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Multi-granulation Fuzzy Rough Set Model on Tolerance Relations

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***Abstract*— Based on the analysis of rough set model on a tolerance relation and considering of the theory of fuzzy rough set, two types of new generalized fuzzy rough set models are constructed, which are multi-granulation fuzzy rough sets on tolerance relations. It follows the research on the properties of the lower and upper approximations of the new multi-granulation fuzzy rough set models on tolerance relations. The fuzzy rough set model and rough set model on a tolerance relation are special cases of the new one from the perspective of the considered concepts and granular computing. The relationships among the fuzzy rough set model, the first MGFRS and the second MGFRS all on tolerance relations are investigated.**

I. Introduction

ROUGH set theory, proposed by Pawlak [1], is a theory for the research of uncertainty management in a wide variety of applications related to artificial intelligence. The theory has been applied successfully in the fields of pattern recognition, medical diagnosis, data mining, conflict analysis, algebra, which related an amount of imprecise, vague and uncertain information. In recent years, the rough set theory has generated a great deal of interest among more and more researchers. The generalization of classical rough set model is one of the most important study spotlights.

As we know, the classification of objects in the classical approximation space is based on the approximation classification of equivalence relations. This kind of classification is very restrictive. So it is necessary to relax the equivalence relations to tolerance ones for the need of some practical issues. On the basis of this point, some researchers extended the classical approximation space to tolerance one and discussed the reduction approach of objects sets [2]-[4]. Besides, rough set theory is generalized by combining with other theories that deal with uncertainty knowledge such as fuzzy set. It has been acknowledged by different studies that fuzzy set theory and rough set theory are complementary because of handling different kinds of uncertainty. Dubois and Prade proposed concepts of rough fuzzy sets and fuzzy rough sets based on approximations of fuzzy sets by crisp approximations spaces, and crisp sets by fuzzy approximation spaces, respectively [5]. Yao proposed a unified model for both rough fuzzy sets and fuzzy rough sets based on the analysis of level sets of fuzzy sets in [6]. A rough fuzzy set is a pair of fuzzy sets resulting from the approximation of a fuzzy set in a crisp approximation space, and a fuzzy rough set is a pair of fuzzy sets resulting from the approximation of a crisp set in a fuzzy approximation space.

Rough set theory was also discussed with the point view of granular computing. Information granules refers to pieces, classes and groups divided in accordance with characteristics and performances of complex information in the process of human understanding, reasoning and decision-making. Zadeh firstly proposed the concept of granular computing and discussed issues of fuzzy information granulation in 1979 [7]. In the point view of granulation computing, the classical Pawlak rough set is based on a single granulation induced from an indiscernibility relation. And an equivalence relation on the universe can be regarded as a granulation. For the need of some practical issues, Qian and Xu extended the Pawlak rough set to multi-granulation rough set models where the approximation operators are defined by multiple equivalence relations on the universe [8]-[**?**].

Associated tolerant rough set with the theory of fuzzy rough set with granular computing point view, we will propose two types of multi-granulation fuzzy rough set models on tolerance relations. The main objective of this paper is to extend J. Järinen's tolerance rough set model determined by single tolerance relation to multi-granulation fuzzy rough sets where set approximations are defined by multiple tolerance relations. The rest of this paper is organized as follows. Some preliminary concepts of tolerance rough set theory and fuzzy rough sets theory are showed in Section II. In Section III, for a fuzzy target information system, based on multiple ordinary tolerance relations, two types of multi-granulation fuzzy rough approximation operators of a fuzzy concept are constructed and a number of important properties of them are discussed in detail. Especially, one can find that the definition of lower and upper approximation operators proposed in this paper are the generalized model of other formats, not only from the aspects of the considered concepts but also from the perspective of granulation. Then it follows the comparison and relations among the properties of the two types of multi-granulation fuzzy rough sets on tolerance relations and fuzzy rough set. And finally, the paper is concluded by a summary in Section IV.

II. Preliminaries

In this section, we will first review some basic concepts and notions in the theory of rough set on tolerance relation and fuzzy rough set on the basis of equivalence relations. More can be found in Ref. [2].

A. Rough Set On A Tolerance Relation

The notion of information system provides a convenient tool for the representation of objects in terms of their attribute

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values.

A tolerance information system [4] is an ordered triple $\mathcal{I} = (U, AT, \tau)$, where U is the non-empty finite set of objects known as universe; A is the non-empty finite set of attributes. τ is the mapping from powerset AT into the family set $\mathcal{R}$ of tolerance relations satisfying reflexivity and symmetry on universe U.

Let $I = (U, AT, \tau)$ be a tolerance information system. The lower approximation and the upper approximation of a set $X \subseteq U$ on a tolerance relation R with respect to $A \subseteq AT$ are respectively defined by

$$\begin{aligned} \underline{R_A}(X) &= \{x \in U | R_A(x) \subseteq X\}, \\ \overline{R_A}(X) &= \{x \in U | R_A(x) \cap X \neq \emptyset\}. \end{aligned} \tag{1}$$

where $R_B(x)$ is the tolerance class of x with respect to the tolerance relation R_B. Notice that a tolerance relation can construct a covering instead of a partition of the universe U. The set $Bnd_R(X) = \overline{R_A}(X) - \underline{R_A}(X)$ is called the boundary of X.

The set $\underline{R_A}(X)$ consists of elements which surely belong to X in view of the knowledge provided by R, while $\overline{R_A}(X)$ consists of elements which possibly belong to X. The boundary is the actual area of uncertainty. It consists of elements whose membership in X can not be decided when R-related objects can not be distinguished from each other.

The properties of the lower approximation and the upper approximation of sets with respect to a tolerance relation R_A are as follows, if $X, Y \subseteq U$ and $\sim X$ is the complement of X.

(1)$\underline{R_A}(X) \subseteq X \subseteq \overline{R_A}(X)$;
(2)$\underline{R_A}(\emptyset) = \overline{R_A}(\emptyset) = \emptyset, \underline{R_A}(U) = \overline{R_A}(U) = U$;
(3) $\sim \underline{R_A}(X) = \overline{R_A}(\sim X), \sim \overline{R_A}(X) = \underline{R_A}(\sim X)$;
(4)$Bnd_R(X) = Bnd_R(\sim X)$;
(5)$X \subseteq Y \Rightarrow \underline{R_A}(X) \subseteq \underline{R_A}(Y)$ and $\overline{R_A}(X) \subseteq \overline{R_A}(Y)$.

B. Fuzzy Set And Fuzzy Rough Set

We will first introduce some basic concepts of fuzzy set. Let U be a finite and non-empty set called universe. A fuzzy set A is a mapping from U into the unit interval $[0,\ 1] : \mu : \rightarrow [0,\ 1]$, where each $x \in U$ is the membership degree of x in A. Practically, we may consider U as a set of objects of concern and crisp subset of U represents a "non-vague" concept imposed on objects in U. Then a fuzzy set A of U is thought of as a mathematical representation of "vague" concept described linguistically. The set of all the fuzzy sets defined on U is denoted by $F(U)$.

Let A be a fuzzy set on U, for any $\alpha \in [0, 1]$, if denote

$$A_\alpha = \{x \in U \mid A(x) \geq \alpha\}.$$

then A_α is the α-cut set of A.

Let U be the universe, R be an equivalence relation, for a fuzzy set A on U, if take

$$\begin{aligned} \underline{R}(A)(x) &= \wedge\{A(y)|y \in [x]_R\}, \\ \overline{R}(A)(x) &= \vee\{A(y)|y \in [x]_R\}, \end{aligned} \tag{2}$$

then $\underline{R}(A)$ and $\overline{R}(A)$ are called the lower and upper approximation of the fuzzy set A with respect to the relation R, where "$\wedge$" means "min" and "$\vee$" means "max" and $[x]_R$ is the equivalence class of x with respect to equivalence relation R. A is a fuzzy definable set if and only if A satisfies $\underline{R}(A) = \overline{R}(A)$. Otherwise, A is called a fuzzy rough set.

Let $\mathcal{I} = (U, AT, F)$ be an information system, $D_j : U \rightarrow [0, 1] (j \leq r)$, if denote

$$D = \{D_j \mid j \leq r\},$$

then (U, AT, F, D) is a fuzzy target information system. In a fuzzy target information system, we can defined the approximation operators with respect the decision attribute D similarly.

Because of the limitation of the paper length, the properties of the above set approximation and measures have been showed in the reference [10].

III. Two types of multi-granulation fuzzy rough sets on tolerance relations

In this section, we will make researches about multi-granulation fuzzy rough set on tolerance relations which are on the problem of the rough approximations of a fuzzy set on multiple tolerance relations.

At first, we will propose a fuzzy rough set model (in brief FRS) on a tolerance relation in the following.

Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\begin{aligned} \underline{R_A}(X)(x) &= \wedge\{X(y) \mid y \in R_A(x)\}, \\ \overline{R_A}(X)(x) &= \vee\{X(y) \mid y \in R_A(x)\}, \end{aligned} \tag{3}$$

where " $\vee$ " means "max" and " $\wedge$ " means "min", then $\underline{R_A}(X)$ and $\overline{R_A}(X)$ are the lower and upper approximation of the fuzzy set X over the tolerance relation R with respect to the subset of attributes A. If $\underline{R_A}(X) \neq \overline{R_A}(X)$, then the fuzzy set X is a fuzzy rough set on the tolerance relation. We can easily find that this model will be the fuzzy rough set model we have introduced if the above relation R is an equivalence relation.

A. The First Type Of Multi-granulation Fuzzy Rough Set On Tolerance Relations

First, the first type of two-granulation fuzzy rough set (in brief 1st TGFRS) on tolerance relations of a fuzzy set is defined.

Definition 3.1: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\begin{aligned} \underline{FR_{A+B}}(X)(x) = \ & \{\wedge\{X(y) \mid y \in R_A(x)\}\} \\ & \vee\{\wedge\{X(y) \mid y \in R_B(x)\}\}, \\ \overline{FR_{A+B}}(X)(x) = \ & \{\vee\{X(y) \mid y \in R_A(x)\}\} \\ & \wedge\{\vee\{X(y) \mid y \in R_B(x)\}\}, \end{aligned} \tag{4}$$

where " $\vee$ " means "max" and " $\wedge$ " means "min", then $\underline{FR_{A+B}}(X)$ and $\overline{FR_{A+B}}(X)$ are respectively called the first type of two-granulation lower approximation and upper approximation of X on tolerance relations R with respect to the subsets of attributes A and B. X is a two-granulation fuzzy rough set on tolerance relations if and

TABLE I

A FUZZY TARGET INFORMATION SYSTEM

U	a_1	a_2	a_3	d
x_1	2	1	3	0.6
x_2	3	2	1	0.7
x_3	2	3	3	0.7
x_4	2	2	3	0.9
x_5	1	1	4	0.5
x_6	1	3	2	0.4
x_7	3	2	1	0.7
x_8	1	1	4	0.5
x_9	2	1	2	0.8
x_{10}	3	1	2	0.7

only if $\underline{FR_{A+B}}(X) \neq \overline{FR_{A+B}}(X)$. Otherwise, X is a two-granulation fuzzy definable set on tolerance relations. The boundary of the set X is defined as

$$Bnd^F_{R_{A+B}}(X) = \overline{FR_{A+B}}(X) \cap (\sim \underline{FR_{A+B}}(X)). \quad (5)$$

It can be found that the 1st TGFRS on tolerance relations will be degenerated into fuzzy rough set when $A = B$ and $R_A(x)$ and $R_B(x)$ are equivalence classes with respect to the subsets of attributes A and B. That is to say, a fuzzy rough set model is a special instance of the 1st TGFRS on tolerance relations. What's more, the 1st TGFRS on tolerance relations will be degenerated into a rough set model on a tolerance relation if $A = B$ and the considered concept X is a crisp set.

In the following, we employ an example to illustrate the above concepts.

Example 3.1: A fuzzy target information system are given in Table I. The universe $U = \{x_1, x_2, \cdots, x_{10}\}$, the set of condition attributes $AT = \{a_1, a_2, a_3\}$, the set of decision attribute $D = \{d\}$. If suppose $A_1 = \{a_1, a_2\}$ and $A_2 = \{a_1, a_3\}$, we consider the first type of two-granulation lower and upper approximation of D with respect to A_1 and A_2, where the tolerance relation is defined as $R_{A_i} = \{(x_i, y_j) \in U \times U \mid |f(x_i, a) - f(x_j, a)| \leq 1, a \in A_i\}$. For the fuzzy set

$$D = \{0.6, 0.7, 0.7, 0.9, 0.5, 0.4, 0.7, 0.7, 0.8, 0.7\},$$

the single granulation lower and upper approximation on a tolerance relation are

$$\underline{R_{A_1}}(D) = \{0.5, 0.6, 0.4, 0.4, 0.5, 0.4, 0.6, 0.5, 0.5, 0.6\},$$
$$\overline{R_{A_1}}(D) = \{0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9\};$$
$$\underline{R_{A_2}}(D) = \{0.4, 0.7, 0.4, 0.4, 0.5, 0.4, 0.7, 0.5, 0.4, 0.6\},$$
$$\overline{R_{A_2}}(D) = \{0.9, 0.8, 0.9, 0.9, 0.9, 0.9, 0.8, 0.9, 0.9, 0.9\};$$
$$\underline{R_{A_1 \cup A_2}}(D) = \{0.5, 0.7, 0.4, 0.4, 0.5, 0.4, 0.7, 0.5, 0.6, 0.6\},$$
$$\overline{R_{A_1 \cup A_2}}(D) = \{0.9, 0.8, 0.9, 0.9, 0.9, 0.9, 0.8, 0.9, 0.9, 0.9\}.$$

From Definition 3.1, we can compute the first type of two-granulation lower and upper approximation of D on tolerance relations are

$$\underline{FR_{A_1+A_2}}(D) = \{0.5, 0.7, 0.4, 0.4, 0.5, 0.4, 0.7, 0.5, 0.5, 0.6\},$$
$$\overline{FR_{A_1+A_2}}(D) = \{0.9, 0.8, 0.9, 0.9, 0.9, 0.9, 0.8, 0.9, 0.9, 0.9\}.$$

Obviously, the following can be found

$$\underline{FR_{A_1+A_2}}(D) = \underline{R_{A_1}}(D) \cup \underline{R_{A_2}}(D),$$
$$\overline{FR_{A_1+A_2}}(D) = \overline{R_{A_1}}(D) \cap \overline{R_{A_2}}(D),$$
$$\underline{FR_{A_1+A_2}}(D) \subseteq \underline{R_{A_1 \cup A_2}}(D) \subseteq D$$
$$\subseteq \overline{R_{A_1 \cup A_2}}(D) \subseteq \overline{FR_{A_1+A_2}}(D).$$

Just from Definition 3.1, we can obtain some properties of the 1st TGFRS in a tolerance information system.

Proposition 3.1: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $B, A \subseteq AT$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{FR_{A+B}}(X) \subseteq X$,
(2) $\overline{FR_{A+B}}(X) \supseteq X$;
(3) $\underline{FR_{A+B}}(\sim X) = \sim \overline{FR_{A+B}}(X)$,
(4) $\overline{FR_{A+B}}(\sim X) = \sim \underline{FR_{A+B}}(X)$;
(5) $\underline{FR_{A+B}}(U) = \overline{FR_{A+B}}(U) = U$,
(6) $\underline{FR_{A+B}}(\emptyset) = \overline{FR_{A+B}}(\emptyset) = \emptyset$;
(7) $\overline{FR_{A+B}}(X) \supseteq \underline{FR_{A+B}}(\underline{FR_{A+B}}(X))$,
(8) $\overline{FR_{A+B}}(X) \subseteq \overline{FR_{A+B}}(\overline{FR_{A+B}}(X))$.

Proof: We only prove (1),(2) and (3) and the rest can been proved by Definition 3.1. It is obvious that all terms hold when $A = B$. When $A \neq B$, the proposition can be proved as follows.

(1) For any $x \in U$ and $A,\ B \subseteq AT$, since $\underline{R_A}(X) \subseteq X$ and $\underline{R_B}(X) \subseteq X$, we know

$$\wedge\{X(y) \mid y \in R_A(x)\} \leq X(y)$$

and

$$\wedge\{X(y) \mid y \in R_B(x)\} \leq X(y)$$

Therefore,

$$\{\wedge\{X(y) \mid y \in R_A(x)\}\} \vee \{\wedge\{X(y) \mid y \in R_B(x)\}\} \leq X(y).$$

i.e., $\underline{FR_{A+B}}(X) \subseteq X$.

(2) For any $x \in U$ and $A,\ B \subseteq AT$, since $X \subseteq \overline{R_A}(X)$ and $X \subseteq \overline{R_A}(X)$, we know

$$X(y) \leq \vee\{X(y) \mid y \in R_A(x)\}$$

and

$$X(y) \leq \vee\{X(y) \mid y \in R_B(x)\}$$

Therefore,

$$X(y) \leq \{\vee\{X(y) \mid y \in R_A(x)\}\} \wedge \{\vee\{X(y) \mid y \in R_B(x)\}\}.$$

i.e., $X \subseteq \overline{FR_{A+B}}(X)$.

(3) For any $x \in U$ and $A,\ B \subseteq AT$, since $\underline{R_A}(\sim X) = \sim \overline{R_A}(X)$ and $\underline{R_B}(\sim X) = \sim \overline{R_B}(X)$, then we have

$$\begin{aligned}
&\underline{FR_{A+B}}(\sim X) \\
&= \{\wedge\{1 - X(y) \mid y \in R_A(x)\}\} \\
&\quad \vee \{\wedge\{1 - X(y) \mid y \in R_B(x)\}\} \\
&= \{1 - \vee\{X(y) \mid y \in R_A(x)\}\} \\
&\quad \vee \{1 - \vee\{X(y) \mid y \in R_B(x)\}\} \\
&= 1 - \{\vee\{X(y) \mid y \in R_A(x)\}\} \\
&\quad \wedge \{\vee\{X(y) \mid y \in R_B(x)\}\} \\
&= \sim \overline{FR_{A+B}}(X).
\end{aligned}$$

Proposition 3.2: Let $\mathcal{I} = (U, AT, F)$ be an information system, $B, A \subseteq AT$, $X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{FR_{A+B}}(X \cap Y) \subseteq \underline{FR_{A+B}}(X) \cap \underline{FR_{A+B}}(Y)$,

(2) $\overline{FR_{A+B}}(X \cup Y) \supseteq \overline{FR_{A+B}}(X) \cup \overline{FR_{A+B}}(Y)$;

(3) $X \subseteq Y \Rightarrow \underline{FR_{A+B}}(X) \subseteq \underline{FR_{A+B}}(Y)$,

(4) $X \subseteq Y \Rightarrow \overline{FR_{A+B}}(X) \subseteq \overline{FR_{A+B}}(Y)$;

(5) $\underline{FR_{A+B}}(X \cup Y) \supseteq \underline{FR_{A+B}}(X) \cup \underline{FR_{A+B}}(Y)$,

(6) $\overline{FR_{A+B}}(X \cap Y) \subseteq \overline{FR_{A+B}}(X) \cap \overline{FR_{A+B}}(Y)$.

proof: According to the similarity of the properties, we only prove the odd items. All terms hold obviously when $A = B$ or $X = Y$. If $A \neq B$ and $X \neq Y$, the proposition can be proved as follows.

(1) For any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
&\underline{FR_{A+B}}(X \cap Y)(x) \\
&= \{\wedge\{(X \cap Y)(y) \mid y \in R_A(x)\}\} \\
&\quad \vee \{\wedge\{(X \cap Y)(y) \mid y \in R_B(x)\}\} \\
&= \{\wedge\{X(y) \wedge Y(y) \mid y \in R_A(x)\}\} \\
&\quad \vee \{\wedge\{X(y) \wedge Y(y) \mid y \in R_B(x)\}\} \\
&= \{\underline{R_A}(X)(x) \wedge \underline{R_A}(Y)(x)\} \vee \{\underline{R_B}(X)(x) \wedge \underline{R_B}(Y)(x)\} \\
&\leq \{\underline{R_A}(X)(x) \vee \underline{R_B}(X)(x)\} \wedge \{\underline{R_A}(Y)(x) \vee \underline{R_B}(Y)(x)\} \\
&= \underline{FR_{A+B}}(X)(x) \wedge \underline{FR_{A+B}}(Y)(x).
\end{aligned}$$

Then $\underline{FR_{A+B}}(X \cap Y) \subseteq \underline{FR_{A+B}}(X) \cap \underline{FR_{A+B}}(Y)$.

(3) Since for any $x \in U$, we have $X(y) \leq Y(y)$. Then the properties hold obviously by Definition 3.1.

(5) Since $X \subseteq X \cup Y$, and $Y \subseteq X \cup Y$, then $\underline{FR_{A+B}}(X) \subseteq \underline{FR_{A+B}}(X \cup Y)$ and $\underline{FR_{A+B}}(Y) \subseteq \underline{FR_{A+B}}(X \cup Y)$. So the property $\underline{FR_{A+B}}(X \cup Y) \supseteq \underline{FR_{A+B}}(X) \cup \underline{FR_{A+B}}(Y)$ obviously holds.

The proposition was proved.

The lower and upper approximation in Definition 3.1 are a pair of fuzzy sets. If we associate the cut set of a fuzzy set, we can make a description of a fuzzy set X by a classical set in an information system.

Definition 3.2: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$ and $X \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, the lower approximation $\underline{FR_{A+B}}(X)$ and upper approximation $\overline{FR_{A+B}}(X)$ of X about the α, β cut sets on tolerance relations R_A and R_B are defined, respectively, as follows

$$\begin{aligned}
\underline{FR_{A+B}}(X)_\alpha &= \{x \in U \mid \underline{FR_{A+B}}(X)(x) \geq \alpha\}, \\
\overline{FR_{A+B}}(X)_\beta &= \{x \in U \mid \overline{FR_{A+B}}(X)(x) \geq \beta\}.
\end{aligned} \tag{6}$$

$\underline{FR_{A+B}}(X)_\alpha$ can be explained as the set of objects in U which surely belong to X on tolerance relations R_A and R_B and the memberships of which are more than α, while $\overline{FR_{A+B}}(X)_\beta$ is the set of objects in U which possibly belong to X on tolerance relations R_A and R_B and the memberships of which are more than β.

Proposition 3.3: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$ and $X, Y \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, we have

(1) $\underline{FR_{A+B}}(X \cap Y)_\alpha \subseteq \underline{FR_{A+B}}(X)_\alpha \cap \underline{FR_{A+B}}(Y)_\alpha$,

(2) $\overline{FR_{A+B}}(X \cup Y)_\beta \supseteq \overline{FR_{A+B}}(X)_\beta \cup \overline{FR_{A+B}}(Y)_\beta$;

(3) $X \subseteq Y \Rightarrow \underline{FR_{A+B}}(X)_\alpha \subseteq \underline{FR_{A+B}}(Y)_\alpha$,

(4) $X \subseteq Y \Rightarrow \overline{FR_{A+B}}(X)_\beta \subseteq \overline{FR_{A+B}}(Y)_\beta$;

(5) $\underline{FR_{A+B}}(X \cup Y)_\alpha \supseteq \underline{FR_{A+B}}(X)_\alpha \cup \underline{FR_{A+B}}(Y)_\alpha$,

(6) $\overline{FR_{A+B}}(X \cap Y)_\beta \subseteq \overline{FR_{A+B}}(X)_\beta \cap \overline{FR_{A+B}}(Y)_\beta$.

proof: It is easy to prove by Definition 3.2 and Proposition 3.2.

In the following, we will introduce the first type of multi-granulation fuzzy rough set (in brief 1st MGFRS) on tolerance relations and its corresponding properties by extending the 1st two-granulation fuzzy rough set on tolerance relations.

Definition 3.3: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT, i = 1, \cdots, m$. For the fuzzy set $X \in F(U)$, denote

$$\begin{aligned}
\underline{FR_{\sum_{i=1}^m A_i}}(X)(x) &= \bigvee_{i=1}^m \{\bigwedge\{X(y) \mid y \in R_{A_i}(x)\}\}, \\
\overline{FR_{\sum_{i=1}^m A_i}}(X)(x) &= \bigwedge_{i=1}^m \{\bigvee\{X(y) \mid y \in R_{A_i}(x)\}\},
\end{aligned} \tag{7}$$

where "$\bigvee$" means "max" and "$\bigwedge$" means "min", then $\underline{FR_{\sum_{i=1}^m A_i}}(X)$ and $\overline{FR_{\sum_{i=1}^m A_i}}(X)$ are respectively called the first type of multi-granulation lower approximation and upper approximation of X on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$. X is a multi-granulation fuzzy rough set on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ if and only if $\underline{FR_{\sum_{i=1}^m A_i}}(X) \neq \overline{FR_{\sum_{i=1}^m A_i}}(X)$. Otherwise, X is a multi-granulation fuzzy definable set on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$. The boundary of the set X is defined as

$$Bnd^F_{R_{\sum_{i=1}^m A_i}}(X) = \overline{FR_{\sum_{i=1}^m A_i}}(X) \cap (\sim \underline{FR_{\sum_{i=1}^m A_i}}(X)). \tag{8}$$

It can be found that the 1st MGFRS on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ will be degenerated into fuzzy rough set when $A_i = A_j,\ i \neq j$ and $R_{A_i}(x)$ are equivalence classes with respect to the subsets of attributes $A_i(i = 1, \cdots, m)$. That is to say, a fuzzy rough set model is a special instance of the 1st MGFRS on the tolerance relations. What's more, the 1st MGFRS on tolerance relations will be degenerated into a rough set model on tolerance relation if $A_i = A_j,\ i \neq j$ and the considered concept X is a crisp set.

The properties about 1st MGFRS on tolerance relations are listed in the following which can be extended from the 1st TGFRS model on tolerance relations.

Proposition 3.4: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT,\ i = 1, \cdots, m$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{FR_{\sum_{i=1}^m A_i}}(X) \subseteq X$,

(2) $\overline{FR_{\sum_{i=1}^m A_i}}(X) \supseteq X$;

(3) $\underline{FR_{\sum_{i=1}^m A_i}}(\sim X) = \sim \overline{R_{\sum_{i=1}^m A_i}}(X)$,

(4) $\overline{FR_{\sum_{i=1}^{m}A_i}}(\sim X) = \sim \underline{FR_{\sum_{i=1}^{m}A_i}}(X)$;

(5) $\underline{FR_{\sum_{i=1}^{m}A_i}}(U) = \overline{FR_{\sum_{i=1}^{m}A_i}}(U) = U$,

(6) $\underline{FR_{\sum_{i=1}^{m}A_i}}(\emptyset) = \overline{FR_{\sum_{i=1}^{m}A_i}}(\emptyset) = \emptyset$;

(7) $\underline{FR_{\sum_{i=1}^{m}A_i}}(X) \supseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(\underline{FR_{\sum_{i=1}^{m}A_i}}(X))$,

(8) $\overline{FR_{\sum_{i=1}^{m}A_i}}(X) \subseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(\overline{FR_{\sum_{i=1}^{m}A_i}}(X))$.

proof: The proof is similar to Proposition 3.1.

Proposition 3.5: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT$, $i = 1, \cdots, m$, $X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{FR_{\sum_{i=1}^{m}A_i}}(X \cap Y) \subseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(X) \cap \underline{FR_{\sum_{i=1}^{m}A_i}}(Y)$,

(2) $\overline{FR_{\sum_{i=1}^{m}A_i}}(X \cup Y) \supseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(X) \cup \overline{FR_{\sum_{i=1}^{m}A_i}}(Y)$;

(3) $X \subseteq Y \Rightarrow \underline{FR_{\sum_{i=1}^{m}A_i}}(X) \subseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(Y)$,

(4) $X \subseteq Y \Rightarrow \overline{FR_{\sum_{i=1}^{m}A_i}}(X) \subseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(Y)$;

(5) $\underline{FR_{\sum_{i=1}^{m}A_i}}(X \cup Y) \supseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(X) \cup \underline{FR_{\sum_{i=1}^{m}A_i}}(Y)$,

(6) $\overline{FR_{\sum_{i=1}^{m}A_i}}(X \cap Y) \subseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(X) \cap \overline{FR_{\sum_{i=1}^{m}A_i}}(Y)$.

proof The proof of this proposition is similar to Proposition 3.2.

Definition 3.4: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT$, $1 \le i \le m$, and $X \subseteq U$. For any $0 < \beta \le \alpha \le 1$, the lower approximation $\underline{FR_{\sum_{i=1}^{m}A_i}}(X)$ and upper approximation $\overline{FR_{\sum_{i=1}^{m}A_i}}(X)$ of X about the α, β cut sets on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ are defined, respectively, as follows

$$\begin{aligned} \underline{FR_{\sum_{i=1}^{m}A_i}}(X)_\alpha &= \{x \in U \mid \underline{FR_{\sum_{i=1}^{m}A_i}}(X)(x) \ge \alpha\}, \\ \overline{FR_{\sum_{i=1}^{m}A_i}}(X)_\beta &= \{x \in U \mid \overline{FR_{\sum_{i=1}^{m}A_i}}(X)(x) \ge \beta\}. \end{aligned} \tag{9}$$

$\underline{FR_{\sum_{i=1}^{m}A_i}}(X)_\alpha$ can be explained as the set of objects in U which surely belong to X on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ and the memberships of which are more than α, while $\overline{FR_{\sum_{i=1}^{m}A_i}}(X)_\beta$ is the set of objects in U which possibly belong to X on the tolerance relations $R_{A_i}(i = 1, \cdots, m)$ and the memberships of which are more than β.

Proposition 3.6: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT$, $i = 1, \cdots, m$, and $X, Y \subseteq U$. For any $0 < \beta \le \alpha \le 1$, we have

(1) $\underline{FR_{\sum_{i=1}^{m}A_i}}(X \cap Y)_\alpha \subseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(X)_\alpha \cap \underline{FR_{\sum_{i=1}^{m}A_i}}(Y)_\alpha$,

(2) $\overline{FR_{\sum_{i=1}^{m}A_i}}(X \cup Y)_\beta \supseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(X)_\beta \cup \overline{FR_{\sum_{i=1}^{m}A_i}}(Y)_\beta$;

(3) $X \subseteq Y \Rightarrow \underline{FR_{\sum_{i=1}^{m}A_i}}(X)_\alpha \subseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(Y)_\alpha$,

(4) $X \subseteq Y \Rightarrow \overline{FR_{\sum_{i=1}^{m}A_i}}(X)_\beta \subseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(Y)_\beta$;

(5) $\underline{FR_{\sum_{i=1}^{m}A_i}}(X \cup Y)_\alpha \supseteq \underline{FR_{\sum_{i=1}^{m}A_i}}(X)_\alpha \cup \underline{FR_{\sum_{i=1}^{m}A_i}}(Y)_\alpha$,

(6) $\overline{FR_{\sum_{i=1}^{m}A_i}}(X \cap Y)_\beta \subseteq \overline{FR_{\sum_{i=1}^{m}A_i}}(X)_\beta \cap \overline{FR_{\sum_{i=1}^{m}A_i}}(Y)_\beta$.

proof It is easy to prove by Definition 3.4 and Proposition 3.5.

B. The Second Type Of Multi-granulation Fuzzy Rough Set On Tolerance Relations

In this subsection, we will propose another type of MGFRS on tolerance relations. We first define the second type of two-granulation fuzzy rough set (in brief 2nd TGFRS)on tolerance relations.

Definition 3.5: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$. For the fuzzy set $X \in F(U)$, denote

$$\begin{aligned} \underline{SR_{A+B}}(X)(x) = \quad & \{\wedge\{X(y) \mid y \in R_A(x)\}\} \\ & \wedge\{\wedge\{X(y) \mid y \in R_B(x)\}\}, \\ \overline{SR_{A+B}}(X)(x) = \quad & \{\vee\{X(y) \mid y \in R_A(x)\}\} \\ & \vee\{\vee\{X(y) \mid y \in R_B(x)\}\}, \end{aligned} \tag{10}$$

then $\underline{SR_{A+B}}(X)$ and $\overline{SR_{A+B}}(X)$ are respectively called the second type of two-granulation lower approximation and upper approximation of X on the tolerance relations R with respect to the subsets of attributes A and B. X is the second type of two-granulation fuzzy rough set on the tolerance relations if and only if $\underline{SR_{A+B}}(X) \ne \overline{SR_{A+B}}(X)$. Otherwise, X is the second type of two-granulation fuzzy definable set on the tolerance relations. The boundary of the set X is defined as

$$Bnd^S_{R_{A+B}}(X) = \overline{SR_{A+B}}(X) \cap (\sim \underline{SR_{A+B}}(X)). \tag{11}$$

It can be found that the 2nd TGFRS on tolerance relations will be degenerated into the fuzzy rough set model when $A = B$ and $R_A(x)$ and $R_B(x)$ are equivalence classes with respect to the subsets of attributes A and B. That is, a fuzzy rough set model is a special instance of the 2nd TGFRS on tolerance relations. What's more, the 2nd TGFRS on tolerance relations will be degenerated into a rough set model on a tolerance relation if $A = B$ and the considered concept X is a crisp set.

In the following, we employ an example to illustrate the above concepts.

Example 3.2: (Continued from Example 3.1) From Definition 3.2, we can compute the second type of two-granulation lower and upper approximation of D on the tolerance relation R_{A_1} and R_{A_2} are

$$\begin{aligned} \underline{SR_{A_1+A_2}}(D) &= \{0.4, 0.6, 0.4, 0.4, 0.5, 0.4, 0.6, 0.5, 0.4, 0.6\}, \\ \overline{SR_{A_1+A_2}}(D) &= \{0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9, 0.9\}. \end{aligned}$$

Obviously, the following can be found

$$\begin{aligned}
\underline{SR_{A_1+A_2}}(D) &= \underline{R_{A_1}}(D) \cap \underline{R_{A_2}}(D), \\
\overline{SR_{A_1+A_2}}(D) &= \overline{R_{A_1}}(D) \cup \overline{R_{A_2}}(D), \\
\underline{SR_{A_1+A_2}}(D) &\subseteq \underline{R_{A_1 \cup A_2}}(D) \subseteq D \\
&\subseteq \overline{R_{A_1 \cup A_2}}(D) \subseteq \overline{SR_{A_1+A_2}}(D).
\end{aligned}$$

Proposition 3.7 Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $B, A \subseteq AT$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{SR_{A+B}}(X) \subseteq X$,

(2) $\overline{SR_{A+B}}(X) \supseteq X$;

(3) $\underline{SR_{A+B}}(\sim X) = \sim \overline{SR_{A+B}}(X)$,

(4) $\overline{SR_{A+B}}(\sim X) = \sim \underline{SR_{A+B}}(X)$;

(5) $\underline{SR_{A+B}}(U) = \overline{SR_{A+B}}(U) = U$,

(6) $\underline{SR_{A+B}}(\emptyset) = \overline{SR_{A+B}}(\emptyset) = \emptyset$;

(7) $\underline{SR_{A+B}}(X) \supseteq \underline{SR_{A+B}}(\underline{SR_{A+B}}(X))$,

(8) $\overline{SR_{A+B}}(X) \subseteq \overline{SR_{A+B}}(\overline{SR_{A+B}}(X))$.

proof: The methods of the proof is similar to Proposition 3.1.

Proposition 3.8: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $B, A \subseteq AT$, $X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{SR_{A+B}}(X \cap Y) = \underline{SR_{A+B}}(X) \cap \underline{SR_{A+B}}(Y)$,

(2) $\overline{SR_{A+B}}(X \cup Y) = \overline{SR_{A+B}}(X) \cup \overline{SR_{A+B}}(Y)$;

(3) $X \subseteq Y \Rightarrow \underline{SR_{A+B}}(X) \subseteq \underline{SR_{A+B}}(Y)$,

(4) $X \subseteq Y \Rightarrow \overline{SR_{A+B}}(X) \subseteq \overline{SR_{A+B}}(Y)$;

(5) $\underline{SR_{A+B}}(X \cup Y) \supseteq \underline{SR_{A+B}}(X) \cup \underline{SR_{A+B}}(Y)$,

(6) $\overline{SR_{A+B}}(X \cap Y) \subseteq \overline{SR_{A+B}}(X) \cap \overline{SR_{A+B}}(Y)$.

proof: Here we prove (1) and (2). All terms hold obviously when $A = B$ or $X = Y$. If $A \neq B$ and $X \neq Y$, the proposition can be proved as follows.

(1) For any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
&\underline{SR_{A+B}}(X \cap Y)(x) \\
&= \{\wedge\{(X \cap Y)(y) \mid y \in R_A(x)\}\} \\
&\quad \wedge \{\wedge\{(X \cap Y)(y) \mid y \in R_B(x)\}\} \\
&= \{\wedge\{X(y) \wedge Y(y) \mid y \in R_A(x)\}\} \\
&\quad \wedge \{\wedge\{X(y) \wedge Y(y) \mid y \in R_B(x)\}\} \\
&= \{\underline{R_A}(X)(x) \wedge \underline{R_A}(Y)(x)\} \wedge \{\underline{R_B}(X)(x) \wedge \underline{R_B}(Y)(x)\} \\
&= \{\underline{R_A}(X)(x) \wedge \underline{R_B}(X)(x)\} \wedge \{\underline{R_A}(Y)(x) \wedge \underline{R_B}(Y)(x)\} \\
&= \underline{R_{A+B}}(X)(x) \wedge \underline{R_{A+B}}(Y)(x).
\end{aligned}$$

Then $\underline{SR_{A+B}}(X \cap Y) = \underline{SR_{A+B}}(X) \cap \underline{SR_{A+B}}(Y)$.

(2) Similarly, for any $x \in U$, $A, B \subseteq AT$ and $X, Y \in F(U)$,

$$\begin{aligned}
&\overline{SR_{A+B}}(X \cup Y)(x) \\
&= \{\vee\{(X \cup Y)(y) \mid y \in R_A(x)\}\} \\
&\quad \vee \{\vee\{(X \cup Y)(y) \mid y \in R_B(x)\}\} \\
&= \{\vee\{X(y) \vee Y(y) \mid y \in R_A(x)\}\} \\
&\quad \vee \{\vee\{X(y) \vee Y(y) \mid y \in R_B(x)\}\} \\
&= \{\overline{R_A}(X)(x) \vee \overline{R_A}(Y)(x)\} \vee \{\overline{R_B}(X)(x) \vee \overline{R_B}(Y)(x)\} \\
&= \{\overline{R_A}(X)(x) \vee \overline{R_B}(X)(x)\} \vee \{\overline{R_A}(Y)(x) \vee \overline{R_B}(Y)(x)\} \\
&= \overline{SR_{A+B}}(X)(x) \vee \overline{SR_{A+B}}(Y)(x).
\end{aligned}$$

Then $\overline{SR_{A+B}}(X \cup Y) = \overline{SR_{A+B}}(X) \cup \overline{SR_{A+B}}(Y)$.

Definition 3.6: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$ and $X \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, the lower approximation $\underline{SR_{A+B}}(X)$ and upper approximation $\overline{SR_{A+B}}(X)$ of X about the α, β cut sets on tolerance relations R_A and R_B are defined, respectively, as follows

$$\begin{aligned}
\underline{\underline{SR_{A+B}}}(X)_\alpha &= \{x \in U \mid \underline{SR_{A+B}}(X)(x) \geq \alpha\}, \\
\overline{\overline{SR_{A+B}}}(X)_\beta &= \{x \in U \mid \overline{SR_{A+B}}(X)(x) \geq \beta\}.
\end{aligned} \tag{12}$$

$\underline{\underline{SR_{A+B}}}(X)_\alpha$ can be explained as the set of objects in U which surely belong to X on tolerance relations R_A and R_B and the memberships of which are more than α, while $\overline{\overline{SR_{A+B}}}(X)_\beta$ is the set of objects in U which possibly belong to X on tolerance relations R_A and R_B and the memberships of which are more than β.

Proposition 3.9: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A, B \subseteq AT$ and $X, Y \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, we have

(1) $\underline{\underline{SR_{A+B}}}(X \cap Y)_\alpha = \underline{\underline{SR_{A+B}}}(X)_\alpha \cap \underline{\underline{SR_{A+B}}}(Y)_\alpha$,

(2) $\overline{\overline{SR_{A+B}}}(X \cup Y)_\beta = \overline{\overline{SR_{A+B}}}(X)_\beta \cup \overline{\overline{SR_{A+B}}}(Y)_\beta$;

(3) $X \subseteq Y \Rightarrow \underline{\underline{SR_{A+B}}}(X)_\alpha \subseteq \underline{\underline{SR_{A+B}}}(Y)_\alpha$,

(4) $X \subseteq Y \Rightarrow \overline{\overline{SR_{A+B}}}(X)_\beta \subseteq \overline{\overline{SR_{A+B}}}(Y)_\beta$;

(5) $\underline{\underline{SR_{A+B}}}(X \cup Y)_\alpha \supseteq \underline{\underline{SR_{A+B}}}(X)_\alpha \cup \underline{\underline{SR_{A+B}}}(Y)_\alpha$,

(6) $\overline{\overline{SR_{A+B}}}(X \cap Y)_\beta \subseteq \overline{\overline{SR_{A+B}}}(X)_\beta \cap \overline{\overline{SR_{A+B}}}(Y)_\beta$.

proof It is easy to prove by Definition 3.6 and Proposition 3.8.

In the following, we will introduce the second type of multi-granulation fuzzy rough set (in brief 2nd MGFRS) on tolerance relations and its corresponding properties by extending the second type of two-granulation fuzzy rough set on tolerance relations.

Definition 3.7: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT, i = 1, \cdots, m$. For the fuzzy set $X \in F(U)$, denote

$$\begin{aligned}
\underline{SR_{\sum_{i=1}^{m} A_i}}(X)(x) &= \bigwedge_{i=1}^{m} \{\bigwedge\{X(y) \mid y \in R_{A_i}(x)\}\}, \\
\overline{SR_{\sum_{i=1}^{m} A_i}}(X)(x) &= \bigvee_{i=1}^{m} \{\bigvee\{X(y) \mid y \in R_{A_i}(x)\}\},
\end{aligned} \tag{13}$$

where "$\bigvee$" means "max" and "$\bigwedge$" means "min", then $\underline{SR_{\sum_{i=1}^{m} A_i}}(X)$ and $\overline{SR_{\sum_{i=1}^{m} A_i}}(X)$ are respectively called the second type of multi-granulation lower approximation and upper approximation of X on tolerance relations R with respect to the subsets of attributes $A_i(i = 1, \cdots, m)$. X is the second type of multi-granulation fuzzy rough set if and only if $\underline{SR_{\sum_{i=1}^{m} A_i}}(X) \neq \overline{SR_{\sum_{i=1}^{m} A_i}}(X)$. Otherwise, X is the second type of multi-granulation fuzzy definable set on tolerance relations. The boundary of the set X is defined as

$$Bnd^S_{R_{\sum_{i=1}^{m} A_i}}(X) = \overline{SR_{\sum_{i=1}^{m} A_i}}(X) \cap (\sim \underline{SR_{\sum_{i=1}^{m} A_i}}(X)). \tag{14}$$

It can be found that the 2nd MGFRS will be degenerated into fuzzy rough set when $A_i = A_j,\ i \neq j$ and $R_{A_i}(x)$ are equivalence classes with respect to the subsets of attributes $A_i(i = 1, \cdots, m)$. That is , a fuzzy rough set model is also a special instance of the 2nd MGFRS on tolerance relations. What's more, the MGFRS 2nd will be degenerated into a rough set model on a tolerance relation if $A_i = A_j,\ i \neq j$ and the considered concept X is a crisp set.

The properties about the 2nd MGFRS on tolerance relations are listed in the following which can be extended from the 2nd TGFRS model on tolerance relations.

Proposition 3.10: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT,\ 1 \leq i \leq m$ and $X \in F(U)$. Then the following properties hold.

(1) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X) \subseteq X$,

(2) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X) \supseteq X$;

(3) $\underline{SR_{\sum_{i=1}^{m} A_i}}(\sim X) = \sim \overline{SR_{\sum_{i=1}^{m} A_i}}(X)$,

(4) $\overline{SR_{\sum_{i=1}^{m} A_i}}(\sim X) = \sim \underline{SR_{\sum_{i=1}^{m} A_i}}(X)$;

(5) $\underline{SR_{\sum_{i=1}^{m} A_i}}(U) = \overline{SR_{\sum_{i=1}^{m} A_i}}(U) = U$,

(6) $\underline{SR_{\sum_{i=1}^{m} A_i}}(\emptyset) = \overline{SR_{\sum_{i=1}^{m} A_i}}(\emptyset) = \emptyset$.

(7) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \underline{SR_{\sum_{i=1}^{m} A_i}}(\underline{SR_{A+B}}(X))$,

(8) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \overline{SR_{\sum_{i=1}^{m} A_i}}(\overline{SR_{A+B}}(X))$.

proof: The proof is similar to Proposition 3.7.

Proposition 3.11: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT,\ 1 \leq i \leq m,\ X, Y \in F(U)$. Then the following properties hold.

(1) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X \cap Y) = \underline{SR_{\sum_{i=1}^{m} A_i}}(X) \cap \underline{SR_{\sum_{i=1}^{m} A_i}}(Y)$,

(2) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X \cup Y) = \overline{SR_{\sum_{i=1}^{m} A_i}}(X) \cup \overline{SR_{\sum_{i=1}^{m} A_i}}(Y)$;

(3) $X \subseteq Y \Rightarrow \underline{SR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{SR_{\sum_{i=1}^{m} A_i}}(Y)$,

(4) $X \subseteq Y \Rightarrow \overline{SR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \overline{SR_{\sum_{i=1}^{m} A_i}}(Y)$;

(5) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X \cup Y) \supseteq \underline{SR_{\sum_{i=1}^{m} A_i}}(X) \cup \underline{SR_{\sum_{i=1}^{m} A_i}}(Y)$,

(6) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X \cap Y) \subseteq \overline{SR_{\sum_{i=1}^{m} A_i}}(X) \cap \overline{SR_{\sum_{i=1}^{m} A_i}}(Y)$.

proof: The proof of this proposition is similar to Proposition 3.8.

Definition 3.8: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT,\ 1 \leq i \leq m$, and $X \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, the lower approximation $\underline{SR_{\sum_{i=1}^{m} A_i}}(X)$ and upper approximation $\overline{SR_{\sum_{i=1}^{m} A_i}}(X)$ of X about the α, β cut sets on tolerance relations $R_{A_i}(i = 1, \cdots, m)$ are defined, respectively, as follows

$$\begin{aligned} \underline{SR_{\sum_{i=1}^{m} A_i}}(X)_\alpha &= \{x \in U \mid \underline{SR_{\sum_{i=1}^{m} A_i}}(X)(x) \geq \alpha\}, \\ \overline{SR_{\sum_{i=1}^{m} A_i}}(X)_\beta &= \{x \in U \mid \overline{SR_{\sum_{i=1}^{m} A_i}}(X)(x) \geq \beta\}. \end{aligned} \tag{15}$$

$\underline{SR_{\sum_{i=1}^{m} A_i}}(X)_\alpha$ can be explained as the set of objects in U which surely belong to X on tolerance relations $R_{A_i}(i = 1, \cdots, m)$ and the memberships of which are more than α, while $\overline{SR_{\sum_{i=1}^{m} A_i}}(X)_\beta$ is the set of objects in U which possibly belong to X on tolerance relations $R_{A_i}(i = 1, \cdots, m)$ and the memberships of which are more than β.

Proposition 3.12: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT,\ 1 \leq i \leq m$, and $X, Y \subseteq U$. For any $0 < \beta \leq \alpha \leq 1$, we have

(1) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X \cap Y)_\alpha = \underline{SR_{\sum_{i=1}^{m} A_i}}(X)_\alpha \cap \underline{SR_{\sum_{i=1}^{m} A_i}}(Y)_\alpha$,

(2) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X \cup Y)_\beta = \overline{SR_{\sum_{i=1}^{m} A_i}}(X)_\beta \cup \overline{SR_{\sum_{i=1}^{m} A_i}}(Y)_\beta$;

(3) $X \subseteq Y \Rightarrow \underline{SR_{\sum_{i=1}^{m} A_i}}(X)_\alpha \subseteq \underline{SR_{\sum_{i=1}^{m} A_i}}(Y)_\alpha$,

(4) $X \subseteq Y \Rightarrow \overline{SR_{\sum_{i=1}^{m} A_i}}(X)_\beta \subseteq \overline{SR_{\sum_{i=1}^{m} A_i}}(Y)_\beta$;

(5) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X \cup Y)_\alpha \supseteq \underline{SR_{\sum_{i=1}^{m} A_i}}(X)_\alpha \cup \underline{SR_{\sum_{i=1}^{m} A_i}}(Y)_\alpha$,

(6) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X \cap Y)_\beta \subseteq \overline{SR_{\sum_{i=1}^{m} A_i}}(X)_\beta \cap \overline{SR_{\sum_{i=1}^{m} A_i}}(Y)_\beta$.

proof: It is easy to prove by Definition 3.8 and Proposition 3.11.

On the basis of tolerance relations, we will investigate the interrelationship among SGFRS, the 1st MGFRS and the 2nd MGFRS in this section after the discussion of the properties of them.

Proposition 3.13: Let $\mathcal{I} = (U, AT, \tau)$ be a tolerance information system, $A_i \subseteq AT,\ 1 \leq i \leq m,\ X \in F(U)$. Then the following properties hold.

(1) $\underline{FR_{\sum_{i=1}^{m} A_i}}(X) = \bigcup_{i=1}^{m} \underline{R_{A_i}}(X)$,

(2) $\overline{FR_{\sum_{i=1}^{m} A_i}}(X) = \bigcap_{i=1}^{m} \overline{R_{A_i}}(X)$;

(3) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X) = \bigcap_{i=1}^{m} \underline{R_{A_i}}(X)$,

(4) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X) = \bigcup_{i=1}^{m} \overline{R_{A_i}}(X)$;

(5) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{FR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{R_{\bigcup_{i=1}^{m} A_i}}(X)$;

(6) $\overline{SR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{FR_{\sum_{i=1}^{m} A_i}}(X) \supseteq \overline{R_{\bigcup_{i=1}^{m} A_i}}(X)$.

(7) $\underline{SR_{\sum_{i=1}^{m} A_i}}(X) \subseteq \underline{R_{A_i}}(X) \subseteq \underline{FR_{\sum_{i=1}^{m} A_i}}(X)$;

(8) $\overline{\overline{SR_{\sum_{i=1}^{m} A_i}}}(X) \supseteq \overline{R_{A_i}}(X) \supseteq \overline{\overline{FR_{\sum_{i=1}^{m} A_i}}}(X)$.

proof: The proof can be obtained by Definition 3.3, 3.7.

IV. CONCLUSIONS

The theories of rough sets and fuzzy sets both extended the classical set theory in terms of dealing with uncertainty and imprecision. However, the theory of fuzzy set pay more attention to the fuzziness of knowledge while the theory of rough set to the roughness of knowledge in the point view of granular of knowledge. For the complement of the two types of theory, fuzzy rough set models are investigated to solve practical problems. Given that the equivalence relations in the fuzzy rough set theory is too rigorous for some practical application, it is necessary to weaken the equivalence relations to tolerance relations. The contribution of this paper is having constructed two new types of fuzzy rough sets on tolerance relations associated with granular computing called multi-granulation fuzzy rough set models on tolerance relations, in which the set approximation operators are defined on the basis of multiple tolerance relations. What's more, we make conclusions that fuzzy rough set model and rough set model on a tolerance relation are special cases of the two types of multi-granulation fuzzy rough set on tolerance relations by analyzing the definition of them. More properties of the two types of fuzzy rough set on tolerance relations are discussed and comparison are made with fuzzy rough set on a tolerance relation. The construction of the new types of fuzzy rough set models on tolerance relations is an extension in the point view of granular computing and is meaningful in terms of the generalization of rough set theory. We will investigate the knowledge reduction in depth.

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