hence,  $\overline{\text{OM}}_{A+B}^{\geq}(X) \subseteq X$ .

(OU<sub>1</sub>) For any  $u \in X$ , we have  $u \in [u]_A^{\geq}$  and  $u \in [u]_B^{\geq}$ . So  $[u]_A^{\geq} \cap X \neq \emptyset$  and  $[u]_B^{\geq} \cap X \neq \emptyset$ , that is to say  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X)$ . Hence,  $X \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X)$ .

(OL<sub>2</sub>) For any  $u \in \overline{\text{OM}}_{A+B}^{\geq}(\sim X)$ , then

$$\begin{aligned} u \in \overline{\text{OM}}_{A+B}^{\geq}(\sim X) &\Leftrightarrow [u]_A^{\geq} \subseteq \sim X \text{ or } [u]_B^{\geq} \subseteq \sim X \\ &\Leftrightarrow [u]_A^{\geq} \cap X = \emptyset \text{ or } [u]_B^{\geq} \cap X = \emptyset \\ &\Leftrightarrow u \notin \overline{\text{OM}}_{A+B}^{\geq}(X) \\ &\Leftrightarrow u \in \sim \overline{\text{OM}}_{A+B}^{\geq}(X). \end{aligned}$$

Hence,  $\overline{\text{OM}}_{A+B}^{\geq}(\sim X) = \sim \overline{\text{OM}}_{A+B}^{\geq}(X)$ .

(OU<sub>2</sub>) By (OL<sub>2</sub>), we have  $\overline{\text{OM}}_{A+B}^{\geq}(X) = \sim \overline{\text{OM}}_{A+B}^{\geq}(\sim X)$ . So it can be obtain that  $\sim \overline{\text{OM}}_{A+B}^{\geq}(X) = \overline{\text{OM}}_{A+B}^{\geq}(\sim X)$ .

(OL<sub>3</sub>) From (OL<sub>1</sub>), we have  $\overline{\text{OM}}_{A+B}^{\geq}(\emptyset) \subseteq \emptyset$ . Also, it is well known that  $\emptyset \subseteq \overline{\text{OM}}_{A+B}^{\geq}(\emptyset)$ . So,  $\overline{\text{OM}}_{A+B}^{\geq}(\emptyset) = \emptyset$ .

(OU<sub>3</sub>) If  $\overline{\text{OM}}_{A+B}^{\geq}(\emptyset) \neq \emptyset$ , then there must exist a  $u \in \overline{\text{OM}}_{A+B}^{\geq}(\emptyset)$ . So we can find that  $[u]_A^{\geq} \cap \emptyset \neq \emptyset$  and  $[u]_B^{\geq} \cap \emptyset \neq \emptyset$ . Obviously, this is a contradiction. Thus,  $\overline{\text{OM}}_{A+B}^{\geq}(\emptyset) = \emptyset$ .

$$(OL_4) \overline{\text{OM}}_{A+B}^{\geq}(U) = \overline{\text{OM}}_{A+B}^{\geq}(\sim \emptyset) = \sim \overline{\text{OM}}_{A+B}^{\geq}(\emptyset) = \sim \emptyset = U$$

$$(OU_4) \overline{\text{OM}}_{A+B}^{\geq}(U) = \overline{\text{OM}}_{A+B}^{\geq}(\sim \emptyset) = \sim \overline{\text{OM}}_{A+B}^{\geq}(\emptyset) = \sim \emptyset = U. \quad \square$$

PROPOSITION 3.2. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$  and  $X, Y \subseteq U$ . Then the following properties hold:

$$(OL_5) \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X \cap Y) \subseteq \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \cap \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (L\text{-multiplication}),$$

$$(OU_5) \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X \cup Y) \supseteq \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \cup \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (L\text{-addition}),$$

$$(OL_6) X \subseteq Y \Rightarrow \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (\text{Granularity}),$$

$$(OU_6) X \subseteq Y \Rightarrow \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (\text{Granularity}),$$

$$(OL_7) \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X \cup Y) \supseteq \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \cup \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (U\text{-addition}),$$

$$(OU_7) \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X \cap Y) \subseteq \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \cap \overline{\text{OM}}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (U\text{-multiplication}).$$

*Proof.* Since the number of the granulations is finite, we only prove the results are true when the ordered information system has two dominance relations ( $A, B \subseteq AT$ ) for convenience. It is obvious that all terms hold when  $A = B$  or  $X = Y$ . When  $A \neq B$  and  $X \neq Y$ , the proposition can be proved as follows:

(OL<sub>5</sub>) For any  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X \cap Y)$ , we have that  $[u]_A^{\geq} \subseteq (X \cap Y)$  or  $[u]_B^{\geq} \subseteq (X \cap Y)$  by Definition 3.1. Then, it can be obtained that  $[u]_A^{\geq} \subseteq X$  and  $[u]_A^{\geq} \subseteq Y$  hold at the same time or  $[u]_B^{\geq} \subseteq X$  and  $[u]_B^{\geq} \subseteq Y$  hold at the same time. So, not only  $[u]_A^{\geq} \subseteq X$  or  $[u]_B^{\geq} \subseteq X$

hold, but  $[u]_A^{\geq} \subseteq Y$  or  $[u]_B^{\geq} \subseteq Y$  hold at the same time. That is to say that  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X)$  and  $u \in \overline{\text{OM}}_{A+B}^{\geq}(Y)$ , i.e.  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X) \cap \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

Hence,  $\overline{\text{OM}}_{A+B}^{\geq}(X \cap Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cap \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

(OU<sub>5</sub>) For any  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X) \cup \overline{\text{OM}}_{A+B}^{\geq}(Y)$ , we have that  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X)$  or  $u \in \overline{\text{OM}}_{A+B}^{\geq}(Y)$ . Then  $[u]_A^{\geq} \cap X \neq \emptyset$  and  $[u]_B^{\geq} \cap X \neq \emptyset$  hold at the same time or  $[u]_A^{\geq} \cap Y \neq \emptyset$  and  $[u]_B^{\geq} \cap Y \neq \emptyset$  hold at the same time. So, not only  $[u]_A^{\geq} \cap (X \cup Y) \neq \emptyset$  hold, but also  $[u]_B^{\geq} \cap (X \cup Y) \neq \emptyset$  hold. That is to say  $u \in \overline{\text{OM}}_{A+B}^{\geq}(X \cup Y)$ .

Hence,  $\overline{\text{OM}}_{A+B}^{\geq}(X \cap Y) \supseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cup \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

(OL<sub>6</sub>) Since  $X \subseteq Y$ , one can have  $X \cap Y = X$ . Then,  $\overline{\text{OM}}_{A+B}^{\geq}(X \cap Y) = \overline{\text{OM}}_{A+B}^{\geq}(X)$ .

Also, it can be found that  $\overline{\text{OM}}_{A+B}^{\geq}(X \cap Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cap \overline{\text{OM}}_{A+B}^{\geq}(Y)$  by (OL<sub>5</sub>). So, we can obtain that  $\overline{\text{OM}}_{A+B}^{\geq}(X) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cap \overline{\text{OM}}_{A+B}^{\geq}(Y)$ , that is to say that  $\overline{\text{OM}}_{A+B}^{\geq}(X) = \overline{\text{OM}}_{A+B}^{\geq}(X) \cap \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

Thus,  $\overline{\text{OM}}_{A+B}^{\geq}(X) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

(OU<sub>6</sub>) Since  $X \subseteq Y$ , one can have  $X \cup Y = Y$ . Then,  $\overline{\text{OM}}_{A+B}^{\geq}(X \cup Y) = \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

Also, it can be found that  $\overline{\text{OM}}_{A+B}^{\geq}(X \cup Y) \supseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cup \overline{\text{OM}}_{A+B}^{\geq}(Y)$  by (OU<sub>5</sub>). So, we can obtain that  $\overline{\text{OM}}_{A+B}^{\geq}(Y) \supseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cup \overline{\text{OM}}_{A+B}^{\geq}(Y)$  that is to say that  $\overline{\text{OM}}_{A+B}^{\geq}(Y) = \overline{\text{OM}}_{A+B}^{\geq}(X) \cup \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

Thus,  $\overline{\text{OM}}_{A+B}^{\geq}(X) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(Y)$ .

(OL<sub>7</sub>) Since  $X \subseteq X \cup Y$  and  $Y \subseteq X \cup Y$ , by (OL<sub>6</sub>) it can be obtained that

$$\overline{\text{OM}}_{A+B}^{\geq}(X) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X \cup Y), \quad \overline{\text{OM}}_{A+B}^{\geq}(Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X \cup Y).$$

So, we have  $\overline{\text{OM}}_{A+B}^{\geq}(X) \cup \overline{\text{OM}}_{A+B}^{\geq}(Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X \cup Y)$ .

(OU<sub>7</sub>) Since  $X \cap Y \subseteq X$  and  $X \cap Y \subseteq Y$ , by (OU<sub>6</sub>) it can be obtained that

$$\overline{\text{OM}}_{A+B}^{\geq}(X \cap Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X), \quad \overline{\text{OM}}_{A+B}^{\geq}(Y \cap Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(Y).$$

So, we have  $\overline{\text{OM}}_{A+B}^{\geq}(X \cap Y) \subseteq \overline{\text{OM}}_{A+B}^{\geq}(X) \cap \overline{\text{OM}}_{A+B}^{\geq}(X)$ . □

The proposition is proved.

#### 4. The pessimistic MGRS in OIS

In this section, we consider another MGRS is in an ordered information system.

DEFINITION 4.1. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_1, A_2, \dots, A_s \subseteq AT$  be attribute subsets ( $s \leq 2^{|AT|}$ ), and  $R_{A_1}^{\geq}, R_{A_2}^{\geq}, \dots, R_{A_s}^{\geq}$  be dominant relations, respectively. The operators  $\overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}$  and  $\overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$  are defined as follows: for  $\forall X \in \mathcal{P}(U)$ ,

$$\overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) = \left\{ u \mid \bigvee_{i=1}^m ([u]_{A_i}^{\geq} \subseteq X) \right\}, \quad \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) = \left\{ u \mid \bigwedge_{i=1}^m ([u]_{A_i}^{\geq} \cap X \neq \emptyset) \right\},$$

where ‘ $\vee$ ’ means ‘or’ and ‘ $\wedge$ ’ means ‘and’. We call them the pessimistic multiple granulation lower and upper approximation operators, and call  $\overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X)$  and  $\overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X)$  the pessimistic multiple granulation lower approximation set and upper approximation set of  $X$ , respectively.

Moreover, if  $\underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \neq \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X)$ , we say that  $X$  is the pessimistic rough set with respect to multiple granulation spaces  $A_1, A_2, \dots, A_s$ . Otherwise, we say that  $X$  is the pessimistic definable set with respect to these multiple granulation spaces in ordered information systems.

Similarly, the area of uncertainty or boundary region of this rough set is defined as

$$\text{Bn}_{\sum_{i=1}^s A_i}^p(X) = \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) - \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X).$$

To describe conveniently in our context, we express the pessimistic MGRS by using the pessimistic multiple granulation rough set (PMGRS) in the ordered information system. Moreover, one can obtain the following properties of the PMGRS approximations in the ordered information systems.

*Example 4.1.* (Continued from Example 3.1) In Example 3.1, we have raised some questions, here we have also proposed other questions, such as

Question 3: Which student must be excellent whatever condition we consider?

Question 4: If we consider one of these conditions at least, who may be excellent?

According to the definition of PMGRS, we can have

$$\underline{\text{PM}}_{1+2}^{\geq}(X) = \emptyset, \quad \overline{\text{PM}}_{1+2}^{\geq}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}\}.$$

We can find out that no one must be excellent if we consider all of these conditions, but every one may be an excellent student expect  $x_7$  when we consider one of these conditions at least.

Furthermore, one can check the following properties:

$$\underline{R}_1^{\geq}(X) \cap \underline{R}_2^{\geq}(X) = \underline{\text{PM}}_{1+2}^{\geq}(X), \quad \overline{R}_1^{\geq}(X) \cup \overline{R}_2^{\geq}(X) = \overline{\text{PM}}_{1+2}^{\geq}(X).$$

**PROPOSITION 4.1.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then the following properties hold:

$$(PL_1) \quad \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq X \quad (\text{Contraction}),$$

$$(PU_1) \quad \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \supseteq X \quad (\text{Extension}),$$

$$(PL_2) \quad \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(\sim X) = \sim \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \quad (\text{Duality}),$$

$$(PU_2) \quad \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(\sim X) = \sim \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \quad (\text{Duality}),$$

$$(PL_3) \quad \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(\emptyset) = \emptyset \quad (\text{Normality}),$$

$$(PU_3) \quad \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(\emptyset) = \emptyset \emptyset \quad (\text{Normality}),$$

$$(PL_4) \quad \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(U) = U \quad \emptyset \quad (\text{Co-normality}),$$

$$(PU_4) \quad \overline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(U) = U \quad (\text{Co-normality}).$$

*Proof.* Since the number of the granulations is finite, we only prove the results are true when the ordered information system has two dominance relations  $(A, B \subseteq AT)$  for convenience. It is obvious that all terms hold when  $A = B$ . When  $A \neq B$ , the proposition can be proved as follows:

(PL<sub>1</sub>) For any  $u \in \underline{PM}_{A+B}^{\geq}(X)$ , it can be known that  $[u]_A^{\geq} \subseteq X$  and  $[u]_B^{\geq} \subseteq X$  by Definition 4.1. However,  $u \in \underline{[u]_A^{\geq}}$  and  $u \in \underline{[u]_B^{\geq}}$ . So we can have that  $u \in X$ . Hence,  $\underline{PM}_{A+B}^{\geq}(X) \subseteq X$ .

(PU<sub>1</sub>) For any  $u \in X$ , we have  $u \in \underline{[u]_A^{\geq}}$  and  $u \in \underline{[u]_B^{\geq}}$ . So  $[u]_A^{\geq} \cap X \neq \emptyset$  and  $[u]_B^{\geq} \cap X \neq \emptyset$ , which imply that  $u \in \underline{PM}_{A+B}^{\geq}(X)$ . Hence,  $X \subseteq \underline{PM}_{A+B}^{\geq}(X)$ .

(PL<sub>2</sub>) For any  $u \in \underline{PM}_{A+B}^{\geq}(\sim X)$ , then

$$\begin{aligned} u \in \underline{PM}_{A+B}^{\geq}(\sim X) &\Leftrightarrow [u]_A^{\geq} \subseteq \sim X \text{ and } [u]_B^{\geq} \subseteq \sim X \\ &\Leftrightarrow [u]_A^{\geq} \cap X = \emptyset \text{ and } [u]_B^{\geq} \cap X = \emptyset \Leftrightarrow u \notin \overline{PM}_{A+B}^{\geq}(X) \Leftrightarrow u \in \sim \overline{PM}_{A+B}^{\geq}(X). \end{aligned}$$

Hence,  $\underline{PM}_{A+B}^{\geq}(\sim X) = \sim \overline{PM}_{A+B}^{\geq}(X)$ .

(PU<sub>2</sub>) By (PL<sub>2</sub>), we have  $\underline{PM}_{A+B}^{\geq}(X) = \sim \overline{PM}_{A+B}^{\geq}(\sim X)$ . So it can be obtained that  $\sim \underline{PM}_{A+B}^{\geq}(X) = \overline{PM}_{A+B}^{\geq}(\sim X)$ .

(PL<sub>3</sub>) From (PL<sub>1</sub>), we have  $\underline{PM}_{A+B}^{\geq}(\emptyset) \subseteq \emptyset$ . Also, it is well known that  $\emptyset \subseteq \underline{PM}_{A+B}^{\geq}(\emptyset)$ . So,  $\underline{PM}_{A+B}^{\geq}(\emptyset) = \emptyset$ .

(PU<sub>3</sub>) If  $\underline{PM}_{A+B}^{\geq}(\emptyset) \neq \emptyset$ , then there must exist  $u \in \underline{PM}_{A+B}^{\geq}(\emptyset)$ . So we can find that  $[u]_A^{\geq} \cap \emptyset \neq \emptyset$  or  $[u]_B^{\geq} \cap \emptyset \neq \emptyset$ . Obviously, this is a contradiction. Thus,  $\underline{PM}_{A+B}^{\geq}(\emptyset) = \emptyset$ .

$$(PL_4) \underline{PM}_{A+B}^{\geq}(U) = \underline{PM}_{A+B}^{\geq}(\sim \emptyset) = \sim \overline{PM}_{A+B}^{\geq}(\emptyset) = \sim \emptyset = U.$$

$$(PU_{4s}) \overline{PM}_{A+B}^{\geq}(U) = \overline{PM}_{A+B}^{\geq}(\sim \emptyset) = \sim \underline{PM}_{A+B}^{\geq}(\emptyset) = \sim \emptyset = U.$$

□

PROPOSITION 4.2. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X, Y \subseteq U$ . Then the following properties hold:

$$(PL_5) \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X \cap Y) \subseteq \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \cap \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (L\text{-multiplication}),$$

$$(PU_5) \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X \cup Y) \supseteq \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \cup \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (L\text{-addition}),$$

$$(PL_6) X \subseteq Y \Rightarrow \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (\text{Granularity}),$$

$$(PU_6) X \subseteq Y \Rightarrow \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \supseteq \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (\text{Granularity}),$$

$$(PL_7) \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X \cup Y) \supseteq \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \cup \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (U\text{-addition}),$$

$$(PU_7) \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X \cap Y) \subseteq \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \cap \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(Y) \quad (U\text{-multiplication}).$$

*Proof.* Since the number of the granulations is finite, we prove the results are true only when the ordered information system has two dominance relations  $(A, B \subseteq AT)$  for convenience. It is obvious that all terms hold when  $A = B$  or  $X = Y$ . When  $A \neq B$  and  $X \neq Y$ , the proposition can be proved as follows:

(PL<sub>5</sub>) For any  $u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y)$ , by Definition 4.1 we have

$$\begin{aligned} u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y) &\Leftrightarrow [u]_A^{\succcurlyeq} \subseteq (X \cap Y) \text{ and } [u]_B^{\succcurlyeq} \subseteq (X \cap Y) \\ &\Leftrightarrow [u]_A^{\succcurlyeq} \subseteq X, [u]_A^{\succcurlyeq} \subseteq Y, [u]_B^{\succcurlyeq} \subseteq X \text{ and } [u]_B^{\succcurlyeq} \subseteq Y \\ &\Leftrightarrow [u]_A^{\succcurlyeq} \subseteq X, [u]_B^{\succcurlyeq} \subseteq X, [u]_A^{\succcurlyeq} \subseteq Y \text{ and } [u]_B^{\succcurlyeq} \subseteq Y \\ &\Leftrightarrow u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \text{ and } u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y) \Leftrightarrow u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cap \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y). \end{aligned}$$

Hence,  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cap \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ .

(PU<sub>5</sub>) For any  $u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y)$ , by Definition 4.1 we have

$$\begin{aligned} u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y) &\Leftrightarrow [u]_A^{\succcurlyeq} \cap (X \cup Y) \neq \emptyset \text{ or } [u]_B^{\succcurlyeq} \cap (X \cup Y) \neq \emptyset \\ &\Leftrightarrow [u]_A^{\succcurlyeq} \cap X \neq \emptyset \text{ or } [u]_A^{\succcurlyeq} \cap Y \neq \emptyset, \text{ or } [u]_B^{\succcurlyeq} \cap X \neq \emptyset \text{ or } [u]_B^{\succcurlyeq} \cap Y \neq \emptyset \\ &\Leftrightarrow [u]_A^{\succcurlyeq} \cap X \neq \emptyset \text{ or } [u]_B^{\succcurlyeq} \cap X \neq \emptyset, \text{ or } [u]_A^{\succcurlyeq} \cap Y \neq \emptyset \text{ or } [u]_B^{\succcurlyeq} \cap Y \neq \emptyset \\ &\Leftrightarrow u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \text{ or } u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y) \Leftrightarrow u \in \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cup \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y). \end{aligned}$$

Hence,  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cup \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ .

(PL<sub>6</sub>) Since  $X \subseteq Y$ , one can have  $X \cap Y = X$ . Then,  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X)$ .

Also, it can be found that  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cap \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$  by (PL<sub>5</sub>). So, we can obtain that  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cap \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ , that is to say that  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \subseteq \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ .

(PU<sub>6</sub>) Since  $X \subseteq Y$ , one can have  $X \cup Y = Y$ . Then,  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ .

Also, it can be found that  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cup \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$  by (PU<sub>5</sub>). So, we can obtain that  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y) = \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cup \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ , that is to say that  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \subseteq \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y)$ .

(PU<sub>7</sub>) Since  $X \subseteq X \cup Y$  and  $Y \subseteq X \cup Y$ , by (PL<sub>6</sub>) we can obtain that

$$\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \subseteq \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y), \quad \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y) \subseteq \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y).$$

So, we have  $\underline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cup \underline{\text{PM}}_{A+B}^{\succcurlyeq}(Y) \subseteq \underline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cup Y)$ .

(PU<sub>7</sub>) Since  $\overline{X \cap Y} \subseteq \overline{X}$  and  $\overline{X \cap Y} \subseteq \overline{Y}$ , by (PU<sub>6</sub>), it can be obtained that

$$\overline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y) \subseteq \overline{\text{PM}}_{A+B}^{\succcurlyeq}(X), \quad \overline{\text{PM}}_{A+B}^{\succcurlyeq}(Y \cap Y) \subseteq \overline{\text{PM}}_{A+B}^{\succcurlyeq}(Y).$$

So, we have  $\overline{\text{PM}}_{A+B}^{\succcurlyeq}(X \cap Y) \subseteq \overline{\text{PM}}_{A+B}^{\succcurlyeq}(X) \cap \overline{\text{PM}}_{A+B}^{\succcurlyeq}(X)$ .

The proposition is proved. □

### 5. Differences and relationships among Greco's rough set, the OMGRS, the PMGRS in OIS

From the above sections, we have known the concepts and properties of the OMGRS and the PMGRS. We will investigate the differences and relationships among Greco's rough set, the OMGRS, the PMGRS in an ordered information system in this section.

**PROPOSITION 5.1.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then, the following properties hold:

- (1)  $\overline{\overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}(X)}}} \subseteq \overline{\overline{R_{\cup_{i=1}^s A_i}^{\geq}(X)}}$ .
- (2)  $\overline{\overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}(X)}}} \supseteq \overline{\overline{R_{\cup_{i=1}^s A_i}^{\geq}(X)}}$ .

*Proof.* Because the number of the granulations which was discussed in the ordered information system is finite, we need to prove these properties only in the ordered information system which has two dominance relations ( $A, B \subseteq AT$ ) for convenience.

- (1) For any  $u \in \overline{\overline{\text{OM}_{A+B}^{\geq}(X)}}$ , it can be known that  $[u]_A^{\geq} \subseteq X$  or  $[u]_B^{\geq} \subseteq X$  by Definition 3.1. On the other hand, since  $A \subseteq A \cup B$  and  $B \subseteq A \cup B$ , we have  $[u]_{A \cup B}^{\geq} \subseteq [u]_A^{\geq}$  and  $[u]_{A \cup B}^{\geq} \subseteq [u]_B^{\geq}$  by Proposition 2.2. So we can obtain that  $[u]_{A \cup B}^{\geq} \subseteq X$ . That is to say that  $u \in \overline{\overline{R_{A \cup B}^{\geq}(X)}}$ . Hence,  $\overline{\overline{\text{OM}_{A+B}^{\geq}(X)}}} \subseteq \overline{\overline{R_{A \cup B}^{\geq}(X)}}$ .
- (2) For  $B, A \subseteq AT$  and  $X \subseteq U$ , we have  $\overline{\overline{R_{A \cup B}^{\geq}(X)}}} = \sim \overline{\overline{R_{A \cup B}^{\geq}(\sim X)}}$ . Then, it can be obtained that  $\overline{\overline{\text{OM}_{A+B}^{\geq}(\sim X)}}} \subseteq \overline{\overline{R_{A \cup B}^{\geq}(\sim X)}}$  by the conclusion of (1). So, one can obtain that  $\sim \overline{\overline{\text{OM}_{A+B}^{\geq}(\sim X)}}} \supseteq \sim \overline{\overline{R_{A \cup B}^{\geq}(\sim X)}}$ . Hence,  $\overline{\overline{\text{OM}_{A+B}^{\geq}(X)}}} \supseteq \overline{\overline{R_{A \cup B}^{\geq}(X)}}$ .

The proof is complete. □

**PROPOSITION 5.2.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then, the following properties hold:

- (1)  $\overline{\overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}(X)}}} \subseteq \overline{\overline{R_{\cup_{i=1}^s A_i}^{\geq}(X)}}$ .
- (2)  $\overline{\overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}(X)}}} \supseteq \overline{\overline{R_{\cup_{i=1}^s A_i}^{\geq}(X)}}$ .

*Proof.* Because the number of granulations which was discussed in the ordered information system is finite, we need to prove these properties only in the ordered information system which has two dominance relations ( $A, B \subseteq AT$ ) for convenience.

- (1) For any  $u \in \overline{\overline{\text{PM}_{A+B}^{\geq}(X)}}$ , it can be known that  $[u]_A^{\geq} \subseteq X$  and  $[u]_B^{\geq} \subseteq X$  by Definition 4.1. On the other hand, since  $A \subseteq A \cup B$  and  $B \subseteq A \cup B$ , we have  $[u]_{A \cup B}^{\geq} \subseteq [u]_A^{\geq}$  and  $[u]_{A \cup B}^{\geq} \subseteq [u]_B^{\geq}$  by Proposition 2.2. So we can obtain that  $[u]_{A \cup B}^{\geq} \subseteq X$ . That is to say that  $u \in \overline{\overline{R_{A \cup B}^{\geq}(X)}}$ . Hence,  $\overline{\overline{\text{PM}_{A+B}^{\geq}(X)}}} \subseteq \overline{\overline{R_{A \cup B}^{\geq}(X)}}$ .
- (2) For  $B, A \subseteq AT$  and  $X \subseteq U$ , we have  $\overline{\overline{R_{A \cup B}^{\geq}(X)}}} = \sim \overline{\overline{R_{A \cup B}^{\geq}(\sim X)}}$ . Then, it can be obtained that  $\overline{\overline{\text{PM}_{A+B}^{\geq}(\sim X)}}} \subseteq \overline{\overline{R_{A \cup B}^{\geq}(\sim X)}}$  by the conclusion of (1). So, one can obtain that  $\sim \overline{\overline{\text{PM}_{A+B}^{\geq}(\sim X)}}} \supseteq \sim \overline{\overline{R_{A \cup B}^{\geq}(\sim X)}}$ . Hence,  $\overline{\overline{\text{PM}_{A+B}^{\geq}(X)}}} \supseteq \overline{\overline{R_{A \cup B}^{\geq}(X)}}$ .



The proof is complete. □

PROPOSITION 5.3. Let  $\mathcal{I}^\succcurlyeq = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then, the following properties hold:

- (1)  $\overline{\text{OM}}_{\sum_i A_i}^\succcurlyeq(X) = \cup_{i=1}^s \overline{R}_{A_i}^\succcurlyeq(X)$ .
- (2)  $\overline{\text{OM}}_{\sum_i A_i}^\succcurlyeq(X) = \cap_{i=1}^s \overline{R}_{A_i}^\succcurlyeq(X)$ .

*Proof.* Because the number of granulations which was discussed in the ordered information system is finite, we need to prove these properties only in the ordered information system which has two dominance relations ( $A, B \subseteq AT$ ) for convenience.

- (1) For any  $u \in \overline{\text{OM}}_{A+B}^\succcurlyeq(X)$ , we have

$$\begin{aligned} u \in \overline{\text{OM}}_{A+B}^\succcurlyeq(X) &\Leftrightarrow [u]_A^\succcurlyeq \subseteq X \text{ or } [u]_B^\succcurlyeq \subseteq X \\ &\Leftrightarrow u \in \overline{R}_A^\succcurlyeq(X) \text{ or } u \in \overline{R}_B^\succcurlyeq(X) \\ &\Leftrightarrow u \in \overline{R}_A^\succcurlyeq(X) \cup \overline{R}_B^\succcurlyeq(X). \end{aligned}$$

Hence,  $\overline{\text{OM}}_{A+B}^\succcurlyeq(X) = \overline{R}_A^\succcurlyeq(X) \cup \overline{R}_B^\succcurlyeq(X)$ .

- (2) For any  $u \in \overline{\text{OM}}_{A+B}^\succcurlyeq(X)$ , we have

$$\begin{aligned} u \in \overline{\text{OM}}_{A+B}^\succcurlyeq(X) &\Leftrightarrow [u]_A^\succcurlyeq \cap X \neq \emptyset \text{ and } [u]_B^\succcurlyeq \cap X \neq \emptyset \\ &\Leftrightarrow u \in \overline{R}_A^\succcurlyeq(X) \text{ and } u \in \overline{R}_B^\succcurlyeq(X) \\ &\Leftrightarrow u \in \overline{R}_A^\succcurlyeq(X) \cap \overline{R}_B^\succcurlyeq(X). \end{aligned}$$

Hence,  $\overline{\text{OM}}_{A+B}^\succcurlyeq(X) = \overline{R}_A^\succcurlyeq(X) \cap \overline{R}_B^\succcurlyeq(X)$ .

The proof is complete. □

PROPOSITION 5.4. Let  $\mathcal{I}^\succcurlyeq = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then, the following properties hold:

- (1)  $\overline{\text{PM}}_{\sum_i A_i}^\succcurlyeq(X) = \cap_{i=1}^s \overline{R}_{A_i}^\succcurlyeq(X)$ .
- (2)  $\overline{\text{PM}}_{\sum_i A_i}^\succcurlyeq(X) = \cup_{i=1}^s \overline{R}_{A_i}^\succcurlyeq(X)$ .

*Proof.* Because the number of granulations which was discussed in the ordered information system is finite, we need to prove these properties only in the ordered information system which has two dominance relations ( $A, B \subseteq AT$ ) for convenience.

- (1) For any  $u \in \overline{\text{PM}}_{A+B}^\succcurlyeq(X)$ , we have

$$\begin{aligned} u \in \overline{\text{PM}}_{A+B}^\succcurlyeq(X) &\Leftrightarrow [u]_A^\succcurlyeq \subseteq X \text{ and } [u]_B^\succcurlyeq \subseteq X \\ &\Leftrightarrow u \in \overline{R}_A^\succcurlyeq(X) \text{ and } u \in \overline{R}_B^\succcurlyeq(X) \\ &\Leftrightarrow u \in \overline{R}_A^\succcurlyeq(X) \cap \overline{R}_B^\succcurlyeq(X). \end{aligned}$$

Hence,  $\overline{\text{PM}}_{A+B}^\succcurlyeq(X) = \overline{R}_A^\succcurlyeq(X) \cap \overline{R}_B^\succcurlyeq(X)$ .

(2) For any  $u \in \overline{\text{PM}_{A+B}^{\geq}}(X)$ , we have

$$\begin{aligned} u \in \overline{\text{PM}_{A+B}^{\geq}}(X) &\Leftrightarrow [u]_A^{\geq} \cap X \neq \emptyset \text{ or } [u]_B^{\geq} \cap X \neq \emptyset \\ &\Leftrightarrow u \in \overline{R_A^{\geq}}(X) \text{ or } u \in \overline{R_B^{\geq}}(X) \\ &\Leftrightarrow u \in \overline{R_A^{\geq}}(X) \cup \overline{R_B^{\geq}}(X). \end{aligned}$$

Hence,  $\overline{\text{PM}_{A+B}^{\geq}}(X) = \overline{R_A^{\geq}}(X) \cup \overline{R_B^{\geq}}(X)$ .

The proof is complete.  $\square$

**PROPOSITION 5.5.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X, Y \subseteq U$ . Then, the following properties hold:

$$\begin{aligned} (1) \quad \overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}}(X \cap Y) &= \cup_{i=1}^s \left( \overline{R_{A_i}^{\geq}}(X) \cap \overline{R_{A_i}^{\geq}}(Y) \right). \\ (2) \quad \overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}}(X \cup Y) &= \cap_{i=1}^s \left( \overline{R_{A_i}^{\geq}}(X) \cup \overline{R_{A_i}^{\geq}}(Y) \right). \end{aligned}$$

*Proof.* It can be obtained easily by Proposition 5.3.  $\square$

**PROPOSITION 5.6.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X, Y \subseteq U$ . Then, the following properties hold:

$$\begin{aligned} (1) \quad \overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}}(X \cap Y) &= \cap_{i=1}^s \left( \overline{R_{A_i}^{\geq}}(X) \cap \overline{R_{A_i}^{\geq}}(Y) \right). \\ (2) \quad \overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}}(X \cup Y) &= \cup_{i=1}^s \left( \overline{R_{A_i}^{\geq}}(X) \cup \overline{R_{A_i}^{\geq}}(Y) \right). \end{aligned}$$

*Proof.* It can be obtained directly by Proposition 5.4.  $\square$

**PROPOSITION 5.7.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then, the following properties hold:

$$\begin{aligned} (1) \quad \overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}}(X) &\subseteq \overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}}(X) \subseteq \underline{R_{\cup_{i=1}^s A_i}^{\geq}}(X), \\ (2) \quad \overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}}(X) &\supseteq \overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}}(X) \supseteq \underline{R_{\cup_{i=1}^s A_i}^{\geq}}(X). \end{aligned}$$

*Proof.* It can be obtained easily by Definitions 3.1 and 4.1 and Propositions 5.1 and 5.2.  $\square$

**PROPOSITION 5.8.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then, the following properties hold:

$$\begin{aligned} (1) \quad \overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}}(X) &\subseteq \underline{R_{A_i}^{\geq}}(X) \subseteq \overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}}(X) \subseteq \underline{R_{\cup_{i=1}^s A_i}^{\geq}}(X). \\ (2) \quad \overline{\text{PM}_{\sum_{i=1}^s A_i}^{\geq}}(X) &\supseteq \underline{R_{A_i}^{\geq}}(X) \supseteq \overline{\text{OM}_{\sum_{i=1}^s A_i}^{\geq}}(X) \supseteq \underline{R_{\cup_{i=1}^s A_i}^{\geq}}(X). \end{aligned}$$

*Proof.* It can be obtained easily by Propositions 5.3 and 5.4.  $\square$

Example 5.1. (Continued from Examples 3.1 and 4.1) In Examples 3.1 and 4.1, we have obviously obtained that

$$\underline{R}_{1 \cup 2} = \{x_2, x_5, x_6\}, \quad \bar{R}_{1 \cup 2} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9, x_{10}\},$$

and

$$\underline{PM}_{1+2}^{\geq}(X) \subseteq \underline{OM}_{1+2}^{\geq}(X) \subseteq \underline{R}_{1 \cup 2}^{\geq}(X) \subseteq X \subseteq \overline{R}_{1 \cup 2}^{\geq}(X) \subseteq \overline{OM}_{1+2}^{\geq}(X) \subseteq \overline{PM}_{1+2}^{\geq}(X).$$

### 6. Measures of the two types of MGRS models in OIS

In this section, we investigate several elementary measures in the MGRS and their properties in ordered information systems.

At first, we discuss the elementary measures of OMGRS and their properties in ordered information systems.

Uncertainty of a set (category) is due to the existence of a borderline region. The bigger the borderline region of a set is, the lower the accuracy of the set is (and vice versa). To more precisely express this idea, we introduce the accuracy measure to the OMGRS in ordered information systems as follows.

DEFINITION 6.1. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . The optimistic rough measure of  $X$  by  $\sum_{i=1}^s A_i$  is defined as

$$\rho_{\sum_{i=1}^s A_i}^O(X) = 1 - \frac{\left| \underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) \right|}{\left| \overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) \right|},$$

where  $X \neq \emptyset$ .

From the definition, one can derive the following properties.

PROPOSITION 6.1. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then

$$\rho_{A_i}(X) \geq \rho_{\sum_{i=1}^s A_i}^O(X) \geq \rho_{\cup_{i=1}^s A_i}(X).$$

*Proof.* By Proposition 5.3, we have

$$\underline{R}_{A_i}^{\geq}(X) \subseteq \underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X), \quad \overline{R}_{A_i}^{\geq}(X) \supseteq \overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X).$$

And by Proposition 5.7, we have

$$\underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq \underline{R}_{\cup_{i=1}^s A_i}^{\geq}(X), \quad \overline{OM}_{\sum_{i=1}^s A_i}^{\geq}(X) \supseteq \overline{R}_{\cup_{i=1}^s A_i}^{\geq}(X).$$

So, the following holds:

$$\frac{|R_{A_i}^{\geq}(X)|}{|R_{A_i}^{\leq}(X)|} \leq \frac{|OM_{\sum_{i=1}^s A_i}^{\geq}(X)|}{|OM_{\sum_{i=1}^s A_i}^{\leq}(X)|} \leq \frac{|R_{U_{i=1}^s A_i}^{\geq}(X)|}{|R_{U_{i=1}^s A_i}^{\leq}(X)|}.$$

Hence, by Definition 6.1, we have

$$\rho_{A_i}(X) \geq \rho_{\sum_{i=1}^s A_i}^O(X) \geq \rho_{U_{i=1}^s A_i}(X).$$

The proof is complete. □

*Example 6.1.* (Continued from Example 3.1) Computing the optimistic rough measures of  $X = \{x_1, x_2, x_3, x_5, x_6, x_8, x_{10}\}$  by using the results in Example 3.1, it follows that

$$\rho_1(X) = 1 - \frac{|R_1^{\geq}(X)|}{|R_1^{\leq}(X)|} = \frac{8}{9}, \quad \rho_2(X) = 1 - \frac{|R_2^{\geq}(X)|}{|R_2^{\leq}(X)|} = \frac{3}{4},$$

$$\rho_{1U_2}(X) = 1 - \frac{|R_{1U_2}^{\geq}(X)|}{|R_{1U_2}^{\leq}(X)|} = \frac{5}{8}, \quad \rho_{1+2}^O(X) = 1 - \frac{|OM_{1+2}^{\geq}(X)|}{|OM_{1+2}^{\leq}(X)|} = \frac{5}{8}.$$

Clearly, it follows from the earlier computation that

$$\rho_1(X) \geq \rho_{1+2}^O(X) \geq \rho_{1U_2}(X),$$

and

$$\rho_2(X) \geq \rho_{1+2}^O(X) \geq \rho_{1U_2}(X).$$

Note that the rough measure of a target concept defined by using multiple granulations is always much better than that defined by using a single granulation, which is suitable for more precisely characterizing a target concept and problem solving according to user requirements in ordered information systems.

**DEFINITION 6.2.** Let  $\mathcal{T}^{\geq} = (U, C \cup \{d\}, V, f)$  be an ordered decision table,  $A_i \subseteq C$ ,  $i = 1, 2, \dots, s$ , and  $\{D_1, D_2, \dots, D_k\}$  be all decision classes induced by the decision attribute  $d$ . Approximation quality of  $d$  by  $\sum_{i=1}^s A_i$ , called the optimistic degree of dependence, is defined as

$$\gamma_O\left(\sum_{i=1}^s A_i, d\right) = \frac{1}{|U|} \sum_{j=1}^k \left( \frac{|OM_{\sum_{i=1}^s A_i}^{\geq}(D_j)|}{|D_j|} \right).$$

This measure can be used to evaluate the deterministic part of the rough set description of  $U/d$  by counting those objects which can be reclassified as blocks of  $U/d$  with the knowledge given by  $\sum_{i=1}^s A_i$ . Moreover, we have the following properties with respect to the above definition.

PROPOSITION 6.2. Let  $\mathcal{T}^{\geq} = (U, C \cup \{d\}, V, f)$  be an ordered decision table,  $A_i \subseteq C$ ,  $i = 1, 2, \dots, s$ , and  $\{D_1, D_2, \dots, D_k\}$  be all decision classes induced by the decision attribute  $d$ . Then

$$\gamma(A_i, d) \leq \gamma_o\left(\sum_{i=1}^s A_i, d\right) \leq \gamma\left(\bigcup_{i=1}^s A_i, d\right).$$

*Proof.* For every  $D_j, j = 1, 2, \dots, k$ , by Propositions 5.3 and 5.7, we have

$$\underline{R}_{A_i}^{\geq}(D_j) \subseteq \underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(D_j) \subseteq \underline{R}_{\bigcup_{i=1}^s A_i}^{\geq}(D_j).$$

So,

$$\left| \underline{R}_{A_i}^{\geq}(D_j) \right| \leq \left| \underline{OM}_{\sum_{i=1}^s A_i}^{\geq}(D_j) \right| \leq \left| \underline{R}_{\bigcup_{i=1}^s A_i}^{\geq}(D_j) \right|.$$

Hence, by Definition 6.2, we have

$$\gamma(A_i, d) \leq \gamma_o\left(\sum_{i=1}^s A_i, d\right) \leq \gamma\left(\bigcup_{i=1}^s A_i, d\right).$$

The proof is completed. □

*Example 6.2.* (Continued from Example 3.1) Computing the degree of dependence by using the single granulation and multiple granulations.

From Table 1, we can have  $U/D = \{D_Y, D_N\}$  and

$$D_Y = \{x_1, x_2, x_3, x_5, x_6, x_8, x_{10}\}, \quad D_N = \{x_4, x_7, x_9\}.$$

Moreover, the following can be computed by Table 1 and the results of Example 3.1:

$$\begin{aligned} \underline{R}_1^{\geq}(D_Y) &= \{x_5\}, & \underline{R}_2^{\geq}(D_Y) &= \{x_2, x_6\}, & \underline{R}_{1 \cup 2}^{\geq}(D_Y) &= \{x_2, x_5, x_6\}, \\ \underline{OM}_{1+2}^{\geq}(D_Y) &= \{x_2, x_5, x_6\}, & \underline{R}_1^{\geq}(D_N) &= \{x_7\}, & \underline{R}_2^{\geq}(D_N) &= \{x_4, x_7\}, \\ \underline{R}_{1 \cup 2}^{\geq}(D_N) &= \{x_4, x_7\}, & \underline{OM}_{1+2}^{\geq}(D_N) &= \{x_4, x_7\}. \end{aligned}$$

So, we have

$$\begin{aligned} \gamma(1, D) &= \frac{1}{|U|} \left( \left| \underline{R}_1^{\geq}(D_Y) \right| + \left| \underline{R}_1^{\geq}(D_N) \right| \right) = \frac{1}{5}, \\ \gamma(2, D) &= \frac{1}{|U|} \left( \left| \underline{R}_2^{\geq}(D_Y) \right| + \left| \underline{R}_2^{\geq}(D_N) \right| \right) = \frac{2}{5}, \\ \gamma(1 \cup 2, D) &= \frac{1}{|U|} \left( \left| \underline{R}_{1 \cup 2}^{\geq}(D_Y) \right| + \left| \underline{R}_{1 \cup 2}^{\geq}(D_N) \right| \right) = \frac{1}{2}, \\ \gamma_{o(1+2), D} &= \frac{1}{|U|} \left( \left| \underline{OM}_{1+2}^{\geq}(D_Y) \right| + \left| \underline{OM}_{1+2}^{\geq}(D_N) \right| \right) = \frac{1}{2}. \end{aligned}$$

Hence, it can be found that

$$\gamma(1, D) \leq \gamma_{o(1+2), D} \leq \gamma(1 \cup 2, D),$$

and

$$\gamma(2, D) \leq \gamma_{o(1+2), D} \leq \gamma(1 \cup 2, D).$$

Next, we will investigate several elementary measures in the PMGRS and their properties in ordered information systems.

Similarly, we introduce the accuracy measure to the PMGRS in ordered information systems as follows.

**DEFINITION 6.3.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . The pessimistic rough measure of  $X$  by  $\sum_{i=1}^s A_i$  is defined as

$$\rho_{\sum_{i=1}^s A_i}^p(X) = 1 - \frac{\left| \underline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \right|}{\left| \overline{PM}_{\sum_{i=1}^s A_i}^{\geq}(X) \right|},$$

where  $X \neq \emptyset$ .

From the definitions, one can derive the following properties.

**PROPOSITION 6.3.** Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then

$$\rho_{\sum_{i=1}^s A_i}^p(X) \geq \rho_{A_i}(X) \geq \rho_{\cup_{i=1}^s A_i}(X).$$

*Proof.* By Proposition 5.4, we have

$$\underline{\underline{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \subseteq \underline{R}_{A_i}^{\geq}(X), \quad \overline{\overline{PM}}_{\sum_{i=1}^s A_i}^{\geq}(X) \supseteq \overline{R}_{A_i}^{\geq}(X).$$

And, we have known

$$\underline{R}_{A_i}^{\succcurlyeq}(X) \subseteq \underline{R}_{\bigcup_{i=1}^s A_i}^{\succcurlyeq}(X), \quad \overline{R}_{A_i}^{\succcurlyeq}(X) \supseteq \overline{R}_{\bigcup_{i=1}^s A_i}^{\succcurlyeq}(X).$$

So, the following holds:

$$\frac{\left| \underline{PM}_{\sum_{i=1}^s A_i}^{\succcurlyeq}(X) \right|}{\left| \overline{PM}_{\sum_{i=1}^s A_i}^{\succcurlyeq}(X) \right|} \leq \frac{\left| \underline{R}_{A_i}^{\succcurlyeq}(X) \right|}{\left| \overline{R}_{A_i}^{\succcurlyeq}(X) \right|} \leq \frac{\left| \underline{R}_{\bigcup_{i=1}^s A_i}^{\succcurlyeq}(X) \right|}{\left| \overline{R}_{\bigcup_{i=1}^s A_i}^{\succcurlyeq}(X) \right|}.$$

Hence, by Definition 6.3, we have

$$\rho_{\sum_{i=1}^s A_i}^P(X) \geq \rho_{A_i}(X) \geq \rho_{\bigcup_{i=1}^s A_i}(X).$$

The proof is complete. □

*Example 6.3.* (Continued from Examples 3.1 and 4.1) Computing the pessimistic rough measures of  $X = \{x_1, x_2, x_3, x_5, x_6, x_8, x_{10}\}$  in the system given in Example 3.1. By Example 4.1, it follows that

$$\rho_{1+2}^P(X) = 1 - \frac{\left| \underline{PM}_{1+2}^{\succcurlyeq}(X) \right|}{\left| \overline{PM}_{1+2}^{\succcurlyeq}(X) \right|} = 1.$$

Clearly, it follows from the earlier computation that

$$\rho_{1+2}^P(X) \geq \rho_1(X) \geq \rho_{1 \cup 2}(X)$$

and

$$\rho_{1+2}^P(X) \geq \rho_2(X) \geq \rho_{1 \cup 2}(X).$$

Similarly to the OMGRS, in the following, we will discuss the pessimistic degree of dependence.

**DEFINITION 6.4.** Let  $\mathcal{I}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$  be an ordered decision table,  $A_i \subseteq C$ ,  $i = 1, 2, \dots, s$ , and  $\{D_1, D_2, \dots, D_k\}$  be all decision classes induced by decision attribute  $d$ . Approximation quality of  $d$  by  $\sum_{i=1}^s A_i$ , called the pessimistic degree of dependence, is defined as

$$\gamma_P \left( \sum_{i=1}^s A_i, d \right) = \frac{1}{|U|} \sum_{j=1}^k \left( \left| \underline{PM}_{\sum_{i=1}^s A_i}^{\succcurlyeq}(D_j) \right| \right).$$

Moreover, we have the following properties with respect to the above definition.

**PROPOSITION 6.4.** Let  $\mathcal{I}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$  be an ordered decision table,  $A_i \subseteq C$ ,  $i = 1, 2, \dots, s$ , and  $\{D_1, D_2, \dots, D_k\}$  be all decision classes induced by decision attribute



d. Then

$$\gamma_P \left( \sum_{i=1}^s A_i, d \right) \leq \gamma(A_i, d) \leq \gamma \left( \bigcup_{i=1}^s A_i, d \right).$$

*Proof.* For every  $D_j, j = 1, 2, \dots, k$ , by Propositions 5.3 and 5.7, we have

$$\underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(D_j) \subseteq \underline{R}_{A_i}^{\geq}(D_j) \subseteq \underline{R}_{\bigcup_{i=1}^s A_i}^{\geq}(D_j).$$

So,

$$\left| \underline{\text{PM}}_{\sum_{i=1}^s A_i}^{\geq}(D_j) \right| \leq \left| \underline{R}_{A_i}^{\geq}(D_j) \right| \leq \left| \underline{R}_{\bigcup_{i=1}^s A_i}^{\geq}(D_j) \right|.$$

Hence, by Definition 6.4, we have

$$\gamma_P \left( \sum_{i=1}^s A_i, d \right) \leq \gamma(A_i, d) \leq \gamma \left( \bigcup_{i=1}^s A_i, d \right).$$

The proof is complete. □

*Example 6.4.* (Continued from Examples 3.2 and 4.1) Computing the pessimistic degree of dependence in the ordered system given in Table 1.

In Example 3.2, we have known that  $U/D = \{D_Y, D_N\}$  and

$$\gamma(1, D) = \frac{1}{5}, \quad \gamma(2, D) = \frac{2}{5}, \quad \gamma(1 \cup 2, D) = \frac{1}{2}.$$

Moreover, the following can be computed by Table 1 and the results of Example 4.1:

$$\underline{\text{PM}}_{1+2}^{\geq}(D_Y) = \emptyset, \quad \underline{\text{PM}}_{1+2}^{\geq}(D_N) = \{x_7\}.$$

So, we have

$$\gamma_P(1 + 2, D) = \frac{1}{|U|} \left( \left| \underline{\text{PM}}_{1+2}^{\geq}(D_Y) \right| + \left| \underline{\text{PM}}_{1+2}^{\geq}(D_N) \right| \right) = \frac{1}{10}.$$

Hence, it can be found that

$$\gamma_P(1 + 2, D) \leq \gamma(1, D) \leq \gamma(1 \cup 2, D)$$

and

$$\gamma_P(1 + 2, D) \leq \gamma(2, D) \leq \gamma(1 \cup 2, D).$$

PROPOSITION 6.5. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then

$$\rho_{\sum_{i=1}^s A_i}^P(X) \geq \rho_{A_i}(X) \geq \rho_{\sum_{i=1}^s A_i}^O(X) \geq \rho_{\sum_{i=1}^s A_i}(X).$$

*Proof.* It can be obtained directly by Definitions 6.1 and 6.3 and Proposition 5.8.  $\square$

PROPOSITION 6.6. Let  $\mathcal{I}^{\geq} = (U, AT, V, f)$  be an ordered information system,  $A_i \subseteq AT$ ,  $i = 1, 2, \dots, s$ , and  $X \subseteq U$ . Then

$$\gamma_P\left(\sum_{i=1}^s A_i, d\right) \leq \gamma(A_i, d) \leq \gamma_O\left(\sum_{i=1}^s A_i, d\right) \leq \gamma\left(\bigcup_{i=1}^s A_i, d\right).$$

*Proof.* It can be obtained directly by Definitions 6.2 and 6.4 and Proposition 5.8.  $\square$

## 7. Conclusion

The original dominance-based rough set model cannot be used to deal with ordered information systems with complicated context. Nevertheless, by relaxing the indiscernibility relation to more general binary relations, many improved rough set models have been successfully applied in the information systems with complicated context for knowledge acquisition. The contribution of this paper is to extend Greco's single granulation dominance-based rough set model to two new types of MGRS models in an ordered information system. In this paper, two new types of MGRS models have been constructed, respectively, based on multiple dominant relations for an ordered information system. In the two new types of MGRS models, a target concept was approximated from two different kinds of views by using the dominant classes induced from multiple granulations. In particular, some important properties of the two types of MGRS models were investigated and the relationships and differences among Greco's rough set and two new types of MGRS models in an ordered information system were shown. Moreover, several important measures have been developed in two types of MGRS models, such as rough measure and quality of approximation. From the contribution, it can be found that when two attribute sets in ordered information systems possess a contradiction or inconsistent relationship, or where efficient computation is required, the two new types of MGRS models will display its advantage for rule extraction and knowledge discovery in an ordered information system.

In our further research, we will extend other rough set methods in the context of multiple granulations such as viable precision rough set model, rough set induced by covering, and so on.

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