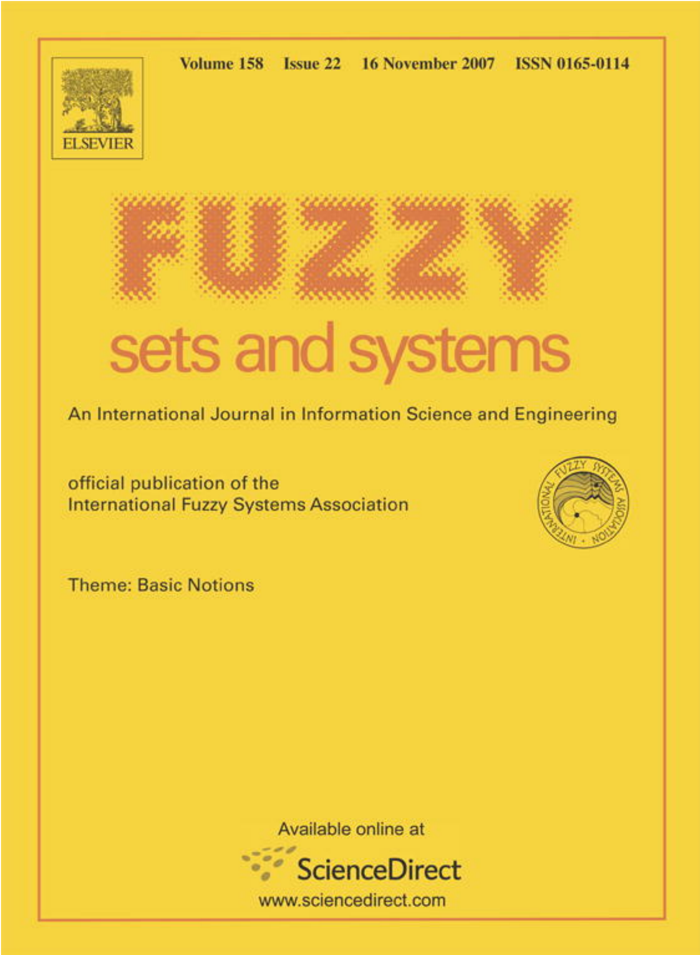
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Measuring roughness of generalized rough sets induced by a covering☆

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Abstract

In this paper, we propose new lower and upper approximations and obtain some important properties in generalized rough set induced by a covering. Especially, these properties are compared with ones of Pawlak's rough sets and Bonikowski's covering generalized rough sets, respectively. Moreover, we define a measure of roughness based on generalized rough sets with the new approximations and discuss some significant properties of the measure.

Keywords: Rough set; Covering lower and upper approximations; Fuzzy set; Degree of rough membership

1. Introduction

Rough set theory, proposed by Pawlak in the early 1980s [13], is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Moreover, the theory may serve as a new mathematical tool to soft computing besides fuzzy set theory [20], and has been successfully applied in machine learning, pattern recognition, expert systems, data analysis, and so on. Recently, lots of researchers are interested in the theory.

In Pawlak's original rough set theory, partition or equivalence (indiscernibility) relation is an important and primitive concept. But, partition or equivalence relation is still restrictive for many applications. To address this issue, several interesting and meaningful extensions to equivalence relation have been proposed in the past, such as tolerance relations [7,15], similarity relations [16], others [17–19]. Particularly, Zakowski has used coverings of an universe for establishing the generalized rough set [21]. And an extensive body of research works has been developed [4–6,14]. In 1990, Dubois and Prade [9] combined fuzzy sets with rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Furthermore, Banerjee and Pal [1] have characterized a measure of roughness of a fuzzy set making use of the concept of rough fuzzy sets in 1995. They also suggested some possible applications of the measure in pattern recognition and image analysis problems. Some results are obtained about rough sets and fuzzy sets in [2,3,8,10–12].

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In this paper, we investigate some important and basic issues of generalized rough sets induced by a covering. The plan of this paper is as follows:

In Section 2, we recall the basic concepts and properties of the Pawlak's rough set theory. In Section 3, some new concepts and main results are considered in generalized rough sets induced by a covering. Here, these results are compared with ones of Pawlak's rough sets and Bonikowski's covering generalized rough sets, respectively. In Section 4, we define a measure of roughness based on generalized rough sets with the new approximations, and prove some properties of the measure. Finally, we give an example in order to indicate the use of the measure in Section 5.

2. Some relevant concepts

Let U be a finite and nonempty set called universe of discourse. We use $P(U)$ $(F(U))$ to denote the class of all subsets (fuzzy subsets) of U. For any $\tilde{A} \in F(U)$, denote the α-cut of $\tilde{A}$ by $\tilde{A}_\alpha$. That is to say $\tilde{A}_\alpha = \{x \in U | \tilde{A}(x) \geqslant \alpha\}$, where $\alpha \in [0, 1]$.

Let $\tilde{A} \in F(U)$ and $u \in U$, we define:

$$N(\tilde{A})(u) = \begin{cases} 0 & \text{if } \tilde{A}(u) < 0.5, \\ 1 & \text{if } \tilde{A}(u) \geqslant 0.5. \end{cases}$$

It is easy to see that $N(\tilde{A}) = \tilde{A}_{0.5}$. Therefore,

$$|\tilde{A}(u) - N(\tilde{A})(u)| = |\tilde{A}(u) - \tilde{A}_{0.5}(u)|.$$

Obviously, $N(\tilde{A})$ is an ordinary set nearest to $\tilde{A}$. It is called the nearest ordinary set [8].

Definition 2.1. Let the support set of fuzzy set $\tilde{A}$ have n elements. The index of fuzziness of $\tilde{A}$ is defined as follows:

$$\upsilon_p(\tilde{A}) = (2/n^p) \cdot d(\tilde{A}, N(\tilde{A})),$$

where $d(\tilde{A}, N(\tilde{A}))$ denotes the distance between the fuzzy set $\tilde{A}$ and its nearest ordinary set $N(\tilde{A})$. The value of p depends on the type of distance function used. E.g. $p = 1$ for a generalized Hamming distance whereas $p = 0.5$ for an Euclidean distance. When $p = 1$, $\upsilon_1(\tilde{A})$ is called the linear index of fuzziness of $\tilde{A}$, denoted by $\upsilon_l(\tilde{A})$. When $p = 0.5$, $\upsilon_{0.5}(\tilde{A})$ is called the quadratic index of fuzziness of $\tilde{A}$, denoted by $\upsilon_q(\tilde{A})$ [8].

A Pawlak approximation space is an ordered pair (U, R), where U is a nonempty finite set of objects called the universe and R is an equivalence relation on U.

For any nonempty subset X of U, the sets

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\},$$

and

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \phi\}$$

are, respectively, called the lower and upper approximations of X in (U, R), where $[x]_R$ denotes the equivalence class of the relation R containing the element x.

Pawlak studies the group of subsets of U with the upper and lower approximations in (U, R). Using upper and lower approximations, an equivalence relation $\approx_R$ can be defined on the power set of U:

$$X \approx_R Y \quad \Leftrightarrow \quad \underline{R}(X) = \underline{R}(Y) \quad \text{and} \quad \overline{R}(X) = \overline{R}(Y),$$

where $X, Y \in P(U)$, and R is an equivalence relation on U.

In addition, this equivalence relation induces a partition on the power set $P(U)$. An equivalence class of such partition is called a rough set. Specifically, a rough set can be defined as follows [13]:

Definition 2.2. Given the Pawlak approximation space (U, R) and two sets $A_1, A_2 \in P(U)$, with $A_1 \subseteq A_2$, a Pawlak rough set is the family of subset of U described as follows:

$$(A_1, A_2) = \{X \in P(U) | \underline{R}(X) = A_1, \overline{R}(X) = A_2\}.$$

Equivalently, a Pawlak rough set containing $X \in P(U)$ can be defined by

$$[X]_{\approx_R} = \{Y \in P(U) | \underline{R}(X) = \underline{R}(Y), \overline{R}(X) = \overline{R}(Y)\}.$$

$[X]_{\approx_R}$ is the set of subsets of U. In order to emphasize $\underline{R}(X)$ and $\overline{R}(X)$, denote

$$[X]_{\approx_R} = (\underline{R}(X), \overline{R}(X)).$$

If $\underline{R}(X) = \overline{R}(X)$, then X is said to be exact.

For a fixed nonempty subset X of U, the rough set of X is unique. Moreover, let ϕ be the empty set, and $\sim X$ be the complement of X in U, then the following conclusions have been established for Pawlak's rough sets [13]:

(1L)	$\underline{R}(X) \subseteq X$	(Contraction)
(1U)	$X \subseteq \overline{R}(X)$	(Extension)
(2)	$\underline{R}(\sim X) = \sim \overline{R}(X)$	(Duality)
	$\overline{R}(\sim X) = \sim \underline{R}(X)$	(Duality)
(3L)	$\underline{R}(\phi) = \phi$	(Normality)
(3U)	$\overline{R}(\phi) = \phi$	(Normality)
(4L)	$\underline{R}(U) = U$	(Co-normality)
(4U)	$\overline{R}(U) = U$	(Co-normality)
(5L)	$\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$	(Multiplication)
(5U)	$\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$	(Addition)
(6L)	$X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$	(Monotone)
(6U)	$X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$	(Monotone)
(7L)	$\underline{R}(\underline{R}(X)) = \underline{R}(X)$	(Idempotency)
(7U)	$\overline{R}(\overline{R}(X)) = \overline{R}(X)$	(Idempotency)

3. Generalized rough sets induced by a covering

Let U be an universe of discourse, and $\{\mathcal{C}_i\} (i = 1, 2, \ldots, n)$ be a family of subsets of U. If $\mathcal{C}_i \neq \phi$, and $\bigcup \mathcal{C}_i = U$, then $\{\mathcal{C}_i\}$ is called a covering of U, denoted by $\mathcal{C}$.

It is clear that a partition of U is certainly a covering of U, so the concept of a covering is an extension of the concept of a partition.

A covering approximation space is an ordered pair $S = (U, \mathcal{C})$ where U is the universe, and $\mathcal{C}$ is a covering of U.

3.1. Bonikowski's generalized rough sets induced by a covering

In this section we will recall some definitions and results about Bonikowski's generalized rough sets induced by a covering, which can be found in [4–6].

Definition 3.1. Let $S = (U, \mathcal{C})$ be a covering approximation space. For $u \in U$, the family of sets

$$Md(u) = \{K \in \mathcal{C} | u \in K \wedge (\forall H \in \mathcal{C} \wedge u \in H \wedge H \subseteq K \Rightarrow K = H)\}$$

is called the minimal description of u.

Definition 3.2. For a set $X \subseteq U$, the family of sets $\underline{S\mathcal{C}}(X) = \{K \in \mathcal{C} | K \subseteq X\}$ is called the covering lower approximation the family of sets of X.

Set $\underline{\mathcal{C}}(X) = \bigcup \underline{S\mathcal{C}}(X)$ is called the covering lower approximation of X.

The family of sets $Bn(X) = \bigcup \{Md(x) | x \in X - \underline{\mathcal{C}}(X)\}$ is called the covering boundary approximation the family of sets of X.

The family of sets $\overline{S\mathcal{C}}(X) = \underline{S\mathcal{C}}(X) \bigcup Bn(X)$ is called the covering upper approximation the family of sets of X.

Set $\overline{\mathcal{C}}(X) = \bigcup \overline{S\mathcal{C}}(X)$ is called the covering upper approximation of X.

In Pawlak's rough set theory, the lower and upper approximations are dual to each other. However, in the generalized rough set induced by a covering theory, the covering lower and upper approximations are no longer dual from the above definition.

Remark 3.1. Another natural definition of the covering upper approximation could be $\bigcap\{K \in \mathcal{C} | X \subseteq K\}$ from Definition 3.2. But it is not equal to the covering upper approximation of Bonikowski.

For example, let $U = \{a, b, c, d\}$ be an universe and $\mathcal{C} = \{\{a, b\}, \{a, c, d\}\}$ be a covering of U. If denote $\mathcal{C}^\sharp(X) = \bigcap\{K \in \mathcal{C} | X \subseteq K\}$ and take $X_1 = \{a\}$, then we have

$$\mathcal{C}^\sharp(X_1) = \{a, b\} \cap \{a, c, d\} = \{a\}.$$

However, the covering lower approximation of Bonikowski is $\underline{\mathcal{C}}(X_1) = \phi$. So $X_1 - \underline{\mathcal{C}}(X_1) = \{a\}$. Therefore $Md(a) = \{\{a, b\}, \{a, c, d\}\}$. That is to say $Bn(X_1) = U$. Hence, we know that the covering upper approximation of Bonikowski is $\overline{\mathcal{C}}(X_1) = U$.

Obviously, $\overline{\mathcal{C}}(X_1) \neq \mathcal{C}^\sharp(X)$.

By covering lower and upper approximations of Bonikowski, an equivalence relation $\approx_{\mathcal{C}}$ can be defined on the power set of U:

$$X \approx_{\mathcal{C}} Y \quad \Leftrightarrow \quad \underline{\mathcal{C}}(X) = \underline{\mathcal{C}}(Y) \quad \text{and} \quad \overline{\mathcal{C}}(X) = \overline{\mathcal{C}}(Y),$$

where $X, Y \in P(U)$, and $\mathcal{C}$ is a covering of U.

In addition, this equivalence relation induces a partition on the power set $P(U)$. An equivalence class of such partition is called a generalized rough set induced by a covering. Moreover, the concept can be defined by:

Definition 3.3. Given the covering approximation space $S = (U, \mathcal{C})$ and two sets $A_1, A_2 \in U$, with $A_1 \subseteq A_2$, a generalized rough set induced by a covering is the family of subset of U described as follows:

$$(A_1, A_2) = \{X \in P(U) | \underline{\mathcal{C}}(X) = A_1, \overline{\mathcal{C}}(X) = A_2\}.$$

Equivalently, a generalized rough set induced by a covering containing $X \in P(U)$ can be defined by

$$[X]_{\approx_{\mathcal{C}}} = \{Y \in P(U) | \underline{\mathcal{C}}(X) = \underline{\mathcal{C}}(Y), \overline{\mathcal{C}}(X) = \overline{\mathcal{C}}(Y)\}.$$

$[X]_{\approx_{\mathcal{C}}}$ is the set of subsets of U. In order to emphasize $\underline{\mathcal{C}}(X)$ and $\overline{\mathcal{C}}(X)$, denote:

$$[X]_{\approx_{\mathcal{C}}} = (\underline{\mathcal{C}}(X), \overline{\mathcal{C}}(X)).$$

If $\underline{\mathcal{C}}(X) = \overline{\mathcal{C}}(X)$, then X is said to be exact.

Obviously, generalized rough sets induced by a covering have the following results [5,6] from the above definitions.

Proposition 3.1. *If $\mathcal{C}$ is a partition, then $\underline{\mathcal{C}}(X)$ and $\overline{\mathcal{C}}(X)$ are the Pawlak's lower and upper approximations of X.*

From Proposition 3.1, it can be found that a generalized rough set induced by a covering will become a classical rough set when the covering is restricted a partition.

Proposition 3.2. *For a covering $\mathcal{C}$, the covering lower and upper approximations have the following properties*:

$(1L)$ $\underline{\mathcal{C}}(X) \subseteq X$	(Contraction)
$(1U)$ $X \subseteq \overline{\mathcal{C}}(X)$	(Extension)
$(3L)$ $\underline{\mathcal{C}}(\phi) = \phi$	(Normality)
$(3U)$ $\overline{\mathcal{C}}(\phi) = \phi$	(Normality)
$(4L)$ $\underline{\mathcal{C}}(U) = U$	(Co-normality)
$(4U)$ $\overline{\mathcal{C}}(U) = U$	(Co-normality)
$(6L)$ $X \subseteq Y \Rightarrow \underline{\mathcal{C}}(X) \subseteq \underline{\mathcal{C}}(Y)$	(Monotone)
$(7L)$ $\underline{\mathcal{C}}(\underline{\mathcal{C}}(X)) = \underline{\mathcal{C}}(X)$	(Idempotency)
$(7U)$ $\overline{\mathcal{C}}(\overline{\mathcal{C}}(X)) = \overline{\mathcal{C}}(X)$	(Idempotency)

Remark 3.2. The following properties of Pawlak's lower and upper approximations do not hold for the covering lower and upper approximations:

(2)	$\underline{\mathcal{C}}(\sim X) = \sim \overline{\mathcal{C}}(X)$	(Duality)
	$\overline{\mathcal{C}}(\sim X) = \sim \underline{\mathcal{C}}(X)$	(Duality)
(5L)	$\underline{\mathcal{C}}(X \cap Y) = \underline{\mathcal{C}}(X) \cap \underline{\mathcal{C}}(Y)$	(Multiplication)
(5U)	$\overline{\mathcal{C}}(X \cup Y) = \overline{\mathcal{C}}(X) \cup \overline{\mathcal{C}}(Y)$	(Addition)
(6U)	$X \subseteq Y \Rightarrow \overline{\mathcal{C}}(X) \subseteq \overline{\mathcal{C}}(Y)$	(Monotone)

3.2. New approximations of generalized rough sets induced by a covering

From 3.1, we can find that some important properties of Pawlak's lower and upper approximations do not hold for the covering lower and upper approximation. So we need introduce another covering lower and upper approximations. The new generalized rough sets will be established in the following.

Definition 3.4. Let U be a nonempty set called universe, and $\mathcal{C}$ be a covering of U. For any $X \subseteq U$, the lower and upper approximations of X with respect to approximation space $(U, \mathcal{C})$ are defined as follows:

$$\mathcal{C}_*(X) = \{x \in U | (\cap Md(x)) \subseteq X\};$$
$$\mathcal{C}^*(X) = \{x \in U | (\cap Md(x)) \cap X \neq \phi\}.$$

Definition 3.5. Given the covering approximation space $S = (U, \mathcal{C})$ and two sets $A_1, A_2 \in U$, with $A_1 \subseteq A_2$, a generalized rough set induced by a covering is the family of subset of U described as follows:

$$(A_1, A_2) = \{X \in P(U) | \mathcal{C}_*(X) = A_1, \mathcal{C}^*(X) = A_2\}.$$

Equivalently, a generalized rough set induced by a covering containing $X \in P(U)$ can be defined by

$$[X]_{\approx_{\mathcal{C}}} = \{Y \in P(U) | \mathcal{C}_*(X) = \mathcal{C}_*(Y), \mathcal{C}^*(X) = \mathcal{C}^*(Y)\}.$$

$[X]_{\approx_{\mathcal{C}}}$ is the set of subsets of U. In order to emphasize $\mathcal{C}_*(X)$ and $\mathcal{C}^*(X)$, denote:

$$[X]_{\approx_{\mathcal{C}}} = (\mathcal{C}_*(X), \mathcal{C}^*(X)).$$

If $\mathcal{C}_*(X) = \mathcal{C}^*(X)$, then X is said to be exact.

From the above new covering approximations, we can find that when $\mathcal{C}$ is a partition, they will be the Pawlak's lower and upper approximations. That is to say, the generalized rough set with new covering approximations will also become a classical rough set when the covering is a restricted partition.

Furthermore, we have the following conclusions.

Proposition 3.3. *For a covering $\mathcal{C}$, the covering lower approximation $\mathcal{C}_*$ and upper approximation $\mathcal{C}^*$ have the following properties*:

(1L)	$\mathcal{C}_*(X) \subseteq X$	(Contraction)
(1U)	$X \subseteq \mathcal{C}^*(X)$	(Extension)
(2)	$\mathcal{C}_*(\sim X) = \sim \mathcal{C}^*(X)$	(Duality)
	$\mathcal{C}^*(\sim X) = \sim \mathcal{C}_*(X)$	(Duality)
(3L)	$\mathcal{C}_*(\phi) = \phi$	(Normality)
(3U)	$\mathcal{C}^*(\phi) = \phi$	(Normality)
(4L)	$\mathcal{C}_*(U) = U$	(Co-normality)
(4U)	$\mathcal{C}^*(U) = U$	(Co-normality)
(5L)	$\mathcal{C}_*(X \cap Y) = \mathcal{C}_*(X) \cap \mathcal{C}_*(Y)$	(Multiplication)
(5U)	$\mathcal{C}^*(X \cup Y) = \mathcal{C}^*(X) \cup \mathcal{C}^*(Y)$	(Addition)
(6L)	$X \subseteq Y \Rightarrow \mathcal{C}_*(X) \subseteq \mathcal{C}_*(Y)$	(Monotone)
(6U)	$X \subseteq Y \Rightarrow \mathcal{C}^*(X) \subseteq \mathcal{C}^*(Y)$	(Monotone)
(7L)	$\mathcal{C}_*(\mathcal{C}_*(X)) = \mathcal{C}_*(X)$	(Idempotency)
(7U)	$\mathcal{C}^*(\mathcal{C}^*(X)) = \mathcal{C}^*(X)$	(Idempotency)

Proof. (1L) For any $x \in \mathcal{C}_*(X)$, we have $\cap Md(x) \subseteq X$. Since $x \in \cap Md(x)$, it follows $x \in X$. Hence, $\mathcal{C}_*(X) \subseteq X$.

(1U) For any $x \in X$, we have $(\cap Md(x)) \cap X \neq \phi$. So $x \in \mathcal{C}^*(X)$. Hence, $X \subseteq \mathcal{C}^*(X)$.

(2) For any $x \in \mathcal{C}_*(X)$, we have

$$\begin{aligned} x \in \mathcal{C}_*(X) &\Leftrightarrow \ \cap Md(x) \subseteq X \\ &\Leftrightarrow (\cap Md(x)) \cap \sim X = \phi \\ &\Leftrightarrow x \notin \mathcal{C}^*(\sim X) \\ &\Leftrightarrow x \in \sim \mathcal{C}^*(\sim X). \end{aligned}$$

Hence, $\mathcal{C}_*(X) = \sim \mathcal{C}^*(\sim X)$. That is to say, $\mathcal{C}^*(\sim X) = \sim \mathcal{C}_*(X)$.

Another result can be proved similarly.

(3L) By (1L) and (1U), we can obtain $\mathcal{C}_*(\phi) \subseteq \phi$. Obviously, $\phi \subseteq \mathcal{C}_*(\phi)$. So we have $\mathcal{C}_*(\phi) = \phi$.

(3U) If $\mathcal{C}^*(\phi) \neq \phi$, then there exists a $x \in \mathcal{C}^*(\phi)$. Therefore, $(\cap Md(x)) \cap \phi \neq \phi$. But $(\cap Md(x)) \cap \phi = \phi$, which is a contradiction. Hence $\mathcal{C}^*(\phi) = \phi$.

(4L) From (2) and (3L), we directly have

$$\mathcal{C}_*(U) = \mathcal{C}_*(\sim \phi) = \sim \mathcal{C}^*(\phi) = \sim \phi = U.$$

(4U) It can be proved in a similar way as (4L).

(5L) For any $x \in \mathcal{C}_*(X \cap Y)$, we have

$$\begin{aligned} x \in \mathcal{C}_*(X \cap Y) &\Leftrightarrow (\cap Md(x)) \subseteq X \cap Y \\ &\Leftrightarrow (\cap Md(x)) \subseteq X \quad \text{and} \quad (\cap Md(x)) \subseteq Y \\ &\Leftrightarrow x \in \mathcal{C}_*(X) \quad \text{and} \quad x \in \mathcal{C}_*(Y) \\ &\Leftrightarrow x \in \mathcal{C}_*(X) \cap x \in \mathcal{C}_*(Y). \end{aligned}$$

So we can obtain $\mathcal{C}_*(X \cap Y) = \mathcal{C}_*(X) \cap \mathcal{C}_*(Y)$.

(5U) It can be directly shown by (2) and (5L).

(6L) For any $x \in \mathcal{C}_*(X)$, we have $\cap Md(x) \subseteq X$. Since $X \subseteq Y$, it follows $\cap Mdx \subseteq X \subseteq Y$. Thus $x \in \mathcal{C}_*(Y)$. Hence, we obtain $\mathcal{C}_*(X) \subseteq \mathcal{C}_*(Y)$. That is to say, $X \subseteq Y \Rightarrow \mathcal{C}_*(X) \subseteq \mathcal{C}_*(Y)$.

(6U) It can be proved in the same way as (6L).

(7L) For any $x \in \mathcal{C}_*(X)$, we have $\cap Md(x) \subseteq X$. By (6L), we can obtain $\mathcal{C}_*(\cap Md(x)) \subseteq \mathcal{C}_*(X)$. Let $y \in \cap Md(x)$, then $y \in K$ for any $K \in Md(x)$. Therefore, $\cap Md(y) \subseteq \cap Md(x)$. So $y \in \mathcal{C}_*(\cap Md(x))$. Thus, $\cap Md(x) \subseteq \mathcal{C}_*(\cap Md(x)) \subseteq \mathcal{C}_*(X)$. That is to say $x \in \mathcal{C}_*(\mathcal{C}_*(X))$. Hence, $\mathcal{C}_*(X) \subseteq \mathcal{C}_*(\mathcal{C}_*(X))$. Obviously, we can know $\mathcal{C}_*(\mathcal{C}_*(X)) \subseteq \mathcal{C}_*(X)$. Consequently, we have $\mathcal{C}_*(\mathcal{C}_*(X)) = \mathcal{C}_*(X)$.

Table 1

U	a_1	a_2	a_3
x_1	1	2	1
x_2	3	2	2
x_3	1	1	2
x_4	2	1	3
x_5	3	3	2
x_6	3	2	3

(7U) For any $X \subseteq U$, we have $\mathcal{C}_*(\mathcal{C}_*(\sim X)) = \mathcal{C}_*(\sim X)$ by (7L). So, by (2) we can know

$$\begin{aligned}\mathcal{C}_*(\mathcal{C}_*(\sim X)) = \mathcal{C}_*(\sim X) &\Rightarrow \mathcal{C}_*(\mathcal{C}_*(\sim X)) \supseteq \mathcal{C}_*(\sim X)\\ &\Rightarrow \mathcal{C}_*(\sim \mathcal{C}^*(X)) \supseteq \sim \mathcal{C}^*(X)\\ &\Rightarrow \sim\mathcal{C}^*(\mathcal{C}^*(X)) \supseteq \sim \mathcal{C}^*(X)\\ &\Rightarrow \mathcal{C}^*(\mathcal{C}^*(X)) \subseteq \mathcal{C}^*(X).\end{aligned}$$

Thus, $\mathcal{C}^*(\mathcal{C}^*(X)) = \mathcal{C}^*(X)$ holds.
Hence, the proposition is proved. □

From the above, we can find the 14 important properties of Pawlak's lower and upper approximations all hold for the new covering lower and upper approximations. But, they do not hold for the Bonikowski's covering approximation.

4. A measure of roughness in generalized rough sets induced by a covering

Let $S = (U, \mathcal{C})$ be a covering approximation space and $X \subseteq U$. In the covering $\mathcal{C}$, the rough set of X is $(\mathcal{C}_*(X), \mathcal{C}^*(X))$. Thus in the space $S = (U, \mathcal{C})$, the set X is approximated by two approximations, one from the inner side called the lower approximation of X, and another from the outer side called the upper approximation of X.

Definition 4.1. For a given covering $\mathcal{C}$ of U and the rough set $(\mathcal{C}_*(X), \mathcal{C}^*(X))$ of $X \subseteq U$, degree of rough membership of u in X with respect to $\mathcal{C}$, denoted by $D_{\mathcal{C}}(u, X)$, is defined by

$$D_{\mathcal{C}}(u, X) = \frac{|(\cap Md(u)) \cap X|}{|\cap Md(u)|} \quad \forall u \in U.$$

Clearly, for any $u \in U$, $D_{\mathcal{C}}(u, X) \in [0, 1]$. Then, $D_{\mathcal{C}}(u, X)$ is a fuzzy set of U, and we denote it by $\widetilde{D_X^{\mathcal{C}}}$. Thus $D_{\mathcal{C}}(u, X) = \widetilde{D_X^{\mathcal{C}}}(u)$.

Example 4.1. Let an information table be based on a dominance relation in Table 1, where $U = \{x_i | i = 1, 2, \ldots, 6\}$ is a nonempty finite set of objects and $A = \{a_1, a_2, a_3\}$ denotes the set of attributes.

And dominance relation $R_A^{\leqslant}$ of information Table 1 is defined by

$$R_A^{\leqslant} = \{(x_i, x_j) \in U \times U | f_l(x_i) \leqslant f_l(x_j) \forall a_l \in A\},$$

where $f_l(x)$ is the value of a_l on $x \in U$. Moreover, if we denote

$$[x_i]_A^{\leqslant} = \{x_j \in U | (x_i, x_j) \in R_A^{\leqslant}\} = \{x_j \in U | f_l(x_i) \leqslant f_l(x_j) \forall a_l \in A\},$$

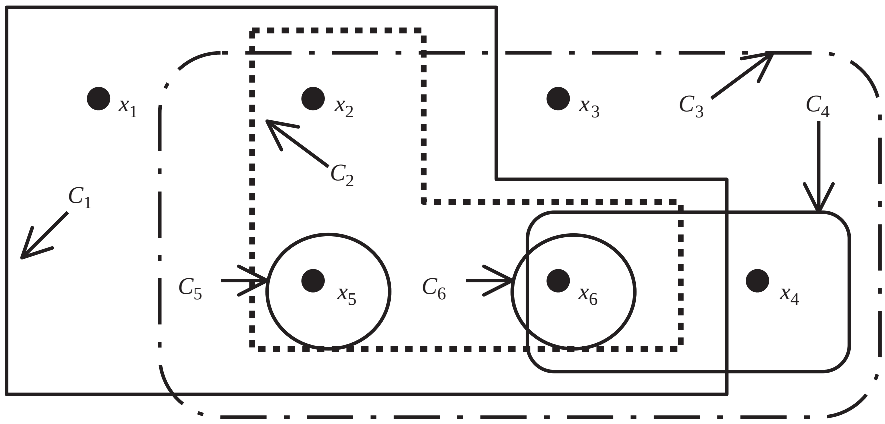

Fig. 1. Covering of S in Table 1.

then from Table 1 we have

$$\mathcal{C}_1 = [x_1]_A^{\leqslant} = \{x_1, x_2, x_5, x_6\};$$
$$\mathcal{C}_2 = [x_2]_A^{\leqslant} = \{x_2, x_5, x_6\};$$
$$\mathcal{C}_3 = [x_3]_A^{\leqslant} = \{x_2, x_3, x_4, x_5, x_6\};$$
$$\mathcal{C}_4 = [x_4]_A^{\leqslant} = \{x_4, x_6\};$$
$$\mathcal{C}_5 = [x_5]_A^{\leqslant} = \{x_5\};$$
$$\mathcal{C}_6 = [x_6]_A^{\leqslant} = \{x_6\};$$

and $\mathcal{C}_i \neq \phi; \cup\mathcal{C}_i = U, (i = 1, 2, \ldots, 6)$. Thus $\mathcal{C}$ is a covering of U and $S = (U, \mathcal{C})$ is a covering approximation space. Moreover, we can obtain $Md(x_i) = \mathcal{C}_i(i = 1, 2, \ldots, 6)$ by the following Fig. 1.

Now let's consider a subset $X = \{x_1, x_2, x_5\}$ of U. It is easy to obtain $\mathcal{C}_*(X) = \mathcal{C}_5 = \{x_5\}$, and $\mathcal{C}^*(X) = \{x_1, x_2, x_3, x_5\}$.

By the above definitions we can obtain

$$D_{\mathcal{C}}(x_1, X) = \tfrac{3}{4}; \quad D_{\mathcal{C}}(x_2, X) = \tfrac{2}{3};$$
$$D_{\mathcal{C}}(x_3, X) = \tfrac{2}{5}; \quad D_{\mathcal{C}}(x_4, X) = 0;$$
$$D_{\mathcal{C}}(x_5, X) = 1; \quad D_{\mathcal{C}}(x_6, X) = 0.$$

Thus,

$$\widetilde{D_X^{\mathcal{C}}} = \{\tfrac{3}{4}/x_1, \tfrac{2}{3}/x_2, \tfrac{2}{5}/x_3, 0/x_4, 1/x_5, 0/x_6\}.$$

In another way we have

$$N(\widetilde{D_X^{\mathcal{C}}}) = \{1/x_1, 1/x_2, 0/x_3, 0/x_4, 1/x_5, 0/x_6\}.$$

Therefore, we have

$$\begin{aligned} v_1(\widetilde{D_X^{\mathcal{C}}}) &= (\tfrac{2}{6}) \cdot d(\widetilde{D_X^{\mathcal{C}}}, N(\widetilde{D_X^{\mathcal{C}}})) \\ &= \tfrac{1}{3} \cdot (\tfrac{1}{4} + \tfrac{1}{3} + \tfrac{2}{5}) \\ &= 0.32778. \end{aligned}$$

and

$$\begin{aligned}\upsilon_q(\widetilde{D_X^{\mathcal{C}}}) &= (\tfrac{2}{\sqrt{6}})\cdot d(\widetilde{D_X^{\mathcal{C}}}, N(\widetilde{D_X^{\mathcal{C}}}))\\ &= (\tfrac{2}{\sqrt{6}})\cdot\sqrt{\frac{1}{16}+\frac{1}{9}+\frac{4}{25}}\\ &= 0.47161.\end{aligned}$$

In the next, we will discuss some properties of the measure of roughness in generalized rough sets induced by a covering.

Proposition 4.1. *Let $S=(U,\mathcal{C})$ be a covering approximation space. The following properties always hold*:

(1) $D_{\mathcal{C}}(u,X)=1$ *if and only if* $u\in\mathcal{C}_*(X)$;
(2) $D_{\mathcal{C}}(u,X)=0$ *if and only if* $u\in U-\mathcal{C}^*(X)$; *and*
(3) $0<D_{\mathcal{C}}(u,X)<1$ *if and only if* $u\in\mathcal{C}^*(X)-\mathcal{C}_*(X)$.

Proof. (1) By definitions, we directly obtain

$$D_{\mathcal{C}}(u,X)=1 \;\Leftrightarrow\; \cap Md(u)\subseteq X \;\Leftrightarrow\; u\in\mathcal{C}_*(X).$$

(2) From definitions, we have

$$\begin{aligned}D_{\mathcal{C}}(u,X)=0 &\Leftrightarrow (\cap Md(u))\cap X=\phi\\ &\Leftrightarrow u\notin\mathcal{C}^*(X)\\ &\Leftrightarrow U\in U-\mathcal{C}^*(X).\end{aligned}$$

(3) It can be easily known by (1) and (2).
The proof is completed. □

From this proposition, we see that $D_{\mathcal{C}}(u,X)$ indicates degree of rough membership of u in X with respect to $\mathcal{C}$. In particular, $D_{\mathcal{C}}(u,X)$ will become characteristic function, when $\mathcal{C}_*(X)=\mathcal{C}^*(X)$, i.e., X is classical set. So we have the following proposition.

Proposition 4.2. *Let $S=(U,\mathcal{C})$ be a covering approximation space. If X is an exact set of U, then*:

(1) $D_{\mathcal{C}}(u,X)=1$ *if and only if* $u\in X$; *and*
(2) $D_{\mathcal{C}}(u,X)=0$ *if and only if* $u\notin X$.

Corollary 4.1. *If both* (1) *and* (2) *hold in Proposition* 4.2, *then X is an exact set*.

Proof. Obviously, $\mathcal{C}_*(X)\subseteq\mathcal{C}^*(X)$. We need only prove $\mathcal{C}^*(X)\subseteq\mathcal{C}_*(X)$.

For $\forall a\in\mathcal{C}^*(X)$, $D_{\mathcal{C}}(a,X)\neq 0$ by Proposition 4.1. And, either $a\in X$ or $a\notin X$ holds. At the same time, both (1) and (2) hold in Proposition 4.2. Therefore $D_{\mathcal{C}}(a,X)=1$. If we assume $a\notin\mathcal{C}_*(X)$, then $D_{\mathcal{C}}(a,X)\neq 1$, which is a contradiction. Thus $\mathcal{C}^*(X)\subseteq\mathcal{C}_*(X)$.
The proof is completed. □

So we can obtain the following result from Proposition 4.2 and Corollary 4.1.

Corollary 4.2. *Let $S=(U,\mathcal{C})$ be a covering approximation space. X is an exact set of U if and only if the following are all true*:

(1) $D_{\mathcal{C}}(u,X)=1\Leftrightarrow u\in X$; *and*
(2) $D_{\mathcal{C}}(u,X)=0\Leftrightarrow u\notin X$.

Proposition 4.3. *Let $S = (U, \mathcal{C})$ be a covering approximation space. The following properties always hold*:

(1) $v_l(\widetilde{D_U^{\mathcal{C}}}) = 0$;
(2) $v_l(\widetilde{D_\phi^{\mathcal{C}}}) = 0$;
(3) $v_q(\widetilde{D_U^{\mathcal{C}}}) = 0$; *and*
(4) $v_q(\widetilde{D_\phi^{\mathcal{C}}}) = 0$;

Proof. Without loss of generality, we may only prove (1).

For any $u \in U$, we have

$$\widetilde{D_U^{\mathcal{C}}}(u) = \frac{|(\cap Md(u)) \cap U|}{|\cap Md(u)|} = \frac{|\cap Md(u)|}{|\cap Md(u)|} = 1.$$

Therefore, $\widetilde{D_U^{\mathcal{C}}} \cap (\sim \widetilde{D_U^{\mathcal{C}}})(u) = 0$, where $(\sim \widetilde{D_U^{\mathcal{C}}})$ denotes the complement of the fuzzy set $\widetilde{D_U^{\mathcal{C}}}$. Thus, $v_l(\widetilde{D_U^{\mathcal{C}}}) = (2/n) \cdot d(\widetilde{D_U^{\mathcal{C}}}, N(\widetilde{D_U^{\mathcal{C}}})) = (2/n) \cdot \sum |\widetilde{D_U^{\mathcal{C}}}(u) - N(\widetilde{D_U^{\mathcal{C}}})(u)| = (2/n) \cdot \sum \widetilde{D_U^{\mathcal{C}}} \cap (\sim \widetilde{D_U^{\mathcal{C}}})(u) = 0$.

The proof is completed. □

Proposition 4.4. *For any two sets X and Y in a covering approximation space $S = (U, \mathcal{C})$. If $X \subseteq Y$, then $D_{\mathcal{C}}(u, X) \subseteq D_{\mathcal{C}}(u, Y)$ for any $u \in U$.*

Proof. Obviously, for any $u \in U$, $X \subseteq Y$ implies $|(\cap Md(u)) \cap X| \leqslant |(\cap Md(u)) \cap Y|$. So we have

$$\frac{|(\cap Md(u)) \cap X|}{|\cap Md(u)|} \leqslant \frac{|(\cap Md(u)) \cap Y|}{|\cap Md(u)|}.$$

That means $D_{\mathcal{C}}(u, X) \subseteq D_{\mathcal{C}}(u, Y)$.

The proposition is proved. □

Proposition 4.5. *For any two sets X and Y in a covering approximation space $S = (U, \mathcal{C})$, the following hold*:

(1) $\widetilde{D_{X\cup Y}^{\mathcal{C}}} \supseteq \widetilde{D_X^{\mathcal{C}}} \cup \widetilde{D_Y^{\mathcal{C}}}$; *and*
(2) $\widetilde{D_{X\cup Y}^{\mathcal{C}}} = \widetilde{D_X^{\mathcal{C}}} \cup \widetilde{D_Y^{\mathcal{C}}}$, *if either $X \subseteq Y$ or $Y \subseteq X$.*

Proof. (1) For any $u \in U$, we have

$$\begin{aligned}
\widetilde{D_{X\cup Y}^{\mathcal{C}}}(u) &= \frac{|(\cap Md(u)) \cap (X \cup Y)|}{|\cap Md(u)|} \\
&= \frac{|((\cap Md(u)) \cap X) \cup ((\cap Md(u)) \cap Y)|}{|\cap Md(u)|} \\
&\geqslant \frac{\max\{|((\cap Md(u)) \cap X)|, |((\cap Md(u)) \cap Y)|\}}{|\cap Md(u)|} \\
&= \max\{\frac{|((\cap Md(u)) \cap X)|}{|\cap Md(u)|}, \frac{|((\cap Md(u)) \cap Y)|}{|\cap Md(u)|}\} \\
&= \max\{\widetilde{D_X^{\mathcal{C}}}(u), \widetilde{D_Y^{\mathcal{C}}}(u)\} \\
&= \widetilde{D_X^{\mathcal{C}} \cup \widetilde{D_Y^{\mathcal{C}}}}(u).
\end{aligned}$$

Thus the proof is completed.

(2) The proof is trivial. □

In a similar way, the following proposition can be obtained.

Table 2

Cars	P	M	S	X	d
x_1	High	High	Full	Low	Good
x_2	Low	*	Full	Low	Good
x_3	*	*	Compact	High	Poor
x_4	High	*	Full	High	Good
x_5	*	*	Full	High	Excel
x_6	Low	High	Full	*	Good

Proposition 4.6. *For any two sets X and Y in a covering approximation space* $S = (U, \mathcal{C})$, *the following hold*:

(1) $\widetilde{D^{\mathcal{C}}_{X\cap Y}} \subseteq \widetilde{D^{\mathcal{C}}_X} \cap \widetilde{D^{\mathcal{C}}_Y}$; *and*
(2) $\widetilde{D^{\mathcal{C}}_{X\cap Y}} = \widetilde{D^{\mathcal{C}}_X} \cap \widetilde{D^{\mathcal{C}}_Y}$, *if either* $X \subseteq Y$ *or* $Y \subseteq X$.

Proposition 4.7. *For any set X in a covering approximation space* $S = (U, \mathcal{C})$, $D_{\mathcal{C}}(u, X) + D_{\mathcal{C}}(u, \sim X) = 1$ *holds*.

Proof. For any $u \in U$, we have

$$D_{\mathcal{C}}(u, X) + D_{\mathcal{C}}(u, \sim X) = \frac{|(\cap Md(u)) \cap X| + |(\cap Md(u)) \cap (\sim X)|}{|\cap Md(u)|} = \frac{|\cap Md(u)|}{|\cap Md(u)|} = 1.$$

Hence, it is proved. □

Proposition 4.8. *Let* $S = (U, \mathcal{C})$ *be a covering approximation space. If* $\{X_1, X_2, \ldots, X_n\}$ *is a partition of U, then for any* $x \in U$,

$$D_{\mathcal{C}}(u, X_1) + D_{\mathcal{C}}(u, X_2) + \cdots + D_{\mathcal{C}}(u, X_n) = 1.$$

Proof. For any $x \in U$,

$$\begin{aligned} &D_{\mathcal{C}}(u, X_1) + D_{\mathcal{C}}(u, X_2) + \cdots + D_{\mathcal{C}}(u, X_n) \\ &= \frac{|(\cap Md(u)) \cap X_1|}{|\cap Md(u)|} + \frac{|(\cap Md(u)) \cap X_2|}{|\cap Md(u)|} + \cdots + \frac{|(\cap Md(u)) \cap X_n|}{|\cap Md(u)|} \\ &= \frac{|(\cap Md(u)) \cap U|}{|\cap Md(u)|} = 1. \end{aligned}$$

Thus the proposition is proved. □

5. Example

Example 5.1. Table 2 describes an incomplete decision table about cars [10], where $U = \{x_i | i = 1, 2, \ldots, 6\}$ is set of cars, and $A = \{P, M, S, X, d\}$ is set of attributes. Moreover, P, M, S, X stand for Price, Mileage, Size and Max-Speed, respectively, and d is the decision attribute.

For $B \subseteq A$, a similarity relation $R^{\simeq}_B$ can be defined on U [10]:

$$R^{\simeq}_B = \{(x, y) \in U \times U | f_l(x) = f_l(y) \text{ or } f_l(x) = * \text{ or } * = f_l(y) \forall a_l \in B\},$$

where $f_l(x)$ is the value of a_l on $x \in U$, and "*" indicates unknown values.

Table 3

Cars	{Good}	{Poor}	{Excel}
x_1	1	0	0
x_2	1	0	0
x_3	0	1	0
x_4	2/3	0	1/3
x_5	2/3	0	1/3
x_6	4/5	0	1/5

Moreover, for $B = \{S, X\}$, we write

$$[x]_B^{\approx} = \{y \in U \mid (x, y) \in R_B^{\approx}\}.$$

By Table 2, it is easy to verify that

$$\begin{aligned}
\mathcal{C}_1 &= [x_1]_B^{\approx} = [x_2]_B^{\approx} = \{x_1, x_2, x_6\};\\
\mathcal{C}_2 &= [x_3]_B^{\approx} = \{x_3\};\\
\mathcal{C}_3 &= [x_4]_B^{\approx} = [x_5]_B^{\approx} = \{x_4, x_5, x_6\};\\
\mathcal{C}_4 &= [x_6]_B^{\approx} = \{x_1, x_2, x_4, x_5, x_6\}.
\end{aligned}$$

So $\{\mathcal{C}_i\}(i = 1, 2, 3, 4)$ is a covering of U, denoted by $\mathcal{C}$. And we can see that

$$Md(x_1) = Md(x_2) = \mathcal{C}_1, \quad Md(x_3) = \mathcal{C}_2,$$

$$Md(x_4) = Md(x_5) = \mathcal{C}_3, \quad Md(x_6) = \mathcal{C}_4.$$

Let

$$\begin{aligned}
\{\text{Good}\} &= \{x \in U \mid f_d(x) = \text{Good}\} = \{x_1, x_2, x_4, x_6\};\\
\{\text{Poor}\} &= \{x \in U \mid f_d(x) = \text{Poor}\} = \{x_3\};\\
\{\text{Excel}\} &= \{x \in U \mid f_d(x) = \text{Excel}\} = \{x_5\}.
\end{aligned}$$

Then sets {Good}, {Poor}, {Excel} constitute a partition of U. Moreover, {Good} means cars with "Good" property.

Hence, we can easily calculate that $\mathcal{C}_*(\{\text{Good}\}) = \{x_1, x_2\}$ and $\mathcal{C}^*(\{\text{Good}\}) = \{x_1, x_2, x_4, x_5, x_6\}$. Furthermore, for any $u \in U$, the degree of rough membership in X with respect to the covering can be directly obtained, which is as follows:

$$\begin{aligned}
&D_{\mathcal{C}}(x_1, \{\text{Good}\}) = 1; \quad D_{\mathcal{C}}(x_2, \{\text{Good}\}) = 1;\\
&D_{\mathcal{C}}(x_3, \{\text{Good}\}) = 0; \quad D_{\mathcal{C}}(x_4, \{\text{Good}\}) = \tfrac{2}{3};\\
&D_{\mathcal{C}}(x_5, \{\text{Good}\}) = \tfrac{2}{3}; \quad D_{\mathcal{C}}(x_6, \{\text{Good}\}) = \tfrac{4}{5}.
\end{aligned}$$

Similarly, all degree of rough membership in different decisions with respect to the covering $\mathcal{C}$ induced by $B = \{S, X\}$ are depicted in Table 3.

It is found that the degree of the third car belonging to {Good} is 0, which means that it is not a "Good" one with respect to $B = \{S, X\}$, although the other two attributes are unknown. The membership degree of the first or the second in "Good" decision is 1. The fourth, the fifth and the sixth approximately belong to "Good" cars because the membership degrees are $\frac{2}{3}$, $\frac{2}{3}$, $\frac{4}{5}$, respectively.

Every car belongs to {Good} or {Poor} or {Excel}, because $U = \{\text{Good}\} \cup \{\text{Poor}\} \cup \{\text{Excel}\}$. However the membership degrees of the car in different decisions may be different. So we can decide to which category a car belongs according to the memberships degrees.

On the other hand, the linear and the quadratic indices of fuzzy set of the membership degree are

$$v_l(\widetilde{D^{\mathcal{C}}_{\{G\}}}) = 0.2889; \quad v_q(\widetilde{D^{\mathcal{C}}_{\{G\}}}) = 0.4181;$$

$$v_l(\widetilde{D^{\mathcal{C}}_{\{P\}}}) = 0; \quad v_q(\widetilde{D^{\mathcal{C}}_{\{G\}}}) = 0;$$

$$v_l(\widetilde{D^{\mathcal{C}}_{\{G\}}}) = 0.2889; \quad v_q(\widetilde{D^{\mathcal{C}}_{\{G\}}}) = 0.4181,$$

where {G}, {P}, {E} is {Good}, {Poor}, {Excel}, respectively.

So, we can see that v_l, v_q may character the fuzziness of corresponding degree of rough membership to some extent.

In the above example, given a car, according to the conditional attributes, we can make a decision from the degree of rough membership, because we can decide to which category a car belongs by the degree of rough membership.

6. Conclusions

It is well-known that rough set theory has been regarded as a generalization of classical set theory in one way. Furthermore, this is an important mathematical tool to deal with uncertainty. As a natural need, it is a fruitful way to extend classical rough sets to generalized rough sets induced by a covering. In this paper, new lower and upper approximations are proposed in generalized rough set induced by a covering, and some important properties are obtained. Also, we define the concept of a rough membership function in covering approximation spaces. It is a generalization of classical rough membership function of Pawlak rough sets. The rough membership function can be used to analyze which decision should be made according to a conditional attribute in decision table.

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