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Research Article

Rough Set Approach to Approximation Reduction in Ordered Decision Table with Fuzzy Decision

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In practice, some of information systems are based on dominance relations, and values of decision attribute are fuzzy. So, it is meaningful to study attribute reductions in ordered decision tables with fuzzy decision. In this paper, upper and lower approximation reductions are proposed in this kind of complicated decision table, respectively. Some important properties are discussed. The judgement theorems and discernibility matrices associated with two reductions are obtained from which the theory of attribute reductions is provided in ordered decision tables with fuzzy decision. Moreover, rough set approach to upper and lower approximation reductions is presented in ordered decision tables with fuzzy decision as well. An example illustrates the validity of the approach, and results show that it is an efficient tool for knowledge discovery in ordered decision tables with fuzzy decision.

1. Introduction

Rough set theory, which was first proposed by Pawlak in the early 1980s [1], can describe knowledge via set-theoretic analysis based on equivalence classification for the universe of discourse. It provides a theoretical foundation for inference reasoning about data analysis and has extensive applications in areas of artificial intelligence and knowledge acquisition.

A primary use of rough set theory is to reduce the number of attributes in databases thereby improving the performance of applications in a number of aspects including speed, storage, and accuracy. For a data set with discrete attribute values, this can be done by reducing the number of redundant attributes and find a subset of the original attributes that are the most informative. As is well known, an information system may usually has more than one reduct. It means that the set of rules derived from knowledge reduction is not unique. In practice, it is always hoped to obtain the set of the most concise rules. Therefore, people have been attempting to find the minimal reduct of information systems, which means that the number of attributes contained in the reduction is minimal. Unfortunately, it has been proven that finding the minimal reduct of an information system is an NP-hard problem.

Recently, some new theories and reduction methods have been developed. Many types of knowledge reduction have been proposed in the area of rough sets [2–8]. Possible rules and reducts have been proposed as a way to deal with inconsistence in an inconsistent decision table [9]. Approximation rules [10] are also used as an alternative to possible rules. On the other hand, generalized decision rules and reducts [9] provide a decision with more flexible selection of decision behavior. In [11], the notions of α-reduct and α-relative reduct for decision tables are defined. The α-reduct allows occurrence of additional inconsistency that is controlled by means of a parameter. In [12], Slezak presented a new concept of attribute reduction that keeps the class membership distribution unchanging for all objects in the information system. It was shown by Slezak [13] that the knowledge reduction preserving the membership distribution is equivalent to the knowledge reduction preserving the value of generalized inference measure function. A generalized knowledge reduction was also introduced in [13] that allows the value of generalized inference measure function after the attribute reduction to be different from the original one by user-specified threshold. By eliminating the rigorous conditions required by distribution reduct, maximum distribution reduct was introduced by Zhang et al. in [14].

Partition or equivalence (indiscernibility relation) is an important and primitive concept in Pawlak's original rough set theory. However, partition or equivalence relation is still restrictive for many applications. To overcome this limitation, classical rough sets have been extended to several interesting and meaningful general models in recent years by proposing other binary relations, like tolerance relations [15], neighborhood operators [16], and others [17–24]. However, the original rough set theory does not consider attributes with preference ordered domain, that is, criteria. Particularly, in many real situations, we are often faced with the problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco et al. [25–31] proposed an extension rough set theory, called the dominance-based rough set approach (DRSA), to take into account the ordering properties of criteria. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. In DRSA, condition attributes are criteria and classes are preference ordered; the knowledge approximated is a collection of upward, and downward unions of classes and the dominance classed are sets of objects defined by using a dominance relation. In recent years, several studies have been made about properties and algorithmic implementations of DRSA [23, 32–35].

Nevertheless, only a limited number of methods using DRSA to acquire knowledge from in inconsistent ordered information systems have been proposed. Pioneering work on inconsistent ordered information systems with the DRSA has been proposed by Greco, Matarazzo, and Slowinski [1, 4–6, 17, 18], but they did not clearly point out the semantic explanation of unknown values. Shao and Zhang [36] further proposed an extension of the dominance relation in the IOIS. Yao [37] introduced the notion of high-order decision rules. While a standard decision rule expresses connections between attribute values of the same object, a high-order decision rule expresses connections of different objects in terms of their attribute values. Various types of relationships can be used, such as ordering relations, closeness relations, similarity relations, and neighborhood systems on attribute values. The introduction of semantics information on attribute values leads to information tables with added semantics. Depending on the decision rules to be mined, one can transform the original table into another information table. Yao et al. [38] provided a granular computing-based interpretation of rules representing two levels of knowledge, which was done by adopting and adapting the decision logic language for granular computing. The language provided a formal method for describing and interpreting conditions in rules as granules and rules as relationships between granules. As examples, they examined rules in the standard rough set analysis and dominance-based rough set analysis. For the modeling of ordering problems, Sai et al. [39] generalized the notion of information tables to ordered information tables by adding order relations on attribute values. For mining ordering rules, they first transformed an ordered information table into binary information.

Despite this, these contributions did not mention the underlying concept of attribute reduction in ordered decision table with fuzzy decision and only proposed an approach to attribute reduction in consistent ordered information systems. Therefore, the purpose of this paper is to study approaches to attribute reductions in ordered decision tables with fuzzy decision.

The rest of this paper is organized as follows. To facilitate our discussion, some preliminary concepts are briefly recalled in Section 2. In Section 3, the concept of ordered decision table with fuzzy decision are introduced, and some important properties are discussed. In Section 4, upper approximation reduction and lower approximation are proposed for the complicated decision table. Moreover, the judgement theorems and discernibility matrices associated with two reductions are obtained, from which we can provide an approach to attributes reductions in ordered decision tables with fuzzy decision. In Section 5, the practical approaches to upper and lower approximation reduction are provided in ordered decision tables with fuzzy decision. Finally, we conclude the paper with a summary and outlook for further research.

2. Preliminaries

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in the source papers [25–31, 34]. A description has also been made in [14].

The notion of information system (sometimes called data tables, attribute-value systems, knowledge representation systems, etc.) provides a convenient tool for the representation of objects in terms of their attribute values.

An information system is an quadruple $\mathcal{I} = (U, AT, V, f)$, where

(i) $U = \{u_1, u_2, \ldots, u_n\}$ is a nonempty finite set of objects,

(ii) $AT = \{a_1, a_2, \ldots, a_p\}$ is a nonempty finite set of attributes,

(iii) $V = \bigcup_{a \in AT} V_a$ and V_a is a domain of attribute a,

(iv) $f : U \times AT \to V$ is a function such that $f(x, a) \in V_a$, for every $a \in AT$, $u \in U$, called an information function.

An information system with decision, which is also called to a decision table (DT), is a special case of an information system $\mathcal{I} = (U, C \cup \{d\}, V, f)$ and $C \cap \{d\} = \phi$, where set C and $\{d\}$ be condition attributes set and the decision attribute set, respectively.

In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion.

An information system is called an ordered information system (OIS) if all condition attributes are criteria. Moreover, a decision table (ODT) is called an ordered decision table if all attributes are criteria (See [25–31]).

In general, we denote an ordered information system by $\mathcal{I}^{\succcurlyeq} = (U, AT, V, f)$ and denote an ordered decision table by $\mathcal{T}_d^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$.

Assumed that the domain of a criterion $a \in AT$ is complete preordered by an outranking relation $\succcurlyeq_a$, then $u \succcurlyeq_a v$ means that u is at least as good as v with respect to criterion a for $a \in AT, u, v \in U$. And we can say that u dominates v. In the following, without any loss of generality, we consider criterions having a numerical domain; that is, $V_a \subseteq \mathcal{R}$ ($\mathcal{R}$ denotes the set of real numbers). Being of type gain; that is, $u \succcurlyeq_a v \Leftrightarrow f(u, a) \geq f(v, a)$ (according to increasing preference) or $u \succcurlyeq_a v \Leftrightarrow f(u, a) \leq f(v, a)$ (according to decreasing preference).

Without any loss of generality and for simplicity, in the following, we only consider condition attributes with increasing preference.

For a subset of attributes $B \subseteq AT$, we define $u \succcurlyeq_B v \Leftrightarrow$ for all $a \in B, f(u, a) \geq f(v, a)$, and that is to say that u dominates v with respect to all attributes in B.

For an ordered information system with decision, we say that u dominates v with respect to $B \subseteq C$ if $u \succcurlyeq_B v$, and denote by $u R_B^{\succcurlyeq} v$. That is

$$\begin{aligned} R_B^{\succcurlyeq} &= \{(u, v) \in U \times U \mid u \succcurlyeq_B v\} \\ &= \left\{(u, v) \in U \times U \mid f(u, a) \geq f(v, a), \forall a \in B\right\}, \end{aligned} \tag{2.1}$$

$R_B^{\succcurlyeq}$ is called dominance relation of ordered information system $\mathcal{I}_d^{\succcurlyeq}$.

If we denote

$$\begin{aligned} [u_i]_B^{\succcurlyeq} &= \left\{u_j \in U \mid (u_j, u_i) \in R_B^{\succcurlyeq}\right\} \\ &= \left\{u_j \in U \mid f(u_j, a) \geq f(u_i, a), \forall a \in B\right\}, \end{aligned} \tag{2.2}$$

then the following properties of a dominance relation are trivial by above definition.

Let $R_A^{\succcurlyeq}$ be a dominance relation. The following hold ([25–31]):

(i) $R_A^{\succcurlyeq}$ is reflexive, transitive, but not symmetric, so it is not an equivalence relation

(ii) if $B \subseteq A$, then $R_A^{\succcurlyeq} \subseteq R_B^{\succcurlyeq}$,

(iii) if $B \subseteq A$, then $[u_i]_A^{\succcurlyeq} \subseteq [u_i]_B^{\succcurlyeq}$,

(iv) if $u_j \in [u_i]_A^{\succcurlyeq}$, then $[u_j]_A^{\succcurlyeq} \subseteq [u_i]_A^{\succcurlyeq}$ and $[u_i]_A^{\succcurlyeq} = \cup\{[u_j]_A^{\succcurlyeq} | u_j \in [u_i]_A^{\succcurlyeq}\}$,

(v) $[u_j]_A^{\succcurlyeq} = [u_i]_A^{\succcurlyeq}$ if and only if $f(u_i, a) = f(u_j, a)$ for all $a \in A$,

(vi) $|[u_i]_B^{\succcurlyeq}| \geq 1$ for any $u_i \in U$,

(vii) $U/R_B^{\succcurlyeq}$ constitute a covering of U, that is, for every $u \in U$ we have that $[u]_B^{\succcurlyeq} \neq \phi$ and $\bigcup_{u \in U} [u]_B^{\succcurlyeq} = U$,

where $|\cdot|$ denotes cardinality of the set.

Table 1: A fuzzy decision table about investment projects $\mathcal{T}^{\succcurlyeq}_{\tilde{d}}$.

U	a_1	a_2	a_3	d
u_1	F	M	F	S
u_2	N	M	M	NS
u_3	F	F	M	VS
u_4	M	F	N	VS
u_5	N	N	M	NS
u_6	N	M	N	S

F: far; N: near; M: medium; S: suitable; Vs: very suitable; Ns: not suitable.

For any subset $X \subseteq U$ and $A \subseteq AT$ in $\mathcal{I}^{\succcurlyeq}$, the lower and upper approximation of X with respect to a dominance relation $R_A^{\succcurlyeq}$ could be defined as following (see [25–31]):

$$\underline{R_A^{\succcurlyeq}}(X) = \left\{ u \in U \mid [u]_A^{\succcurlyeq} \subseteq X \right\},$$
$$\overline{R_A^{\succcurlyeq}}(X) = \left\{ u \in U \mid [u]_A^{\succcurlyeq} \cap X \neq \phi \right\}. \tag{2.3}$$

Unlike classical rough set theory, one can find easily that $\underline{R_A^{\succcurlyeq}}(X) = \bigcup\{ [u]_A^{\succcurlyeq} \mid [u]_A^{\succcurlyeq} \subseteq X\}$ and $\overline{R_A^{\succcurlyeq}}(X) = \bigcup\{ [u]_A^{\succcurlyeq} \mid [u]_A^{\succcurlyeq} \cap X \neq \phi\}$ do not hold.

3. Ordered Decision Table with Fuzzy Decision

Traditionally, decision tables (DTs) are crisp, indicating that the conditions are specified in an exact manner. In many real worlds, crisp decision tables may be stringent. A potential problem of such DTs is that any measurement error is not taken into account. Fuzzy decision tables (FDTs) offer a solution to this problem. A fuzzy decision table (FDT) is an extended version of a crisp DT in order to deal with imprecise and vague decision situations [40, 41]. The extension amounts to the introduction of fuzzy sets in the condition and decision attributes sets of the crisp DT; the crisp condition and decision states are replaced with fuzzy conditions and decisions. A membership function need be given, so the decision maker can judge the extent to which a particular attribute level meets a particular condition by using the membership function.

A familiar example of a fuzzy decision table is shown in order to illustrate the meaning in Table 1. This table is a fuzzy decision table which describes the investment program of a real estate company, where $U = \{u_1, u_2, \ldots, u_6\}$ is the set of projects considered by the company, $C = \{a_1, a_2, a_3\}$ is the set of condition attributes, and d is a decision attribute, which means the suitability of the project. Moreover, $a_1, a_2,$ and a_3 stand for the distance between site of the project and *Hospital*, *School*, and *Central Business District*, respectively. And interpretations of values of condition and objective attributes are represented as follows.

It is obvious that condition and decision attributes are fuzzy environment in the DT of Table 1, that is to say that the DT is a fuzzy decision table.

In this contribution, we focus on the DT, in which values of contribution attributes set is crisp and value of decision attribute set is fuzzy. Moreover, we will consider the increasing preference of condition and decision attributes. In other words, we will discuss approximation reduction of an ordered decision table with fuzzy decision based on the rough set theory.

Definition 3.1. Let $\mathcal{T}_d^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table. If the value of decision attribute is fuzzy, in other words, $f(x,d) \in [0,1]$ for all $u \in U$, then the ordered decision table is called ordered decision table with fuzzy decision.

From above definition, one can find easily that the set of decision attribute is a fuzzy set on U, which reflects the degree of fuzzy decision in ordered decision table. So, these systems are different from the systems in which a decision is numeric value. For convenience, we denote set of decision attribute by $\widetilde{d}$ and denote an ordered decision table with fuzzy decision by $\mathcal{T}_{\widetilde{d}}^{\succcurlyeq}$. Thus, that is to say $\widetilde{d}(x) = f(x,d)$ for all $u \in U$ in an ordered decision table with fuzzy decision $\mathcal{T}_{\widetilde{d}}^{\succcurlyeq}$.

Let $\mathcal{T}_{\widetilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table and $B \subseteq C$, then the lower and lower approximation sets of $\widetilde{d}$ with respect to B, which are fuzzy sets, are denoted by $\underline{R_B^{\succcurlyeq}(\widetilde{d})}$ and $\overline{R_B^{\succcurlyeq}(\widetilde{d})}$, respectively. And their membership functions are defined as follows:

$$\underline{R_B^{\succcurlyeq}\left(\widetilde{d}\right)}(u_i) = \min\left\{\widetilde{d}(u_j) \mid u_j \in [u_i]_B^{\succcurlyeq}\right\},$$
$$\overline{R_B^{\succcurlyeq}\left(\widetilde{d}\right)}(u_i) = \max\left\{\widetilde{d}(u_j) \mid u_j \in [u_i]_B^{\succcurlyeq}\right\}. \tag{3.1}$$

It is obvious that these two approximation sets are fuzzy set.

Moreover, by the above definition and rough set theory, one can easily consider that the upper and lower approximation of $\widetilde{d}$ have the following properties in an ordered decision table with fuzzy decision.

Proposition 3.2. *Let $\mathcal{T}_{\widetilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision, then the following propositions hold:*

(i) *if $B \subseteq C$, then $\overline{R_C^{\succcurlyeq}(\widetilde{d})}(u) \le \overline{R_B^{\succcurlyeq}(\widetilde{d})}(u)$,*

(ii) *if $B \subseteq C$, then $\underline{R_B^{\succcurlyeq}(\widetilde{d})}(u) \le \underline{R_C^{\succcurlyeq}(\widetilde{d})}(u)$,*

(iii) *if $u_j \in [u_i]_B^{\succcurlyeq}$, then $\overline{R_B^{\succcurlyeq}(\widetilde{d})}(u_j) \le \overline{R_B^{\succcurlyeq}(\widetilde{d})}(u_i)$, where $u_i, u_j \in U$ and $B \subseteq C$,*

(iv) *if $u_j \in [u_i]_B^{\succcurlyeq}$, then $\underline{R_B^{\succcurlyeq}(\widetilde{d})}(u_i) \le \underline{R_B^{\succcurlyeq}(\widetilde{d})}(u_j)$, where $u_i, u_j \in U$ and $B \subseteq C$.*

Example 3.3. We still consider the fuzzy DT in Table 1. To depict the degree of decision attribute, we take the values as Table 2 according to opinions of some experts.

Naturally, the increasing preference of condition attribute a_i is

$$N \succcurlyeq_{a_i} M \succcurlyeq_{a_i} F. \tag{3.2}$$

Table 2: An ordered DT with fuzzy decision about investment projects $\mathcal{T}_{\tilde{d}}^{\succcurlyeq}$.

U	a_1	a_2	a_3	d
u_1	F	M	F	0.5
u_2	N	M	M	0.3
u_3	F	F	M	0.7
u_4	M	F	N	0.9
u_5	N	N	M	0.1
u_6	N	M	N	0.6

So, from the table, one can find that

$$\begin{aligned}
[u_1]_C^{\succcurlyeq} &= \{u_1, u_2, u_5, u_6\}, \\
[u_2]_C^{\succcurlyeq} &= \{u_2, u_5, u_6\}, \\
[u_3]_C^{\succcurlyeq} &= \{u_2, u_3, u_4, u_5, u_6\}, \\
[u_4]_C^{\succcurlyeq} &= \{u_4, u_6\}, \\
[u_5]_C^{\succcurlyeq} &= \{u_5\}, \\
[u_6]_C^{\succcurlyeq} &= \{u_6\}.
\end{aligned} \tag{3.3}$$

If take $X = \{u_2, u_3, u_5\}$, then

$$\begin{aligned}
\underline{R_C^{\succcurlyeq}}(X) &= \{u_5\}, \\
\overline{R_C^{\succcurlyeq}}(X) &= \{u_1, u_2, u_3, u_5\}.
\end{aligned} \tag{3.4}$$

Moreover, we can have that

$$\begin{aligned}
\underline{R_C^{\succcurlyeq}\left(\tilde{d}\right)} &= \frac{0.1}{u_1} + \frac{0.1}{u_2} + \frac{0.1}{u_3} + \frac{0.6}{u_4} + \frac{0.1}{u_5} + \frac{0.6}{u_6}, \\
\overline{R_C^{\succcurlyeq}\left(\tilde{d}\right)} &= \frac{0.6}{u_1} + \frac{0.6}{u_2} + \frac{0.9}{u_3} + \frac{0.9}{u_4} + \frac{0.1}{u_5} + \frac{0.6}{u_6}.
\end{aligned} \tag{3.5}$$

4. Theory of Approximation Reduction of ODT with Fuzzy Decision

The approximation reduction proposed by Mi et al. is an important attribute reduction, which can be used to simplify an inconsistent classical decision table [4]. So far, however, there is not any practical approach to attribute reduction in ordered decision tables with fuzzy decision. In this section, we present the notions of a lower approximation reduction and an upper approximation reduction in an ordered decision table with fuzzy decision and then deduce their some important properties, from which we introduce the theory of approximation reductions to this complicated decision table.

Definition 4.1. Let $\mathcal{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $B \subseteq C$.

(i) If $\overline{R_B^{\succcurlyeq}}(\tilde{d})(u_i) = \overline{R_C^{\succcurlyeq}}(\tilde{d})(u_i)$ for any $u_i \in U$, then we say that B is upper approximation consistent set of this information system. Moreover, if any proper subset of B is not the upper approximation set, then B is called to one upper approximation reduction of this information system.

(ii) If $\underline{R_B^{\succcurlyeq}}(\tilde{d})(u_i) = \underline{R_C^{\succcurlyeq}}(\tilde{d})(u_i)$ for any $u_i \in U$, then we say that B is lower approximation consistent set of this information system. Moreover, if any proper subset of B is not a lower approximation consistent set, then B is called to one lower approximation reduction of this information system.

From above definition, one can find that upper and lower approximation consistent sets preserve the upper and lower approximations of the fuzzy decision, respectively.

Example 4.2 (Continued from Example 3.3). Let us consider the upper and under approximation reductions of the ordered decision table with fuzzy decision in Table 2.

If take $A = \{a_2, a_3\}$, it can be easily checked that $[u_i]_A^{\succcurlyeq} = [u_i]_C^{\succcurlyeq}$, for $u_i \in U$. Hence, $\overline{R_A^{\succcurlyeq}}(\tilde{d})(u_i) = \overline{R_C^{\succcurlyeq}}(\tilde{d})(u_i)$ and $\underline{R_A^{\succcurlyeq}}(\tilde{d})(u_i) = \underline{R_C^{\succcurlyeq}}(\tilde{d})(u_i)$, that is to say that $\{a_2, a_3\}$ is one upper approximation consistent set, and it is also one under approximation consistent set.

Moreover, if take $B = \{a_1, a_2\}$, then we have

$$
\begin{aligned}
&[u_1]_B^{\succcurlyeq} = \{u_1, u_2, u_5, u_6\},\\
&[u_2]_B^{\succcurlyeq} = \{u_2, u_5, u_6\},\\
&[u_3]_B^{\succcurlyeq} = \{u_1, u_2, u_3, u_4, u_5, u_6\},\\
&[u_4]_B^{\succcurlyeq} = \{u_2, u_4, u_5, u_6\},\\
&[u_5]_B^{\succcurlyeq} = \{u_5\},\\
&[u_6]_B^{\succcurlyeq} = \{u_2, u_5, u_6\},\\
&\overline{R_B^{\succcurlyeq}}\left(\tilde{d}\right) = \frac{0.6}{u_1} + \frac{0.6}{u_2} + \frac{0.9}{u_3} + \frac{0.9}{u_4} + \frac{0.1}{u_5} + \frac{0.6}{u_6}.
\end{aligned}
\tag{4.1}
$$

Hence, $\overline{R_B^{\succcurlyeq}}(\tilde{d}) = \overline{R_C^{\succcurlyeq}}(\tilde{d})$.

Thus, $\{a_1, a_2\}$ is also one upper approximation consistent set of the system.

Moreover, we can examine that $\{a_1\}$ and $\{a_3\}$ are not the upper approximation consistent sets by computing and $\{a_2\}$ is the upper approximation consistent set of the system. Hence, we conclude that this decision table has unique upper approximation reduction $\{a_2\}$.

Furthermore, if we take $B' = \{a_1, a_3\}$, by computing, we have

$$\begin{aligned}
[u_1]_{B'}^{\succcurlyeq} &= \{u_1, u_2, u_3, u_4, u_5, u_6\},\\
[u_2]_{B'}^{\succcurlyeq} &= \{u_2, u_5, u_6\},\\
[u_3]_{B'}^{\succcurlyeq} &= \{u_2, u_3, u_4, u_5, u_6\},\\
[u_4]_{B'}^{\succcurlyeq} &= \{u_4, u_6\},\\
[u_5]_{B'}^{\succcurlyeq} &= \{u_2, u_5, u_6\},\\
[u_6]_{B'}^{\succcurlyeq} &= \{u_6\},\\
\underline{R_{B'}^{\succcurlyeq}\left(\tilde{d}\right)} &= \frac{0.1}{u_1} + \frac{0.1}{u_2} + \frac{0.1}{u_3} + \frac{0.6}{u_4} + \frac{0.1}{u_5} + \frac{0.6}{u_6}.
\end{aligned} \tag{4.2}$$

Hence, $\underline{R_{B'}^{\succcurlyeq}(\tilde{d})} = \underline{R_C^{\succcurlyeq}(\tilde{d})}$. Therefore, $\{a_1, a_3\}$ is also another lower approximation consistent set of the decision table. Moreover, we can find that $\{a_3\}$ is lower approximation consistent sets by computing. Hence, we conclude that $\{a_3\}$ is unique lower approximation reduction of this ordered decision table.

Detailed judgment theorems of upper approximation reductions will be proposed in the following.

Theorem 4.3. *Let $\mathfrak{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $B \subseteq C$. Attribute set B is an upper approximation consistent set if and only if if $u_i, u_j \in U$ such that $\overline{R_C^{\succcurlyeq}(\tilde{d})}(u_i) < \overline{R_C^{\succcurlyeq}(\tilde{d})}(u_j)$, then there must exist $a \in B$ such that $f(u_i, a) > f(u_j, a)$.*

Proof. "⇒" Suppose that the conclusion does not holds, that is to say that if $u_i, u_j \in U$ such that $\overline{R_C^{\succcurlyeq}(\tilde{d})}(u_i) < \overline{R_C^{\succcurlyeq}(\tilde{d})}(u_j)$, then $f(u_i, a) \le f(u_j, a)$ for any $a \in B$. So, we can obtain $u_j \in [u_i]_B^{\succcurlyeq}$, which implies that $[u_j]_B^{\succcurlyeq} \subseteq [u_i]_B^{\succcurlyeq}$. And by the upper approximation definition, we have known that

$$\begin{aligned}
\overline{R_B^{\succcurlyeq}\left(\tilde{d}\right)}(u_i) &= \max\left\{\tilde{d}(u) \mid u \in [u_i]_B^{\succcurlyeq}\right\},\\
\overline{R_B^{\succcurlyeq}\left(\tilde{d}\right)}(u_j) &= \max\left\{\tilde{d}(u) \mid u \in [u_j]_B^{\succcurlyeq}\right\}.
\end{aligned} \tag{4.3}$$

Therefore, one can get $\overline{R_B^{\succcurlyeq}(\tilde{d})}(u_i) \ge \overline{R_B^{\succcurlyeq}(\tilde{d})}(u_j)$.

On the other hand, B is an upper approximation consistent set, then we have that

$$\begin{aligned}
\overline{R_C^{\succcurlyeq}\left(\tilde{d}\right)}(u_j) &= \overline{R_B^{\succcurlyeq}\left(\tilde{d}\right)}(u_j),\\
\overline{R_C^{\succcurlyeq}\left(\tilde{d}\right)}(u_i) &= \overline{R_B^{\succcurlyeq}\left(\tilde{d}\right)}(u_i).
\end{aligned} \tag{4.4}$$

Hence, we can obtain $\overline{R_C^{\succcurlyeq}(\tilde{d})}(u_i) \ge \overline{R_C^{\succcurlyeq}(\tilde{d})}(u_j)$. Obviously, this is a contradiction.

"⇐" Suppose that B is not an upper approximation consistent set, then there exist certainly one $u_{i_0} \in U$ such that $\overline{R_C^{\succeq}}(\tilde{d})(u_{i_0}) \neq \overline{R_B^{\succeq}}(\tilde{d})(u_{i_0})$. So, we have that $\overline{R_C^{\succeq}}(\tilde{d})(u_{i_0}) < \overline{R_B^{\succeq}}(\tilde{d})(u_{i_0})$ according to Proposition 3.2.

Since $\overline{R_B^{\succeq}}(\tilde{d})(u_{i_0}) = \max\{\tilde{d}(u) \mid u \in [u_{i_0}]_B^{\succeq}\}$, we take $u_{j_0} \in [u_{i_0}]_B^{\succeq}$ such that $\tilde{d}(u_{j_0}) = \overline{R_B^{\succeq}}(\tilde{d})(u_{i_0}) = \max\{\tilde{d}(u) \mid u \in [u_{i_0}]_B^{\succeq}\}$. At the same time, we easily observe that $u_{j_0} \in [u_{j_0}]_C^{\succeq}$, then we can obtain $\max\{\tilde{d}(u) \mid u \in [u_{j_0}]_C^{\succeq}\} \geq \tilde{d}(u_{j_0})$. That is to say $\overline{R_C^{\succeq}}(\tilde{d})(u_{j_0}) \geq \tilde{d}(u_{j_0})$.

Therefore, from the above, the following inequality holds:

$$\overline{R_C^{\succeq}}\left(\tilde{d}\right)(u_{j_0}) \geq \tilde{d}(u_{j_0}) = \overline{R_B^{\succeq}}\left(\tilde{d}\right)(u_{i_0}) > \overline{R_C^{\succeq}}\left(\tilde{d}\right)(u_{i_0}). \tag{4.5}$$

Thus, there exist certainly $a \in B$ such that $f(u_{i_0}, a) > f(u_{j_0}, a)$, which is a contradiction with $u_{j_0} \in [u_{i_0}]_B^{\succeq}$.

The theorem is proved. □

In the following, judgment theorems of lower approximation reductions will be presented.

Theorem 4.4. *Let $\mathcal{T}_{\tilde{d}}^{\succeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $B \subseteq C$. Attribute set B is a lower approximation consistent set if and only if if $u_i, u_j \in U$ such that $\underline{R_C^{\succeq}}(\tilde{d})(u_i) < \underline{R_C^{\succeq}}(\tilde{d})(u_j)$, then there exist certainly $a \in B$ such that $f(u_i, a) < f(u_j, a)$.*

Proof. "⇒" Suppose that the conclusion does not holds, that is to say that if $u_i, u_j \in U$ such that $\underline{R_C^{\succeq}}(\tilde{d})(u_i) < \underline{R_C^{\succeq}}(\tilde{d})(u_j)$, then $f(u_i, a) \geq f(u_j, a)$ for any $a \in B$. So we can obtain $u_i \in [u_j]_B^{\succeq}$, which implies that $[u_i]_B^{\succeq} \subseteq [u_j]_B^{\succeq}$. And, by the lower approximation definition, we have known that

$$\begin{gathered}\underline{R_B^{\succeq}}\left(\tilde{d}\right)(u_i) = \min\left\{\tilde{d}(u) \mid u \in [u_i]_B^{\succeq}\right\},\\ \underline{R_B^{\succeq}}\left(\tilde{d}\right)(u_j) = \min\left\{\tilde{d}(u) \mid u \in [u_j]_B^{\succeq}\right\}.\end{gathered} \tag{4.6}$$

Therefore, one can get $\underline{R_B^{\succeq}}(\tilde{d})(u_i) \geq \underline{R_B^{\succeq}}(\tilde{d})(u_j)$.

On the other hand, B is an lower approximation consistent set, then we have that

$$\begin{gathered}\underline{R_C^{\succeq}}\left(\tilde{d}\right)(u_j) = \underline{R_B^{\succeq}}\left(\tilde{d}\right)(u_j)\\ \underline{R_C^{\succeq}}\left(\tilde{d}\right)(u_i) = \underline{R_B^{\succeq}}\left(\tilde{d}\right)(u_i).\end{gathered} \tag{4.7}$$

Hence, we can obtain $\underline{R_C^{\succeq}}(\tilde{d})(u_i) \geq \underline{R_C^{\succeq}}(\tilde{d})(u_j)$. Obviously, this is a contradiction.

"⇐" Suppose that B is not an lower approximation consistent set, then there exist certainly one $u_{i_0} \in U$ such that $\underline{R_C^{\succeq}}(\tilde{d})(u_{i_0}) \neq \underline{R_B^{\succeq}}(\tilde{d})(u_{i_0})$. So we have that $\underline{R_C^{\succeq}}(\tilde{d})(u_{i_0}) > \underline{R_B^{\succeq}}(\tilde{d})(u_{i_0})$ according to Proposition 3.2.

Since $\underline{R_B^{\succcurlyeq}}(\tilde{d})(u_{i_0}) = \min\{\tilde{d}(u) \mid u \in [u_{i_0}]_B^{\succcurlyeq}\}$, we take $u_{j_0} \in [u_{i_0}]_B^{\succcurlyeq}$ such that $\tilde{d}(u_{j_0}) = \underline{R_B^{\succcurlyeq}}(\tilde{d})(u_{i_0}) = \min\{\tilde{d}(u) \mid u \in [u_{i_0}]_B^{\succcurlyeq}\}$. At the same time, we easily observe that $u_{j_0} \in [u_{j_0}]_C^{\succcurlyeq}$, then we can obtain $\min\{\tilde{d}(u) \mid u \in [u_{j_0}]_C^{\succcurlyeq}\} \le \tilde{d}(u_{j_0})$. That is to say $\underline{R_C^{\succcurlyeq}}(\tilde{d})(u_{j_0}) \le \tilde{d}(u_{j_0})$.

Therefore, from the above, the following inequality holds:

$$\underline{R_C^{\succcurlyeq}}\left(\tilde{d}\right)(u_{j_0}) \le \tilde{d}(u_{j_0}) = \underline{R_B^{\succcurlyeq}}\left(\tilde{d}\right)(u_{i_0}) < \underline{R_C^{\succcurlyeq}}\left(\tilde{d}\right)(u_{i_0}). \tag{4.8}$$

Thus, there exist certainly $a \in B$ such that $f(u_{i_0}, a) > f(u_{j_0}, a)$, which is a contradiction with $u_{j_0} \in [u_{i_0}]_B^{\succcurlyeq}$.

The theorem is proved. □

Theorems 4.3 and 4.4 provide an approach to judge whether a subset of condition attributes is a lower and upper approximation consistent set or not, respectively.

5. Approach to Approximation Reduction of ODT with Fuzzy Decision

In this section, we can further obtain practical approaches to upper and lower reductions in ordered decision tables with fuzzy decision, and an illustrative example is also employed to show their mechanisms. We first give the following notions.

Definition 5.1. Let $\mathcal{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $B \subseteq C$. If we denote, for $x_i, x_j \in U$,

$$\begin{gathered}
UD_{\tilde{d}}^{*} = \left\{(u_i, u_j) \mid \overline{R_C^{\succcurlyeq}}\left(\tilde{d}\right)(u_i) < \overline{R_C^{\succcurlyeq}}\left(\tilde{d}\right)(u_j)\right\}, \\
LD_{\tilde{d}}^{*} = \left\{(u_i, u_j) \mid \underline{R_C^{\succcurlyeq}}\left(\tilde{d}\right)(u_i) < \underline{R_C^{\succcurlyeq}}\left(\tilde{d}\right)(u_j)\right\}, \\
UD_{\tilde{d}}(x_i, x_j) = \begin{cases} \{a \in C \mid f(x_i, a) > f(x_j, a)\}, & (x_i, x_j) \in UD_{\tilde{d}}^{*}, \\ \emptyset, & (x_i, x_j) \notin UD_{\tilde{d}}^{*}, \end{cases} \\
LD_{\tilde{d}}(x_i, x_j) = \begin{cases} \{a \in C \mid f(x_i, a) < f(x_j, a)\}, & (x_i, x_j) \in LD_{\tilde{d}}^{*}, \\ \emptyset, & (x_i, x_j) \notin LD_{\tilde{d}}^{*}, \end{cases} \\
UM_{\tilde{d}} = (u_{ij})_{n \times n}, \\
LM_{\tilde{d}} = (v_{ij})_{n \times n},
\end{gathered} \tag{5.1}$$

where $u_{ij} = UD_{\tilde{d}}(x_i, x_j)$, $v_{ij} = LD_{\tilde{d}}(x_i, x_j)$, then $UD_{\tilde{d}}(x_i, x_j)$ and $LD_{\tilde{d}}(x_i, x_j)$ are said to be upper and lower approximation discernibility attributes set between objects x_i and x_j, respectively. And matrices $UM_{\tilde{d}}$ and $LM_{\tilde{d}}$ are referred as to upper and lower approximation discernibility matrix of the decision table $\mathcal{T}_{\tilde{d}}^{\succcurlyeq}$, respectively.

Theorem 5.2. *Let $\mathcal{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $B \subseteq C$. Subset B is upper approximation consistent set if and only if $B \cap UD_{\tilde{d}}(u_i, u_j) \neq \emptyset$ for all $(u_i, u_j) \in UD_{\tilde{d}}^*$.*

Proof. "⇒" From $(u_i, u_j) \in UD_{\tilde{d}}^*$, we have $\overline{R_C^{\succcurlyeq}(\tilde{d})}(u_i) < \overline{R_C^{\succcurlyeq}(\tilde{d})}(u_j)$. By Theorem 4.3, we can know that there exist certainly $a \in B$ such that $f(u_i, a) > f(u_j, a)$. So, $a \in UD_{\tilde{d}}(u_i, u_j)$ according to the above definition. Hence, $B \cap UD_{\tilde{d}}(u_i, u_j)(u_i, u_j) \neq \emptyset$.

"⇐" For $(u_i, u_j) \in UD_{\tilde{d}}^*$, if $B \cap UD_{\tilde{d}}(u_i, u_j) \neq \emptyset$, then there exist certainly $a \in B$ such that $a \in UD_{\tilde{d}}(u_i, u_j)$, which implies that $f(u_i, a) > f(u_j, a)$. By Theorem 4.3, we can obtain that B is an upper approximation consistent set of the decision table $\mathcal{T}_{\tilde{d}}^{\succcurlyeq}$. □

Theorem 5.3. *Let $\mathcal{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $B \subseteq C$. Subset B is lower approximation consistent set if and only if $B \cap LD_{\tilde{d}}(u_i, u_j) \neq \emptyset$ for all $(u_i, u_j) \in LD_{\tilde{d}}^*$.*

Proof. It is similar to Theorem 5.2. □

Definition 5.4. Let $\mathcal{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision and $UM_{\tilde{d}}$ and $LM_{\tilde{d}}$ upper and lower approximation discernibility matrices of $\mathcal{T}_{\tilde{d}}^{\succcurlyeq}$, respectively. If denote

$$\begin{aligned} UF_{\tilde{d}} &= \wedge\{\vee\{a \mid a \in UD_{\tilde{d}}(x_i, x_j)\}, x_i, x_j \in U\} \\ &= \wedge\left\{\vee\{a \mid a \in UD_{\tilde{d}}(x_i, x_j)\}, (x_i, x_j) \in UD_{\tilde{d}}^*\right\}, \\ LF_{\tilde{d}} &= \wedge\{\vee\{a \mid a \in LD_{\tilde{d}}(x_i, x_j)\}, x_i, x_j \in U\} \\ &= \wedge\left\{\vee\{a \mid a \in LD_{\tilde{d}}(x_i, x_j)\}, (x_i, x_j) \in LD_{\tilde{d}}^*\right\}, \end{aligned} \tag{5.2}$$

then $UF_{\tilde{d}}$ and $LF_{\tilde{d}}$ are called discernibility formulas of upper and lower approximation of the decision table $\mathcal{T}_{\tilde{d}}^{\succcurlyeq}$, respectively.

Theorem 5.5. *Let $\mathcal{T}_{\tilde{d}}^{\succcurlyeq} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision. The minimal disjunctive normal form of discernibility formula of upper approximation is*

$$UF_{\tilde{d}} = \bigvee_{k=1}^{p}\left(\bigwedge_{s=1}^{q_k}\right) a_s. \tag{5.3}$$

Denote $UB_{\tilde{d}}^k = \{a_s \mid s = 1, 2, \ldots, q_k\}$, then $\{UB_{\tilde{d}}^k \mid k = 1, 2, \ldots, p\}$ is just set of all upper approximation reductions of $\mathcal{T}_{\tilde{d}}^{\succcurlyeq}$.

Proof. For any $(u_i, u_j) \in UD_{\tilde{d}}^*$, by the definition of minimum alternative normal form, we have that $UB_{\tilde{d}}^k$ is upper approximation consistent set. If one element of $UB_{\tilde{d}}^k$ is reduced in $UF_{\tilde{d}} = \vee_{k=1}^{p}(UB_{\tilde{d}}^k)$, without loss of generality and the result denoted by $UB_{\tilde{d}}^{k'}$, then there

Table 3: Upper approximation discernibility matrix $UM_{\tilde{d}}$ of the ODT.

u_i, u_j	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$\emptyset$	$\emptyset$	a_2	a_2	$\emptyset$	$\emptyset$
u_2	$\emptyset$	$\emptyset$	a_1, a_2	a_2	$\emptyset$	$\emptyset$
u_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_5	a_1, a_2, a_3	a_2	a_1, a_2	a_1, a_2	$\emptyset$	a_2
u_6	$\emptyset$	$\emptyset$	a_1, a_2	a_1, a_2	$\emptyset$	$\emptyset$

Table 4: Lower approximation discernibility matrix $UM_{\tilde{d}}$ of the ODT.

u_i, u_j	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$\emptyset$	$\emptyset$	$\emptyset$	a_1, a_3	$\emptyset$	a_1, a_3
u_2	$\emptyset$	$\emptyset$	$\emptyset$	a_3	$\emptyset$	a_3
u_3	$\emptyset$	$\emptyset$	$\emptyset$	a_1, a_3	$\emptyset$	a_1, a_2, a_3
u_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u_5	$\emptyset$	$\emptyset$	$\emptyset$	a_3	$\emptyset$	a_3
u_6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

exist certainly $(u_{i_0}, u_{j_0}) \in UD^*_{\tilde{d}}$ such that $UB^{k'}_{\tilde{d}} \cap UD_{\tilde{d}}(u_{i_0}, u_{j_0}) = \emptyset$. So, $UB^{k'}_{\tilde{d}}$ is not an upper approximation consistent set. So, $UB^{k}_{\tilde{d}}$ is an upper approximation reduction of the ordered decision table $\mathcal{T}^{\succcurlyeq}_{\tilde{d}}$.

On the other hand, we have known that the discernibility formula of upper approximation includes all $UD_{\tilde{d}}(u_i, u_j)$. Thus, there is not other upper approximation reduction besides of $UB^{k}_{\tilde{d}}$.

The proof is completed. □

Theorem 5.6. *Let $\mathcal{T}^{\succcurlyeq}_{\tilde{d}} = (U, C \cup \{d\}, V, f)$ be an ordered decision table with fuzzy decision. The minimal disjunctive normal form of discernibility formula of lower approximation is*

$$LF_{\tilde{d}} = \bigvee_{k=1}^{p} \left(\bigwedge_{s=1}^{q_k} \right) a'_s. \tag{5.4}$$

Denote $LB^{k}_{\tilde{d}} = \{a'_s \mid s = 1, 2, \ldots, q_k\}$, then $\{LB^{k}_{\tilde{d}} \mid k = 1, 2, \ldots, p\}$ is just set of all lower approximation reductions of $\mathcal{T}^{\succcurlyeq}_{\tilde{d}}$.

Proof. It is similar to Theorem 5.5. □

Example 5.7 (Continued from Example 3.3). Compute the upper approximation reduction and lower approximation reduction of the ordered decision table with fuzzy decision in Table 2.

By computing, we can easily obtain the upper and lower approximation discernibility matrices in Tables 3 and 4.

Therefore, by Theorems 5.5 and 5.6, we have

$$UF_{\tilde{d}} = (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_2) \wedge a_2 = a_2,$$
$$LF_{\tilde{d}} = (a_1 \vee a_2 \vee a_3) \wedge (a_1 \vee a_3) \wedge a_3 = a_3. \tag{5.5}$$

Thus, we can conclude that $\{a_2\}$ is the unique upper approximation reduction and $\{a_3\}$ is the unique lower approximation reduction of the ordered decision table with fuzzy decision, which accord with the result of Example 4.2.

Remark 5.8. By the example, one can easily find that $\{a_2\}$ is the upper approximation consistent set, but it is not a lower approximation consistent set, and $\{a_3\}$ is the lower approximation consistent set, but it is not a upper approximation consistent set. Hence, there is no static relationship between upper approximation consistent sets and lower approximation consistent set.

6. Conclusions

Attributes reduction, as one research problem, has played an important role in rough set theory. Many types of attribute reductions have been proposed based on the rough set theory, each of them aimed at a different requirement. In practise, some of information systems are based on dominance relations, and values of decision attribute are fuzzy. Therefore, it is meaningful to study the attribute reductions in ordered decision table with fuzzy decision. In this paper, upper approximation reduction and lower approximation reduction were proposed for this kind of complicated decision table. Some important properties were discussed. The judgement theorems and discernibility matrices associated with the two reductions have been obtained, from which we can provide an approach to attribute reductions in ordered decision tables with fuzzy decision. Then, the practical approaches to upper and lower approximation reduction in ordered decision tables with fuzzy decision have been provided as well. Finally, an illustrative example has been employed to explain the mechanism of this method.

Though ordered decision table with fuzzy decision are discussed here, they are all complete. Because incomplete decision table are more complicated than complete one, we will develop the proposed approaches to more generalized and more complicated ODTs such as incomplete ODTs with fuzzy decision and ordered decision tables with interval values or intuitionistic fuzzy values.

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