Multi-granulation Fuzzy Rough Sets in a Fuzzy Tolerance Approximation Space

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Abstract

Based on the analysis of the rough set model on a tolerance relation and the fuzzy rough set, two types of fuzzy rough sets models on tolerance relations are constructed and researched. Then we propose the optimistic and pessimistic multi-granulation fuzzy rough sets models in a fuzzy tolerance approximation space with the point view of granular computing. In these models, the fuzzy lower and upper approximations of a fuzzy set are defined on multiple fuzzy tolerance relations. It follows the research on the properties of the fuzzy lower and upper approximations of the new models. The fuzzy rough set model, rough set model on a tolerance relation and multi-granulation rough sets models are special cases of the new ones from the perspective of the considered concepts, related relations and granular computing. The new models are meaningful generalizations of the classical rough sets.

Keywords: Approximation operators, Fuzzy rough set, Fuzzy tolerance relation, Multi-granulation.

1. Introduction

Rough set theory, proposed by Pawlak [11-13], is a theory for the research of uncertainty management in a wide variety of applications related to artificial intelligence [2, 5, 8]. The theory has been applied successfully in the fields of pattern recognition, medical diagnosis, data mining, conflict analysis, algebra [1, 14, 20], which relate an amount of imprecise, vague and uncertain information. In recent years, the rough set theory generated a great deal of interest among more and more researchers. The generalization of the classical rough set model from the perspective of granular computing is one of the study spotlights.

Information granules refer to pieces, classes and groups divided in accordance with characteristics and performances of complex information in the process of human understanding, reasoning and decision-making. Zadeh firstly proposed the concept of granular computing and discussed issues of fuzzy information granulation in 1979 [29]. Then the basic idea of information granulation had been applied to many fields including rough set [11, 12]. In 1985, Hobbs proposed the concept of granularity [4]. Granular computing played a more and more important role gradually in soft computing, knowledge discovery, data mining and many excellent results were achieved [9, 15, 16, 19, 22, 24-28]. In the point view of granular computing, the classical Pawlak rough set and some expanded rough sets like tolerant models [6, 10, 21] are based on a single granulation. And an equivalence relation or a tolerance relation on the universe can be regarded as a granulation. This approach to describing a concept is mainly based on the following assumption [16]:

If R_A and R_B are two relations induced by the attributes subsets A and B and $X \subseteq U$ is a target concept, then the rough set of X is derived from the quotient set.

$$U/(R_A \cup R_B) = \{[x]_{R_A} \cup [x]_{R_B} \mid [x]_{R_A} \in U/R_A, [x]_{R_B} \in U/R_B, [x]_{R_A} \cap [x]_{R_B} \neq \varnothing\},$$

which suggests that we can perform an intersection operation between $[x]_{R_A}$ and $[x]_{R_B}$ and the target concept is approximately described by using the quotient set $U/(R_A \cup R_B)$. Then the target concept is described by a finer granulation (partitions or coverings) formed through combining two known granulations (partitions or coverings) induced from two attribute subsets. However, the combination that generates a much finer granulation and more knowledge destroys the original granulation structure.

In fact, the above assumption cannot always be satisfied or required generally. The following cases illustrate the restrictions.

Case 1. In some data analysis issues, for the same object, there is a contradiction or inconsistent relationship between its values under one attribute set R_A and those under another attribute set R_B. In other words, we cannot perform the intersection operations between their quotient sets and the target concept cannot be approximated by using $U/(R_A \cup R_B)$.

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Case 2. In the process of some decision making, the decision or the view of each of decision makers may be independent for the same project.

Case 3. To extract decision rules from distributive information systems and groups of intelligent agents, for the reduction of the time complexity of rule extractions, it is unnecessary for us to perform the intersection operations in between all the sites in the context of distributive information systems.

To overcome these limitations, Qian and Xu et al. extended the Pawlak rough set to multi-granulation rough set models where the approximation operators are defined by multiple equivalence relations on the universe [7, 15–18, 25]. However, the kind of classification is still restrictive for it pays too much attention to the differences between objects and ignores the similarities between objects. So we will extend the models by relaxing the equivalent relations to tolerance relations or fuzzy tolerance relations and by combining with other theories that deal with uncertainty knowledge such as fuzzy sets.

Associated tolerant rough set with the theory of fuzzy rough set with granular computing point view, we constructed two types of multi-granulation fuzzy rough set models on tolerance relations [26]. Moreover, there exist issues which have to be solved by fuzzy relations. Therefore, we propose the optimistic and pessimistic multi-granulation fuzzy rough sets models in a fuzzy tolerance approximation space in which the needed relations are fuzzy ones, to extend A. Skowron's [21] and J. Järinen's [6] tolerance rough set model determined by single tolerance relation. And these models are fuzzy cases from the perspective of both relations and concepts. The rest of this paper is organized as follows. Some preliminary concepts of tolerance rough set theory, fuzzy rough sets theory and multi-granulation rough sets are showed in Section 2. In Section 3, we represent our achievements on two types of multi-granulation fuzzy rough approximation operators of a fuzzy concept based on multiple ordinary tolerance relations. Then we define the fuzzy approximations of the fuzzy rough sets in a fuzzy tolerance approximation space and analyze the properties of them in Section 4. Especially, one can find that the definition of lower and upper approximation operators proposed in this paper are the generalized model of other formats in Section 2, not only from the aspects of the considered concepts but also from the perspective of granulation. And finally, the paper is concluded by a summary in Section 5.

2. Preliminaries

In this section, we will first review some basic concepts and notions in the theory of rough set on a tolerance relation, fuzzy rough sets and multi-granulation rough sets on the basis of equivalence relations. More can be found in ref [3, 6, 16, 25, 30].

A. The Rough Set on a Tolerance Relation

The notion of information system provides a convenient tool for the representation of objects in terms of their attribute values.

A tolerance information system [23] is an ordered triple $\mathcal{I}=(U,AT,\tau)$, where U is the non-empty finite set of objects known as universe; AT is the non-empty finite set of attributes. t is the mapping from power set AT into the family set R of tolerance relations satisfying reflexivity and symmetry on universe U.

Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, for $B\subseteq AT, X\subseteq U$, $R_B\in R$ is a relation with respect to the attributes set B. Denote

$$R_B(x_i)=\left\{x_j\in U\mid (x_i,x_j)\in R_B\right\}, \tag{1}$$

where $i\in\left\{1,2,\cdots,|U|\right\}$, then $R_B(x_i)$ will be called the tolerance class of x_i with respect to the tolerance relation R_B. Note that a tolerance relation can construct a covering instead of a partition of the universe U.

Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system. The lower approximation and upper approximation of a set $X\subseteq U$ with respect to a tolerance relation R_A are respectively defined by

$$\begin{aligned}\underline{R_A}(X)&=\left\{x\in U\mid R_A(x)\subseteq X\right\},\\ \overline{R_A}(X)&=\left\{x\in U\mid R_A(x)\cap X\neq\varnothing\right\}.\end{aligned} \tag{2}$$

The set $Bnd_{R_A}(X)=\overline{R_A}(X)-\underline{R_A}(X)$ is called the boundary of X.

The set $\underline{R_A}(X)$ consists of elements which surely belong to X in view of the knowledge provided by R_A, while $\overline{R_A}(X)$ consists A of elements which possibly belong to X. The boundary is the actual area of uncertainty. It consists of elements whose membership in X can not be decided when R-related objects can not be distinguished from each other.

Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system. The properties of the lower approximation and the upper approximation of sets with respect to a tolerance relation R_A are as follows, if $X,Y\subseteq U$ and $\sim X$ is the complement of X.

(1). $\underline{R_A}(X)\subseteq X\subseteq\overline{R_A}(X)$,

(2). $\underline{R_A}(\varnothing)=\overline{R_A}(\varnothing)=\varnothing,\underline{R_A}(U)=\overline{R}_A(U)=U$,

(3). $\sim\underline{R_A}(X)=\overline{R_A}(\sim X),\sim\overline{R_A}(X)=\underline{R_A}(\sim X)$,

(4). $Bnd_R(X)=Bnd_R(\sim X)$,

(5). $X\subseteq Y\Rightarrow\underline{R_A}(X)\subseteq\underline{R_A}(Y),\overline{R_A}(X)\subseteq\overline{R_A}(Y)$.

B. The Fuzzy Set and Fuzzy Rough Set

We will firstly introduce some basic concepts of fuzzy set [3]. Let U be a finite and non-empty set called universe. A fuzzy set A is a mapping from U into the unit interval $[0,1]:\mu:\rightarrow[0,1]$; where each $x\in U$ is the membership degree of x in A. Practically, we may consider U as a set of objects of concern and crisp subset of U represents a "non-vague" concept imposed on objects in U. Then a fuzzy set A of U is thought of as a mathematical representation of "vague" concept described linguistically. The set of all the fuzzy sets defined on U is denoted by $F(U)$.

Let A and B be two fuzzy sets on U, the operation between them are defined as

$$\begin{aligned}&(A\cup B)(x)=\max\{A(x),B(x)\},\\&(A\cap B)(x)=\min\{A(x),B(x)\},\\&\sim A(x)=1-A(x),\\&A\subseteq B\Leftrightarrow A(x)\le B(x)(x\in U).\end{aligned}\tag{3}$$

Let U be the universe, R be an equivalence relation, for a fuzzy set A on U, if take

$$\begin{aligned}&\underline{R}(A)(x)=\wedge\{A(y)\mid y\in[x]_R\},\\&\overline{R}(A)(x)=\vee\{A(y)\mid y\in[x]_R\},\end{aligned}\tag{4}$$

then $\underline{R}(A)$ and $\overline{R}(A)$ are called the lower and upper approximation of the fuzzy set A with respect to the relation R, where "$\wedge$" means "min" and "$\vee$" means "max" and $[x]_R$ is the equivalence class of x with respect to equivalence relation R [3, 29]. A is a fuzzy definable set if and only if A satisfies $\underline{R}(A)=\overline{R}(A)$. Otherwise, A is called a fuzzy rough set.

Let $\mathcal{I}=(U,AT,F)$ be an information system [29]. $F=\{f_j\mid j\le n\}$ is a set of relationship between U and AT, where $f_j:U\rightarrow V_j(j\le n)$, V_j is the domain of attribute a_j, and n is the number of the attributes. $D_j:U\rightarrow[0,1](j\le r)$, if denote

$$D=\{D_j\mid j\le r\},$$

then (U,AT,F,D) is a fuzzy target information system. In a fuzzy target information system, we can define the approximation operators with respect the decision attribute D similarly.

Let U be the universe, be an equivalence relation, $A,B\in F(U)$, the fuzzy lower and upper approximation with respect to relation R satisfy the following properties.

(1). $\underline{R}(A)\subseteq A\subseteq\overline{R}(A)$,

(2). $\underline{R}(A\cap B)=\underline{R}(A)\cap\underline{R}(B),\overline{R}(A\cup B)=\overline{R}(A)\cup\overline{R}(B)$,

(3). $\underline{R}(A)=\sim\overline{R}(\sim A),\overline{R}(A)=\sim\underline{R}(\sim A)$,

(4). $\underline{R}(A\cup B)\supseteq\underline{R}(A)\cup\underline{R}(B),\overline{R}(A\cap B)\subseteq\overline{R}(A)\cap\overline{R}(B)$,

(5). $\overline{R}(\overline{R}(A))=\underline{R}(\overline{R}(A))=\overline{R}(A)$,

(6). $\overline{R}(\underline{R}(A))=\underline{R}(\underline{R}(A))=\underline{R}(A)$,

(7). $\underline{R}(U)=U,\overline{R}(\varnothing)=\varnothing$,

(8). $A\subseteq B\Rightarrow\underline{R}(A)\subseteq\underline{R}(B)$, and $\overline{R}(A)\subseteq\overline{R}(B)$.

C. Multi-granulation Rough Sets

For simplicity, we just recall the models of multi-granulation rough sets and details can be seen in ref [16-18, 25].

Let $\mathcal{I}=(U,AT,F)$ be an information system, $A_i\subseteq AT,i=1,2,\cdots m$, m is the number of the considered attribute sets. The optimistic lower and upper approximations of the set $X\in U$ with respect to $A_i\subseteq AT,(i=1,2,\cdots m)$ are

$$\begin{aligned}&\underline{OR_{\sum_{i=1}^{m}A_i}}(X)=\left\{x\in U\mid \overset{m}{\underset{i=1}{\vee}}[x]_{A_i}\subseteq X,1\le i\le m\right\},\\&\overline{OR_{\sum_{i=1}^{m}A_i}}(X)=\left\{x\in U\mid \overset{m}{\underset{i=1}{\wedge}}[x]_{A_i}\cap X\ne\varnothing,1\le i\le m\right\},\end{aligned}\tag{5}$$

where $[x]_{A_i}=\{x\in U\mid(x,y)\in R_{A_i}\}$ and R_{A_i} is an equivalent relation with respect to the attributes set A_i. Moreover, $\underline{OR_{\sum_{i=1}^{m}A_i}}(X)\ne\overline{OR_{\sum_{i=1}^{m}A_i}}(X)$, we say that X is the optimistic multi-granulation rough set. Otherwise, we say that X is the optimistic multi-granulation definable set.

Let $\mathcal{I}=(U,AT,F)$ be an information system, $A_i\subseteq AT,i=1,2,\cdots m$, m is the number of the considered attribute sets. The pessimistic lower and upper approximations of the set $X\in U$ with respect to $A_i\subseteq AT,(i=1,2,\cdots m)$ are

$$\begin{aligned}&\underline{PR_{\sum_{i=1}^{m}A_i}}(X)=\left\{x\in U\mid \overset{m}{\underset{i=1}{\wedge}}[x]_{A_i}\subseteq X,1\le i\le m\right\},\\&\overline{PR_{\sum_{i=1}^{m}A_i}}(X)=\left\{x\in U\mid \overset{m}{\underset{i=1}{\vee}}[x]_{A_i}\cap X\ne\varphi,1\le i\le m\right\}.\end{aligned}\tag{6}$$

Moreover, $\underline{PR_{\sum_{i=1}^{m}A_i}}(X)\ne\overline{PR_{\sum_{i=1}^{m}A_i}}(X)$, we say that X is the pessimistic multi-granulation rough set. Otherwise, we say that X is the pessimistic multi-granulation definable set.

3. Two Types of Multi-granulation Fuzzy Rough Sets on Tolerance Relations

In this section, we make researches about multigranulation fuzzy rough set which are on the problem of the rough approximations of a fuzzy set on multiple tolerance relations.

At first, we propose a fuzzy rough set model (in brief FRS) on a tolerance relation in the following.

Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A\subseteq AT$. For the fuzzy set $X\in F(U)$, denote

$$\underline{R_A}(X)(x)=\wedge\left\{X(y)\mid y\in R_A(x)\right\},\quad \overline{R_A}(X)(x)=\vee\left\{X(y)\mid y\in R_A(x)\right\}, \tag{7}$$

where "$\wedge$" means "min" and "$\vee$" means "max", then $\underline{R_A}(X)$ and $\overline{R_A}(X)$ are the lower and upper approximation of the fuzzy set X on the tolerance relation R with respect to the subset of attributes A. If $\underline{R_A}(X)\neq\overline{R_A}(X)$, then the fuzzy set X is a fuzzy rough set on the tolerance relation. We can easily find that this model will be the fuzzy rough set model we have introduced in the part B of Section 2, if the above relation R is an equivalence relation.

A. The Optimistic of Multi-granulation Fuzzy Rough Set on Tolerance Relations

First, the optimistic two-granulation fuzzy rough set (in brief optimistic TGFRS) on tolerance relations of a fuzzy set is defined.

Definition 3.1[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A,B\subseteq AT$. For the fuzzy set $X\in F(U)$, denote

$$\underline{OR_{A+B}}(X)(x)=\left\{\wedge\left\{X(y)\mid y\in R_A(x)\right\}\right\}\vee\left\{\wedge\left\{X(y)\mid y\in R_B(x)\right\}\right\},$$
$$\overline{OR_{A+B}}(X)(x)=\left\{\vee\left\{X(y)\mid y\in R_A(x)\right\}\right\}\wedge\left\{\vee\left\{X(y)\mid y\in R_B(x)\right\}\right\}, \tag{8}$$

where "$\vee$" means "max" and "$\wedge$" means "min", then $\underline{OR_{A+B}}(X)$ and $\overline{OR_{A+B}}(X)$ are respectively called the optimistic two-granulation lower approximation and upper approximation of X on tolerance relations R with respect to the subsets of attributes A and B. X is a two-granulation fuzzy rough set on tolerance relations if and only if $\underline{OR_{A+B}}(X)\neq\overline{OR_{A+B}}(X)$. Otherwise, X is a two-granulation fuzzy definable set on two tolerance relations. The boundary of the set is X defined as

$$Bnd^F_{R_{A+B}}(X)=\overline{OR_{A+B}}(X)\cap(\sim\underline{OR_{A+B}}(X)). \tag{9}$$

It can be found that the optimistic TGFRS on tolerance relations will be degenerated into fuzzy rough set when $A=B$ and $R_A(X)$ and $R_B(X)$ are equivalence classes with respect to the subsets of attributes A and B. That is to say, a fuzzy rough set model is a special instance of the optimistic TGFRS on tolerance relations. What's more, the optimistic TGFRS on tolerance relations will be degenerated into a rough set model on a tolerance relation if $A=B$ and the considered concept X is a crisp set.

In the following, we employ an example to illustrate the above concepts.

Example 3.1: A fuzzy target information system is given in Table 1. The universe $U=\left\{x_1,x_2,\cdots,x_{10}\right\}$, the set of condition attributes $AT=\left\{a_1,a_2,a_3\right\}$, the set of decision attribute $D=\left\{d\right\}$. If suppose $A_1=\left\{a_1,a_2\right\}$ and $A_2=\left\{a_1,a_3\right\}$, we consider the optimistic two-granulation lower and upper approximation of D with respect to A_1 and A_2, where the tolerance relation is defined as

$$R_{A_i}=\left\{(x_i,x_j)\in U\times U\mid f(x_i,a)-f(x_j,a)|\leq 1, a\in A_i\right\}.$$

For the fuzzy set

$$D=(0.6,0.7,0.7,0.9,0.5,0.4,0.7,0.7,0.8,0.7),$$

the single-granulation lower and upper approximations on a tolerance relation are

$$\underline{R_{A_1}}(D)=(0.5,0.6,0.4,0.4,0.5,0.4,0.6,0.5,0.5,0.6),$$
$$\overline{R_{A_1}}(D)=(0.9,0.9,0.9,0.9,0.9,0.9,0.9,0.9,0.9,0.9);$$
$$\underline{R_{A_2}}(D)=(0.4,0.7,0.4,0.4,0.5,0.4,0.7,0.5,0.4,0.6),$$
$$\overline{R_{A_2}}(D)=(0.9,0.8,0.9,0.9,0.9,0.9,0.8,0.9,0.9,0.9);$$
$$\underline{R_{A_1\cup A_2}}(D)=(0.5,0.7,0.4,0.4,0.5,0.4,0.7,0.5,0.6,0.6),$$
$$\overline{R_{A_1\cup A_2}}(D)=(0.9,0.8,0.9,0.9,0.9,0.9,0.8,0.9,0.9,0.9).$$

From Definition 3.1, we can compute the optimistic two-granulation lower and upper approximation of D on tolerance relations are

$$\underline{OR_{A_1+A_2}}(D)=(0.5,0.7,0.4,0.4,0.5,0.4,0.7,0.5,0.5,0.6),$$
$$\overline{OR_{A_1+A_2}}(D)=(0.9,0.8,0.9,0.9,0.9,0.9,0.8,0.9,0.9,0.9).$$

Obviously, the following can be found

$$\underline{OR_{A_1+A_2}}(D)=\underline{R_{A_1}}(D)\cup\underline{R_{A_2}}(D),$$
$$\overline{OR_{A_1+A_2}}(D)=\overline{R_{A_1}}(D)\cap\overline{R_{A_2}}(D),$$
$$\underline{OR_{A_1+A_2}}(D)\subseteq\underline{R_{A_1\cup A_2}}(D)\subseteq D\subseteq\overline{R_{A_1\cup A_2}}(D)\subseteq\overline{OR_{A_1+A_2}}(D).$$

Just from Definition 3.1, we can obtain some properties of the optimistic TGFRS in a tolerance information system.

Table 1. A fuzzy target information system.

U	*a_1*	*a_2*	*a_3*	*d*
x_1	2	1	3	0.6
x_2	3	2	1	0.7
x_3	2	3	3	0.7
x_4	2	2	3	0.9
x_5	1	1	4	0.5
x_6	1	3	2	0.4
x_7	3	2	1	0.7
x_8	1	1	4	0.7
x_9	2	1	2	0.8
x_{10}	3	1	2	0.7

Proposition 3.1[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A,B\subseteq AT$ and $X\in F(U)$. Then

the following properties hold.

(1). $\underline{OR_{A+B}}(X) \subseteq X$,

(2). $\overline{OR_{A+B}}(X) \supseteq X$;

(3). $\underline{OR_{A+B}}(\sim X) = \sim \overline{OR_{A+B}}(X)$,

(4). $\overline{OR_{A+B}}(\sim X) = \sim \underline{OR_{A+B}}(X)$;

(5). $\underline{OR_{A+B}}(U) = \overline{OR_{A+B}}(U) = U$,

(6). $\underline{OR_{A+B}}(\varnothing) = \overline{OR_{A+B}}(\varnothing) = \varnothing$;

(7). $\underline{OR_{A+B}}(X) \supseteq \underline{OR_{A+B}}(\underline{OR_{A+B}}(X))$,

(8). $\overline{OR_{A+B}}(X) \subseteq \overline{OR_{A+B}}(\overline{OR_{A+B}}(X))$.

Proposition 3.2[26]: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$, $X, Y \in F(U)$. Then the following properties hold.

(1). $\underline{OR_{A+B}}(X \cap Y) \subseteq \underline{OR_{A+B}}(X) \cap \underline{OR_{A+B}}(Y)$,

(2). $\overline{OR_{A+B}}(X \cup Y) \supseteq \overline{OR_{A+B}}(X) \cup \overline{OR_{A+B}}(Y)$;

(3). $X \subseteq Y \Rightarrow \underline{OR_{A+B}}(X) \subseteq \underline{OR_{A+B}}(Y)$

(4). $X \subseteq Y \Rightarrow \overline{OR_{A+B}}(X) \subseteq \overline{OR_{A+B}}(Y)$;

(5). $\underline{OR_{A+B}}(X \cup Y) \supseteq \underline{OR_{A+B}}(X) \cup \underline{OR_{A+B}}(Y)$,

(6). $\overline{OR_{A+B}}(X \cap Y) \subseteq \overline{OR_{A+B}}(X) \cap \overline{OR_{A+B}}(Y)$.

Definition 3.2[26]: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\begin{aligned} \underline{OR_{\sum_{i=1}^{m} A_i}}(X)(x) &= \vee_{i=1}^{m} \left\{ \wedge \left\{ (X)(y) \mid y \in R_{A_i}(x) \right\} \right\}, \\ \overline{OR_{\sum_{i=1}^{m} A_i}}(X)(x) &= \wedge_{i=1}^{m} \left\{ \vee \left\{ (X)(y) \mid y \in R_{A_i}(x) \right\} \right\}. \end{aligned} \tag{10}$$

where " $\vee$ " means "max" and " $\wedge$ " means "min", then $\underline{OR_{\sum_{i=1}^{m} A_i}}(X)$ and $\overline{OR_{\sum_{i=1}^{m} A_i}}(X)$ are respectively called the optimistic multi-granulation lower approximation and upper approximation of X on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$. X is a multi-granulation fuzzy rough set on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ if and only if $\underline{OR_{\sum_{i=1}^{m} A_i}}(X) \neq \overline{OR_{\sum_{i=1}^{m} A_i}}(X)$. Otherwise, X is a multi-granulation fuzzy definable set on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ .The boundary of the set X is defined as

$$Bnd^{F}_{R_{\sum_{i=1}^{m} A_i}}(X) = \overline{OR_{\sum_{i=1}^{m} A_i}}(X) \cap (\sim \underline{OR_{\sum_{i=1}^{m} A_i}}(X)). \tag{11}$$

It can be found that the optimistic MGFRS on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ will be degenerated into fuzzy rough set when $A_i = A_j (i \neq j)$, and $R_{A_i}(x)$ are equivalence classes with respect to the subsets of attributes $A_i (i = 1, \cdots, m)$. That is to say, a fuzzy rough set model is a special instance of the optimistic MGFRS on the tolerance relations. Besides, this model can also be turned the optimistic MGRS if the relations are equivalence relations and the considered set is a crisp one. What's more, the optimistic MGFRS on tolerance relations will be degenerated into a rough set model on tolerance relations if $A_i = A_j (i \neq j)$, and the considered concept X is a crisp set.

The properties about optimistic MGFRS on tolerance relations are listed in the following which can be extended from the optimistic TGFRS model on tolerance relations.

Proposition 3.3[26]: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT, 1 \leq i \leq m$ and $X \in F(U)$. Then the following properties hold.

(1). $\underline{OR_{\sum_{i=1}^{m} A_i}}(X) \subseteq X$,

(2). $\overline{OR_{\sum_{i=1}^{m} A_i}}(X) \supseteq X$;

(3). $\underline{OR_{\sum_{i=1}^{m} A_i}}(\sim X) = \sim \overline{R_{\sum_{i=1}^{m} A_i}}(X)$,

(4). $\overline{OR_{\sum_{i=1}^{m} A_i}}(\sim X) = \sim \underline{OR_{\sum_{i=1}^{m} A_i}}(X)$;

(5). $\underline{OR_{\sum_{i=1}^{m} A_i}}(U) = \overline{OR_{\sum_{i=1}^{m} A_i}}(U) = U$,

(6). $\underline{OR_{\sum_{i=1}^{m} A_i}}(\varnothing) = \overline{OR_{\sum_{i=1}^{m} A_i}}(\varnothing) = \varnothing$;

(7). $\underline{OR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \underline{OR_{\sum_{i=1}^{m} A_i}}(\underline{OR_{\sum_{i=1}^{m} A_i}}(X))$,

(8). $\overline{OR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \overline{OR_{\sum_{i=1}^{m} A_i}}(\overline{OR_{\sum_{i=1}^{m} A_i}}(X))$.

Proposition 3.4[26]: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT, 1 \leq i \leq m,$ $X, Y \in F(U)$. Then the following properties hold.

(1). $\underline{OR_{\sum_{i=1}^{m} A_i}}(X \cap Y) \subseteq \underline{OR_{\sum_{i=1}^{m} A_i}}(X) \cap \underline{OR_{\sum_{i=1}^{m} A_i}}(Y)$,

(2). $\overline{OR_{\sum_{i=1}^{m} A_i}}(X \cup Y) \supseteq \overline{OR_{\sum_{i=1}^{m} A_i}}(X) \cup \overline{OR_{\sum_{i=1}^{m} A_i}}(Y)$;

(3). $X \subseteq Y \Rightarrow \underline{OR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{OR_{\sum_{i=1}^{m} A_i}}(Y)$,

(4). $X \subseteq Y \Rightarrow \overline{OR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \overline{OR_{\sum_{i=1}^{m} A_i}}(Y)$;

(5). $\underline{OR_{\sum_{i=1}^{m} A_i}}(X \cup Y) \supseteq \underline{OR_{\sum_{i=1}^{m} A_i}}(X) \cup \underline{OR_{\sum_{i=1}^{m} A_i}}(Y)$,

(6). $\overline{OR_{\sum_{i=1}^{m} A_i}}(X \cap Y) \subseteq \overline{OR_{\sum_{i=1}^{m} A_i}}(X) \cap \overline{OR_{\sum_{i=1}^{m} A_i}}(Y)$.

B. The Pessimistic Multi-granulation Fuzzy Rough Set *on Tolerance Relations*

In this subsection, we will propose a pessimistic MGFRS on tolerance relations. We first define the pessimistic two-granulation fuzzy rough set (in brief pessimistic TGFRS) on tolerance relations.

Definition 3.3[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A,B\subseteq AT$. For the fuzzy set $X\in F(U)$, denote

$$\begin{aligned}\underline{PR_{A+B}}(X)(x)&=\{\wedge\{(X)(y)\mid y\in R_A(x)\}\}\\&\quad\wedge\{\wedge\{(X)(y)\mid y\in R_B(x)\}\},\\\overline{PR_{A+B}}(X)(x)&=\{\vee\{(X)(y)\mid y\in R_A(x)\}\}\\&\quad\vee\{\vee\{(X)(y)\mid y\in R_B(x)\}\}.\end{aligned}\tag{12}$$

then $\underline{PR_{A+B}}(X)$ and $\overline{PR_{A+B}}(X)$ are respectively called the pessimistic two-granulation lower approximation and upper approximation of X on the tolerance relations R with respect to the subsets of attributes A and B. X is the pessimistic two-granulation fuzzy rough set on the tolerance relations if and only if $\underline{PR_{A+B}}(X)\neq\overline{PR_{A+B}}(X)$. Otherwise, X is the pessimistic two-granulation fuzzy definable set on these tolerance relations. The boundary of the set X is defined as

$$Bnd^S_{R_{A+B}}(X)=\overline{PR_{A+B}}(X)\cap\left(\sim\underline{PR_{A+B}}(X)\right).\tag{13}$$

It can be found that the pessimistic TGFRS on tolerance relations will be degenerated into the fuzzy rough set model when $A=B$ and $R_A(x)$ and $R_B(x)$ are equivalence classes with respect to the subsets of attributes A and B. That is, a fuzzy rough set model is a special instance of the pessimistic TGFRS on tolerance relations. What's more, the pessimistic TGFRS on tolerance relations will be degenerated into a rough set model on a tolerance relation if $A=B$ and the considered concept X is a crisp set.

In the following, we continue e.g. 3.1 to illustrate the above concepts.

Example 3.2: (Continued from e.g. 3.1) From Definition 3.2, we can compute the pessimistic two-granulation lower and upper approximation of D on the tolerance relation R_{A_1} and R_{A_2} are

$$\underline{PR_{A_1+A_2}}(D)=(0.4,0.6,0.4,0.4,0.5,0.4,0.6,0.5,0.4,0.6),$$
$$\overline{PR_{A_1+A_2}}(D)=(0.9,0.9,0.9,0.9,0.9,0.9,0.9,0.9,0.9,0.9).$$

Obviously, the following can be found

$$\begin{aligned}&\underline{PR_{A_1+A_2}}(D)=\underline{R_{A_1}}(D)\cap\underline{R_{A_2}}(D),\\&\overline{PR_{A_1+A_2}}(D)=\overline{R_{A_1}}(D)\cup\overline{R_{A_2}}(D),\\&\underline{PR_{A_1+A_2}}(D)\subseteq\underline{R_{A_1\cup A_2}}(D)\subseteq D\\&\qquad\subseteq\overline{R_{A_1\cup A_2}}(D)\subseteq\overline{PR_{A_1+A_2}}(D).\end{aligned}$$

Proposition 3.5[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $B,A\subseteq AT$ and $X\in F(U)$. Then the following properties hold.

(1). $\underline{PR_{A+B}}(X)\subseteq X$,

(2). $\overline{PR_{A+B}}(X)\supseteq X$;

(3). $\underline{PR_{A+B}}(\sim X)=\sim\overline{PR_{A+B}}(X)$,

(4). $\overline{PR_{A+B}}(\sim X)=\sim\underline{PR_{A+B}}(X)$;

(5). $\underline{PR_{A+B}}(U)=\overline{PR_{A+B}}(U)=U$,

(6). $\underline{PR_{A+B}}(\varnothing)=\overline{PR_{A+B}}(\varnothing)=\varnothing$;

(7). $\underline{PR_{A+B}}(X)\supseteq\underline{PR_{A+B}}(\underline{PR_{A+B}}(X))$,

(8). $\overline{PR_{A+B}}(X)\subseteq\overline{PR_{A+B}}(\overline{PR_{A+B}}(X))$.

Proposition 3.6[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $B,A\subseteq AT$, $X,Y\in F(U)$. Then the following properties hold.

(1). $\underline{PR_{A+B}}(X\cap Y)=\underline{PR_{A+B}}(X)\cap\underline{PR_{A+B}}(Y)$,

(2). $\overline{PR_{A+B}}(X\cup Y)=\overline{PR_{A+B}}(X)\cup\overline{PR_{A+B}}(Y)$;

(3). $X\subseteq Y\Rightarrow\underline{PR_{A+B}}(X)\subseteq\underline{PR_{A+B}}(Y)$,

(4). $X\subseteq Y\Rightarrow\overline{PR_{A+B}}(X)\subseteq\overline{PR_{A+B}}(Y)$;

(5). $\underline{PR_{A+B}}(X\cup Y)\supseteq\underline{PR_{A+B}}(X)\cup\underline{PR_{A+B}}(Y)$,

(6). $\overline{PR_{A+B}}(X\cap Y)\subseteq\overline{PR_{A+B}}(X)\cap\overline{PR_{A+B}}(Y)$.

In the following, we will introduce the pessimistic multi-granulation fuzzy rough set (in brief pessimistic MGFRS) on tolerance relations and its corresponding properties by extending the pessimistic TGFRS on tolerance relations.

Definition 3.4[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A,B\subseteq AT$. For the fuzzy set $X\in F(U)$, denote

$$\begin{aligned}\underline{PR_{\sum_{i=1}^m A_i}}(X)(x)&=\mathop{\wedge}_{i=1}^{m}\{\wedge\{(X)(y)\mid y\in R_{A_i}(x)\}\},\\\overline{PR_{\sum_{i=1}^m A_i}}(X)(x)&=\mathop{\vee}_{i=1}^{m}\{\vee\{(X)(y)\mid y\in R_{A_i}(x)\}\},\end{aligned}\tag{14}$$

where "$\vee$" means "max" and "$\wedge$" means "min", then $\underline{PR_{\sum_{i=1}^m A_i}}(X)$ and $\overline{PR_{\sum_{i=1}^m A_i}}(X)$ are respectively called the pessimistic multi-granulation lower approximation and upper approximation of X on tolerance relations R with respect to the subsets of attributes $A_i(i=1,\cdots,m)$. X is the pessimistic multi-granulation fuzzy rough set if and only if $\underline{PR_{\sum_{i=1}^m A_i}}(X)\neq\overline{PR_{\sum_{i=1}^m A_i}}(X)$. Otherwise, X is the pessimistic multi-granulation fuzzy definable set on many tolerance relations. The boundary of the set X is defined as

$$Bnd^{F}_{R_{\sum_{i=1}^{m}A_i}}(X)=\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\cap(\sim \underline{PR}_{\sum_{i=1}^{m}A_i}(X)). \quad (15)$$

It can be found that the pessimistic MGFRS will be degenerated into fuzzy rough set when $A_i = A_j (i \neq j)$ and $R_{A_i}(x)$ are equivalence classes with respect to the subsets of attributes $A_i (i=1,\cdots,m)$. That is, a fuzzy rough set model is also a special instance of the pessimistic MGFRS on tolerance relations. Besides, this model can also been turned the pessimistic MGRS if the relations are equivalence relations and the considered set is a crisp one. What's more, the pessimistic MGFRS model will be degenerated into a rough set model on a tolerance relation if $A_i = A_j (i \neq j)$ and the considered concept X is a crisp set.

The properties about the pessimistic MGFRS on tolerance relations are listed in the following which can be extended from the pessimistic TGFRS model on tolerance relations.

Proposition 3.7[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A_i \subseteq AT, 1 \le i \le m$ and $X \in F(U)$. Then the following properties hold.

(1). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\subseteq X$,

(2). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\supseteq X$;

(3). $\underline{PR}_{\sum_{i=1}^{m}A_i}(\sim X)=\sim\overline{PR}_{\sum_{i=1}^{m}A_i}(X)$,

(4). $\overline{PR}_{\sum_{i=1}^{m}A_i}(\sim X)=\sim\underline{PR}_{\sum_{i=1}^{m}A_i}(X)$;

(5). $\underline{PR}_{\sum_{i=1}^{m}A_i}(U)=\overline{PR}_{\sum_{i=1}^{m}A_i}(U)=U$,

(6). $\underline{PR}_{\sum_{i=1}^{m}A_i}(\varnothing)=\overline{PR}_{\sum_{i=1}^{m}A_i}(\varnothing)=\varnothing$;

(7). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\supseteq\underline{PR}_{\sum_{i=1}^{m}A_i}(\underline{PR}_{\sum_{i=1}^{m}A_i}(X))$,

(8). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\subseteq\overline{PR}_{\sum_{i=1}^{m}A_i}(\overline{PR}_{\sum_{i=1}^{m}A_i}(X))$.

Proposition 3.8[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A_i \subseteq AT, 1 \le i \le m$, $X,Y \in F(U)$. Then the following properties hold.

(1). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X\cap Y)=\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\cap\underline{PR}_{\sum_{i=1}^{m}A_i}(Y)$,

(2). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X\cup Y)=\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\cup\overline{PR}_{\sum_{i=1}^{m}A_i}(Y)$;

(3). $X\subseteq Y\Rightarrow\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\subseteq\underline{PR}_{\sum_{i=1}^{m}A_i}(Y)$,

(4). $X\subseteq Y\Rightarrow\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\subseteq\overline{PR}_{\sum_{i=1}^{m}A_i}(Y)$;

(5). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X\cup Y)\supseteq\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\cup\underline{PR}_{\sum_{i=1}^{m}A_i}(Y)$,

(6). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X\cap Y)\subseteq\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\cap\overline{PR}_{\sum_{i=1}^{m}A_i}(Y)$.

On the basis of tolerance relations, we will investigate the interrelationships among SGFRS, the optimistic MGFRS and the pessimistic MGFRS after the discussion of the properties of them.

Proposition 3.9[26]: Let $\mathcal{I}=(U,AT,\tau)$ be a tolerance information system, $A_i \subseteq AT, 1 \le i \le m$, $X \in F(U)$. Then the following properties hold.

(1). $\underline{OR}_{\sum_{i=1}^{m}A_i}(X)=\bigcup_{i=1}^{m}\underline{R_{A_i}}(X)$,

(2). $\overline{OR}_{\sum_{i=1}^{m}A_i}(X)=\bigcap_{i=1}^{m}\overline{R_{A_i}}(X)$;

(3). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X)=\bigcap_{i=1}^{m}\underline{R_{A_i}}(X)$,

(4). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X)=\bigcup_{i=1}^{m}\overline{R_{A_i}}(X)$;

(5). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\subseteq\underline{OR}_{\sum_{i=1}^{m}A_i}(X)\subseteq\underline{R}_{\bigcup_{i=1}^{m}A_i}(X)$;

(6). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\supseteq\overline{OR}_{\sum_{i=1}^{m}A_i}(X)\supseteq\overline{R}_{\bigcup_{i=1}^{m}A_i}(X)$.

(7). $\underline{PR}_{\sum_{i=1}^{m}A_i}(X)\subseteq\underline{R_{A_i}}(X)\subseteq\underline{OR}_{\sum_{i=1}^{m}A_i}(X)$;

(8). $\overline{PR}_{\sum_{i=1}^{m}A_i}(X)\supseteq\overline{R_{A_i}}(X)\supseteq\overline{OR}_{\sum_{i=1}^{m}A_i}(X)$.

4. Two Type of Multi-granulation Fuzzy Rough Sets in a Fuzzy Tolerance Approximation Space

In this section, we will propose two types of MGFRS models in a tolerance approximation space, in which the relations are fuzzy tolerance relations and the characterized concepts are fuzzy sets. The properties of the approximations will be showed. Then the optimistic Multi-granulation fuzzy rough sets will be introduced firstly.

A. *The Optimistic Multi-granulation Fuzzy Rough Sets in a Fuzzy Tolerance Approximation Space*

(U,R) is a fuzzy tolerance approximation space, where U is a non-empty finite set of objects known as universe, $R\subseteq U\times U$ is a family set of fuzzy tolerance relations [10] satisfying

(1). Reflexivity: $R(x,x)=1,\forall x\in U$, (2). Symmetry: $R(x,y)=R(y,x), \forall x,y \in U$.

In the following, we will firstly define the SGFRS in a fuzzy tolerance approximation space.

Definition 4.1: Let (U,R) be a fuzzy tolerance fuzzy approximation space, R be a fuzzy tolerance relation. For the fuzzy set $X \subseteq F(U)$, denote

$$\begin{aligned} &\underline{R}(X)(x)= \wedge_{y\in U} \{X(y)\vee(1-R(x,y))\},\\ &\overline{R}(X)(x)= \vee_{y\in U} \{X(y)\wedge R(x,y)\}, \end{aligned} \tag{16}$$

then $\underline{R}(X)$ and $\overline{R}(X)$ are respectively called the single-granulation fuzzy lower and fuzzy upper approximations of X on a fuzzy tolerance relation. X is a single-granulation fuzzy rough set on a fuzzy tolerance relation if and only if $\underline{R}(X)=\overline{R}(X)$. Otherwise, X is a single-granulation fuzzy definable set on the fuzzy tolerance relation.

Proposition 4.1: Let (U,R) be a fuzzy tolerance fuzzy approximation space, R be a fuzzy tolerance relation. For the fuzzy set $X,Y \subseteq F(U)$, we have

(1). $\underline{R}(X)\subseteq X$,

(2). $\overline{R}(X)\supseteq X$;

(3). $\underline{R}(\sim X)=\sim\overline{R}(X)$,

(4). $\overline{R}(\sim X)=\sim\underline{R}(X)$;

(5). $\underline{R}(U)=\overline{R}(U)=U$,

(6). $\underline{R}(\varnothing)=\overline{R}(\varnothing)=\varnothing$;

(7). $\underline{R}(X\cap Y)=\underline{R}(X)\cap\underline{R}(Y)$,

(8). $\overline{R}(X\cup Y)=\overline{R}(X)\cup\overline{R}(Y)$.

Proof: We only prove the odd items, the rest can be proved similarly.

(1). Since $R(x,x)=1$, then

$$\wedge_{y\in U}\{X(y)\vee(1-R(x,y))\}\le X(x)\vee(1-R(x,x))=X(x) .$$

Thus, $\underline{R}(X)\subseteq X$.

(3). From Definition 4.1, we have

$$\begin{aligned} \underline{R}(\sim X)(x)&= \wedge_{y\in U}\{(1-X(y))\vee(1-R(x,y))\}\\ &=1-\vee_{y\in U}\{X(y)\wedge R(x,y)\}\\ &=\sim\overline{R}(X)(x), \end{aligned}$$

that is $\underline{R}(\sim X)=\sim\overline{R}(X)$.

(5). $\forall x\in U, U(x)=1$ and $R(x,x)=1$, we have

$$\underline{R}(U)(x)=\wedge_{y\in U}\{U(y)\vee(1-R(x,y))\}=\wedge_{y\in U}U(y)=1=U(x),$$

and

$$\overline{R}(U)(x)=\vee_{y\in U}\{U(y)\wedge R(x,y)\}=\vee_{y\in U}R(x,y)=1=U(x).$$

Therefore, $\underline{R}(U)=\overline{R}(U)=U$.

(7). $\forall x\in U, X,Y\subseteq F(U)$,

$$\begin{aligned} \underline{R}(X\cap Y)(x)&= \wedge_{y\in U}\{(X\cap Y)(y)\vee(1-R(x,y))\}\\ &=\wedge_{y\in U}\{\left(X(y)\vee\left(1-R(x,y)\right)\right)\wedge\left(Y(y)\vee\left(1-R(x,y)\right)\right)\}\\ &=\left\{\wedge_{y\in U}\left(X(y)\vee\left(1-R(x,y)\right)\right)\right\}\wedge\left\{\wedge_{y\in U}\left(Y(y)\vee\left(1-R(x,y)\right)\right)\right\}\\ &=\underline{R}(X)\cap\underline{R}(Y). \end{aligned}$$

Now, we will give the definition of the optimistic multi-granulation fuzzy rough set (in brief OMGFRS) in a fuzzy tolerance approximation space.

Definition 4.2: Let (U,R) be a fuzzy tolerance fuzzy approximation space, $R_i(1\le i\le m)$ be fuzzy tolerance relations. For the fuzzy set $X\subseteq F(U)$, denote

$$\begin{aligned} &\underline{\sum_{i=1}^{m}R_i^O}(X)(x)=\vee_{i=1}^{m}\left\{\wedge_{y\in U}\{X(y)\vee(1-R_i(x,y))\}\right\},\\ &\overline{\sum_{i=1}^{m}R_i^O}(X)(x)=\wedge_{i=1}^{m}\left\{\vee_{y\in U}\{X(y)\wedge R_i(x,y)\}\right\}, \end{aligned} \tag{17}$$

then $\underline{\sum_{i=1}^{m}R_i^O}(X)$ and $\overline{\sum_{i=1}^{m}R_i^O}(X)$ are respectively called the optimistic multi-granulation fuzzy lower and fuzzy upper approximations of X on fuzzy tolerance relations. X is an optimistic multi-granulation fuzzy rough set in a fuzzy tolerance approximation space if and only if $\underline{\sum_{i=1}^{m}R_i^O}(X)\neq\overline{\sum_{i=1}^{m}R_i^O}(X)$. Otherwise, X is a multi-granulation fuzzy definable set in a fuzzy tolerance approximation space.

We can obtain some special cases from Definition 4.2:

(1). If $X\subseteq F(U)$, i.e., X is a fuzzy set and fuzzy tolerance relations $R_i=R_j(1\le i,j\le m)$, then

$$\begin{aligned} &\underline{\sum_{i=1}^{m}R_i^O}(X)(x)=\wedge_{y\in U}\{X(y)\vee(1-R_i(x,y))\}=\underline{R_i}(X)(x),\\ &\overline{\sum_{i=1}^{m}R_i^O}(X)(x)=\vee_{y\in U}\{X(y)\wedge R_i(x,y)\}=\overline{R_i}(X)(x). \end{aligned}$$

Hence, the approximations can be degenerated into the ones in Definition 4.1.

(2). If $X\subseteq U, R_i\subseteq U\times U$, i.e., X is a crisp set and $R_i(1\le i\le m)$ are ordinary tolerance relations, $R_i=R_j(i\neq j)$, then

$$\begin{aligned} &\underline{\sum_{i=1}^{m}R_i^O}(X)(x)=\underline{R_i}(X)(x)=1\\ &\Leftrightarrow \forall y\in U, X(y)\vee(1-R_i(x,y))=1\\ &\Leftrightarrow \forall y\in U,(y\notin X\Rightarrow(x,y)\notin R_i\\ &\Leftrightarrow (\forall y\notin X\Rightarrow y\notin R_i(x))\\ &\Leftrightarrow R_i(x)\subseteq X,\\ &\overline{\sum_{i=1}^{m}R_i^O}(X)(x)=\overline{R_i}(X)(x)=1\\ &\Leftrightarrow \exists y\in U, s.t.X(y)=1, R_i(x,y)=1\\ &\Leftrightarrow X\cap R_i(x)\neq\varnothing. \end{aligned}$$

That is to say, they can be changed to (2).

(3). If $X \subseteq F(U)$, $R_i(1 \le i \le m)$ are ordinary tolerance relations, and $R_i = R_j(i \ne j)$, then

$$\underline{\sum_{i=1}^{m} R_i^O}(X)(x) = \underline{R_i}(X)(x)$$
$$= \wedge_{y \in U}\{X(y) \vee (1 - R_i(x,y))\}$$
$$= \wedge_{y \in U}\{X(y) \mid x, y) \in R_i\}$$
$$= \wedge_{y \in U}\{X(y) \mid y \in R_i(x)\},$$
$$\overline{\sum_{i=1}^{m} R_i^O}(X)(x) = \overline{R_i}(X)(x)$$
$$= \vee_{y \in U}\{X(y) \wedge R_i(x,y)\}$$
$$= \vee_{y \in U}\{X(y) \mid (x,y) \in R_i\}$$
$$= \vee_{y \in U}\{X(y) \mid y \in R_i(x)\}.$$

Thus they are consistent to the approximations in (7).

(4). If $X \subseteq U$, $R_i(1 \le i \le m)$ are classical equivalence relations,

$$\underline{\sum_{i=1}^{m} R_i^O}(X)(x) = 1$$
$$\Leftrightarrow \exists R_i, \forall y \in U, X(y) \vee (1 - R_i(x,y)) = 1$$
$$\Leftrightarrow \exists R_i, \forall y \in U, (y \notin X \Rightarrow (x,y) \notin R_i)$$
$$\Leftrightarrow \exists R_i, (\forall y \notin X \Rightarrow y \notin [x]_{R_i})$$
$$\Leftrightarrow \exists R_i, [x]_{R_i} \subseteq X,$$
$$\Leftrightarrow \underline{\sum_{i=1}^{m} R_i^O}(X) = \left\{x \in U \mid \vee_{i=1}^{m}[x]_{R_i} \subseteq X\right\},$$
$$\overline{\sum_{i=1}^{m} R_i^O}(X)(x) = 1$$
$$\Leftrightarrow \forall R_i, \exists y \in U, s.t X(y) = 1, R_i(x,y) = 1$$
$$\Leftrightarrow \forall R_i, X \cap R_i(x) \ne \varnothing$$
$$\Leftrightarrow \overline{\sum_{i=1}^{m} R_i^O}(X) = \left\{x \in U \mid \wedge_{i=1}^{m}[x]_{R_i} \cap X \ne \varnothing\right\}.$$

So the approximations are the same as the ones in (5).

Thus, it is easy to find that the model as Definition 4.2 showed is a reasonable generalization of models presented in Section 2.

In the following, we will give an example to illustrate the above definition.

Example 4.1: Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $X = (0.50, 0.65, 0.69, 0.34, 0.71, 0.67)$ and tolerance relations R, S on $U \times U$ are defined as

$$R = \begin{pmatrix} 1 & 0.85 & 0.50 & 0.65 & 0.61 & 0.81 \\ 0.85 & 1 & 0.43 & 0.67 & 0.69 & 0.84 \\ 0.50 & 0.43 & 1 & 0.50 & 0.35 & 0.36 \\ 0.65 & 0.67 & 0.50 & 1 & 0.91 & 0.50 \\ 0.61 & 0.69 & 0.35 & 0.91 & 1 & 0.58 \\ 0.81 & 0.84 & 0.36 & 0.50 & 0.58 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 0.75 & 0.50 & 0.67 & 0.61 & 0.84 \\ 0.75 & 1 & 0.45 & 0.61 & 0.65 & 0.81 \\ 0.50 & 0.45 & 1 & 0.50 & 0.36 & 0.35 \\ 0.67 & 0.61 & 0.50 & 1 & 0.91 & 0.50 \\ 0.61 & 0.65 & 0.36 & 0.91 & 1 & 0.58 \\ 0.84 & 0.81 & 0.35 & 0.50 & 0.58 & 1 \end{pmatrix}.$$

Then the single-granulation fuzzy lower and fuzzy upper approximations of X on fuzzy tolerance relation R, S are respectively

$$\underline{R}(X) = (0.35, 0.34, 0.50, 0.34, 0.34, 0.50),$$
$$\overline{R}(X) = (0.67, 0.69, 0.69, 0.71, 0.71, 0.67);$$
$$\underline{S}(X) = (0.34, 0.39, 0.50, 0.34, 0.34, 0.50),$$
$$\overline{S}(X) = (0.67, 0.67, 0.69, 0.71, 0.71, 0.67);$$
$$\underline{R \cup S}(X) = (0.34, 0.34, 0.50, 0.34, 0.34, 0.50),$$
$$\overline{R \cup S}(X) = (0.67, 0.69, 0.69, 0.71, 0.71, 0.67);$$
$$\underline{R \cap S}(X) = (0.35, 0.39, 0.50, 0.34, 0.34, 0.50),$$
$$\overline{R \cap S}(X) = (0.67, 0.67, 0.69, 0.71, 0.71, 0.67);$$

And the optimistic multi-granulation fuzzy lower and fuzzy upper approximations of X on fuzzy tolerance relations R,S are

$$\underline{R + S}^O(X) = (0.35, 0.39, 0.50, 0.34, 0.34, 0.50),$$
$$\overline{R + S}^O(X) = (0.67, 0.67, 0.69, 0.71, 0.71, 0.67).$$

We find that

$$\underline{R + S}^O(X) = \underline{R}(X) \cup \underline{S}(X),$$
$$\overline{R + S}^O(X) = \overline{R}(X) \cap \overline{S}(X),$$
$$\underline{R \cup S}(X) \subseteq \underline{R + S}^O(X) \subseteq X \subseteq \overline{R + S}^O(X) \subseteq \overline{R \cup S}(X).$$

Just from Definition 4.2, we can prove the following properties about the optimistic multi-granulation fuzzy rough set in fuzzy tolerance approximation space hold.

Proposition 4.2: Let (U, R) be a fuzzy tolerance fuzzy approximation space, $R_i(1 \le i \le m)$ be fuzzy tolerance relations. For the fuzzy set $X \subseteq F(U)$, the following properties hold.

(1). $\underline{\sum_{i=1}^{m} R_i^O}(X) \subseteq X$,

(2). $\overline{\sum_{i=1}^{m} R_i^O}(X) \supseteq X$;

(3). $\underline{\sum_{i=1}^{m} R_i^O}(\sim X) = \sim \overline{\sum_{i=1}^{m} R_i^O}(X)$,

(4). $\overline{\sum_{i=1}^{m} R_i^O}(\sim X) = \sim \underline{\sum_{i=1}^{m} R_i^O}(X)$;

(5). $\underline{\sum_{i=1}^{m} R_i^O}(U) = \overline{\sum_{i=1}^{m} R_i^O}(U) = U$,

(6). $\underline{\sum_{i=1}^{m} R_i^O}(\varnothing) = \overline{\sum_{i=1}^{m} R_i^O}(\varnothing) = \varnothing$;

(7). $\underline{\sum_{i=1}^{m} R_i^O}(X) = \bigcup_{i=1}^{m} \underline{R_i}(X)$,

(8). $\overline{\sum_{i=1}^{m} R_i^O}(X) = \bigcap_{i=1}^{m} \overline{R_i}(X)$.

Proof: They are straightforward from Definition 4.2 and Proposition 4.1.

Proposition 4.3: Let (U, R) be a fuzzy tolerance fuzzy approximation space, $R_i (1 \le i \le m)$ be fuzzy tolerance relations. For the fuzzy set $X, Y \subseteq F(U)$, the following properties hold.

(1). $\underline{\sum_{i=1}^{m} R_i^O}(X \cap Y) = \bigcup_{i=1}^{m} \left(\underline{R_i}(X) \cap \underline{R_i}(Y) \right)$,

(2). $\overline{\sum_{i=1}^{m} R_i^O}(X \cup Y) = \bigcap_{i=1}^{m} \left(\overline{R_i}(X) \cup \overline{R_i}(Y) \right)$;

(3). $\underline{\sum_{i=1}^{m} R_i^O}(X \cap Y) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(X) \cap \underline{\sum_{i=1}^{m} R_i^O}(Y)$,

(4). $\overline{\sum_{i=1}^{m} R_i^O}(X \cup Y) \supseteq \overline{\sum_{i=1}^{m} R_i^O}(X) \cup \overline{\sum_{i=1}^{m} R_i^O}(Y)$;

(5). $X \subseteq Y \Rightarrow \underline{\sum_{i=1}^{m} R_i^O}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(Y)$,

(6). $X \subseteq Y \Rightarrow \overline{\sum_{i=1}^{m} R_i^O}(X) \subseteq \overline{\sum_{i=1}^{m} R_i^O}(Y)$;

(7). $\underline{\sum_{i=1}^{m} R_i^O}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^O}(X) \cup \underline{\sum_{i=1}^{m} R_i^O}(Y)$,

(8). $\overline{\sum_{i=1}^{m} R_i^O}(X \cap Y) \subseteq \overline{\sum_{i=1}^{m} R_i^O}(X) \cap \overline{\sum_{i=1}^{m} R_i^O}(Y)$.

Proof: We only prove the odd items, and the others can be proved similarly.

(1). $\forall X, Y \subseteq F(U)$,

$$\underline{\sum_{i=1}^{m} R_i^O}(X \cap Y) = \bigcup_{i=1}^{m} \left(\underline{R_i}(X \cap Y) \right) = \bigcup_{i=1}^{m} \left(\underline{R_i}(X) \cap \underline{R_i}(Y) \right).$$

(3). $\forall x \in U, X, Y \subseteq F(U)$,

$$\begin{aligned} &\underline{\sum_{i=1}^{m} R_i^O}(X \cap Y) \\ &= \bigcup_{i=1}^{m} \left(\underline{R_i}(X) \cap \underline{R_i}(Y) \right) \\ &\subseteq \left(\bigcup_{i=1}^{m} \underline{R_i}(X) \right) \cap \left(\bigcup_{i=1}^{m} \underline{R_i}(Y) \right) \end{aligned}$$

(5). If $X \subseteq Y$, then $X \cap Y = X$. It follows from (3) that

$$\begin{aligned} &\underline{\sum_{i=1}^{m} R_i^O}(X \cap Y) = \underline{\sum_{i=1}^{m} R_i^O}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(X) \cap \underline{\sum_{i=1}^{m} R_i^O}(Y) \\ &\Rightarrow \underline{\sum_{i=1}^{m} R_i^O}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(X) \cap \underline{\sum_{i=1}^{m} R_i^O}(Y) \\ &\Rightarrow \underline{\sum_{i=1}^{m} R_i^O}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(Y). \end{aligned}$$

(7). It is clear that $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$. It follows that

$$\underline{\sum_{i=1}^{m} R_i^O}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^O}(X),$$

and

$$\underline{\sum_{i=1}^{m} R_i^O}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^O}(Y).$$

Hence, $\underline{\sum_{i=1}^{m} R_i^O}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^O}(X) \cup \underline{\sum_{i=1}^{m} R_i^O}(Y).$

B. *The Pessimistic Multi-granulation Fuzzy Rough Sets in a Fuzzy Tolerance Approximation Space*

In this subsection, we will give the definition of the pessimistic multi-granulation fuzzy rough set (in brief PMGFRS) in a fuzzy tolerance approximation space.

Definition 4.3: Let (U, R) be a fuzzy tolerance fuzzy approximation space, $R_i (1 \le i \le m)$ be fuzzy tolerance relations. For the fuzzy set $X \subseteq F(U)$, denote

$$\begin{aligned} &\underline{\sum_{i=1}^{m} R_i^P}(X)(x) = \mathop{\wedge}_{i=1}^{m} \left\{ \mathop{\wedge}_{y \in U} \left\{ X(y) \vee (1 - R_i(x, y)) \right\} \right\}, \\ &\overline{\sum_{i=1}^{m} R_i^P}(X)(x) = \mathop{\vee}_{i=1}^{m} \left\{ \mathop{\vee}_{y \in U} \left\{ X(y) \wedge R_i(x, y) \right\} \right\}, \end{aligned} \tag{18}$$

then $\underline{\sum_{i=1}^{m} R_i^P}(X)$ and $\overline{\sum_{i=1}^{m} R_i^P}(X)$ are respectively called the pessimistic multi-granulation fuzzy lower and fuzzy upper approximations of X on fuzzy tolerance relations. X is a PMGFRS in a fuzzy tolerance approximation space if and only if $\underline{\sum_{i=1}^{m} R_i^P}(X) \neq \overline{\sum_{i=1}^{m} R_i^P}(X)$. Otherwise, X is a multi-granulation fuzzy definable set in a fuzzy tolerance approximation space.

We can obtain some special cases from Definition 4.3:

(1). If $X \subseteq F(U)$, i.e., X is a fuzzy set and fuzzy tolerance relations $R_i = R_j (1 \le i, j \le m)$, then

$$\begin{aligned} &\underline{\sum_{i=1}^{m} R_i^P}(X)(x) = \mathop{\wedge}_{y \in U} \left\{ X(y) \vee (1 - R_i(x, y)) \right\} = \underline{R_i}(X)(x), \\ &\overline{\sum_{i=1}^{m} R_i^P}(X)(x) = \mathop{\vee}_{y \in U} \left\{ X(y) \wedge R_i(x, y) \right\} = \overline{R_i}(X)(x). \end{aligned}$$

Hence, the approximations can also be degenerated into the ones in Definition 4.1.

(2). If $X \subseteq U, R_i \subseteq U \times U$, i.e., X is a crisp set and $R_i (1 \le i \le m)$ are ordinary tolerance relations, $R_i = R_j (i \neq j)$, then

$$\begin{aligned} &\underline{\sum_{i=1}^{m} R_i^P}(X)(x) = \underline{R_i}(X)(x) = 1 \\ &\Leftrightarrow \forall y \in U, X(y) \vee (1 - R_i(x, y)) = 1 \\ &\Leftrightarrow \forall y \in U, (y \notin X \Rightarrow (x, y) \notin R_i \\ &\Leftrightarrow (\forall y \notin X \Rightarrow y \notin R_i(x)) \\ &\Leftrightarrow R_i(x) \subseteq X, \end{aligned}$$

$$\overline{\sum_{i=1}^{m} R_i^P}(X)(x)=\overline{R_i}(X)(x)=1$$
$$\Leftrightarrow \exists y\in U, s.t. X(y)=1, R_i(x,y)=1$$
$$\Leftrightarrow X\cap R_i(x)\neq\varnothing.$$

That is to say, they can be changed to (2).

(3). If $X\subseteq F(U)$, $R_i(1\le i\le m)$ are ordinary tolerance relations, and $R_i=R_j(i\neq j)$, then

$$\underline{\sum_{i=1}^{m} R_i^P}(X)(x)=\underline{R_i}(X)(x)$$
$$=\wedge_{y\in U}\{X(y)\vee(1-R_i(x,y))\}$$
$$=\wedge_{y\in U}\{X(y)\mid x,y)\in R_i\}$$
$$=\wedge_{y\in U}\{X(y)\mid y\in R_i(x)\},$$
$$\overline{\sum_{i=1}^{m} R_i^P}(X)(x)=\overline{R_i}(X)(x)$$
$$=\vee_{y\in U}\{X(y)\wedge R_i(x,y)\}$$
$$=\vee_{y\in U}\{X(y)\mid(x,y)\in R_i\}$$
$$=\vee_{y\in U}\{X(y)\mid y\in R_i(x)\}.$$

Thus they are consistent to the approximations in (7)

(4). If $X\subseteq U$, $R_i(1\le i\le m)$ are classical equivalence relations,

$$\underline{\sum_{i=1}^{m} R_i^P}(X)(x)=1$$
$$\Leftrightarrow \forall R_i, y\in U, X(y)\vee(1-R_i(x,y))=1$$
$$\Leftrightarrow \forall R_i, y\in U, (y\notin X\Rightarrow (x,y)\notin R_i)$$
$$\Leftrightarrow \forall R_i, (\forall y\notin X\Rightarrow y\notin [x]_{R_i})$$
$$\Leftrightarrow \forall R_i, [x]_{R_i}\subseteq X,$$
$$\Leftrightarrow \underline{\sum_{i=1}^{m} R_i^P}(X)=\left\{x\in U\mid \wedge_{i=1}^{m}[x]_{R_i}\subseteq X\right\},$$
$$\overline{\sum_{i=1}^{m} R_i^P}(X)(x)=1$$
$$\Leftrightarrow \exists R_i, y\in U, s.t. X(y)=1, R_i(x,y)=1$$
$$\Leftrightarrow \exists R_i, X\cap R_i(x)\neq\varnothing$$
$$\Leftrightarrow \overline{\sum_{i=1}^{m} R_i^P}(X)=\left\{x\in U\mid \vee_{i=1}^{m}[x]_{R_i}\cap X\neq\varnothing\right\}.$$

So the approximations are the same as the ones in (6).

Thus, it is easy to find that the model as Definition 4.3 showed is also a reasonable generalization of models presented in Section 2.

Example 4.2: (Continued from e.g 4.1) From Definition 4.3, we can compute the pessimistic fuzzy lower and upper approximations of X in the same fuzzy tolerance space as follows.

$$\underline{R+S}^P(X)=(0.34,0.34,0.50,0.34,0.34,0.50),$$
$$\overline{R+S}^P(X)=(0.67,0.69,0.69,0.71,0.71,0.67).$$

We find that

$$\underline{R+S}^P(X)=\underline{R}(X)\cap\underline{S}(X),$$
$$\overline{R+S}^P(X)=\overline{R}(X)\cup\overline{S}(X),$$
$$\underline{R+S}^P(X)\subseteq\underline{R\cap S}(X)\subseteq X\subseteq\overline{R\cap S}(X)\subseteq\overline{R+S}^P(X).$$

From Definition 4.3, we can prove the following properties about the PMGFRS in a fuzzy tolerance approximation space hold.

Proposition 4.4: Let (U,R) be a fuzzy tolerance fuzzy approximation space, $R_i(1\le i\le m)$ be fuzzy tolerance relations. For the fuzzy set $X\subseteq F(U)$, the following properties hold.

(1). $\underline{\sum_{i=1}^{m} R_i^P}(X)\subseteq X$,

(2). $\overline{\sum_{i=1}^{m} R_i^P}(X)\supseteq X$;

(3). $\underline{\sum_{i=1}^{m} R_i^P}(\sim X)=\sim\overline{\sum_{i=1}^{m} R_i^P}(X)$,

(4). $\overline{\sum_{i=1}^{m} R_i^P}(\sim X)=\sim\underline{\sum_{i=1}^{m} R_i^P}(X)$;

(5). $\underline{\sum_{i=1}^{m} R_i^P}(U)=\overline{\sum_{i=1}^{m} R_i^P}(U)=U$,

(6). $\underline{\sum_{i=1}^{m} R_i^P}(\varnothing)=\overline{\sum_{i=1}^{m} R_i^P}(\varnothing)=\varnothing$;

(7). $\underline{\sum_{i=1}^{m} R_i^P}(X)=\bigcap_{i=1}^{m}\underline{R_i}(X)$,

(8). $\overline{\sum_{i=1}^{m} R_i^P}(X)=\bigcup_{i=1}^{m}\overline{R_i}(X)$.

Proof: They are straightforward from Definition 4.3 and Proposition 4.1.

Proposition 4.5: Let (U,R) be a fuzzy tolerance fuzzy approximation space, $R_i(1\le i\le m)$ be fuzzy tolerance relations. For the fuzzy set $X,Y\subseteq F(U)$, the following properties hold.

(1). $\underline{\sum_{i=1}^{m} R_i^P}(X\cap Y)=\bigcap_{i=1}^{m}\left(\underline{R_i}(X)\cap\underline{R_i}(Y)\right)$,

(2). $\overline{\sum_{i=1}^{m} R_i^P}(X\cup Y)=\bigcup_{i=1}^{m}\left(\overline{R_i}(X)\cup\overline{R_i}(Y)\right)$;

(3). $\underline{\sum_{i=1}^{m} R_i^P}(X\cap Y)=\underline{\sum_{i=1}^{m} R_i^P}(X)\cap\underline{\sum_{i=1}^{m} R_i^P}(Y)$,

(4). $\overline{\sum_{i=1}^{m} R_i^P}(X\cup Y)=\overline{\sum_{i=1}^{m} R_i^P}(X)\cup\overline{\sum_{i=1}^{m} R_i^P}(Y)$;

(5). $X\subseteq Y\Rightarrow\underline{\sum_{i=1}^{m} R_i^P}(X)\subseteq\underline{\sum_{i=1}^{m} R_i^P}(Y)$,

(6). $X\subseteq Y\Rightarrow\overline{\sum_{i=1}^{m} R_i^P}(X)\subseteq\overline{\sum_{i=1}^{m} R_i^P}(Y)$;

(7). $\underline{\sum_{i=1}^{m} R_i^P}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^P}(X) \cup \underline{\sum_{i=1}^{m} R_i^P}(Y),$

(8). $\overline{\sum_{i=1}^{m} R_i^P}(X \cap Y) \subseteq \overline{\sum_{i=1}^{m} R_i^P}(X) \cap \overline{\sum_{i=1}^{m} R_i^P}(Y).$

Proof: We only prove the odd items, and the others can be proved similarly.

(1). $\forall X, Y \subseteq F(U)$,

$$\underline{\sum_{i=1}^{m} R_i^P}(X \cap Y) = \bigcap_{i=1}^{m}\left(\underline{R_i}(X \cap Y)\right) = \bigcap_{i=1}^{m}\left(\underline{R_i}(X) \cap \underline{R_i}(Y)\right).$$

(3). $\forall x \in U, X, Y \subseteq F(U)$,

$$\begin{aligned}&\underline{\sum_{i=1}^{m} R_i^P}(X \cap Y)\\ &= \bigcap_{i=1}^{m}\left(\underline{R_i}(X) \cap \underline{R_i}(Y)\right)\\ &= \left(\bigcap_{i=1}^{m}\underline{R_i}(X)\right) \cap \left(\bigcap_{i=1}^{m}\underline{R_i}(Y)\right)\end{aligned}$$

(5). If $X \subseteq Y$, then $X \cap Y = X$. It follows from (3) that

$$\begin{aligned}&\underline{\sum_{i=1}^{m} R_i^P}(X \cap Y) = \underline{\sum_{i=1}^{m} R_i^P}(X) = \underline{\sum_{i=1}^{m} R_i^P}(X) \cap \underline{\sum_{i=1}^{m} R_i^P}(Y)\\ &\Rightarrow \underline{\sum_{i=1}^{m} R_i^P}(X) = \underline{\sum_{i=1}^{m} R_i^P}(X) \cap \underline{\sum_{i=1}^{m} R_i^P}(Y)\\ &\Rightarrow \underline{\sum_{i=1}^{m} R_i^P}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^P}(Y).\end{aligned}$$

(7). It is clear that $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$. It follows that

$$\underline{\sum_{i=1}^{m} R_i^P}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^P}(X)$$

and

$$\underline{\sum_{i=1}^{m} R_i^P}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^P}(Y),$$

Hence, $\underline{\sum_{i=1}^{m} R_i^P}(X \cup Y) \supseteq \underline{\sum_{i=1}^{m} R_i^P}(X) \cup \underline{\sum_{i=1}^{m} R_i^P}(Y).$

Proposition 4.6: Let (U, R) be a fuzzy tolerance fuzzy approximation space, $R_i (1 \le i \le m)$ be fuzzy tolerance relations. For the fuzzy set $X \subseteq F(U)$, the following properties hold.

(1). $\underline{\sum_{i=1}^{m} R_i^P}(X) \subseteq \underline{R_i}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(X)$,

(2). $\overline{\sum_{i=1}^{m} R_i^P}(X) \supseteq \overline{R_i}(X) \supseteq \overline{\sum_{i=1}^{m} R_i^O}(X)$;

(3). $\underline{\bigcup_{i=1}^{m} R_i}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(X)$, $\underline{\sum_{i=1}^{m} R_i^P}(X) \subseteq \underline{\bigcap_{i=1}^{m} R_i}(X)$,

(4). $\overline{\sum_{i=1}^{m} R_i^O}(X) \subseteq \overline{\bigcup_{i=1}^{m} R_i}(X)$, $\overline{\bigcap_{i=1}^{m} R_i}(X) \subseteq \overline{\sum_{i=1}^{m} R_i^P}(X)$.

Proof: The first two items are straightforward to prove by Proposition 4.2 and 4.4. We just prove item (3), and item (4) can be proved similarly by item (2).

Since $\forall x, y \in U$, we have

$$\begin{aligned}&\bigcup_{i=1}^{m} R_i \supseteq R_i \supseteq \bigcap_{i=1}^{m} R_i\\ &\Rightarrow \bigvee_{i=1}^{m} R_i(x,y) \ge R_i(x,y) \ge \bigwedge_{i=1}^{m} R_i(x,y)\\ &\Rightarrow 1 - \bigvee_{i=1}^{m} R_i(x,y) \le 1 - R_i(x,y) \le 1 - \bigwedge_{i=1}^{m} R_i(x,y)\\ &\Rightarrow \underline{\bigcup_{i=1}^{m} R_i}(X)(x) \le \underline{R_i}(X)(x) \le \underline{\bigcap_{i=1}^{m} R_i}(X)(x),\end{aligned}$$

then by item (1), $\underline{\bigcup_{i=1}^{m} R_i}(X) \subseteq \underline{R_i}(X) \subseteq \underline{\sum_{i=1}^{m} R_i^O}(X)$ and $\underline{\sum_{i=1}^{m} R_i^P}(X) \subseteq \underline{R_i}(X) \subseteq \underline{\bigcap_{i=1}^{m} R_i}(X)$ hold.

This proposition exposes the relationships among SGFRS, OMGFRS and PMGFRS in a fuzzy tolerance approximation space.

5. Conclusions

The theory of rough sets has been significantly extended from the point view of granular computing by combining the theory of fuzzy sets. The theory of fuzzy sets pays more attention to the fuzziness of knowledge while the theory of rough sets to the roughness of knowledge in the point view of granular of knowledge. For the complement of the two types of theory, fuzzy rough set models are investigated to solve practical problem. Given that the equivalence relation in the fuzzy rough set theory is too rigorous for some practical application, it is necessary to weaken the equivalence relations to tolerance relations or fuzzy relations. The contribution of this paper is having constructed two new types of fuzzy rough set on fuzzy tolerance relations associated with granular computing called multi-granulation fuzzy rough set models in a fuzzy tolerance approximation space, in which the set approximation operators are defined on the basis of multiple fuzzy tolerance relations. What's more, we make conclusions that fuzzy rough set model, rough set model on a tolerance relation and multi-granulation rough sets models are special cases of the two types of multi-granulation fuzzy rough set in a fuzzy tolerance approximation space by analyzing the definitions of them. More properties of the two models are discussed and comparisons are made with fuzzy rough set on a tolerance relation. The construction of the new types of fuzzy rough set models in a fuzzy tolerance approximation space are extensions in the point view of granular computing and are meaningful in terms of the generalization of rough set theory.

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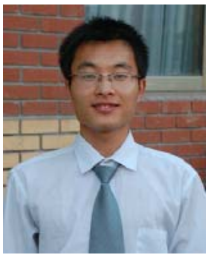

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