Fuzzy Rough Set Based on Dominance Relations

Zhang Xiaoyan and Xu Weihua

Abstract This model for fuzzy rough sets is one of the most important parts in rough set theory. Moreover, it is based on an equivalence relation (indiscernibility relation). However, many systems are not only concerned with fuzzy sets, but also based on a dominance relation because of various factors in practice. To acquire knowledge from the systems, construction of model for fuzzy rough sets based on dominance relations is very necessary. The main aim to this paper is to study this issue. Concepts of the lower and the upper approximations of fuzzy rough sets based on dominance relations are proposed. Furthermore, model for fuzzy rough sets based on dominance relations is constructed, and some properties are discussed.

1 Introduction

The rough set theory [10,11], proposed by Pawlak in the early 1980s, is an extension of set theory for the study of intelligent systems. It can serve as a new mathematical tool to soft computing, and deal with inexact, uncertain or vague information. Moreover, this theory has been applied successfully in discovering hidden patterns in data, recognizing partial or total dependencies in systems, removing redundant knowledge, and many others [7,12,13,15]. Since its introduction, the theory has received wide attention on the research areas in both of the real-life applications and the theory itself.

Theory of fuzzy sets initiated by Zedeh [9] also provides useful ways of describing and modeling vagueness in ill-defined environment. Naturally, Doubois and Prade [8] combined fuzzy sets and rough sets. Attempts to combine these two

Zhang Xiaoyan
School of Mathematics and Statistics, Chongqing University of Technology,
Chongqing 400054, P.R. China
e-mail: zhangxyms@gmail.com

Xu Weihua
School of Management, Xi'an Jiaotong University, P.R. China
e-mail: chxuwh@gmail.com

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theories lead to some new notions [1,5,7], and some progresses were made [2,3,4,5,6,14]. The combination involves many types of approximations and the construction of fuzzy rough sets give a good model for solving this problem [5]. However, most of systems are not only concerned with fuzzy data, but also based on a dominance relation because of various factors. In order to obtain the succinct knowledge from the systems, construction of model for fuzzy rough sets based on dominance relations is needed.

The main aim of the paper is to discuss the issue. In present paper, a dominance relation is introduced and instead of the equivalence relation (discernibility relation) in the standard fuzzy rough set theory. The lower and the upper approximation of a fuzzy rough set based on dominance relations are proposed. Thus a model for fuzzy rough sets based on dominance relations is constructed, and some properties are studied. Finally, we conclude the paper and look ahead the further research.

2 Preliminaries

This section recalls necessary some concepts used in the paper. Detailed description can be found in [15].

Definition 2.1. Let U be a set called universe and let R be an equivalence relation (indiscernibility) on U. The pair $S=(U,R)$ is called a Pawlak approximation space. Then for any non-empty subset X of U, the sets $\underline{R}(X)=\{x\in U: [x]_R\subseteq X\}$ and $\overline{R}(X)=\{x\in U: [x]_R\cap X\neq\varnothing\}$ are respectively, called the lower and the upper approximations of X in S, where $[x]_R$ denotes the equivalence class of the relation R containing the element x.

Then X is said to be definable set, if $\underline{R}(X)=\overline{R}(X)$. Otherwise X is said to be rough set. Pawlak approximation space $S=(U,R)$ derives mainly from an equivalence relation. However, there exist a great of systems based on dominance relations in practice.

Definition 2.2. If we denote $R_B^{\leq}=\{(x_i,x_j)\in U\times U: f_l(x_i)\leq f_l(x_j), \forall a_l\in B\}$ where B is a subset of attributes set, and $f_l(x)$ is the value of attribute a_l, then $R_B^{\leq}$ is referred to as dominance relation of information system S. Moreover, we denote approximation space based on dominance relations by $S^{\leq}=(U,R^{\leq})$.

For any non-empty subset X of U, denote $\underline{R^{\leq}}(X)=\{x\in U: [x]_{R^{\leq}}\subseteq X\}$ and $\overline{R^{\leq}}(X)=\{x\in U: [x]_{R^{\leq}}\cap X\neq\varnothing\}$ $\underline{R^{\leq}}$ and $\overline{R^{\leq}}$ are respectively said to be the lower and the upper approximations of X with respect to a dominance relation $R^{\leq}$.

If we denote $[x_i]_{B^{\leq}}=\{x_j\in U:(x_i,x_j)\in R_B^{\leq}\}=\{x_j\in U: f_l(x_i)\leq f_l(x_j), \forall a_l\in B\}$ then the following properties of a dominance relation are trivial.

Proposition 2.1. Let $R_B^{\leq}$ be a dominance relation.

(1) $R_B^{\leq}$ is reflexive and transitive, but not symmetric, so it isn't an equivalence relation generally.

(2) If $B_1 \subseteq B_2 \subseteq A$, then $R_A^{\leq} \subseteq R_{B_2}^{\leq} \subseteq R_{B_1}^{\leq}$.

(3) If $B_1 \subseteq B_2 \subseteq A$, then $[x_i]_A^{\leq} \subseteq [x_i]_{B_2}^{\leq} \subseteq [x_i]_{B_1}^{\leq}$.

(4) If $x_j \in [x_i]_B^{\leq}$, then $[x_j]_B^{\leq} \subseteq [x_i]_B^{\leq}$.

Next, we will review some notions of fuzzy sets. The notion of fuzzy sets provides a convenient tool for representing vague concepts by allowing partial membership. In fuzzy systems, a fuzzy set can be defined using standard set operators.

Definition 2.3. Let U be a finite and non-empty set called universe. A fuzzy set A of U is defined by a membership function $A: U \to [0,1]$ .The membership value may be interpreted in term of the membership degree. In generally, let $F(U)$ denote the set of all fuzzy sets, i.e., the set of all functions from U to [0, 1].

From above description, we can find that a crisp set can be regarded as a generated fuzzy set in which the membership is restricted to the extreme points $\{0,1\}$ of $[0,1]$.

Definition 2.4. Let $A, B \in F(U)$. For any $x \in U$, if $A(x) \leq B(X)$ is true, then we say that B contain A or A is contained by B, which denoted by $A \subseteq B$.

If both $A \subseteq B$ and $B \subseteq A$ are all true, then we say A is equal to B, denoted by $A = B$. Empty set $\varnothing$ denotes the fuzzy set whose membership function is 0, and set U denotes the fuzzy set whose membership function is 1.

Let denote intersection, union of A and B by $A \cap B$, $A \cup B$ respectively. Moreover, denote complement of A by $\sim A$ or A^C. Membership functions of these fuzzy sets are defined as

$$A \cap B = A(x) \wedge B(x) \quad = \min\{A(x), B(x)\};$$
$$A \cup B = A(x) \vee B(x) \quad = \max\{A(x), B(x)\};$$
$$\sim A = A^C = 1 - A(x).$$

There many properties of these operators in fuzzy sets which similar with crisp sets. Detailed description can be found easily.

Definition 2.5. The λ-level set or λ-cut, denoted by A_λ, A_λ of a fuzzy set A in U comprise all elements of U whose degree membership in A are all greater than or equal to λ, where $0 < \lambda \leq 1$. In other words, $A_\lambda = \{x \in U : A(x) \geq \lambda\}$ is a non-fuzzy set, and be called the λ-level set or λ-cut. Moreover, the set $\{x \in U : A(x) > 0\}$ is defined the supports of fuzzy set A, and denoted by supp A.

3 Fuzzy Rough Sets Based on Dominance Relations

Model for fuzzy rough sets is generalized by the standard Pawlak approximation space, and it is concerned with fuzzy sets on universe U. But the model is still depended on an equivalence relation. In practice, most of systems are not only related to fuzzy sets, but also based on dominance relations. In order to deal with this problem, the model of fuzzy rough sets based on dominance relations is proposed.

Definition 3.1. Let $S^{\leq}=(U,R^{\leq})$ be an approximation space based on dominance relation $R^{\leq}$. For a fuzzy set A of U, the lower and the upper approximation of A denoted by $\underline{R^{\leq}}(A)$ and $\overline{R^{\leq}}(A)$, are defined respectively, by two fuzzy sets, whose membership functions are $\underline{R^{\leq}}(A)(x)=\min\{A(y): y\in[x]_{R^{\leq}}\}, x\in U$ and $\overline{R^{\leq}}(A)(x)= \max\{A(y): y\in[x]_{R^{\leq}}\}, x\in U$ where $[x]_{R^{\leq}}$ denotes the dominance class of the relation $R^{\leq}$.

Fuzzy set A is called fuzzy definable set, if $\underline{R^{\leq}}=\overline{R^{\leq}}$. Otherwise, A is called fuzzy rough set. $\underline{R^{\leq}}$ is called positive field of A in $S^{\leq}=(U,R^{\leq})$, and $\sim\overline{R^{\leq}}$ is called negative field of A in $S^{\leq}=(U,R^{\leq})$. In addition, $\overline{R^{\leq}}(A)\cap(\sim\underline{R^{\leq}}(A))$ is called boundary of A in $S^{\leq}=(U,R^{\leq})$.

It can be easily verified that $\underline{R^{\leq}}(A)$ and $\overline{R^{\leq}}(A)$ will become the lower and the upper approximation of standard approximation space based on dominance relation, when A is a crisp set.

Theorem 3.1. Let $A\in F(U), \underline{R^{\leq}}(A)$ and $\overline{R^{\leq}}(A)$ be the lower and the upper approximation of A respectively. The following always hold.

(1) $\underline{R^{\leq}}(A)\subseteq A\subseteq\overline{R^{\leq}}(A)$.

(2) $\overline{R^{\leq}}(A\cup B)=\overline{R^{\leq}}(A)\cup\overline{R^{\leq}}(B)$; $\underline{R^{\leq}}(A\cap B)=\underline{R^{\leq}}(A)\cap\underline{R^{\leq}}(B)$.

(3) $\underline{R^{\leq}}(A)\cup\underline{R^{\leq}}(B)\subseteq\underline{R^{\leq}}(A\cup B)$; $\overline{R^{\leq}}(A\cap B)\subseteq\overline{R^{\leq}}(A)\cap\overline{R^{\leq}}(B)$.

(4) $\underline{R^{\leq}}(\sim A)=\sim\overline{R^{\leq}}(A); \overline{R^{\leq}}(\sim A)=\sim\underline{R^{\leq}}(A)$;

(5) $\underline{R^{\leq}}(U)=U; \overline{R^{\leq}}(\varnothing)=\varnothing$.

(6) $\underline{R^{\leq}}(A)\subseteq\underline{R^{\leq}}(\underline{R^{\leq}}(A))$; $\overline{R^{\leq}}(\overline{R^{\leq}}(A))\subseteq\overline{R^{\leq}}(A)$.

(7) If $A\subseteq B$, then $\underline{R^{\leq}}(A)\subseteq\underline{R^{\leq}}(B)$ and $\overline{R^{\leq}}(A)\subseteq\overline{R^{\leq}}(B)$.

Proof. These items are obvious from definitions.

Definition 3.2. Let $S^{\leq}=(U,R^{\leq})$ be an approximation space based on dominance relation $R^{\leq}$. For $A\in F(U)$, the lower and the upper approximation with respect to parameters $\alpha,\beta(0<\alpha\leq\beta\leq 1)$, denoted by $\underline{R^{\leq}}(A)_{\alpha}$ and $\overline{R^{\leq}}(A)_{\beta}$, are defined by $\underline{R^{\leq}}(A)_{\alpha}=\{x\in U:\underline{R^{\leq}}(A)(x)\geq\alpha\}$; $\overline{R^{\leq}}(A)_{\beta}=\{x\in U:\overline{R^{\leq}}(A)(x)\geq\beta\}$.

From above definition, we can know that $\underline{R^{\leq}}(A)_{\alpha}$ is the crisp set of some elements of U whose degree of membership in A certainly are not less than α, $\overline{R^{\leq}}(A)_{\beta}$ is the crisp set of some elements of U whose degree of membership in A possibly are not less than β.

Remark. For $\forall\alpha,\beta\in[0,1]$,above definition $\underline{R^{\leq}}(A)_{\alpha}$ and $\overline{R^{\leq}}(A)_{\beta}$ will become $\underline{R^{\leq}}(A)$ and $\overline{R^{\leq}}(A)$ respectively, when A is crisp set.

In fact $\forall\alpha,\beta\in[0,1]$, since A is a crisp set, thus $A(x)\in[0,1]$. So we have

$$\begin{aligned}\underline{R^{\leq}}(A)_{\alpha}&=\{x\in U:\underline{R^{\leq}}(A)(x)\geq\alpha\}\\&=\{x\in U:\underline{R^{\leq}}(A)(x)=1\}\\&=\{x\in U:\underline{R^{\leq}}(A)(y)=1,\forall y\in[x]_{R^{\leq}}\}=\underline{R^{\leq}}(A)\end{aligned}$$

Similarly, we can show that $\overline{R^{\leq}}(A)_{\beta}=\overline{R^{\leq}}(A)$, when A is crisp set.

Theorem 3.2. Let $S^{\leq}=(U,R^{\leq})$ be an approximation space based on dominance relation $R^{\leq}$, and $A,B\in F(U)$. For $\alpha,\beta(0<\alpha\leq\beta\leq1)$, we have

(1) $\overline{R^{\leq}}(A\cup B)_{\beta}=\overline{R^{\leq}}(A)_{\beta}\cup\overline{R^{\leq}}(B)_{\beta}$, $\underline{R^{\leq}}(A\cap B)_{\alpha}=\underline{R^{\leq}}(A)_{\alpha}\cap\underline{R^{\leq}}(B)_{\alpha}$.

(2) $\underline{R^{\leq}}(A)_{\alpha}\cup\underline{R^{\leq}}(B)_{\alpha}\subseteq\underline{R^{\leq}}(A\cup B)_{\alpha}$, $\overline{R^{\leq}}(A\cap B)_{\beta}\subseteq\overline{R^{\leq}}(A)_{\beta}\cap\overline{R^{\leq}}(B)_{\beta}$.

(3) If $A\subseteq B$, then $\overline{R^{\leq}}(A)_{\beta}\subseteq\overline{R^{\leq}}(B)_{\beta}$, $\underline{R^{\leq}}(A)_{\alpha}\subseteq\underline{R^{\leq}}(B)_{\alpha}$.

(4) $\underline{R^{\leq}}(A)_{\alpha}\subseteq\overline{R^{\leq}}(B)_{\beta}$.

(5) $\overline{R^{\leq}}(\sim A)_{\beta}=\sim\underline{R^{\leq}}(A)_{1-\beta}$, $\underline{R^{\leq}}(\sim A)_{\alpha}=\sim\overline{R^{\leq}}(A)_{1-\alpha}$.

Proof. It can be achieved obviously by definitions and Theorem 3.1.

Definition 3.3. Let $S^{\leq}=(U,R^{\leq})$ be an approximation space based dominance relation $R^{\leq}$, and $A\in F(U)$. Roughness measure of A in $S^{\leq}$, denoted by $\rho_{R^{\leq}}(A)$, is be defined as

$$\rho_{R^{\leq}}(A)=1-\frac{|\underline{R^{\leq}}(A)|}{|\overline{R^{\leq}}(A)|}$$

when $|\overline{R^{\leq}}(A)|=0$, $\rho_{R^{\leq}}(A)=0$ is ordered.

Clearly, there is $0\leq\rho_{R^{\leq}}(A)\leq1$. Moreover, $\rho_{R^{\leq}}(A)=0$, if A is definable fuzzy set.

Definition 3.4. Let $S^{\leq}=(U,R^{\leq})$ be an approximation space based dominance relation $R^{\leq}$, and $A\in F(U)$. Roughness measure with respect to parameters $\alpha,\beta(0<\alpha\leq\beta\leq1)$ of A denoted by $\rho_{R^{\leq}}^{\alpha,\beta}(A)$, is be defined as

$$\rho_{R^{\leq}}^{\alpha,\beta}(A)=1-\frac{|\underline{R^{\leq}}(A)_{\alpha}|}{|\overline{R^{\leq}}(A)_{\beta}|}$$

when $|\overline{R^{\leq}}(A)_{\beta}|=0$, $\rho_{R^{\leq}}^{\alpha,\beta}(A)=0$ is ordered.

From the definition, we have easily following properties.

Theorem 3.3. (1) $0\leq\rho_{R^{\leq}}^{\alpha,\beta}(A)\leq 1$.

(2) If β is fixed, then we have that $|\underline{R^{\leq}}(A)_{\alpha}|$ increases. Thus, $\rho_{R^{\leq}}^{\alpha,\beta}(A)$ is to increase, when α increases.

(3) If α is fixed, then we have that $|\overline{R^{\leq}}(A)_{\beta}|$ is to decrease, when β increases. Thus, $\rho_{R^{\leq}}^{\alpha,\beta}(A)$ is to increase, when β increases.

Theorem 3.4. If membership function of fuzzy set A is a constant, i.e., there exists $\delta>0$ such that $A(x)=\delta$ for any $x\in U$, then we have $\rho_{R^{\leq}}^{\alpha,\beta}(A)=1$ when α,β $(0<\alpha\leq\ \beta\leq 1)$. Otherwise, $\rho_{R^{\leq}}^{\alpha,\beta}(A)=0$.

Proof. When $\alpha,\beta(0<\alpha\leq\beta\leq 1)$, it is clear that $|\underline{R^{\leq}}(A)_{\alpha}|=\varnothing$, $|\overline{R^{\leq}}(A)_{\beta}|=1$. So $\rho_{R^{\leq}}^{\alpha,\beta}(A)=1$.

Otherwise, there exist two cases.

Case 1. When $\delta<\beta\leq\alpha$, we can know $|\underline{R^{\leq}}(A)_{\alpha}|=|\overline{R^{\leq}}(A)_{\beta}|=\varnothing$. So $\rho_{R^{\leq}}^{\alpha,\beta}(A)=0$ is true.

Case 2. When $\beta\leq\alpha\leq\delta$, we can know $|\underline{R^{\leq}}(A)_{\alpha}|=|\overline{R^{\leq}}(A)_{\beta}|=U$. So $\rho_{R^{\leq}}^{\alpha,\beta}(A)=0$ is true from the definition.

4 Conclusions

It is well know that there exist most of systems, which are not only concerned with fuzzy sets but also based on dominance relations in practice. Therefore, it is meaningful to study the fuzzy rough set based on dominance relations. In this paper, we discussed this problem mainly. We introduced concepts of the lower and the upper approximations of fuzzy rough sets based on dominance relations, and constructed the model for fuzzy rough sets based on dominance relation additionally. Furthermore some properties are obtained. In next work, we will consider the information systems which are based on dominance relations and with fuzzy decisions.

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