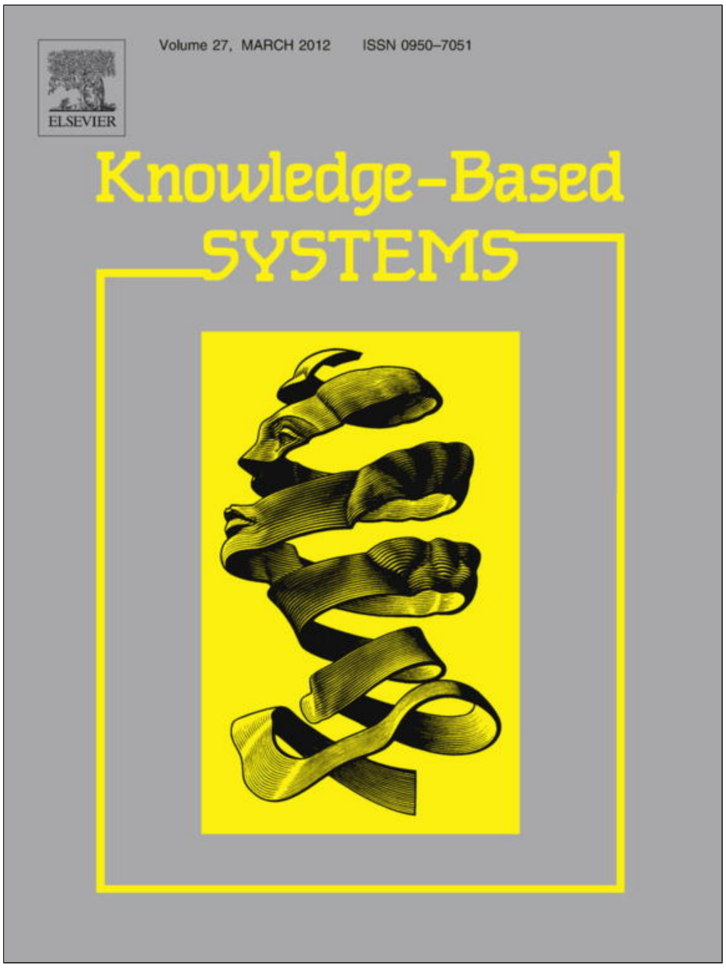
Volume 27, MARCH 2012 ISSN 0950-7051
ELSEVIER
Knowledge-Based
SYSTEMS

Knowledge-Based Systems 27 (2012) 78–91

Contents lists available at SciVerse ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

# Approaches to attribute reductions based on rough set and matrix computation in inconsistent ordered information systems

Weihua Xu [a,b,c,*], Yuan Li [a,b], Xiuwu Liao [a,b]

[a] School of Management, Xi'an Jiaotong University, Xi'an 710049, PR China
[b] The Key Lab of the Ministry of Education for Process Control and Efficiency Engineering, Xi'an 710049, PR China
[c] School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, PR China

ARTICLE INFO

*Article history:*
Received 24 April 2010
Received in revised form 9 November 2011
Accepted 9 November 2011
Available online 20 November 2011

*Keywords:*
Compatible distribution reduction
Matrix computation
Ordered information system
Possible distribution reduction
Rough set

ABSTRACT

In order to conduct classification analysis in inconsistent ordered information systems, notions on possible and compatible distribution reductions are proposed in this paper. The judgement theorems and discernibility matrices associated with the two reductions are examined, from which we can obtain an approach to the two reductions in rough set theory. Furthermore, the dominance matrix, possible and compatible decision distribution matrices are also considered for approach to these two forms of reductions in inconsistent ordered information systems. Algorithms of matrix computation for possible and compatible distribution reductions are constructed, by which we can provide another efficient approach to these two forms of distribution reductions. To interpret and help understand the algorithm, an experimental computing program is designed and two cases are employed as case study. Results of the small-scale case are calculated and compared by the discernibility matrix and the matrix computation to verify the new method we study in this paper. The large-scale case are calculated by the experimental computing program and validated by the definition of the reductions.



## 1. Introduction

Rough set theory, which was first proposed by Pawlak in the early 1980s [20], can describe knowledge via set-theoretic analysis based on equivalence classification to the universe of discourse. It provides a theoretical foundation for inference about data analysis and has extensive applications in areas of artificial intelligence and knowledge acquisition.

A primary use of rough set theory is to process the concepts with vagueness and uncertainty. The attribute reduction is one of the much important research spotlights in database and information systems. For a data set with discrete attribute values, the course of attribute reduction can be done by reducing the number of redundant attributes and find a subset of the original attributes that are the most informative. As is well known that an information system may usually have more than one reduct. This means the set of rules derives from knowledge reduction is not unique. In practice, it is always expected to obtain the set of the most concise rules. Therefore, people have been attempting to find the minimal reduct of information systems, which means that the number of attributes contained in the reduction is minimal. Unfortunately, it has been proved that finding the minimal reduct of an information system is a NP-hard problem [11].

Recently, some new theories and reduction methods have been developed. Many types of knowledge reduction have been proposed in the area of rough sets [1,4,13,14,18,20,23,24,29,30,35]. Possible rules and possible reducts have been proposed as a means to deal with inconsistence in an inconsistent decision table [15,16]. Approximation rules [28] are also used as an alternative to possible rules. On the other hand, generalized decision rules and generalized decision reducts [12,16] provide a decision maker with more flexible selection of decision behavior. In [14], the notions of $\alpha$-reduct and $\alpha$-relative reduct for decision tables are defined. The $\alpha$-reduct allows occurrence of additional inconsistency that is controlled by means of a parameter. In [28], Slezak presented a new concept of attribute reduction that keeps the class membership distribution unchanging for all objects in the information system. It was shown by Slezak [27] that the knowledge reduction preserving the membership distribution is equivalent to the knowledge reduction preserving the value of generalized inference measure function. A generalized knowledge reduction was also introduced in [27] that allows the value of generalized inference measure function after the attribute reduction to be different from the original one by user-specified threshold. By eliminating the rigorous conditions required by distribution reduct, maximum distribution reduct was introduced by Zhang et al. in [41].

* Corresponding author at: School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, PR China.
*E-mail address:* chxuwh@gmail.com (W. Xu).

0950-7051/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.knosys.2011.11.013

Partition or equivalence (indiscernibility) relation is an important and primitive concept in Pawlak's original rough set theory. However, partition or equivalence relation is much restrictive in many applications. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful general models in recent years by proposing other binary relations and operators, such as tolerance relations [26], neighborhood operators [39], and others [17,21,22,32–36,40]. Moreover, the original rough set theory does not consider attributes with preference ordered domain, that is criteria. Particularly, in many concrete situations, we often meet and deal with many problems in which the ranking of the values corresponding to the employed attributes plays a significant and crucial role. One such type of problem is the ordering of objects. To adapt and deal with these cases, Greco, Matarazzo, and Słowinski [5–10] proposed an extension to rough set theory, called the dominance-based rough set approach (DRSA) to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, condition attributes are criteria and classes are preference ordered, the approximation of knowledge is a collection of upward and downward unions of classes and the dominance classes are sets of objects defined via dominance relations. As applications being developed consulting with DRSA, several further studies have been made about properties and algorithmic implementations of DRSA in recent years[2,3,31,37,38].

Nevertheless, only a limited number of methods using DRSA to acquire knowledge in inconsistent ordered information systems has been proposed and studied. Pioneering work on inconsistent ordered information systems with the DRSA has been proposed by Greco, Matarazzo, and Słowinski [17–22], but they did not clearly point out the semantic explanation of unknown values. Shao and Zhang [25] further proposed an extension of the dominance relation in incomplete ordered information systems. Their work was established on the basis of the assumption that all unknown values are lost. Despite this, they did not mention the underlying concept of attribute reduction in inconsistent ordered decision system but an approach to attribute reduction in consistent ordered information systems. Therefore, the purpose of this paper is to develop approaches to attribute reductions in inconsistent ordered information systems. The main contribution of this work is to define two new types of reductions called, respectively, possible and compatible distribution reductions and study the methods to acquire them. On the basis of the definitions, the judgement theorems and discernibility matrices associated with the two reductions are listed to develop several equivalent conditions with the possible and compatible distribution consistent sets. The matrix computing approach and the corresponding algorithm are investigated to extract the above two types of reductions. An small-scale inconsistent information system is employed as an example to interpret what we study in every section and studied by the approaches which is a powerful demonstration of the validity of the algorithm. Furthermore, two cases are employed to verify the feasibility and validity of the matrix algorithm approached in this paper.

The rest of this paper is organized as follows. To facilitate our discussion, some preliminary concepts are briefly recalled in Section 2. In Section 3, theories and approaches on possible and compatible distribution reductions are investigated in inconsistent ordered information systems. The judgement theorems and discernibility matrices associated with the two reductions are examined, from which we can obtain a further approach to these two reductions in rough set theory. In Section 4, we introduce concepts of dominance matrix, possible and compatible decision distribution matrices in inconsistent ordered information systems. Furthermore, algorithm of matrix computation on possible and compatible distribution reductions is designed, from which we can developed another approach to attribute reduction in inconsistent ordered information systems. Moreover, two cases, a small-scale for convenience to compute and a large-scale on Concrete Slump, are employed to verify the feasibility and validity of the matrix algorithm, and still show that the method is effective and efficient in this complicated information system in Section 5. Finally, We then conclude the paper with a summary and outlook for further research.

## 2. Rough sets and ordered information systems

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed descriptions on rough set theory can be found in the source papers [5–10]. A comprehensive description has also been made in [41] and readers who need can look back into the reference.

The notion of information system (sometimes also called data table, information table attribute-value system, knowledge representation system etc.) provides a convenient tool for the representation of objects in terms of their attribute values.

An information system is an ordered quadruple $\mathcal{I} = (U, AT, V, f)$,

- $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects, called the universe;
- $AT = \{a_1, a_2, \ldots, a_p\}$ is a non-empty finite set of attributes, called the attribute set;
- $V = \bigcup_{a \in AT} V_a$ and $V_a$ is a domain of attribute $a$;
- $f : U \times AT \to V$ is a function such that $f(x, a) \in V_a$, for every $a \in AT$, $x \in U$, called an information function.

An information system with decision is a special case of an information systems in which, among the attributes, we distinguish on called a decision attribute. The other attributes are called condition attributes. Therefore, $\mathcal{I} = (U, C \cup \{d\}, V, f)$ and $C \cap \{d\} = \emptyset$, where attribute sets $C$ and $\{d\}$ are condition attributes and the decision attribute respectively.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

**Definition 2.1** (*see [5–10]*). An information system is called an ordered information system (OIS) if all attributes are criteria.

Assume that the domain of a criterion $a \in AT$ is completely preordered by an outranking relation $\succeq_a$, then $x \succeq_a y$ means that $x$ is at least as good as $y$ with respect to criterion $a$. And we can say that $x$ dominates $y$. In the following, for convenience to study, we suppose that criteria have a numerical domain, that is, $V_a \subseteq \mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers). Being of type gain, that is $x \succeq_a y \Leftrightarrow f(x, a) \geqslant f(y, a)$ (according to increasing preference) or $x \succeq_a y \Leftrightarrow f(x, a) \leqslant f(y, a)$ (according to decreasing preference), where $a \in AT$, $x, y \in U$.

Without any loss of generality and for simplicity, we only consider condition attributes with increasing preference in the following.

For a subset of attributes $B \subseteq AT$, we define $x \succeq_B y \Leftrightarrow \forall a \in B$, $f(x, a) \geqslant f(y, a)$, and that is to say $x$ dominates $y$ with respect to all attributes in $B$.

In general, we denote an ordered information system by $\mathcal{I}^{\succeq} = (U, AT, V, f)$, and denote an ordered information system with decision by $\mathcal{I}^{\succeq}_d = (U, C \cup \{d\}, V, f)$, in which the decision attribute is a single one $d$.

For an ordered information system with a single decision attribute, we say that $x$ dominates $y$ with respect to $B \subseteq C$ if $x \succeq_B y$, denoted by $xR^{\succeq}_B y$, and $x$ dominates $y$ with respect to $d$ if $x \succeq_d y$, denoted by $xR^{\succeq}_d y$. That is

$$R_B^{\succeq} = \{(x,y) \in U \times U | x \succeq_B y\} = \{(x,y) \in U \times U | f(x,a) \geqslant f(y,a), \forall a \in B\},$$
$$R_d^{\succeq} = \{(x,y) \in U \times U | x \succeq_d y\} = \{(x,y) \in U \times U | f(x,d) \geqslant f(y,d)\}.$$

$R_B^{\succeq}$ and $R_d^{\succeq}$ are called dominance relations in ordered information system $\mathcal{I}_d^{\succeq}$.

Denote

$$[x_i]_B^{\succeq} = \{x_j \in U | (x_j, x_i) \in R_B^{\succeq}\} = \{x_j \in U | f(x_j, a) \geqslant f(x_i, a), \forall a \in B\},$$
$$[x_i]_d^{\succeq} = \{x_j \in U | (x_j, x_i) \in R_d^{\succeq}\} = \{x_j \in U | f(x_j, d) \geqslant f(x_i, d)\}.$$

And the following properties of dominance relations are trivial by above definitions.

**Proposition 2.1** (see[5–10]). *Let $R_A^{\succeq}$ be a dominance relation. The following properties hold.*

(1) *$R_A^{\succeq}$ is reflexive, transitive, but not symmetric, so it is not an equivalence relation.*
(2) *If $B \subseteq A$, then $R_A^{\succeq} \subseteq R_B^{\succeq}$.*
(3) *If $B \subseteq A$, then $[x_i]_A^{\succeq} \subseteq [x_i]_B^{\succeq}$.*
(4) *If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \cup\{[x_j]_A^{\succeq} | x_j \in [x_i]_A^{\succeq}\}$.*
(5) *$[x_j]_A^{\succeq} = [x_i]_A^{\succeq}$ iff $f(x_i, a) = f(x_j, a)$ for all $a \in A$.*
(6) *$|[x_i]_B^{\succeq}| \geqslant 1$ for any $x_i \in U$.*
(7) *$U/R_B^{\succeq}$ constitute a covering of $U$, i.e., for every $x \in U$ we have that $[x]_B^{\succeq} \neq \emptyset$ and $\bigcup_{x \in U} [x]_B^{\succeq} = U$.*

*where $|\cdot|$ denotes the cardinality of a crisp set.*

*For any subset $X \subseteq U$ and $A \subseteq AT$ in $\mathcal{I}^{\succeq}$, the lower and upper approximation of $X$ with respect to the dominance relation $R_A^{\succeq}$ can be defined as follows (see [5–10]):*

$$\underline{R_A^{\succeq}}(X) = \{x \in U | [x]_A^{\succeq} \subseteq X\}; \quad \overline{R_A^{\succeq}}(X) = \{x \in U | [x]_A^{\succeq} \cap X \neq \emptyset\}.$$

*Unlike classical rough set theory, one can easily find that*

$$\{x \in U | [x]_A^{\succeq} \subseteq X\} \neq \bigcup \{[x]_A^{\succeq} | [x]_A^{\succeq} \subseteq X\}$$

*and*

$$\{x \in U | [x]_A^{\succeq} \cap X \neq \emptyset\} \neq \bigcup \{[x]_A^{\succeq} | [x]_A^{\succeq} \cap X \neq \emptyset\}.$$

**Definition 2.2** (see[5–10]). Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an ordered information system with decision. If $R_C^{\succeq} \subseteq R_d^{\succeq}$, i.e., $U/R_C^{\succeq} \leqslant U/R_d^{\succeq}$, which is defined by $\forall x_i \in U, [x_i]_C^{\succeq} \subseteq [x_i]_d^{\succeq}$, then this information system is consistent. Otherwise, this information system is an inconsistent ordered information system and denoted shortly as IOIS.

**Example 2.1.** An ordered information system with decision is presented in Table 1 and it is denoted by $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$, where $U = \{x_1, x_2, \ldots, x_6\}$, $C = \{a_1, a_2, a_3\}$, $d$ is the decision attribute.

**Table 1**
$\mathcal{I}_d^{\succeq}$: An ordered information system with decision.

| $U$ | $a_1$ | $a_2$ | $a_3$ | $d$ |
|---|---|---|---|---|
| $x_1$ | 1 | 2 | 1 | 3 |
| $x_2$ | 3 | 2 | 2 | 2 |
| $x_3$ | 1 | 1 | 2 | 1 |
| $x_4$ | 2 | 1 | 3 | 2 |
| $x_5$ | 3 | 3 | 2 | 3 |
| $x_6$ | 3 | 2 | 3 | 1 |

From the above table, one can calculate the dominance classes with respect to condition attributes in $C$ and obtain that

$$[x_1]_C^{\succeq} = \{x_1, x_2, x_5, x_6\};$$
$$[x_2]_C^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_C^{\succeq} = \{x_2, x_3, x_4, x_5, x_6\};$$
$$[x_4]_C^{\succeq} = \{x_4, x_6\};$$
$$[x_5]_C^{\succeq} = \{x_5\};$$
$$[x_6]_C^{\succeq} = \{x_6\}.$$

Assume $X = \{x_2, x_3, x_5\}$, then

$$\underline{R_C^{\succeq}}(X) = \{x_5\}, \quad \overline{R_C^{\succeq}}(X) = \{x_1, x_2, x_3, x_5\}.$$

Moreover, we can have the dominance decision classes

$$[x_1]_d^{\succeq} = [x_5]_d^{\succeq} = \{x_1, x_5\};$$
$$[x_2]_d^{\succeq} = [x_4]_d^{\succeq} = \{x_1, x_2, x_4, x_5\};$$
$$[x_3]_d^{\succeq} = [x_6]_d^{\succeq} = \{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

Obviously, $[x_i]_C^{\succeq} \not\subseteq [x_i]_d^{\succeq}$ for $i = 1, 2, 4$, i.e., $R_C^{\succeq} \not\subseteq R_d^{\succeq}$ . So the system in Table 1 is an inconsistent ordered information system.

## 3. Possible and compatible distribution reductions based on rough set in IOIS

### 3.1. Theories of possible and compatible distribution reductions based on rough set in IOIS

Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system. $R_C^{\succeq}, R_d^{\succeq}$ are dominance relations derived, respectively, from condition attribute set $C$ and decision attribute set $\{d\}$. For any $B \subseteq C$, denote

$$U/R_B^{\succeq} = \{[x_i]_B^{\succeq} | x_i \in U\},$$
$$U/R_d^{\succeq} = \{D_1, D_2, \ldots, D_r\},$$
$$\sigma_B^{\succeq}(x) = \{D_j | [x]_B^{\succeq} \cap D_j \neq \emptyset, x \in U\},$$
$$\delta_B^{\succeq}(x) = \{D_j | [x]_B^{\succeq} \subseteq D_j, x \in U\},$$

then $\sigma_B^{\succeq}(x)$ and $\delta_B^{\succeq}(x)$ are called, respectively, possible distribution function and compatible distribution function with respect to attribution set $B$.

From the above, we can have the following properties immediately.

**Proposition 3.1.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system. The following properties hold.*

(1) *If $B \subseteq C$, then $\sigma_C^{\succeq}(x) \subseteq \sigma_B^{\succeq}(x)$ and $\delta_B^{\succeq}(x) \subseteq \delta_C^{\succeq}(x), \forall x \in U$.*
(2) *If $[x]_B^{\succeq} \subseteq [y]_B^{\succeq}$, then $\sigma_B^{\succeq}(x) \subseteq \sigma_B^{\succeq}(y)$ and $\delta_B^{\succeq}(y) \subseteq \delta_B^{\succeq}(x), \forall x, y \in U$.*

**Definition 3.1.** Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system. $B \subseteq C$.

(1) *If $\sigma_B^{\succeq}(x) = \sigma_C^{\succeq}(x)$ for all $x \in U$, we say that $B$ is a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$. If $B$ is a possible distribution consistent set, and no proper subset of $B$ is a possible distribution consistent set, then $B$ is called a possible distribution consistent reduction of $\mathcal{I}_d^{\succeq}$.*
(2) *If $\delta_B^{\succeq}(x) = \delta_C^{\succeq}(x)$ for all $x \in U$, we say that $B$ is a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$. If $B$ is a compatible distribution consistent set, and no proper subset of $B$ is a compatible distribution consistent set, then $B$ is called a compatible distribution reduction of $\mathcal{I}_d^{\succeq}$.*

Example can be more explainable and the inconsistent ordered information system in Table will be still employed in this section to illustrate the possible and compatible distribution reductions.

**Example 3.1.** Continued From Example 2.1For the inconsistent ordered information system in Table 1, if we denote

$$D_1 = [x_1]_d^{\succeq} = [x_5]_d^{\succeq},$$
$$D_2 = [x_2]_d^{\succeq} = [x_4]_d^{\succeq},$$
$$D_3 = [x_3]_d^{\succeq} = [x_6]_d^{\succeq},$$

then we can have

$$\sigma_C^{\succeq}(x_1) = \sigma_C^{\succeq}(x_2) = \sigma_C^{\succeq}(x_3) = \sigma_C^{\succeq}(x_5) = \{D_1, D_2, D_3\};$$
$$\sigma_C^{\succeq}(x_4) = \{D_2, D_3\};$$
$$\sigma_C^{\succeq}(x_6) = \{D_3\}.$$

And

$$\delta_C^{\succeq}(x_1) = \delta_C^{\succeq}(x_2) = \delta_C^{\succeq}(x_3) = \delta_C^{\succeq}(x_4) = \delta_C^{\succeq}(x_6) = \{D_3\},$$
$$\delta_C^{\succeq}(x_5) = \{D_1, D_2, D_3\}.$$

If we take $B = \{a_2,a_3\}$, then it can be easily checked that $[x]_C^{\succeq} = [x]_B^{\succeq}$ holds for all $x \in U$. That is to say, $\sigma_B^{\succeq}(x) = \sigma_C^{\succeq}(x)$ and $\delta_B^{\succeq}(x) = \delta_C^{\succeq}(x)$ hold for all $x \in U$. Thus $B = \{a_2,a_3\}$ is a possible and compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$. Moreover, we can have that $\{a_3\}$ is neither a possible nor a compatible distribution consistent set and $\{a_2\}$ is a compatible but not a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$. Thereupon, we know that $\{a_2,a_3\}$ is a possible distribution reduction and $\{a_2\}$ is a compatible distribution reduction of $\mathcal{I}_d^{\succeq}$.

Furthermore, if we take $B' = \{a_1,a_3\}$, it can be obtained that

$$[x_1]_{B'}^{\succeq} = \{x_1, x_2, x_3, x_4, x_5, x_6\};$$
$$[x_2]_{B'}^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_{B'}^{\succeq} = \{x_2, x_3, x_4, x_5, x_6\};$$
$$[x_4]_{B'}^{\succeq} = \{x_4, x_6\};$$
$$[x_5]_{B'}^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_6]_{B'}^{\succeq} = \{x_6\};$$

and

$$\sigma_{B'}^{\succeq}(x_1) = \sigma_{B'}^{\succeq}(x_2) = \sigma_{B'}^{\succeq}(x_3) = \sigma_{B'}^{\succeq}(x_5) = \{D_1, D_2, D_3\};$$
$$\sigma_{B'}^{\succeq}(x_4) = \{D_2, D_3\};$$
$$\sigma_{B'}^{\succeq}(x_6) = \{D_3\},$$
$$\delta_{B'}^{\succeq}(x_i) = \{D_3\}, \quad i = 1, 2, \ldots, 6.$$

We can easily verify that $\sigma_{B'}^{\succeq}(x) = \sigma_C^{\succeq}(x)$ for all $x \in U$. So $B' = \{a_1,a_3\}$ is also a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$. However, we can find and verify that $\delta_{B'}^{\succeq}(x_5) \neq \delta_C^{\succeq}(x_5)$. Thus, $B'' = \{a_1,a_3\}$ is not a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$. Besides, it can be directly calculated that $\{a_1\}$ is neither a possible distribution consistent set nor a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$. Hence, $B'' = \{a_1,a_3\}$ is also a possible distribution reduction of $\mathcal{I}_d^{\succeq}$.

In addition, if take $B'' = \{a_1,a_2\}$, we have

$$[x_1]_{B''}^{\succeq} = \{x_1, x_2, x_5, x_6\};$$
$$[x_2]_{B''}^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_{B''}^{\succeq} = \{x_1, x_2, x_3, x_4, x_5, x_6\};$$
$$[x_4]_{B''}^{\succeq} = \{x_2, x_4, x_5, x_6\};$$
$$[x_5]_{B''}^{\succeq} = \{x_5\};$$
$$[x_6]_{B''}^{\succeq} = \{x_2, x_5, x_6\},$$

and

$$\delta_{B''}^{\succeq}(x_1) = \delta_{B''}^{\succeq}(x_2) = \delta_{B''}^{\succeq}(x_3) = \delta_{B''}^{\succeq}(x_4) = \delta_{B''}^{\succeq}(x_6) = \{D_3\};$$
$$\delta_{B''}^{\succeq}(x_5) = \{D_1, D_2, D_3\};$$
$$\sigma_{B''}^{\succeq}(x_i) = \{D_1, D_2, D_3\}, \quad i = 1, 2, \ldots 6.$$

It can be easily known that $\delta_{B''}^{\succeq}(x) = \delta_C^{\succeq}(x)$ for all $x \in U$. So $B'' = \{a_1,a_2\}$ is also a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$. However, we can find and check that $\sigma_{B''}^{\succeq}(x_4) \neq \sigma_C^{\succeq}(x_4)$ and $\sigma_{B''}^{\succeq}(x_6) \neq \sigma_C^{\succeq}(x_6)$. Thus, $B'' = \{a_1,a_2\}$ is not a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$.

From the above, one can obtain that there exist two possible distribution reductions of $\mathcal{I}_d^{\succeq}$ of the system, which are $\{a_1,a_3\}$ and $\{a_2,a_3\}$, and there exists only one compatible distribution of the system, which is $\{a_2\}$.

Detailed judgment theorems of possible and compatible distribution reductions will be proposed in the following.

**Theorem 3.1.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, F, G)$ be an inconsistent ordered information system with decision and $B \subseteq C$. B is a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$ if and only if $\sigma_C^{\succeq}(x) \cap \sigma_C^{\succeq}(y) \neq \sigma_C^{\succeq}(y) \Rightarrow [x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ holds for any $x,y \in U$.*

**Proof.** "$\Rightarrow$" Assume that $\sigma_C^{\succeq}(x) \cap \sigma_C^{\succeq}(y) \neq \sigma_C^{\succeq}(y) \Rightarrow [x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ doesn't hold, that implies $[x]_B^{\succeq} \cap [y]_B^{\succeq} = [y]_B^{\succeq}$. Then $[x]_B^{\succeq} \supseteq [y]_B^{\succeq}$ and $\sigma_B^{\succeq}(x) \supseteq \sigma_B^{\succeq}(y)$ can be obtained by Proposition 3.1(3). On the other hand, since $B$ is a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$, we have $\sigma_C^{\succeq}(x) \supseteq \sigma_C^{\succeq}(y)$, which is in contradiction with $\sigma_C^{\succeq}(x) \cap \sigma_C^{\succeq}(y) \neq \sigma_C^{\succeq}(y)$.

"$\Leftarrow$" We only prove $\sigma_B^{\succeq}(x) \subseteq \sigma_C^{\succeq}(x)$ by Proposition 3.1(1).

For any $x, y \in U, \sigma_C^{\succeq}(x) \cap \sigma_C^{\succeq}(y) \neq \sigma_C^{\succeq}(y) \Rightarrow [x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ means that $[x]_B^{\succeq} \cap [y]_B^{\succeq} = [y]_B^{\succeq} \Rightarrow \sigma_C^{\succeq}(x) \cap \sigma_C^{\succeq}(y) = \sigma_C^{\succeq}(y)$, and that is to say $[x]_B^{\succeq} \supseteq [y]_B^{\succeq}$ implies $\sigma_C^{\succeq}(x) \supseteq \sigma_C^{\succeq}(y)$.

On the other hand, suppose $D_k \in \sigma_B^{\succeq}(x)$, that is $[x]_B^{\succeq} \cap D_k \neq \emptyset$. Assume that $y \in [x]_B^{\succeq} \cap D_k$, then $y \in [x]_B^{\succeq}$ and $y \in D_k$. We can obtain that $[x]_B^{\succeq} \supseteq [y]_B^{\succeq}$ is true, which implies $\sigma_C^{\succeq}(x) \supseteq \sigma_C^{\succeq}(y)$. Since $y \in [y]_A^{\succeq}$, we have $y \in [y]_A^{\succeq} \cap D_k$, which means $[y]_A^{\succeq} \cap D_k \neq \emptyset$. So we observe $D_k \in \sigma_C^{\succeq}(y) \subseteq \sigma_C^{\succeq}(x)$, that is $D_k \in \sigma_C^{\succeq}(x)$. Thus, we conclude that $\sigma_B^{\succeq}(x) \subseteq \sigma_C^{\succeq}(x)$, i.e., $B$ is a possible distribution consistent set of $\mathcal{I}_d^{\succeq}$.

The prove is completed. □

**Theorem 3.2.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $B \subseteq C$. B is a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$ if and only if $\delta_C^{\succeq}(x) \cap \delta_C^{\succeq}(y) \neq \delta_C^{\succeq}(x) \Rightarrow [x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ holds for any $x,y \in U$.*

**Proof.** "$\Rightarrow$" Assume that $\delta_C^{\succeq}(x) \cap \delta_C^{\succeq}(y) \neq \delta_C^{\succeq}(x) \Rightarrow [x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ doesn't hold, which implies that there exist $x_0, y_0 \in U$ such that $\delta_C^{\succeq}(x_0) \cap \delta_C^{\succeq}(y_0) \neq \delta_C^{\succeq}(x_0)$ but $[x_0]_B^{\succeq} \cap [y_0]_B^{\succeq} = [y_0]_B^{\succeq}$. That is to say $[y_0]_B^{\succeq} \subseteq [x_0]_B^{\succeq}$. Then we can obtain $\delta_B^{\succeq}(x_0) \subseteq \delta_B^{\succeq}(y_0)$ by Proposition 3.1(2). On the other hand, since $B$ is a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$, we have $\delta_B^{\succeq}(x_0) = \delta_C^{\succeq}(x_0)$ and $\delta_B^{\succeq}(y_0) = \delta_C^{\succeq}(y_0)$. Hence we can get $\delta_C^{\succeq}(x_0) \subseteq \delta_C^{\succeq}(y_0)$. That is to say $\delta_C(x_0)^{\succeq} \cap \delta_C^{\succeq}(y_0) = \delta_C^{\succeq}(x_0)$, which is a contradiction."$\Rightarrow$" We only prove $\delta_C^{\succeq}(x) \subseteq \delta_B^{\succeq}(x)$ by Proposition 3.1(1).

For any $D_k \in \delta_C(x)$ and any $y_0 \in [x]_B^{\succeq}$, we have $[y_0]_B^{\succeq} \subseteq [x]_B^{\succeq}$.

From the fact that $[x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ always holds while $\delta_C^{\succeq}(x) \cap \delta_C^{\succeq}(y) \neq \delta_C^{\succeq}(x)$, we can have the result that $\delta_C^{\succeq}(x) \cap \delta_C^{\succeq}(y) = \delta_C^{\succeq}(x)$ always holds when $[x]_B^{\succeq} \cap [y]_B^{\succeq} = [y]_B^{\succeq}$ for $x, y \in U$. That is to say that if $[y]_B^{\succeq} \subseteq [x]_B^{\succeq}$, then $\delta_C^{\succeq}(x) \subseteq \delta_C^{\succeq}(y)$ holds.

So we can obtain $\delta_C^{\succeq}(x) \subseteq \delta_C^{\succeq}(y_0)$. Therefore $D_k \in \delta_C^{\succeq}(y_0)$, in other words, $[y_0]_C^{\succeq} \subseteq D_k$. Thus we can receive $y_0 \in D_k$, and that is to say $[x]_B^{\succeq} \subseteq D_k$.

Hence, $\delta_C^{\succeq}(x) \subseteq \delta_B^{\succeq}(x)$.

This theorem is proved. □

### 3.2. Approach to possible and compatible distribution reductions in IOIS

This section provides the approach to possible and compatible distribution reductions in inconsistent ordered information systems. Firstly, we present the following notions.

**Definition 3.2.** Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision. For $x_i, x_j \in U$, we denote,

$$D^*_{\sigma^{\succeq}} = \{(x_i, x_j) | \sigma_C^{\succeq}(x_i) \subset \sigma_C^{\succeq}(x_j)\},$$
$$D^*_{\delta^{\succeq}} = \{(x_i, x_j) | \delta_C^{\succeq}(x_i) \supset \delta_C^{\succeq}(x_j)\},$$

$$D_{\sigma^{\succeq}}(x_i, x_j) = \begin{cases} \{a \in C | f(x_i, a) > f(x_j, a)\}, & (x_i, x_j) \in D^*_{\sigma^{\succeq}} \\ C, & (x_i, x_j) \notin D^*_{\sigma^{\succeq}} \end{cases},$$

$$D_{\delta^{\succeq}}(x_i, x_j) = \begin{cases} \{a \in C | f(x_i, a) > f(x_j, a)\}, & (x_i, x_j) \in D^*_{\delta^{\succeq}} \\ C, & (x_i, x_j) \notin D^*_{\delta^{\succeq}} \end{cases},$$

and

$$\mathcal{M}_{\sigma^{\succeq}} = (u_{ij})_{n \times n}, \quad \text{where} \quad u_{ij} = D_{\sigma^{\succeq}}(x_i, x_j),$$
$$\mathcal{M}_{\delta^{\succeq}} = (v_{ij})_{n \times n}, \quad \text{where} \quad v_{ij} = D_{\delta^{\succeq}}(x_i, x_j).$$

Then, $D_{\sigma^{\succeq}}(x_i, x_j)$ and $D_{\delta^{\succeq}}(x_i, x_j)$ are called, respectively, possible and compatible distribution discernibility attributes sets. And matrices $\mathcal{M}_{\sigma^{\succeq}}$ and $\mathcal{M}_{\delta^{\succeq}}$ are referred as possible and compatible distribution discernibility matrix of $\mathcal{I}_d^{\succeq}$ respectively.

**Theorem 3.3.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $B \subseteq C$. B is a compatible distribution consistent set if and only if $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$ holds for any $(x, y) \in D^*_{\delta^{\succeq}}$.*

**Proof.** "⇒" Assume that $B$ is a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$. For any $(x, y) \in D^*_{\delta^{\succeq}}$, we can obtain $\delta_C^{\succeq}(x) \supset \delta_C^{\succeq}(y)$. From Theorem 3.1, we have $[x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$. Thus there exist the following three cases between $[x]_B^{\succeq}$ and $[y]_B^{\succeq}$, which are (1) $[x]_B^{\succeq} \subset [y]_B^{\succeq}$, (2) $[x]_B^{\succeq} \cap [y]_B^{\succeq} = \emptyset$, (3) both $[x]_B^{\succeq} \cap [y]_B^{\succeq} \subset [x]_B^{\succeq}$ and $[x]_B^{\succeq} \cap [y]_B^{\succeq} \subset [y]_B^{\succeq}$. We will prove that $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$ always holds in every case.

Case 1. If $[x]_B^{\succeq} \subset [y]_B^{\succeq}$, then there necessarily exists an element $z \in [y]_B^{\succeq}$, but $z \notin [x]_B^{\succeq}$. From $z \notin [x]_B^{\succeq}$, we can certainly find an element $a \in B$ such that $f(x,a) > f(z,a)$. On the other hand, the fact $f(y,a) \geqslant f(z,a)$ is true according to $z \in [y]_B^{\succeq}$. From the above, we can obtain $f(x,a) > f(y,a)$. Hence, we have $a \in D_{\delta^{\succeq}}(x, y)$, i.e., $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$.

Case 2. If $[x]_B^{\succeq} \cap [y]_B^{\succeq} = \emptyset$, then there exists necessarily an element $a \in B$ such that $f(x,a) > f(y,a)$, i.e. $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$. Otherwise, if for all $a \in B$, $f(x,a) \geqslant f(y,a)$ always holds, then we consider $y \in [x]_B^{\succeq}$. This is contradiction.

Case 3. The proof is similar to Case 1, because it can also be certainly found an element $z \in [y]_B^{\succeq}$, but $z \notin [x]_B^{\succeq}$ in the case.

Thus we can conclude that $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$ for any $(x, y) \in D^*_{\delta^{\succeq}}$.

Hence, if $B$ is a compatible distribution consistent set, then $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$, for all $(x, y) \in D^*_{\delta^{\succeq}}$.

"⇐" If every $(x, y) \in D^*_{\delta^{\succeq}}$ satisfies $B \cap D_{\delta^{\succeq}}(x, y) \neq \emptyset$, then we can select an $a \in B$ such that $a \in D_{\delta^{\succeq}}(x, y)$. That is $f(x,a) > f(y,a)$, further $y \notin [x]_B^{\succeq}$. Since $y \in [y]_B^{\succeq}$ is true, we can obtain $[x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$. On the other hand, since $(x, y) \in D^*_{\delta^{\succeq}}$, we have $\delta_C^{\succeq}(x) \supset \delta_C^{\succeq}(y)$, which means $\delta_C^{\succeq}(x) \cap \delta_C^{\succeq}(y) \neq \delta_C^{\succeq}(x)$. Hence, we find that $\delta_C^{\succeq}(x) \cap \delta_C^{\succeq}(y) \neq \delta_C^{\succeq}(x) \Rightarrow [x]_B^{\succeq} \cap [y]_B^{\succeq} \neq [y]_B^{\succeq}$ holds. Thus we develop that $B$ is a compatible distribution consistent set of $\mathcal{I}_d^{\succeq}$ in term of Theorem 3.1.

The prove is completed. □

**Theorem 3.4.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision ans $B \subseteq C$. B is a possible distribution consistent set if and only if $B \cap D_{\sigma^{\succeq}}(x, y) \neq \emptyset$ holds for any $(x, y) \in D^*_{\sigma^{\succeq}}$.*

**Proof.** It is similar to Theorem 3.3. □

**Definition 3.3.** Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision, $\mathcal{M}_{\sigma^{\succeq}}$ and $\mathcal{M}_{\delta^{\succeq}}$ be possible and compatible distribution discernibility matrices of $\mathcal{I}_d^{\succeq}$ respectively. Denote

$$F_{\sigma^{\succeq}} = \wedge\{\vee\{a_k | a_k \in D_{\sigma^{\succeq}}(x_i, x_j)\}, x_i, x_j \in U\}$$
$$= \wedge\{\vee\{a_k | a_k \in D_{\sigma^{\succeq}}(x_i, x_j)\}, x_i, x_j \in D^*_{\sigma^{\succeq}}\},$$
$$F_{\delta^{\succeq}} = \wedge\{\vee\{a_k | a_k \in D_{\delta^{\succeq}}(x_i, x_j)\}, x_i, x_j \in U\}$$
$$= \wedge\{\vee\{a_k | a_k \in D_{\delta^{\succeq}}(x_i, x_j)\}, x_i, x_j \in D^*_{\delta^{\succeq}}\}.$$

Then $F_{\sigma^{\succeq}}$ and $F_{\delta^{\succeq}}$ are called, respectively, discernibility formulas of possible and compatible distribution.

**Theorem 3.5.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision. The minimal disjunctive normal form of discernibility formula of possible distribution is*

$$F_{\sigma^{\succeq}} = \bigvee_{k=1}^{p} \left( \bigwedge_{s=1}^{q_k} \right) a_s.$$

*Denote $B^k_{\sigma^{\succeq}} = \{a_s | s = 1, 2, \ldots, q_k\}$, then $\left\{B^k_{\sigma^{\succeq}} | k = 1, 2, \ldots, p\right\}$ is the set of all possible distribution reductions of $\mathcal{I}_d^{\succeq}$.*

**Proof.** It follows directly from Theorem 3.4 and the minimal disjunctive normal to the discernibility formula of compatible distribution. □

**Theorem 3.6.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision. The minimal disjunctive normal form of discernibility formula of compatible distribution is*

$$F_{\delta^{\succeq}} = \bigvee_{k=1}^{p} \left( \bigwedge_{s=1}^{q_k} \right) a'_s.$$

*Denote $B^k_{\delta^{\succeq}} = \{a'_s | s = 1, 2, \ldots, q_k\}$, then $\{B^k_{\delta^{\succeq}} | k = 1, 2, \ldots, p\}$ is the set of all compatible distribution reductions of $\mathcal{I}_d^{\succeq}$.*

**Proof.** It is similar to Theorem 3.5. □

Theorems 3.5 and 3.6 provide a practical and effective approach to possible and compatible distribution reductions of inconsistent ordered information systems. The following example will be employed to illustrate the validity of the approach.

**Example 3.2.** Continued From Example 2.1). Considering the inconsistent ordered information system showed in Example 2.1, calculate the possible and compatible reductions of the system by the above approach.

We have got the functions of possible and compatible distribution in Example 3.1. Moreover, we can calculate the possible and compatible distribution discernibility matrices of the system and list them in Tables 2 and 3.

Consequently, we have

$$F_{\sigma^{\succeq}} = (a_1 \vee a_2 \vee a_3 \vee) \wedge (a_1 \vee a_3) \wedge (a_1 \vee a_2) \wedge a_3$$
$$= (a_1 \wedge a_3) \vee (a_2 \wedge a_3),$$

**Table 2**
Possible distribution discernibility matrix $\mathcal{M}_{\sigma^{\succeq}}$.

| $x_i, x_j$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_2$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_3$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_4$ | $a_1, a_3$ | $a_3$ | $a_1, a_3$ | $C$ | $a_3$ | $C$ |
| $x_5$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_6$ | $a_1, a_3$ | $a_3$ | $C$ | $a_1, a_2$ | $a_3$ | $C$ |

**Table 3**
Compatible distribution discernibility matrix $\mathcal{M}_{\delta^{\succeq}}$.

| $x_i, x_j$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_2$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_3$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_4$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_5$ | $a_1, a_2, a_3$ | $a_2$ | $a_1, a_2$ | $a_1, a_2$ | $\emptyset$ | $a_2$ |
| $x_6$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |

$$F_{\delta^{\succeq}} = a_2 \wedge (a_1 \vee a_2) \wedge (a_1 \vee a_2 \vee a_3) = a_2.$$

Therefore, we acquire that $\{a_1,a_3\}$, $\{a_2,a_3\}$ are all possible distribution reductions and $\{a_2\}$ is the noly compatible distribution reductions of the inconsistent ordered information system in Table 1, which accord with the results in Example 3.1.

## 4. Algorithm to acquire possible and compatible reductions in IOIS based on matrix computation

### 4.1. Dominance matrices, possible and compatible distribution decision matrices

In this section, the dominance matrix, possible and compatible distribution decision matrices are proposed, and some important properties are obtained in further.

**Definition 4.1.** Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $B \subseteq C$. Denote

$$M_B = (m_{ij})_{n\times n},$$

where

$$m_{ij} = \begin{cases} 1, & x_j \in [x_i]_B^{\succeq}, \\ 0, & \text{otherwise}. \end{cases}$$

The matrix $M_B$ is called dominance matrix of attribute set $B \subseteq C$. If $|B| = l$, we say that $M_B$ is a $l$th order dominance matrix.

**Definition 4.2.** Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $M_{B_1}, M_{B_2}$ be dominance matrices of attribute sets $B_1, B_2 \subseteq C$. The intersection of $M_{B_1}$ and $M_{B_2}$ is defined by

$$M_{B_1} \cap M_{B_2} = (m_{ij})_{n\times n} \cap (m'_{ij})_{n\times n} = (\min\{m_{ij}, m'_{ij}\})_{n\times n}.$$

From the above definitions, the following properties are obviously.

**Proposition 4.1.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $M_{B_1}, M_{B_2}$ be dominance matrices of attributes sets $B_1, B_2 \subseteq C$. The following results always hold.*

(1) $m_{ii} = 1$.
(2) $M_{B_1 \cup B_2} = M_{B_1} \cap M_{B_2}$.

**Definition 4.3.** Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision. Denote

$$M_d^{\sigma} = \left(s_{ij}^{\sigma}\right)_{n\times n}, \quad M_d^{\delta} = \left(r_{ij}^{\delta}\right)_{n\times n},$$

where

$$s_{ij}^{\sigma} = \begin{cases} 1, & \sigma_C^{\succeq}(x_j) \subseteq \sigma_C^{\succeq}(x_i), \\ 0, & \text{otherwise}. \end{cases}$$

$$r_{ij}^{\delta} = \begin{cases} 1, & \delta_C^{\succeq}(x_i) \subseteq \delta_C^{\succeq}(x_j), \\ 0, & \text{otherwise}. \end{cases}$$

The matrix $M_d^{\sigma}$ and $M_d^{\delta}$ are called, respectively, possible and compatible distribution decision matrices of $\mathcal{I}_d^{\succeq}$.

From the above, we can see that the dominance relation of objects is decided by dominance matrix, and different decision of objects are decided by possible and compatible distribution decision matrices.

**Definition 4.4.** Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be two $n$ dimension vectors. If $a_i \leqslant b_i$, $(i = 1, 2, \ldots, n)$, we say vector $\alpha$ is less than or equal to vector $\beta$, denoted by $\alpha \leqslant \beta$.

**Definition 4.5.** Let $M_A = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ and $M_B = (\beta_1, \beta_2, \ldots, \beta_n)^T$, be two matrices, $\alpha_i$ and $\beta_i$ be row vectors respectively. If $\alpha_i \leqslant \beta_i$, we say $M_A$ is less than or equal to $M_B$, denoted by $M_A \leqslant M_B$.

According to the above two definitions, one can easily find that dominance matrices have the following properties.

**Proposition 4.2.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $B \subseteq C$. $M_C$ and $M_B$ are the dominance matrices, then $M_C \leqslant M_B$.*

### 4.2. Algorithm based on matrix computation

In the following, we develop the method of matrix computation for the acquisition of possible and compatible distribution reductions in inconsistent ordered information systems.

**Theorem 4.1.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent ordered information system with decision and $B \subseteq C$. $B$ is a possible distribution consistent set if and only if $M_B \leqslant M_d^{\sigma}$, where $M_B = (m_{ij})_{n\times n}$ and $M_d^{\sigma} = \left(s_{ij}^{\sigma}\right)_{n\times n}$.*

**Proof.** "$\Rightarrow$" We need to prove that $\sigma_B^{\succeq}(x) = \sigma_C^{\succeq}(x)$ for any $x \in U$ implies $M_B \leqslant M_d^{\sigma}$. Since $m_{ij} = 0 \leqslant s_{ij}^{\sigma}$ is obvious, we only prove $m_{ij} = 1 \Rightarrow s_{ij}^{\sigma} = 1$. In fact, we can have that

$$m_{ij} = 1 \Rightarrow x_j \in [x_i]_B^{\succeq} \Rightarrow [x_j]_B^{\succeq} \subseteq [x_i]_B^{\succeq} \Rightarrow \sigma_B^{\succeq}(x_j) \subseteq \sigma_B^{\succeq}(x_i)$$
$$\Rightarrow \sigma_C^{\succeq}(x_j) \subseteq \sigma_C^{\succeq}(x_i) \Rightarrow s_{ij}^{\sigma} = 1$$

"$\Leftarrow$" From Proposition 3.1, we have known that $\sigma_C^{\succeq}(x_i) \subseteq \sigma_B^{\succeq}(x_i)$ for any $x_i \in U$ and $B \subseteq C$. So,we only prove $\sigma_B^{\succeq}(x_i) \subseteq \sigma_C^{\succeq}(x_i)$ holds for any $x_i \in U$ in the following.

For any $D_k \in \sigma_B^{\succeq}(x_i)$, we have $D_k \cap [x_i]_B^{\succeq} \neq \emptyset$. Take $x_j \in D_k \cap [x_i]_B^{\succeq}$, then $x_j \in D_k$ and $x_j \in [x_i]_B^{\succeq}$. From $x_j \in [x_i]_B^{\succeq}$, we can obtain $m_{ij} = 1$. Since $M_B \leqslant M_d^{\sigma}$, we have $m_{ij} \leqslant s_{ij}^{\sigma}$. Thus, $s_{ij}^{\sigma} = 1$ holds, which can implies $\sigma_C^{\succeq}(x_j) \subseteq \sigma_C^{\succeq}(x_i)$.

On the other hand, it is obvious that $x_j \in [x_j]_C^{\succeq}$. By $x_j \in D_k$, we can observe that $x_j \in D_k \cap [x_j]_C^{\succeq}$. Therefore, $D_k \cap [x_j]_C^{\succeq} \neq \emptyset$ holds, that is to say $D_k \in \sigma_C^{\succeq}(x_j)$.

From the above, we have that $D_k \in \sigma_C^{\succeq}(x_i)$. Hence, $\sigma_B^{\succeq}(x_i) \subseteq \sigma_C^{\succeq}(x_i)$ holds for any $x_i \in U$.

The theorem is proved. □

**Corollary 4.1.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an ordered information system with decision and $B \subseteq C$. B is a possible distribution reduction if and only if $M_B \leqslant M_d^{\sigma}$ holds but $M_{B'} \leqslant M_d^{\sigma}$ does not hold for any proper subset $B''$ of B.*

**Theorem 4.2.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an inconsistent information system with decision and $B \subseteq C$. B is a compatible distribution consistent set if and only if $M_B \leqslant M_d^{\delta}$, where $M_B = (m_{ij})_{n\times n}$ and $M_d^{\delta} = (r_{ij}^{\delta})_{n\times n}$.*

**Proof.** "$\Rightarrow$" It similar to the proof of Theorem 4.1.

"$\Leftarrow$" From Proposition 3.1, we have known that $\delta_B^{\succeq}(x_i) \subseteq \delta_C^{\succeq}(x_i)$ for any $x_i \in U$ and $B \subseteq C$. So, in the following, we only prove $\delta_C^{\succeq}(x_i) \subseteq \delta_B^{\succeq}(x_i)$ for any $x_i \in U$. For $\forall\ D_k \in \delta_C^{\succeq}(x_i)$ and $\forall\ x_j \in [x_i]_B^{\succeq}$, we have

$$x_j \in [x_i]_B^{\succeq} \Rightarrow m_{ij} = 1 \Rightarrow r_{ij} = 1 \Rightarrow \delta_C^{\succeq}(x_i) \subseteq \delta_C^{\succeq}(x_j) \Rightarrow D_k \in \delta_C^{\succeq}(x_j)$$
$$\Rightarrow [x_j]^{\succeq} \subseteq D_k \Rightarrow x_j \in D_k \Rightarrow [x_i]_B^{\succeq} \subseteq D_k \Rightarrow D_k \in \delta_B^{\succeq}(x_i)$$
$$\Rightarrow \delta_C^{\succeq}(x_i) \subseteq \delta_B^{\succeq}(x_i)$$

The theorem is proved. □

**Corollary 4.2.** *Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an ordered information system with decision and $B \subseteq C$. B is a compatible distribution reduction if and only if $M_B \leqslant M_d^{\delta}$ holds but $M_{B'} \leqslant M_d^{\delta}$ does not hold for any proper subset $B''$ of B.*

*From the above theorem and propositions, we can acquire the following algorithm.*

**Algorithm 1.** Algorithm of matrix computation for acquisition of possible and compatible distribution reductions in inconsistent ordered information systems is described as follows:

Input: An inconsistent ordered information system

$$\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f),$$

where $U = \{x_1, x_2, \ldots, x_n\}$ and $C = \{a_1, a_2, \ldots, a_p\}$.

Output: All possible and compatible distribution reductions of $\mathcal{I}_d^{\succeq}$.

Step 1 Simplify the system by combining the objects with same values of every attribute. Check the consistence of $\mathcal{I}_d^{\succeq}$. If it is inconsistent, calculate $\sigma$, $\delta$ and continue; else, terminate the algorithm.

Step 2 Calculate possible and compatible distribution decision matrices of $\mathcal{I}_d^{\succeq}$ and denoted as follows,

$$M_d^{\sigma} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T,$$
$$M_d^{\delta} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T.$$

Step 3 For any $a_l \in C$, $(1 \leqslant l \leqslant p)$, compute 1st order dominance matrix

$$M_{\{a_l\}} = M_{\{a_l\}}^{(1)} = \left(\tau_1^{(1)}, \tau_2^{(1)}, \ldots, \tau_n^{(1)}\right)^T.$$

For $i = 1$ to $n$.

If $0 \neq \tau_i^{(1)} \leqslant \lambda_i$, let $\tau_i^{(1)} = 0$. Denote the new matrix by $FM_{\{a_l\}}^{\sigma(1)}$.

If $0 \neq \tau_i^{(1)} \leqslant \gamma_i$, let $\tau_i^{(1)} = 0$. Denote the new matrix by $FM_{\{a_l\}}^{\delta(1)}$.

Turn into next step.

Step 4 Call matrix $FM_{\{a_l\}}^{\sigma(1)}$ and $FM_{\{a_l\}}^{\delta(1)} a_l \in C, (1 \leqslant l \leqslant p)$ to be 1st order possible and compatible distribution matrices, respectively.

**Algorithm 1** (*continued*)

(1) If $FM_{\{a_l\}}^{\sigma(1)} = 0$, then obtain a 1st order possible distribution reduction: $\{a_l\}$. Otherwise, turn into item (2).

(2) If $FM_{\{a_l\}}^{\delta(1)} = 0$, then obtain a 1st order compatible distribution reduction: $\{a_l\}$. Otherwise, turn into next step.

Step 5 Compute the intersection of two nonzero 1st order possible (or compatible) matrices obtained in Step 3. The new matrices are called 2nd order possible (or compatible) dominance matrices, denoted by $M_{\{a_l a_s\}}^{\sigma(2)} = FM_{\{a_l\}}^{\sigma(1)} \cap FM_{\{a_s\}}^{\sigma(1)}$ (Correspondingly, $M_{\{a_l a_s\}}^{\delta(2)} = FM_{\{a_l\}}^{\delta(1)} \cap FM_{\{a_s\}}^{\delta(1)}$). Let $B = \{a_l, a_s\}$ and $k = |B|$.

Step 6 Denote $k$th ($k \leqslant |C|$) order possible and compatible dominance matrices by $M_B^{\sigma(k)}$ and $M_B^{\delta(k)}$.

$$M_B^{\sigma(k)} = \left(\tau_1^{\sigma(k)}, \tau_2^{\sigma(k)}, \ldots, \tau_n^{\sigma(k)}\right)^T;$$
$$M_B^{\delta(k)} = \left(\tau_1^{\delta(k)}, \tau_2^{\delta(k)}, \ldots, \tau_n^{\delta(k)}\right)^T.$$

For $i = 1$ to $n$.

(1) If $q$th ($q = 1, 2, \ldots, k-1$) order possible reduction is included in $B$, turn into the next item (2); else, continue. For $0 \neq \tau_i^{\sigma(k)} \leqslant \lambda_i$, let $\tau_i^{\sigma(k)} = 0$. Denote the new matrix by $FM_B^{\sigma(k)}$. If $FM_B^{\sigma(k)} = 0$, then obtain a $k$th order possible distribution reduction: $B$.

(2) If $q$th ($q = 1, 2, \ldots, k-1$) order compatible reduction is included in $B$, turn into the next step; else, continue. For $0 \neq \tau_i^{\delta(k)} \leqslant \gamma_i$, let $\tau_i^{\delta(k)} = 0$. Denote the new matrix by $FM_B^{\delta(k)}$. If $FM_B^{\delta(k)} = 0$, we acquire a $k$th order compatible distribution reduction: $B$.

Step 7 For any $|B'| = |B| = k$ and $B' \neq B$, let $B = B'$ (if any). Go back to Step 6 and calculate all $k$th order possible and compatible distribution reductions. If $k < |C|$, compute the intersection of a nonzero $k$th order possible (or compatible) matrix and a nonzero 1st order possible (or compatible) matrix; else, go to the next step. The new matrices are called $(k+1)$th order possible (or compatible) dominance matrices, denoted by $M_{B''}^{\sigma(k+1)} = M_{B\cup\{a_t\}}^{\sigma(k+1)} = FM_B^{\sigma(k)} \cap FM_{\{a_t\}}^{\sigma(1)}$ (Correspondingly, $M_{B''}^{\delta(k+1)} = M_{B\cup\{a_t\}}^{\delta(k+1)} = FM_B^{\delta(k)} \cap FM_{\{a_t\}}^{\delta(1)}$. Let $B = B''$ and $k = k + 1 = |B''| \leqslant |C|$. Go back to Step 6 and compute all $k$th possible and compatible distribution reductions.

Step 8 Collect and output all possible and compatible distribution reductions. Terminate the algorithm. □

**Analysis to time complexity of Algorithm 1**

Let $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$ be an ordered information system. $U = \{x_1, x_2, \ldots, x_n\}$ is the simplified universe. The number of objects in original information system not being simplified is denoted by $n_1$. There are $m$ condition attributes in $C$, i.e., $|C| = m$. The number of compressed decision classes is $r$. We take a variable $t_i$ to stand for the time complexity in an implementation. In the next, we can analyze the time complexity of Algorithm 1 step by step.

The time complexity to simplify the original information system is $n_1^2$ for any two objects being compared and denoted by $t_1 = n_1^2$. Since $|U| = n$, $|C| = m$ and $|\{d\}| = 1$, the time complexity to classify by condition attributes and decision $\{d\}$ are, respectively, $t_2 = |U|^2 \times |C|$ and $t_3 = |U|^2$. For decision classes being merged by comparing classes of any two objects, the time complexity is $t_4 = |U|^2$. Now the consistence of the information system need to be checked by compare the condition class and decision class of any object. If the information system is consistent, the time complexity to check consistence is $|U|$. If the information system is inconsistent, the time complexity to check consistence is less than $|U|$. Thus, the time complexity to check consistence is no more than $|U|$, i.e., it is presented as $t_5 \leqslant |U|$. Then, the possible and compatible distribution functions can be calculated and the time complexity is $t_6 = 2r \times |U|$. The time complexity to calculate each of these two functions is $r \times |U|$ and denoted by $t_6^\sigma = t_6^\delta = r \times |U|$. The analysis to Step 1 is finished.

For Step 2, the time complexity to calculate possible and compatible distribution decision matrices respectively is denoted by $t_7^\sigma = t_7^\delta = |U|^2$. Thus, the time complexity to calculate distribution decision matrices is $t_7 = 2|U^2|$. The time complexity of Step 2 is completed.

The first two steps are preparations to calculate reductions. The next Step 3 to Step 7 are the steps which run the operations. There are $C_m^1 = m$ subsets $\{a_l\}$ and the dominance matrices are with dimensions $n \times n$. In addition, the representation $C_m^i$ is the combinatorial number which means the number of selections to chose $i$ elements from $m$ ones. We consider that the judgement of a vector if it is zero runs one operation and the comparison of two vectors runs according to the dimension of the vectors. Therefore, the time complexities to compare $M_d^\sigma$ and $M_d^\delta$ with $M_{\{a_l\}}$ respectively are $|U|^2$. And the time complexity to compare every line vector of $M_{\{a_l\}}$ with zero is $|U|$. The possible and compatible distribution matrices are obtained by reassignment values $n$ times. And the time complexities to process possible and compatible distribution matrices respectively are both $n$. Then, we have that the total time complexity of Step 3 is $t_8 = C_m^1 \times (3|U|^2 + 3|U|)$. The judgement in Step 4 just need to run according to the number of $\{a_l\}$ and the time complexity is $t_9 = 2C_m^1$.

Since we just need to compute the intersection of nonzero 1st order possible (or compatible) distribution matrices, the maximum time complexities can be analyzed in the next steps but not the true ones in computing. Therefore, the maximum time complexity relies on the number of attribute subsets $2^{|C|}$. The worst case is that no minimum reduction exists in the information system and all $2^{|C|}$ subsets are calculated in the algorithm. Thus, the maximum time complexity of Step 5 is $t_{10} = 2C_m^2 \times |U|^2$.

The time complexities of separate functions in Step 6, Step 7 can be analyzed similarly as the above and they rely on the cardinality of attribute subset $k = |B|$ in the $k$th loop. For Step 6, the operations are judgement if a reduction is included in $B$ and calculation to the new $k$th order distribution matrices. Assume that $k = |B| (k = 2, 3, \ldots, |C|)$ and it is used as superscript to mark the $k$th order. The judgement is briefly treated as a function run one operation since it can be implemented easily by judging if subset. Therefore, the maximum time complexity of Step 6 is $t_{11}^k = 2C_m^k \times (|U|^2 + 2|U| + 2)$. Similarly as Step 5, the time complexity of Step 7 is $t_{12}^{k+1} = 2C_m^{k+1} \times |U|^2$. The time complexity of Step 8 will be not considered since the output runs fast and direct.

From the above analysis, we can know that the maximum time complexity of the main part in the algorithm (Step 3 to Step 7) is

$$t_{main} = t_8 + t_9 + t_{10} + \sum_{k=2}^{|C|} \left(t_{11}^k + t_{12}^{k+1}\right) = (|U|^2 + 2|U| + 3) \times 2^{|C|+2} - |C| \times (|U|^2 + |U| + 2).$$

Hence, the maximum time complexity of the main algorithm is approximately $O((|U|^2 + |U| + 1) \times 2^{|C|+2})$.

If we just compute possible or compatible distribution reductions, the maximum time complexity of the main part is

$$t_{main}^\sigma = t_{main}^\delta = (|U|^2 + |U| + 1) \times 2^{|C|+1} + |C| \times (|U| + 1)^2.$$

Then, the time complexity to compute one form of the distribution reductions is approximately $O((|U|^2 + |U| + 1) \times 2^{|C|+1})$. □

In the following, we will calculate the possible and compatible distribution reductions manually and illustrate the algorithm by the inconsistent ordered information system showed in Table 1.

**Example 4.1** (*Continued From Example 2.1*). Calculate all possible and compatible distribution reductions of the inconsistent ordered information system in Example 2.1.

From Table 1, we can compute all dominance matrices and the distribution decision matrix as follows,

$$M_{\{a_1\}} = \begin{bmatrix} 1&1&1&1&1&1\\ 0&1&0&0&1&1\\ 1&1&1&1&1&1\\ 0&1&0&1&1&1\\ 0&1&0&0&1&1\\ 0&1&0&0&1&1 \end{bmatrix};$$

$$M_{\{a_2\}} = \begin{bmatrix} 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 0&0&0&0&1&0\\ 1&1&0&0&1&1 \end{bmatrix};$$

$$M_{\{a_3\}} = \begin{bmatrix} 1&1&1&1&1&1\\ 0&1&1&1&1&1\\ 0&1&1&1&1&1\\ 0&0&0&1&0&1\\ 0&1&1&1&1&1\\ 0&0&0&1&0&1 \end{bmatrix};$$

$$M_d^\sigma = \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 0&1&1&1&0&1\\ 1&1&1&1&1&1\\ 0&0&1&0&0&1 \end{bmatrix};$$

$$M_d^\delta = \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 0&0&0&0&1&0\\ 1&1&1&1&1&1 \end{bmatrix}.$$

By comparing matrices $M_{\{a_1\}}, M_{\{a_2\}}, M_{\{a_3\}}$ with $M_d^\sigma$, we can find that vectors of 1st, 2nd, 3rd, and 5th rows in matrices $M_{\{a_1\}}, M_{\{a_2\}}, M_{\{a_3\}}$ are less than those in matrix $M_d^\sigma$ respectively but it is not not satisfied for 4th and 6th rows. So this ordered information system doesn't have 1st order possible distribution reduction. And the first order possible distribution matrices are as follows:

$$FM_{\{a_1\}}^{\sigma(1)} = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&1&1&1\\ 0&0&0&0&0&0\\ 0&1&0&0&1&1 \end{bmatrix};$$

$$FM^{\sigma(1)}_{\{a_2\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix};$$

$$FM^{\sigma(1)}_{\{a_3\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Furthermore, 2nd order possible distribution matrices are

$$M^{\sigma(2)}_{\{a_1 a_2\}} = FM^{\sigma(1)}_{\{a_1\}} \cap FM^{\sigma(1)}_{\{a_2\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix};$$

$$M^{\sigma(2)}_{\{a_1 a_3\}} = FM^{\sigma(1)}_{\{a_1\}} \cap FM^{\sigma(1)}_{\{a_3\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

$$M^{\sigma(2)}_{\{a_2 a_3\}} = FM^{\sigma(1)}_{\{a_2\}} \cap FM^{\sigma(1)}_{\{a_3\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we can see that $M^{\sigma(2)}_{\{a_1 a_3\}} = M^{\sigma(2)}_{\{a_2 a_3\}}$, and all row vectors of them are less than those of $M^{\sigma}_{d}$ respectively, by comparing $M^{\sigma(2)}_{\{a_1 a_2\}}, M^{\sigma(2)}_{\{a_1 a_3\}}, M^{\sigma(2)}_{\{a_2 a_3\}}$ and $M^{\sigma}_{d}$. Hence, we can obtain all the second order possible distribution reductions, which are $\{a_1, a_2\}$,$\{a_2, a_3\}$.

From the above, we have

$$FM^{\sigma(2)}_{\{a_1 a_3\}} = FM^{\sigma(2)}_{\{a_2 a_3\}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $k = 2 < 3$ and $k + 1 = 3$, the algorithm is terminated.

Thus, all possible distribution reductions are $\{a_1, a_3\}$, $\{a_2, a_3\}$ in this example.

Similarly, by the above algorithm, we can have that there is only one compatible distribution reduction $\{a_2\}$ in this inconsistent ordered information system.

From this example, we can find that the results are same as those calculated by discernibility matrices in Example 3.2. It can be easily and intuitively concluded that the matrix computation algorithm is not only valid but also easily operated for large-scale information systems since matrix is a very useful and handy tool in computing.

## 5. Experimental computing program and case study

Experimental computing program can be designed and carried out so as to apply the matrix computation algorithm studied in this paper more directly and applicable. The main design process of the program will be introduced by the flow chart in this section and cases are employed to verify the program. The algorithm can be explained and certified by the cases. The small-scale case will be calculated manually by discernibility matrices to compare and verify the results with those calculated by the program. The large-scale case will be used to certified the effective and applicable of the algorithm we discussed in this paper. From Algorithm 1 the above section, the process of the program can be designed and listed in the following Fig. 1: The flow chart of the program.

The small-scale example used in front sections is firstly calculated by the program and the results are same to those calculated manually by discernibility matrices. In the following of this section, the small-scale and large-scale cases will be illustrated and calculated. Results can explain and certify the matrix algorithm we discussed in this paper.

This experimental computing program is running on a personal computer with the following hardware and software configuration.

| Names | Model | Parameters |
|---|---|---|
| CPU | Intel i5–2410 | 2.3 GHz |
| Memory | Samsung DDR3 SDRAM | 2 × 2 GB 1333 MHz |
| Hard disk | West Data | 640 GB |
| System | Windows 7 | 32bit |
| Platform | Matlab | Leasehold |

The following small-scale case is used to test and verify the experimental computing program. The possible and compatible distribution reductions will be calculated manually by discernibility matrices and the program. Results will be compared to verify that the algorithm we studied in this paper is correct and effectivt.

**Case 5.1.** A small-scale inconsistent ordered information system is presented in the following Table 4. Calculate possible and compatible distribution reductions of Table 4.

The condition classes and decision classes are listed, respectively, as follows:

$$[x_1]^{\succeq}_{C} = \{x_1\}; \qquad [x_2]^{\succeq}_{C} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\};$$
$$[x_3]^{\succeq}_{C} = \{x_1, x_3, x_4, x_5, x_8\}; \qquad [x_4]^{\succeq}_{C} = \{x_1, x_4, x_5, x_8\};$$
$$[x_5]^{\succeq}_{C} = \{x_5, x_8\}; \qquad [x_6]^{\succeq}_{C} = \{x_6\};$$
$$[x_7]^{\succeq}_{C} = \{x_1, x_3, x_4, x_5, x_7, x_8\}; \qquad [x_8]^{\succeq}_{C} = \{x_8\}\}.$$

$$D_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} = [x_1]^{\succeq}_{d} = [x_3]^{\succeq}_{d} = [x_5]^{\succeq}_{d};$$
$$D_2 = \{x_2, x_4, x_6, x_7, x_8\} = [x_2]^{\succeq}_{d} = [x_4]^{\succeq}_{d} = [x_7]^{\succeq}_{d};$$
$$D_3 = \{x_6, x_8\} = [x_6]^{\succeq}_{d} = [x_8]^{\succeq}_{d}.$$

Distribution functions are the inception of computation and the foundation to define distribution reductions. From Section 3, the possible distribution function and compatible distribution function can be calculated according to the definition and presented in the following.

$$\sigma^{\succeq}_{C}(x_1) = \{D_1\};$$
$$\sigma^{\succeq}_{C}(x_2) = \sigma^{\succeq}_{C}(x_3) = \sigma^{\succeq}_{C}(x_4) = \sigma^{\succeq}_{C}(x_5) = \sigma^{\succeq}_{C}(x_6) = \sigma^{\succeq}_{C}(x_7) = \sigma^{\succeq}_{C}(x_8) = \{D_1, D_2, D_3\};$$
$$\delta^{\succeq}_{C}(x_1) = \delta^{\succeq}_{C}(x_2) = \delta^{\succeq}_{C}(x_3) = \delta^{\succeq}_{C}(x_4) = \delta^{\succeq}_{C}(x_5) = \delta^{\succeq}_{C}(x_7) = \{D_1\};$$
$$\delta^{\succeq}_{C}(x_6) = \delta^{\succeq}_{C}(x_8) = \{D_1, D_2, D_3\}.$$

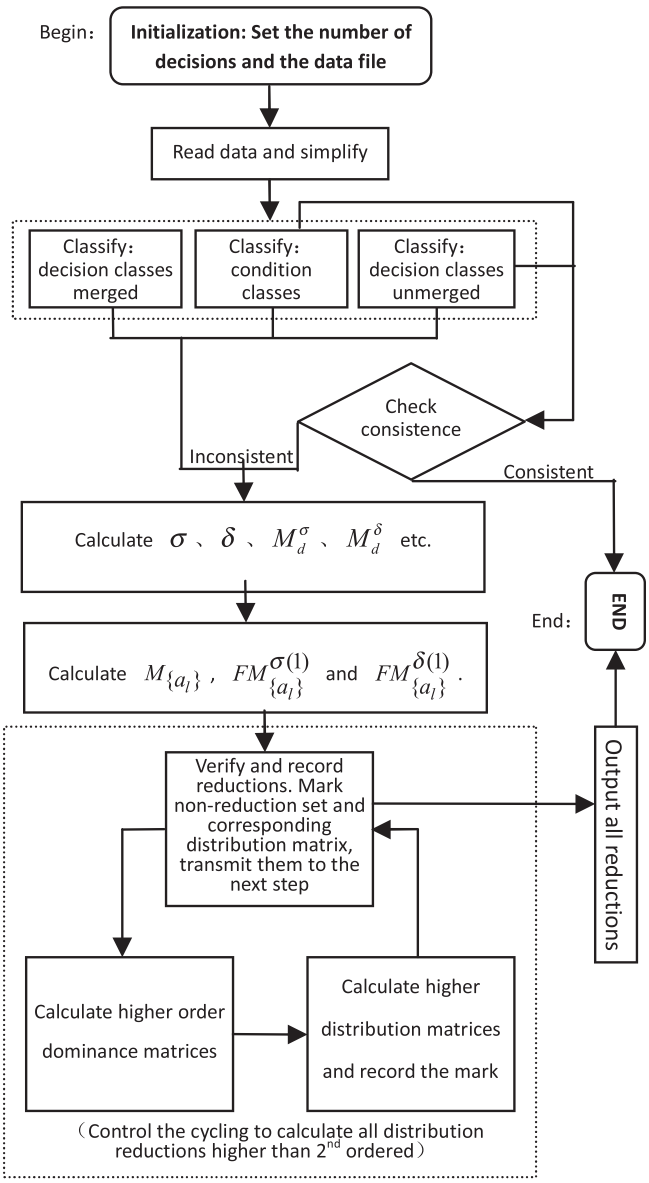


**Fig. 1.** The flow chart of the program.

**Table 4**
$\mathcal{I}_d^{\succeq}$: An ordered information system with decision $d$.

| $(U, C \cup \{d\})$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $d$ |
|---|---|---|---|---|---|---|
| $x_1$ | 2 | 3 | 3 | 2 | 3 | 0 |
| $x_2$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $x_3$ | 2 | 2 | 1 | 2 | 2 | 0 |
| $x_4$ | 2 | 2 | 2 | 2 | 2 | 1 |
| $x_5$ | 3 | 2 | 2 | 3 | 3 | 0 |
| $x_6$ | 2 | 3 | 2 | 3 | 1 | 2 |
| $x_7$ | 1 | 1 | 1 | 1 | 2 | 1 |
| $x_8$ | 3 | 2 | 3 | 3 | 3 | 2 |

Moreover, according to Definition 3.2, we can obtain that

$$D^*_{\sigma^{\succeq}} = \{(x_1,x_2),(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_1,x_6),(x_1,x_7),(x_1,x_8)\};$$
$$D^*_{\delta^{\succeq}} = \{(x_6,x_1),(x_6,x_2),(x_6,x_3),(x_6,x_4),(x_6,x_5),(x_6,x_7),$$
$$(x_8,x_1),(x_8,x_2),(x_8,x_3),(x_8,x_4),(x_8,x_5),(x_8,x_7)\}.$$

The distribution discernibility matrices rely on $D^*_{\sigma^{\succeq}}$ and $D^*_{\delta^{\succeq}}$. Then, according to what we have discussed in Section 3, the possible and compatible distribution discernibility matrices of this inconsistent ordered information system can be calculated from Definition 3.2. These two matrices are showed, respectively, in the following Tables 5 and 6.

**Table 5**
Possible distribution discernibility matrix $\mathcal{M}_{\sigma^{\succeq}}$ of Table 4.

| $D_{\sigma^{\succeq}}(x_i,x_j)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_2$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_3$ | $x_2, x_3, x_5$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_4$ | $x_2, x_3, x_5$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_5$ | $x_2, x_3$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_6$ | $x_3, x_5$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_7$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_8$ | $x_2$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |

**Table 6**
Compatible distribution discernibility matrix $\mathcal{M}_{\delta^{\succeq}}$ of Table 4.

| $D_{\sigma^{\succeq}}(x_i,x_j)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | $C$ | $C$ | $C$ | $C$ | $C$ | $x_4$ | $C$ | $x_1, x_4$ |
| $x_2$ | $C$ | $C$ | $C$ | $C$ | $C$ | $x_1, x_2, x_3, x_4$ | $C$ | $C$ |
| $x_3$ | $C$ | $C$ | $C$ | $C$ | $C$ | $x_2, x_3, x_4$ | $C$ | $x_1, x_3, x_4, x_5$ |
| $x_4$ | $C$ | $C$ | $C$ | $C$ | $C$ | $x_2, x_4$ | $C$ | $x_1, x_3, x_4, x_5$ |
| $x_5$ | $C$ | $C$ | $C$ | $C$ | $C$ | $x_2$ | $C$ | $x_3$ |
| $x_6$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| $x_7$ | $C$ | $C$ | $C$ | $C$ | $C$ | $x_1, x_2, x_3, x_4$ | $C$ | $C$ |
| $x_8$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |

From the above two matrices, the discernibility formulas $F_{\sigma^{\geqslant}}$ and $F_{\delta^{\geqslant}}$ can be calculated. Furthermore, the possible and compatible distribution reductions can be obtained.

$$F_{\sigma^{\succeq}} = (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_2 \vee a_3 \vee a_5) \wedge (a_2 \vee a_3) \wedge (a_3 \vee a_5) \wedge a_2$$
$$= (a_3 \vee a_5) \wedge a_2 = (a_2 \wedge a_3) \vee (a_2 \wedge a_5);$$

$$F_{\delta^{\succeq}} = (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5) \wedge (a_1 \vee a_2 \vee a_3 \vee a_4)$$
$$\wedge (a_1 \vee a_3 \vee a_4 \vee a_5) \wedge (a_2 \vee a_4) \wedge a_2 \wedge a_3 \wedge a_4$$
$$= a_2 \wedge a_3 \wedge a_4.$$

We calculate the distribution reduction by the experimental computing program and the results are the same to the above ones obtained by discernibility matrices. We acquire that $\{a_2, a_3\}$,$\{a_2, a_5\}$ are possible distribution reductions and $\{a_2, a_3, a_4\}$ is the only compatible distribution reduction. The total operating time to compute this case is 0.653156 s.

During the computing of the above case, we can conclude that the algorithm we proposed in this paper is more easily computing than the discernibility matrix method by computers. The only numbers 0 and 1 in the matrix algorithm can be presented and computed more easily and faster than the discernibility matrix method. This is obvious from the binary data representation in computer internal operating.

In the next case, we employ a large-scale ordered information system to verify the effective of the algorithm we studied in this paper.

**Case 5.2.** An inconsistent ordered information system on animals sleep is presented in Table 7.

The information system is denoted by $\mathcal{I}_d^{\succeq} = (U, C \cup \{d\}, V, f)$, where $C$ is the condition attribute set and $d$ is the single dominance decision. There are 42 objects which represent the species of animals and 10 attributes with numerical values in the ordered information system. The animals' names are showed in Table 7 and the interpretations of the attributes will be listed following. The interpretations and the units of attributes are represented as follows:

| | |
|---|---|
| $a_1$—Bodyweight in kg | $a_6$—Maximum life span (years) |
| $a_2$—Brain weight in g | $a_7$—Gestation time (days) |
| $a_3$—Show wave ("non-dreaming") sleep (hrs/day) | $a_8$—Predation index (1–5) |
| $a_4$—Paradoxical ("dreaming") sleep (hrs/day) | $a_9$—Sleep exposure index (1–5) |
| $a_5$—Total sleep (hrs/day) | $d$—Overall danger index (1–5) |

By the experimental computing program, the possible and compatible distribution reductions can be calculated and they are represented in the following. The operating time to compute this case is 11.529617 seconds.

The possible distribution reductions are $\{x_3,x_4,x_6,x_7\}$ and $\{x_4,x_5,x_6,x_7\}$.

The compatible distribution reductions are $\{x_6,x_8,x_9\}$ and $\{x_1,x_2,x_8,x_9\}$.

Denote one of the reductions by $B$. Then, the distribution reductions can be verified by computing the possible distribution functions $\sigma_C^{\succeq}(x), \sigma_B^{\succeq}(x)$ and the compatible distribution functions $\delta_C^{\succeq}(x), \delta_B^{\succeq}(x)$. There are 42 condition classes and 5 decision classes in this ordered information system. Therefore, the distribution functions can be calculated by computers and the results are presented in the following Table 8.

The subsets in Table 8 are, respectively, $B_1 = \{x_3,x_4,x_6,x_7\}$, $B_2 = \{x_4,x_5,x_6,x_7\}$, $B_3 = \{x_6,x_8,x_9\}$ and $B_4 = \{x_1,x_2,x_8,x_9\}$. From Table 8, one can compare the distribution functions and obtain that $B_1,B_2$ are sets keep possible distribution functions invariant and $B_3$, $B_4$ are sets keep compatible distribution functions invariant. That is, for any $x_i \in U, \sigma_{B_1}^{\succeq}(x_i) = \sigma_C^{\succeq}(x_i), \sigma_{B_2}^{\succeq}(x_i) = \sigma_C^{\succeq}(x_i), \delta_{B_3}^{\succeq}(x_i) = \delta_C^{\succeq}(x_i)$, and $\delta_{B_4}^{\succeq}(x_i) = \delta_C^{\succeq}(x_i)$. Furthermore, we can conclude that all proper subset of these sets, correspondingly, are not possible and compatible distribution consistent sets. Hence, $B_1$, $B_2$ are possible distribution reductions and $B_3$, $B_4$ are compatible distribution reductions. The verifying on the results is completed.

By the experimental computing program, we can control the codes to output the possible distribution matrices and the compatible distribution matrices. From the output results, we can obtain that $FM_{B_1}^{\sigma(|B_1|)} = 0, FM_{B_2}^{\sigma(|B_2|)} = 0, FM_{B_3}^{\delta(|B_3|)} = 0$ and $FM_{B_4}^{\delta(|B_4|)} = 0$. Thus, the algorithm presented in the last section is correct and effective.

The computing time of the algorithm to different information systems may be different. And the time relies on the scale of the information system and how to deal with the matrices with only 0 and 1, sometimes with large amount of 0 in this algorithm.

**Table 7**
$\mathcal{I}_d^{\succeq}$: An inconsistent ordered information system on animals sleep.

| | $(U, C \cup \{d\})$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $d$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$: | African giant pouched rat | 1 | 6.6 | 6.3 | 2 | 8.3 | 4.5 | 42 | 3 | 1 | 3 |
| $x_2$: | Asian elephant | 2547 | 4603 | 2.1 | 1.8 | 3.9 | 69 | 624 | 3 | 5 | 4 |
| $x_3$: | Baboon | 10.55 | 179.5 | 9.1 | 0.7 | 9.8 | 27 | 180 | 4 | 4 | 4 |
| $x_4$: | Big brown bat | 0.023 | 0.3 | 15.8 | 3.9 | 19.7 | 19 | 35 | 1 | 1 | 1 |
| $x_5$: | Brazilian tapir | 160 | 169 | 5.2 | 1 | 6.2 | 30.4 | 392 | 4 | 5 | 4 |
| $x_6$: | Cat | 3.3 | 25.6 | 10.9 | 3.6 | 14.5 | 28 | 63 | 1 | 2 | 1 |
| $x_7$: | Chimpanzee | 52.16 | 440 | 8.3 | 1.4 | 9.7 | 50 | 230 | 1 | 1 | 1 |
| $x_8$: | Chinchilla | 0.425 | 6.4 | 11 | 1.5 | 12.5 | 7 | 112 | 5 | 4 | 4 |
| $x_9$: | Cow | 465 | 423 | 3.2 | 0.7 | 3.9 | 30 | 281 | 5 | 5 | 5 |
| $x_{10}$: | Eastern American mole | 0.075 | 1.2 | 6.3 | 2.1 | 8.4 | 3.5 | 42 | 1 | 1 | 1 |
| $x_{11}$: | Echidna | 3 | 25 | 8.6 | 0 | 8.6 | 50 | 28 | 2 | 2 | 2 |
| $x_{12}$: | European hedgehog | 0.785 | 3.5 | 6.6 | 4.1 | 10.7 | 6 | 42 | 2 | 2 | 2 |
| $x_{13}$: | Galago | 0.2 | 5 | 9.5 | 1.2 | 10.7 | 10.4 | 120 | 2 | 2 | 2 |
| $x_{14}$: | Goat | 27.66 | 115 | 3.3 | 0.5 | 3.8 | 20 | 148 | 5 | 5 | 5 |
| $x_{15}$: | Golden hamster | 0.12 | 1 | 11 | 3.4 | 14.4 | 3.9 | 16 | 3 | 1 | 2 |
| $x_{16}$: | Gray seal | 85 | 325 | 4.7 | 1.5 | 6.2 | 41 | 310 | 1 | 3 | 1 |
| $x_{17}$: | Ground squirrel | 0.101 | 4 | 10.4 | 3.4 | 13.8 | 9 | 28 | 5 | 1 | 3 |
| $x_{18}$: | Guinea pig | 1.04 | 5.5 | 7.4 | 0.8 | 8.2 | 7.6 | 68 | 5 | 3 | 4 |
| $x_{19}$: | Horse | 521 | 655 | 2.1 | 0.8 | 2.9 | 46 | 336 | 5 | 5 | 5 |
| $x_{20}$: | Lesser short-tailed shrew | 0.005 | 0.14 | 7.7 | 1.4 | 9.1 | 2.6 | 21.5 | 5 | 2 | 4 |
| $x_{21}$: | Little brown bat | 0.01 | 0.25 | 17.9 | 2 | 19.9 | 24 | 50 | 1 | 1 | 1 |
| $x_{22}$: | Man | 62 | 1320 | 6.1 | 1.9 | 8 | 100 | 267 | 1 | 1 | 1 |
| $x_{23}$: | Mouse | 0.023 | 0.4 | 11.9 | 1.3 | 13.2 | 3.2 | 19 | 4 | 1 | 3 |
| $x_{24}$: | Musk shrew | 0.048 | 0.33 | 10.8 | 2 | 12.8 | 2 | 30 | 4 | 1 | 3 |
| $x_{25}$: | N. American opossum | 1.7 | 6.3 | 13.8 | 5.6 | 19.4 | 5 | 12 | 2 | 1 | 1 |
| $x_{26}$: | Nine-banded armadillo | 3.5 | 10.8 | 14.3 | 3.1 | 17.4 | 6.5 | 120 | 2 | 1 | 1 |
| $x_{27}$: | Owl monkey | 0.48 | 15.5 | 15.2 | 1.8 | 17 | 12 | 140 | 2 | 2 | 2 |
| $x_{28}$: | Patas monkey | 10 | 115 | 10 | 0.9 | 10.9 | 20.2 | 170 | 4 | 4 | 4 |
| $x_{29}$: | Phanlanger | 1.62 | 11.4 | 11.9 | 1.8 | 13.7 | 13 | 17 | 2 | 1 | 2 |
| $x_{30}$: | Pig | 192 | 180 | 6.5 | 1.9 | 8.4 | 27 | 115 | 4 | 4 | 4 |
| $x_{31}$: | Rabbit | 2.5 | 12.1 | 7.5 | 0.9 | 8.4 | 18 | 31 | 5 | 5 | 5 |
| $x_{32}$: | Rat | 0.28 | 1.9 | 10.6 | 2.6 | 13.2 | 4.7 | 21 | 3 | 1 | 3 |
| $x_{33}$: | Red fox | 4.235 | 50.4 | 7.4 | 2.4 | 9.8 | 9.8 | 52 | 1 | 1 | 1 |
| $x_{34}$: | Rhesus monkey | 6.8 | 179 | 8.4 | 1.2 | 9.6 | 29 | 164 | 2 | 3 | 2 |
| $x_{35}$: | Rock hyrax (Hetero.b) | 0.75 | 12.3 | 5.7 | 0.9 | 6.6 | 7 | 225 | 2 | 2 | 2 |
| $x_{36}$: | Rock hyrax (Procavia hab) | 3.6 | 21 | 4.9 | 0.5 | 5.4 | 6 | 225 | 3 | 2 | 3 |
| $x_{37}$: | Sheep | 55.5 | 175 | 3.2 | 0.6 | 3.8 | 20 | 151 | 5 | 5 | 5 |
| $x_{38}$: | Tenrec | 0.9 | 2.6 | 11 | 2.3 | 13.3 | 4.5 | 60 | 2 | 1 | 2 |
| $x_{39}$: | Tree hyrax | 2 | 12.3 | 4.9 | 0.5 | 5.4 | 7.5 | 200 | 3 | 1 | 3 |
| $x_{40}$: | Tree shrew | 0.104 | 2.5 | 13.2 | 2.6 | 15.8 | 2.3 | 46 | 3 | 2 | 2 |
| $x_{41}$: | Vervet | 4.19 | 58 | 9.7 | 0.6 | 10.3 | 24 | 210 | 4 | 3 | 4 |
| $x_{42}$: | Water opossum | 3.5 | 3.9 | 12.8 | 6.6 | 19.4 | 3 | 14 | 2 | 1 | 1 |

**Table 8**
Distribution functions of animals sleep ordered information system.

| $\sigma^{\succeq}_{C}(x_i)$ | $\delta^{\succeq}_{C}(x_i)$ | $\sigma^{\succeq}_{B_1}(x_i)$ | $\delta^{\succeq}_{B_1}(x_i)$ | $\sigma^{\succeq}_{B_2}(x_i)$ | $\delta^{\succeq}_{B_2}(x_i)$ | $\sigma^{\succeq}_{B_3}(x_i)$ | $\delta^{\succeq}_{B_3}(x_i)$ | $\sigma^{\succeq}_{B_4}(x_i)$ | $\delta^{\succeq}_{B_4}(x_i)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ |
| $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ |
| $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_3,D_5\}$ |
| $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_3,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3,D_5\}$ |
| $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_1,D_2,D_3,D_5\}$ |
| $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ | $\{D_1,D_2,D_3,D_4,D_5\}$ | $\{D_3\}$ |

Probable mathematical and computing method can reduce computing time. Then, fast computing programs can be designed and programmed further.

In this section, experimental computing program is designed and the flow chart is presented. Cases are employed to interpret the algorithm we proposed in this paper. A small-scale case is calculated manually by discernibility matrices and the experimental computing program. Results are compared to verify the correction of the algorithm. A large-scale case is calculated by the program and the results are verified through the definitions of possible and compatible distribution reductions. Through these discussion, the algorithm is certified by examples and cases. It is a correct and effective algorithm to acquire possible and compatible distribution reductions in inconsistent ordered information systems.

## 6. Conclusions

To acquire brief decision rules from inconsistent information systems, knowledge reductions are needed. Many types of attribute reductions have been proposed based on the Rough Set Theory, each of them aimed at a different requirement. It is well known that most of information systems are based on dominance relations because of various factors in practise. Therefore, it is meaningful to study the attribute reductions in inconsistent ordered information systems (IOIS). In this paper, possible distribution reduction and compatible distribution reduction were proposed in IOIS. The properties and relationships between them were further discussed. The dominance matrix and decision distribution matrix were proposed in IOIS. Algorithm of matrix computation to acquire possible and compatible distribution reductions were introduced and studied, from which we could provide another approach to attributes reductions in IOIS except discernibility matrix method. An experimental computing program was designed and the flow chart was pictured in this paper. Examples and cases were employed to help interpret and understand what we have studied.

Though ordered information systems we discussed are inconsistent, they are all complete. Since incomplete information systems are more complicated than complete information systems, further research of attribute reductions for different requirements in incomplete ordered information systems are needed. In further research, we will develop the proposed approaches to more generalized and more complicated ordered information systems such as incomplete ordered information systems and fuzzy ordered information systems.

## Acknowledgements

The authors thank the anonymous reviewers for their constructive comments on this study. This work is supported by National Natural Science Foundation of China (No. 61105041, 71071124 and 11001227), Postdoctoral Science Foundation of China (No. 20100481331) and Natural Science Foundation Project of CQ CSTC (No. cstc2011jjA40037).

## References


[1] M. Beynon, Reducts within the variable precision rough sets model: a further investigation, European Journal of Operational Research 134 (2001) 592–605.
[2] K. Dembczynski, R. Pindur, R. Susmaga, Generation of exhaustive set of rules within Dominance-based rough set approach, Electronic Notes Theory Computer Science 82 (4) (2003).
[3] K. Dembczynski, R. Pindur, R. Susmaga, Dominance-based rough set classifier without induction of decision rules, Electronic Notes Theory Computer Science 82 (4) (2003).
[4] L. Feng, T.R. Li, D. Ruan, S.R. Gou, A vague-rough set approach for uncertain knowledge acquisition Original Research Article, Knowledge-Based Systems 6 (2011) 837–843.
[5] S. Greco, B. Matarazzo, R. Slowinski, Rough approximation of a preference relation by dominance relatioin. ICS Research Report 16/ 96, Warsaw University of Technology; 1996 and in Europe Journal of Operation Research 117 (1999) 63–83.
[6] S. Greco, B. Matarazzo, R. Slowinski, A new rough set approach to multicriteria and multiattribute classificatioin, in: L. Polkowsik, A. Skowron (Eds.), Rough sets and current trends in computing (RSCTC'98), Lecture Notes in Artificial Intelligence, vol. 1424, Springer, Verlag, Berlin, 1998, pp. 60–67.
[7] S. Greco, B. Matarazzo, R. Slowinski, A new rough sets approach to evaluation of bankruptcy risk, in: X. Zopounidis (Ed.), Operational Tools in the Management of Financial Risks, Kluwer, Dordrecht, 1999, pp. 121–136.
[8] S. Greco, B. Matarazzo, R. Slowinski, Rough set theory for multicriteria decision analysis, Europe Journal of Operation Research 129 (2001) 11–47.
[9] S. Greco, B. Matarazzo, R. Slowinski, Rough sets methodology for sorting problems in presence of multiple attributes and criteria, Europe Journal of Operation Research 138 (2002) 247–259.
[10] S. Greco, B. Matarazzo, R. Slowinski, Dominance-based Rough Set approach as a proper way of handling graduality in rough set theory, Transaction on Rough Sets VII, Lecture Notes in Computer Science 4400 (2007) 36–52.
[11] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, New York, 1979.
[12] P. Guo, Rough set feature extraction by remarkable degrees with real world decision-making problems, Soft Computing, doi: 10.1007/s00500-009-0494-1
[13] Q. He, C.X. Wu, D.G. Chen, S.Y. Zhao, Fuzzy rough set based attribute reduction for information systems with fuzzy decisions, Knowledge-Based Systems 5 (2011). 689-69.
[14] J. Komorowski, Z. Pawlak, L. Polkowski, A. Skowron, Rough sets: tutorial, in: S.K. Pal, A. Skowron (Eds.), Rough fuzzy hybridization, A New Trend in Decision Making, Springer, 1999, pp. 3–98.
[15] M. Kryszkiewicz, Comparative study of alternative type of knowledge reduction ininconsistent systems, International Journal of Intelligent Systems 16 (2001) 105–120.
[16] M. Kryszkiewicz, Rough set approach to incomplete information systems, Information Sciences 112 (1998) 39–49.
[17] Y. Leuang, W.Z. Wu, W.X. Zhang, Knowledge acquisition in incomplete information systems: a rough set approach, European Journal of Operational Research 168 (1) (2006) 164–180.
[18] H.S. Nguyen, D. Slezak, Approximation reducts and association rules correspondence and complexity results, in: N. Zhong, A. Skowron, S. Oshuga (Eds.), Proceedings of RSFDGrC'99, Yamaguchi, Japan, LNAI 1711, 1999, pp. 137–145.
[19] Z. Pawlak, Rough sets, International Journal of Computer and Information Science 11 (1982) 341–356.
[20] Z. Pawlak, Rough Sets: Theoretical Aspects of Reasoning About Data, Kluwer Academic Publishers, Boston, 1991.
[21] Z. Pawlak, A. Skowron, Rudiments of rough sets, Information Sciences 177 (2007) 3–27.
[22] Z. Pawlak, A. Skowron, Rough sets: some extensions, Information Sciences 177 (2007) 28–40.
[23] Y.H. Qian, J.Y. Liang, D.Y. Li, F. Wang, N.N. Ma, Approximation reduction in inconsistent incomplete decision tables, Knowledge-Based Systems 5 (2010) 427–433.
[24] M. Quafatou, a-RST: a generalization of rough set theory, Information Sciences 124 (2000) 301–316.
[25] M.W. Shao, W.X. Zhang, Dominance relation and relus in an incomplete ordered information system, Int J Intell Syst 20 (2005) 13–27.
[26] A. Skowron, J. Stepaniuk, Tolerance approxximation space, Fundamental Information 27 (1996) 245–253.
[27] D. Slezak, Approximate reducts in decision tables, in: Proceedings of IPMU'96, Granada, Spain, vol. 3, 1996, pp. 1159–1164.
[28] D. Slezak, Searching for dynamic reducts in inconsistent decision tables, in: Proceedings of IPMU'98, Paris, France, vol. 2, 1998, pp. 1362–1369.
[29] R. Slowinski, C. Zopounidis, A.I. Dimitras, Prediction of company acquisition in Greece by means of the rough set approach, European Journal of Operational Research 100 (1997) 1–15.
[30] J. Stefanowski, On rough set based approaches to induction of decision rules, in: J. Polkowski, A. Skowron (Eds.), Rough Sets in Knowledge Discovery, vol. 1, Physica-Verlag, Heidelberg, 1998, pp. 500–529.
[31] R. Susmaga, R. Slowinski, S. Greco, B. Matarazzo, Generation of reducts and rules in multi-attribute and multi-criteria classification, Control and Cybernetics 4 (2000) 969–988.
[32] K. Thangavel, A. Pethalakshmi, Dimensionality reduction based on rough set theory: A review, Applied Soft Computing 9 (2009) 1–12.
[33] W.Z. Wu, Y. Leung, W.X. Zhang, Connections between rough set theory and Dempster-Shafer theory of evidence, International Journal of General Systems 31 (2002) 405–430.
[34] W.Z. Wu, M. Zhang, H.Z. Li, J.S. Mi, Knowledge reduction in random information systems via Dempster-Shafer theory of evidence, Information Sciences 174 (2005) 143–164.
[35] Y.T. Xu, L.S. Wang, R.Y. Zhang, A dynamic attribute reduction algorithm based on 0-1 integer programming Original Research Article, Knowledge-Based Systems 8 (2011) 1341–1347.
[36] W.H. Xu, W.X. Zhang, Measuring roughness of generalized rough sets induced by a covering, Fuzzy Sets and Systems 158 (2007) 2443–2455.
[37] W.H. Xu, X.Y. Zhang, W.X. Zhang, Knowledge granulation, knowledge entropy and knowledge uncertainty measurein ordered information systems, Applied Soft Computing 9 (2009) 1244–1251.

[38] W.H. Xu, X.Y. Zhang, J.M. Zhong, W.X. Zhang, Attribute reduction in ordered information systems based on evidence theory, Knowledge and Information Systems 25 (2010) 169–184.

[39] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators. Information, Sciences 101 (1998) 239–259.

[40] D.R. Yu, Q.H. Hu, C.X. Wu, Uncertainty measures for fuzzy relations and their applications, Applied Soft Computing 7 (2007) 1135–1143.

[41] W.X. Zhang, W.Z. Wu, J.Y. Liang, D.Y. Li, Theory and Method of Rough Sets, Science Press, Beijing, 2001.